$(\alpha, \beta)$-METRICS WITH ISOTROPIC S-CURVATURE

Cheng Xinyue

School of Mathematics and Physics
Chongqing Institute of Technology
Yangjiaping, Chongqing 400050
P. R. of China
chengxy@cqit.edu.cn
1 Introduction

A (positive definite) **Finsler metric** on a manifold $M$ is a $C^\infty$ scalar function $F = F(x, y)$ on $TM\backslash\{0\}$:

$$
F(x, y) > 0, \quad y \neq 0
$$

$$
F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0
$$

$$(g_{ij}(x, y)) \text{ positive definite}
$$

where $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^iy^j}(x, y)$.

**Inner product:** $g_y : T_xM \times T_xM \to \mathbb{R}$:

$$
g_y(u, v) = g_{ij}(x, y)u^iu^j,
$$

where $u = u^i \frac{\partial}{\partial x^i}|_x, v = v^j \frac{\partial}{\partial x^j}|_x$. 

Geodesic Equation:

\[ \frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0, \]

where

\[ G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}. \]
Riemann Curvature $R_y : T_x M \to T_x M,$

$$R_y(u) := R^i_k u^k \frac{\partial}{\partial x^i}|_x, \quad u = u^i \frac{\partial}{\partial x^i}|_x,$$

$$R^i_k := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

Flag Curvature:

$$K(P, y) := \frac{g_y(R_y(u), u)}{g_y(y, y) g_y(u, u) - [g_y(u, y)]^2},$$

where $P := \text{span}\{y, u\} \subset T_x M.$

F is of scalar flag curvature: $K(P, y) = K(y)$  
(independent of $P$)

F is of constant flag curvature: $K(P, y) = \text{constant}$
To characterize Riemannian metrics among Finsler metrics, we introduce the quantity

$$
\tau(x, y) := \ln \left[ \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)} \right],
$$

where

$$
\sigma_F(x) := \frac{Vol(B^n)}{Vol \{ (y^i) \in \mathbb{R}^n | F(x, y) < 1 \}}
$$

characterizes the Busemann-Hausdorff volume form. $\tau$ is called the distortion.

*F is Riemannian if and only if $\tau = constant([Shen-2]).*
$C \rightarrow I: I_i = g^{jk} C_{ijk} = \tau y^i \quad \leftarrow \quad \tau$

$L: L_{ijk} := C_{ijk|m} y^m \rightarrow J: J_i := g^{jk} L_{ijk} = I_{i|m} y^m \quad S := \tau_{i|m} y^m$

where $(g^{ij}(x, y)) := \left( \frac{1}{2} \left[ F^2 \right]_{y^i y^j}(x, y) \right)^{-1}$.

$L: \text{Landsberg curvature} \quad J: \text{mean Landsberg curvature} \quad S: \text{S-curvature (Z. Shen, 1997 [Shen-1])}$
• We say S-curvature is **isotropic** if there exists a scalar function $c(x)$ on $M$ such that

$$S(x, y) = (n + 1)c(x)F(x, y),$$

If $c(x) = constant$, we say that $F$ has constant S-curvature.

• S-curvature $S(x,y)$ is the rate of change of $\tau$ along geodesics and measures the averages rate of change of $(T_x M, F_x)$ in the direction $y \in T_x M$. 
• Let $G^i(x,y)$ be the geodesic coefficients of $F$. By the definition of S-curvature, we have
\[
S = \frac{\partial G^m}{\partial y^m}(x,y) - y^m \frac{\partial}{\partial x^m}(\ln \sigma_F(x)).
\]

• ([Shen-2][Shen-3]) For any Berwald metric, the S-curvature vanishes, $S = 0$.  

