Generalized Ricci flow: Local existence and uniqueness

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Outline

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1. Introduction

- In 1982, R. Hamilton introduced the Ricci flow
  \[
  \frac{\partial g_{ij}}{\partial t} = -2R_{ij}
  \]
  to construct canonical metrics for some manifolds.

- Hamilton, Yau, Perelman and other mathematicians developed many tools and techniques to study the Ricci flow.

- Recently, based on Perelman’s breakthrough, Cao and Zhu offered a complete proof of Poincare’s conjecture and Thurston’s geometrization conjecture.
In Perelman’s work, a key step is to introduce a functional

\[ W(g, f) = \int_{M^3} d^3x \sqrt{g} e^{-f} (R + |\nabla f|^2). \]

The corresponding gradient flow:

\[
\begin{align*}
\dot{g}_{ij} &= -2(R_{ij} + \nabla_i \nabla_j f), \\
\dot{f} &= -(R + \Delta f).
\end{align*}
\]

In this way, we express the Ricci flow as a gradient flow.
Dynamics of a gradient flow is much easier to handle.

- The functional generating the flow is monotone along the orbit of the flow automatically.

- If the flow exists for all time, then it shall flow to a critical point which lead to the existence of a canonical metric.

- If the flow does not exist for all time, the generating functional helps very much in the analysis of singularities.
For a three-manifold $M^3$, J.Gegenberg et al. present a action:

$$S = \int_M d^3 x \sqrt{g} e^{-f} (\chi + R + |\nabla f|^2) - \frac{\epsilon H}{2} e^{-f} H \wedge *H$$

$$- \epsilon F e^{-f} F \wedge *F + \frac{e}{2} A \wedge F.$$

$$F = dA, \ H = dB.$$

- J. Gegenberg and G. Kunstatter, Using 3D stringy gravity to understand the Thurston conjecture, 2003.
The corresponding gradient flow:

\[
\begin{align*}
\frac{\partial g_{ij}}{\partial t} &= -2[R_{ij} + 2\nabla_i \nabla_j \phi - \frac{\epsilon_H}{4} H_{ikl} H_{jl} - \epsilon_F F_i^k F_j^k], \\
\frac{\partial B_{ij}}{\partial t} &= \epsilon_H e^{2\phi} \nabla_k (e^{-2\phi} H_k^{ij}), \\
\frac{\partial A_i}{\partial t} &= -\epsilon_F e^{2\phi} \nabla_k (e^{-2\phi} F_i^k) + \frac{e}{2} e^{2\phi} \eta_{ik} F_{kl}, \\
\frac{\partial \phi}{\partial t} &= -\chi + R(g) + 4 \nabla^2 \phi - 4 |\nabla \phi|^2 - \frac{\epsilon_H}{12} H^2 - \frac{\epsilon_F}{2} F^2. 
\end{align*}
\]

(1)
J. Gegenberg et al. found that Thurston’s eight geometries appear as critical points of the above functional.

Under the condition $e = 0$, they show that there are no other critical points.

Basically critical points of the above functional are eight geometries of Thurston.
We consider a flow for a similar functional for a four-dimension manifold:

\[ S = \int_M d^4x \sqrt{g} e^{-f} (\chi + R + |\nabla f|^2) - \frac{\epsilon H}{2} e^{-f} H \wedge *H - \epsilon_F e^{-f} F \wedge *F + \frac{e}{2} F \wedge F. \]

The generalization to four-manifolds is probably more interesting. It may offer a systematic way to study four-manifolds.
The corresponding gradient flow:

\[
\begin{align*}
\frac{\partial g_{ij}}{\partial t} &= -2[R_{ij} + 2\nabla_i \nabla_j \phi - \frac{\epsilon_H}{4} H_{ikl} H_{j}^{kl} - \epsilon_F F_{ik} F_{jk}], \\
\frac{\partial B_{ij}}{\partial t} &= \epsilon_H e^{2\phi} \nabla_k (e^{-2\phi} H_{ij}^k), \\
\frac{\partial A_i}{\partial t} &= -\epsilon_F e^{2\phi} \nabla_k (e^{-2\phi} F_{ik}^k) + e e^{2\phi} \eta_i^{kjl} \nabla_k (F_{jl}), \\
\frac{\partial \phi}{\partial t} &= -\chi + R(g) + 4 \nabla^2 \phi - 4 |\nabla \phi|^2 - \frac{\epsilon_H}{12} H^2 - \frac{\epsilon_F}{2} F^2. 
\end{align*}
\]
♠ We prove that the system of PDEs (2) are strictly and uniformly parabolic in some sense.

♠ Based on this, we show that the generalized Ricci flow defined on a $n$-dimensional compact Riemannian manifold admits a unique short-time smooth solution.

♠ Moreover, we also derive the evolution equations for the curvatures, which play an important role in our future study.
2. Main results

Christoffel symbols

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\}, \]

The Riemannian curvature tensors

\[ R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip} \Gamma^p_{jl} - \Gamma^k_{jp} \Gamma^p_{il}, \quad R_{ijkl} = g_{kp} R^p_{ijkl} \]

The Ricci tensor

\[ R_{ik} = g^{jl} R_{ijkl} \]

The scalar curvature

\[ R = g^{ij} R_{ij} \]
For each field we shall consider the gauge equivalent classes of fields.

- Two metrics $g_1, g_2$ are in the same equivalent class if and only if they are differ by a diffeomorphism, i.e., there exists a diffeomorphism $f : M \rightarrow M$ such that $g_2 = f^*g_1$.

- Two gauge fields $A_1$ and $A_2$ are equivalent if and only if there exists a function $\alpha$ on $M$ such that $A_2 = A_1 + d\alpha$.

- Two $B$-fields $B_1$ and $B_2$ are equivalent if and only if there exists an one-form $\beta$ on $M$ such that $B_2 = B_1 + d\beta$. 
Theorem 1: (Local existences and uniqueness) Let \((M, g_{ij}(x))\) be a three-dimensional compact Riemannian manifold. Then there exists a constant \(T > 0\) such that the system (1) of the evolution equations has a unique smooth solution on \(M \times [0, T)\) for every initial fields.

Theorem 2: (Local existences and uniqueness) Let \((M, g_{ij}(x))\) be a four-dimensional compact Riemannian manifold. Then there exists a constant \(T > 0\) such that the system (2) of the evolution equations has a unique smooth solution on \(M \times [0, T)\) for every initial fields.
3. Method of proof

Lemma 1. For each gauge equivalent class of a gauge field $A$, there exists an $A'$ such that $d(\ast A') = 0$.

The lemma can be proved by the Hodge decomposition.

Proof. For each one-form $A$, by the Hodge decomposition, there exists an one-form $A_0$, a function $\alpha$ and a two-form $\beta$ such that

$$A = A_0 + d\alpha + d^\ast \beta,$$

$$dA_0 = 0, d^\ast A_0 = 0.$$

Let $A' = A - d\alpha$. $A'$ is in the same gauge equivalent class of $A$. Since $d(\ast A_0) = 0, d(\ast d^\ast \beta) = 0$, then we have $d(\ast A') = 0$. $\square$
Lemma 2. The differential operator of the right hand of (2) with respect to the gauge equivalent class of a gauge field $\mathbf{A}$ is uniformly elliptic.

Proof. Let $\mathbf{A} = A_i dx^i$ be a gauge field. From Lemma 3.1 we can choose an $\mathbf{A}'$ in the gauge equivalent class of $\mathbf{A}$ such that $d(\ast \mathbf{A}') = 0$. Then $dd^* \mathbf{A}' = 0$. We still denote $\mathbf{A}'$ as $\mathbf{A}$.
\[
\frac{\partial A_i}{\partial t} = -\epsilon_F e^{2\phi} \nabla_k (e^{-2\phi} F^k_i) + e e^{2\phi} \eta^{kjl} \nabla_k (F_{jl}) \\
= -\epsilon_F \nabla_k (g^{kl} F_{il}) - \epsilon_F e^{2\phi} F^k_i \nabla_k (e^{-2\phi}) + e e^{2\phi} \eta^{kjl} \nabla_k \left( \frac{\partial A_l}{\partial x^j} - \frac{\partial A_j}{\partial x^l} \right) \\
= \epsilon_F (d^* F)_i + 2\epsilon_F F^k_i \frac{\partial \phi}{\partial x^k} + 2e e^{2\phi} \eta^{kjl} \frac{\partial^2 A_l}{\partial x^k \partial x^j} \\
= \epsilon_F (d^* dA)_i + \epsilon_F (dd^* A)_i + 2\epsilon_F F^k_i \frac{\partial \phi}{\partial x^k} \\
= \epsilon_F \Delta A_i + 2\epsilon_F F^k_i \frac{\partial \phi}{\partial x^k}.
\]

The right hand side is clearly elliptic at point \(x\). If we apply a diffeomorphism to the metric it won’t change the positivity property of the second order operator of the right hand side. \(\Box\)
Lemma 3. For each gauge equivalent class of a B-field $B$, i.e., a two-form $B$ on $M$, there exists a $B'$ such that $d(\ast B') = 0$.