2 Why do we study S-curvature?

S-curvature and flag curvature have many affinities ([Cheng-Mo-Shen][Mo])

\[ J_{k;m} y^m + I_m R^m_k = -\frac{1}{3} \{2R^m_{k,m} + R^m_{m,k}\} \]

\[ S_{k;m} y^m - S_{:k} = -\frac{1}{3} \{2R^m_{k,m} + R^m_{m,k}\} : \]

If \( F \) is of scalar curvature with flag curvature \( K = K(x, y) \):

\[ R^i_k = K F^2 h^i_k, \]

where \( h^i_k := g^{ij} h_{jk} \) and \( h_{jk} := g_{jk} - F^{-2} g_{js} y^s g_{kt} y^t \), then we have

\[ S_{k;l} y^l - S_{:k} = -\frac{n + 1}{3} K_{,k} F^2. \]
S-curvature has important influence on the geometric structure of Finsler metrics

- Finsler metrics of positive flag curvature
  - [Kim-Yim] Finsler manifold \((M, F)\):
    1. reversible \((F(-y) = F(y))\);
    2. \(S = 0\),
    3. flag curvature \(K = \text{constant} > 0\).
    \(\implies F\) is a Riemannian.
• Finsler metrics of negative flag curvature
  ♣ [Akbar-Zadeh 1988] Finsler manifold $(M, F)$:
  (1) closed;
  (2) flag curvature $K=$constant;
  (3) $K < 0$.
  $\implies F$ is Riemannian.

♣ [Shen-6] Finsler manifold $(M, F)$:
  (1) closed;
  (2) $S = (n + 1)cF, \quad c = constant$;
  (3) flag curvature $K(P, y) < 0$.
  $\implies F$ is Riemannian.
• If Randers metric $F = \alpha + \beta$ ($\alpha$: Riemann metric; $\beta$: 1-form) is of constant flag curvature $\implies F$ is of constant S-curvature ([Bao-Robles-Shen])

More general, Many known Finsler metrics of constant/scalar flag curvature actually have constant/isotropic S-curvature ([Cheng-Mo-Shen][Shen-4]).

• (Z. Shen, 1997) the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing S-curvature
3 Why do we study \( (\alpha, \beta) \)-metrics?

Given a Riemannian metric \( \alpha = \sqrt{a_{ij}y^iy^j} \) and a 1-form \( \beta = b_iy^i \) on a manifold \( M \). Let

\[
F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha},
\]

where \( \phi(s) \) is a \( C^\infty \) positive function on \((-b_o, b_o)\). It is known that \( F = \alpha \phi(\beta/\alpha) \) is a Finsler metric for any \( \alpha \) and \( \beta \) with \( \|\beta_x\|_\alpha < b_o \) if and only if \( \phi \) satisfies the following condition:

\[
\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_o).
\]

Such metric is called an \( (\alpha, \beta) \)-metric.
Example 3.1 Some important \((\alpha, \beta)\)-metrics:

- **Randers metric:** \(F = \alpha + \beta; \; \phi = 1 + s\).
  More general,
  \[ F = \sqrt{\alpha^2 + k\beta^2 + \epsilon\beta}. \]

- **Matsumoto metric:** \(F = \frac{\alpha^2}{\alpha - \beta}; \; \phi = \frac{1}{1 - s}\).

- \(F = \alpha + \epsilon \beta + k\beta^2/\alpha; \; \phi = 1 + \epsilon s + ks^2\), where \(\epsilon\) and \(k \neq 0\) are constants.
  In particular,
  \[ F = \frac{(\alpha + \beta)^2}{\alpha}. \]
A. \((\alpha,\beta)\)-metrics are “computable”

B. The study for \((\alpha,\beta)\)-metrics can help us to understand better and deeply geometric properties of Finsler metrics in general case

C. \((\alpha,\beta)\)-metrics have important applications in physics and biology (ecology) ([Antonelli-Miron][Asanov2006])
D. Some important progress of the study on $(\alpha, \beta)$-metrics

(a) Randers metrics of constant/scalar flag curvature

♣ (Z. Shen, 2003) classified locally projectively flat Randers metrics with constant Ricci curvature

Remark: The solutions to the Hilbert’s Fourth Problem in the regular case are projectively flat Finsler metrics

♣ (Cheng-Mo-Shen, 2003) classified locally projectively flat Randers metrics with isotropic S-curvature