Proof. Again we use the Hodge decomposition. For a two-form $B$, there exist a one-form $\alpha$, a two-form $B_0$ and a three-form $\beta$ such that

$$B = B_0 + d\alpha + d^*\beta,$$

$$dB_0 = 0, d^*B_0 = 0.$$ 

Let $B' = B - d\alpha$. $B'$ is in the same gauge equivalent class of $B$, then we have $d(\ast B') = 0$. □
Lemma 4. The differential operator of the right hand side of (2) with respect to the gauge equivalent class of a $B$-field $B$ is uniformly elliptic.

Proof. Let us consider the equation for $B$-field. Without loss of generality, we assume $d(\ast B) = 0$. Then $dd^*B = 0$. We have
\[
\frac{\partial B_{ij}}{\partial t} = \epsilon_H e^{2\phi} \nabla_k (e^{-2\phi} H^k_{ij})
\]
\[
= \epsilon_H \nabla_k (H_{lij} g^{kl}) + \epsilon_H e^{2\phi} H^k_{ij} \nabla_k (e^{-2\phi})
\]
\[
= -\epsilon_H (d^* dB)_{ij} - \epsilon_H (dd^* B)_{ij} - 2\epsilon_H H^k_{ij} \frac{\partial \phi}{\partial x^k}
\]
\[
= -\epsilon_H \Delta B_{ij} - 2\epsilon_H H^k_{ij} \frac{\partial \phi}{\partial x^k}
\]

The right hand side is clearly elliptic at the point \( x \). If we apply a diffeomorphism to the metric it does not change the positivity property of the second order operator of the right hand side. \( \square \)
Suppose \((\hat{g}_{ij}(x, t), \hat{\phi}(x, t))\) is a solution of the equations (2), and \(\varphi_t : M \rightarrow M\) is a family of diffeomorphisms of \(M\). Let

\[
g_{ij}(x, t) = \varphi_t^* \hat{g}_{ij}(x, t), \quad \phi(x, t) = \varphi_t^* \hat{\phi}(x, t),
\]

where \(\varphi_t^*\) is the pull-back operator of \(\varphi_t\). We now want to find the evolution equations for the metric \(g_{ij}(x, t)\) and \(\phi(x, t)\).

Denote

\[
y(x, t) = \varphi_t(x) = \{y^1(x, t), y^2(x, t), \ldots, y^n(x, t)\}
\]

in local coordinates.
If we define \( y(x, t) = \varphi_t(x) \) by the equations

\[
\begin{align*}
\frac{\partial y^\alpha}{\partial t} &= \frac{\partial y^\alpha}{\partial x^k} (2 \nabla_j \phi g^{kj} + g^{jl} (\Gamma^k_{jl} - \tilde{\Gamma}^k_{jl})), \\
y^\alpha(x, 0) &= x^\alpha
\end{align*}
\]

and \( V_i = g_{ik} g^{jl} (\Gamma^k_{jl} - \tilde{\Gamma}^k_{jl}) \), then we have

\[
\begin{align*}
\frac{\partial g_{ij}(x, t)}{\partial t} &= g^{kl} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\epsilon_H}{2} H_{ikl} H^k_{jl} + 2 \epsilon_F F^k_i F_{jk}, \\
\frac{\partial \phi}{\partial t} &= -\frac{1}{2} g^{ij} g_{kl} \frac{\partial^2 \phi}{\partial x^k \partial x^l} + 2 g^{kl} \frac{\partial^2 \phi}{\partial x^k \partial x^l} \\
&\quad - 2 g^{kl} \frac{\partial \phi}{\partial x^k} \frac{\partial \phi}{\partial x^l} - g^{jl} \tilde{\Gamma}^k_{jl} \frac{\partial \phi}{\partial x^k} - \chi - \frac{\epsilon_H}{12} H^2 - \frac{\epsilon_F}{2} F^2.
\end{align*}
\]

(6)
Let

\[ u_1 = g_{11}, u_2 = g_{12}, u_3 = g_{13}, u_4 = g_{14}, u_5 = g_{22}, u_6 = g_{23}, \]
\[ u_7 = g_{24}, u_8 = g_{33}, u_9 = g_{34}, u_{10} = g_{44}, u_{11} = \phi. \]

The above equations can be rewritten as the following form

\[
\frac{\partial u_i}{\partial t} = \sum_{jkl} a_{ikjl} \frac{\partial^2 u_j}{\partial x^k \partial x^l} + (\text{lower order terms}).
\]

So for arbitrary \( \xi \in \mathbb{R}^{4 \times 11}, \) it is easily verified that the eigenvalues of the above quadratic forms \( \sum_{ijkl} a_{ikjl} \xi_i \xi_j \) read

\[
\frac{3}{2} \pm \frac{\sqrt{2}}{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1.
\]

It means that the uniformly parabolic condition of the system (6) holds.
Lemma The differential operator of the right hand side of (6) with respect to the metric $g$ and the dilaton $\phi$ is uniformly elliptic.