More general, we characterized projectively flat Finsler metrics with isotropic S-curvature ([Cheng-Shen2006(1)])

♣ (Bao-Robles-Shen, 2004) classified Randers metrics of constant flag curvature

♣ (Cheng-Shen, 2005) classified Randers metrics of scalar flag curvature with isotropic S-curvature (This class contains all Randers metrics of constant flag curvature)
(b) Projectively flat $(\alpha,\beta)$-metrics

Berwald’s metric (Berwald, 1929)

$$B = \frac{(\alpha + \beta)^2}{\alpha}, \quad y \in T_xB^n,$$

where $\alpha = \lambda\bar{\alpha}, \beta = \lambda\bar{\beta}$ and

$$\bar{\alpha} = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}}{1 - |x|^2}, \quad \bar{\beta} = \frac{\langle x, y \rangle}{1 - |x|^2}, \quad \lambda = \frac{1}{1 - |x|^2}.$$

◊ projectively flat

◊ $K = 0$
• (Shen-Zhao, 2005) determined completely the local structure of a projectively flat Finsler metric $F$ in the form 
$$F = \alpha + C_1 \beta + \frac{1}{2p} \beta^2 - \frac{1}{48p^2} \frac{\beta^4}{\alpha^3}$$
of constant flag curvature

• (Z. Shen, 2006) studied and characterized projectively flat $(\alpha, \beta)$-metrics in dimension $n \geq 3$

• (Shen-Yildirim, 2006) determined completely the local structure of a projectively flat Finsler metric $F$ in the form 
$$F = (\alpha + \beta)^2/\alpha$$
of constant flag curvature

• (Li-Shen, 2006) classified projectively flat $(\alpha, \beta)$-metrics with constant flag curvature in dimension $n \geq 3$:
on of the following holds
  • $\alpha$ is projectively flat and $\beta$ is parallel with respect to $\alpha$
  • $\phi = \sqrt{1 + ks^2 + \epsilon s}$, $k, \epsilon(\neq 0)$: constants; $K < 0$
  • $\phi = (\sqrt{1 + ks^2 + \epsilon s})^2/\sqrt{1 + ks^2}$, $k, \epsilon(\neq 0)$: constants; $K=0$
(c) $(\alpha, \beta)$-metrics of Landsberg type

F is called a **Berwald metric** if its geodesic coefficients

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k$$

are quadratic in $y \in T_x M$, or equivalently

$$[G^i]_{y^j y^k y^l} = 0.$$ 

**Riemannian metrics are Berwald metrics.**

F is called a **Landsberg metric** if its Landsberg curvature

$$L_{ijk} = 0.$$ 

**Fact:** Every Berwald metric is a Landsberg metric

**A Long Existing Open Problem:** Is there any Landsberg metric which is not Berwald metric?
• (Z. Shen, 2006) characterized the Landsberg curvature of \((\alpha, \beta)\)-metrics

a regular \((\alpha, \beta)\)-metric is Landsbergian if and only if it is Berwaldian

• (Li-Shen, 2006) characterized weakly Landsberg (i.e. \(J = 0\)) \((\alpha, \beta)\)-metrics and shown that there exist weakly Landsberg metrics which are not Landsberg metrics in dimension greater than two
4 \((\alpha, \beta)\)-metrics with isotropic S-curvature

Open Problems:

(1) Determine non-Randers \((\alpha, \beta)\)-metrics with vanishing S-curvature and constant flag curvature.

(2) Determine \((\alpha, \beta)\)-metrics of scalar flag curvature and isotropic S-curvature.
Let
\[ F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}. \]

We have the following formula for the spray coefficients \( G^i \) of \( F \):
\[
G^i = \bar{G}^i + \Theta \left\{ -2Q\alpha s_0 + r_{00} \right\} \frac{y^i}{\alpha} + \alpha Q s^i_0 + \Psi \left\{ -2\alpha s_0 + r_{00} \right\} b^i,
\]
where \( \bar{G}^i \) denote the spray coefficients of \( \alpha \) and
\[
Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta = \frac{Q - sQ'}{2\Delta}, \quad \Psi = \frac{Q'}{2\Delta},
\]
where \( \Delta := 1 + sQ + (b^2 - s^2)Q' \) and \( b = \|\beta_x\|_\alpha \).
Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where “$|$” denotes the covariant derivative with respect to $\alpha$.

Let

$$r_{ij} := \frac{1}{2} (b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2} (b_{i|j} - b_{j|i}), \quad s^i_j := a^{ih}s_{hj},$$
$$s_j := b^i s_{ij}, \quad r_j := b^i r_{ij}, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Recall:

$$S = \frac{\partial G^m}{\partial y^m}(x, y) - y^m \frac{\partial}{\partial x^m}(\ln \sigma_F(x)).$$
Proposition 4.1 ([Cheng-Shen2006(2)]) Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. Let $dV = dV_{BH}$ or $dV_{HT}$. Let

$$f(b) := \begin{cases} 
\frac{\int_0^\pi \sin^{n-2}(t)dt}{\int_0^\pi \sin^{n-2}(t)\phi(t)dt} & \text{if } dV = dV_{BH} \\
\frac{\int_0^\pi \sin^{n-2}(t)T(b \cos t)dt}{\int_0^\pi \sin^{n-2}(t)dt} & \text{if } dV = dV_{TH},
\end{cases}$$

where $T(s) := \phi(\phi - s\phi')^{n-2}[\phi - s\phi'] + (b^2 - s^2)\phi'']$. Then the volume form $dV$ is given by

$$dV = f(b)dV_{\alpha},$$

where $dV_{\alpha} = \sqrt{\det(a_{ij})}dx$ denotes the Riemannian volume form of $\alpha$.

**A useful technique in the proof:** take a local coordinate system at $x$ such that

$$\alpha = \sqrt{\sum (y^i)^2}, \quad \beta = by^1,$$

where $b = \|\beta_x\|_\alpha$. Then the volume form $dV_{\alpha} = \sigma_{\alpha}dx$ at $x$ is given by

$$\sigma_{\alpha} = \sqrt{\det(a_{ij})} = 1.$$
In order to evaluate the integrals
\[ \text{Vol}\{(y^i) \in \mathbb{R}^n | F(x,y^i \frac{\partial}{\partial x^i}) < 1\} = \int_{F(x,y)<1} dy = \int_{\alpha \phi(\beta/\alpha) < 1} dy \]
and
\[ \int_{F(x,y)<1} \det(g_{ij}) dy = \int_{\alpha \phi(\beta/\alpha) < 1} \det(g_{ij}) dy, \]
we take the following coordinate transformation, \( \psi : (s,u^a) \rightarrow (y^i) : \)
\[ y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^a = u^a, \]
where \( \bar{\alpha} = \sqrt{\sum_{a=2}^n (u^a)^2}. \) Then
\[ \alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}. \]
Thus
\[ F = \alpha \phi(\beta/\alpha) = \frac{b \phi(s)}{\sqrt{b^2 - s^2}} \bar{\alpha} \]
and the Jacobian of the transformation \( \psi : (s,u^a) \rightarrow (y^i) \) is given by
\[ \frac{b^2}{(b^2 - s^2)^{3/2}} \bar{\alpha}. \]
Q.E.D.
An important formula on S-curvature of \((\alpha, \beta)\)-metrics:

\[
S = \left\{ 2\Psi - \frac{f'(b)}{bf(b)} \right\} (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0),
\]

where

\[
\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''.
\]
Theorem 4.2 ([Cheng-Shen2003]) Randers metric $F = \alpha + \beta$ is of isotropic $S$-curvature, $S = (n+1)c(x)F$, if and only if

$$r_{ij} + b_i s_j + b_j s_i = 2c(a_{ij} - b_i b_j).$$

Theorem 4.3 ([Cheng-Shen2006(2)]) Let

$$F = k_1 \sqrt{\alpha^2 + k_2 \beta^2 + k_3 \beta}$$

be a Finsler metric of Randers type where $k_1 > 0$ and $k_3 \neq 0$. $F$ is of isotropic $S$-curvature, $F = (n+1)cF$ if and only if $\beta$ satisfies