Summarizing the above discussions and by virtue of the standard theory of partial differential equations, we can obtain the existence and uniqueness of solution.
4. Evolution of curvatures

**Theorem 1** Under the gradient flow (2), the curvature tensor satisfies the evolution equation

\[
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})
\]

\[
- g^{pq} (R_{pqkl} R_{ql} + R_{ipkl} R_{qj} + R_{ijpl} R_{qk} + R_{ijkp} R_{ql})
\]

\[
- 2g^{pq} (R_{pqkl} \nabla_q \nabla_i \phi + R_{ipkl} \nabla_q \nabla_j \phi + R_{ijpl} \nabla_q \nabla_k \phi + R_{ijkp} \nabla_q \nabla_l \phi)
\]

\[
- 2g^{pq} \nabla_q \phi \nabla_p R_{ijkl}
\]

\[
+ \epsilon H \left[ \nabla_i \nabla_l (H_{kpq} H_j^{pq}) - \nabla_i \nabla_k (H_{jpq} H_l^{pq}) \right]
\]

\[
- \nabla_j \nabla_l (H_{kpq} H_i^{pq}) + \nabla_j \nabla_k (H_{ipq} H_l^{pq})
\]

\[
+ \epsilon H \frac{1}{4} g^{mn} (H_{kpq} H_m^{pq} R_{ijnl} + H_{mpq} H_l^{pq} R_{ijkn})
\]

\[
+ \epsilon F \left[ \nabla_i \nabla_l (F_k^p F_{jp}) - \nabla_i \nabla_k (F_j^p F_{lp}) - \nabla_j \nabla_l (F_k^p F_{ip}) + \nabla_j \nabla_k (F_i^p F_{lp}) \right]
\]

\[
+ \epsilon F g^{mn} (F_k^p F_m^p R_{ijnl} + F_m^p F_l^p R_{ijkn}),
\]

where \( B_{ijkl} = g^{pr} g^{qs} R_{pijq} R_{rksl} \) and \( \Delta \) is the Laplacian with respect to the evolving metric.
Theorem 2 The Ricci curvature satisfies the following evolution equation

$$\frac{\partial}{\partial t} R_{ik} = \triangle R_{ik} + 2g^{pr} g^{qs} R_{pikq} R_{rs} - 2g^{pq} R_{pi} R_{qk}$$

$$- 2g^{pq} (R_{pk} \nabla_q \nabla_i \phi + R_{ip} \nabla_q \nabla_k \phi) - 2g^{pq} \nabla_q \phi \nabla_p R_{ik}$$

$$+ \frac{\epsilon_H}{4} g^{jl} [\nabla_i \nabla_l (H_{kpq} H^j_{pq}) - \nabla_i \nabla_k (H_{jpq} H^l_{pq})]$$

$$- \nabla_j \nabla_l (H_{kpq} H^i_{pq}) + \nabla_j \nabla_k (H_{ipq} H^l_{pq})$$

$$+ \frac{\epsilon_H}{4} g^{mn} (H_{kpq} H^p_{m} R_{in} - g^{jl} H_{mpq} H^l_{pq} R_{ijnk})$$

$$+ \epsilon_F g^{jl} [\nabla_i \nabla_l (F^p_{k} F^j_{lp}) - \nabla_i \nabla_k (F^p_{j} F^l_{lp})]$$

$$- \nabla_j \nabla_l (F^p_{k} F^i_{lp}) + \nabla_j \nabla_k (F^p_{i} F^l_{lp})]$$

$$+ \epsilon_F g^{mn} (F^p_{k} F^p_{mp} R_{in} - g^{jl} F^p_{m} F^l_{lp} R_{ijnk}).$$
Theorem 3 The scalar curvature satisfies the following evolution equation

$$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2 - 2g^{pq}\nabla_q \phi \nabla_p R$$

$$+ \frac{\epsilon_H}{2} g^{jl} g^{ik} \left[ \nabla_i \nabla_l (H_{kpq} H_{j}^{pq}) - \nabla_i \nabla_k (H_{jpq} H_{l}^{pq}) \right]$$

$$+ 2\epsilon_F g^{jl} g^{ik} \left[\nabla_i \nabla_l (F_{kp}^p F_{jp}) - \nabla_i \nabla_k (F_{jp}^p F_{lp}) \right]$$

$$- g^{ip} R_{ik} \left( \frac{\epsilon_H}{2} H_{pmn} H^{kmn} + 2\epsilon_F F_{pm} F^{km} \right).$$
Thank you!