$$r_{ij} + \tau(s_i b_j + s_j b_i) = \frac{2c(1 + k_2 b^2)k_1^2}{k_3}(a_{ij} - \tau b_i b_j),$$

where

$$\tau := \frac{k_3^2}{k_1^2} - k_2.$$
Lemma 4.4 ([Cheng-Shen2006(2)]) Let $\beta$ be a 1-form on a Riemannian manifold $(M, \alpha)$. Then $b(x) := \|\beta_x\|_\alpha = \text{constnt}$ if and only if $\beta$ satisfies the following equation:

$$r_j + s_j = 0.$$  

In this case, the S-curvature is given by

$$S = -\alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Qs_0).$$

Proof: This follows immediately from $(b^2)_{ij} = 2(r_j + s_j)$ . Q.E.D.
Theorem 4.5 ([Cheng-Shen2006(2)]) Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold. Suppose that

$$\phi \neq k_1 \sqrt{1 + k_2 s^2 + k_3 s}$$

for any constants $k_1, k_2$ and $k_3$.

Then $F$ is of isotropic $S$-curvature, $S = (n + 1)cF$, if and only if one of the following holds

(i) $\beta$ satisfies

$$r_j + s_j = 0$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0.$$  \hspace{1cm} (2)

In this case, $S = 0$. 

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(ii) $\beta$ satisfies
\[ r_{ij} = \epsilon \{ b^2 a_{ij} - b_i b_j \}, \quad s_j = 0, \] (3)
where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies
\[ \Phi = -2(n + 1)k \frac{\phi \Delta^2}{b^2 - s^2}, \] (4)
where $k$ is a constant. In this case, $S = (n + 1)cF$ with $c = k\epsilon$.

(iii) $\beta$ satisfies
\[ r_{ij} = 0, \quad s_j = 0. \] (5)
In this case, $S = 0$, regardless of the choice of a particular $\phi$. 
Remark. It is easy to see that

\[ r_{ij} = 0, \quad s_j = 0 \quad (5) \]

\[ \Downarrow \]

\[ r_{ij} = \epsilon\{b^2a_{ij} - b_ib_j\}, \quad s_j = 0 \quad (3) \]

\[ \Downarrow \]

\[ r_j + s_j = 0 \quad (1) \]

(\(\iff b := \|\beta_x\|_\alpha = constant\))

If an \((\alpha, \beta)\)-metric of non-Randers type is of isotropic S-curvature, then \(b := \|\beta_x\|_\alpha = constant\)
Example 4.1 Let $F = \alpha \phi(\beta/\alpha)$ be an $(\alpha, \beta)$-metric defined on an open subset in $R^3$. At a point $x = (x, y, z) \in R^3$ and in the direction $y = (u, v, w) \in T_x R^3$, $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are given by

$$
\alpha := \sqrt{u^2 + e^{2x}(v^2 + w^2)},
$$

$$
\beta := u.
$$

Then $\beta$ satisfies (3) with $\epsilon = 1, b = 1$. Thus if $\phi = \phi(s)$ satisfies (4) for some constant $k$, then $F = \alpha \phi(\beta/\alpha)$ is of constant S-curvature $S = (n + 1) cF$. 
Example 4.2 For $F = \alpha + \epsilon \beta + k(\beta^2/\alpha)$, $\epsilon, k(\neq 0)$ are constants, the following are equivalent:

(i) $F$ is of isotropic $S$-curvature, $S = (n + 1)cF$;

(ii) $\beta$ is a Killing 1-form with $b$-constant with respect to $\alpha$, i.e.

$$r_{ij} = 0, \quad s_j = 0;$$

(iii) $S = 0$;

(iv) $F$ is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}cF^{-1}h$, where $E$ denotes the mean Berwald curvature of $F$;

(v) $F$ is a weakly Berwald metric, $E = 0$. 

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References


Thank you very much for your attention!