Infinitely many solutions for an elliptic problem involving critical nonlinearity

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1. Background

Yamabe Conjecture

Manifolds without boundary

Let \((M, g)\) be an \(N\)-dimensional, smooth, compact Riemannian manifold without boundary. For \(N \geq 3\), the Yamabe conjecture states that there exist Riemannian metrics which are pointwise conformal to \(g\) and have a constant scalar curvature.

Let \(R_g\) denote the scalar curvature of \(g\) and \(\Delta_g\) denote the Laplace-Beltrami operator of \(g\). For \(N \geq 3\), let \(g' = u^{\frac{4}{N-2}} g\) for some positive function \(u\), we have

\[
R_{g'} = u^{-\frac{N+2}{N-2}} (R_g u - \frac{4(N-1)}{N-2} \Delta_g u),
\]

The Yamabe conjecture is therefore equivalent to the solvability of

\[
-\Delta_g u + \frac{N-2}{4(N-1)} R_g u = \bar{R} u^{\frac{N+2}{N-2}}, \quad u > 0, \quad \text{in } M \tag{1.1}
\]

for \(\bar{R} = 1, 0, \text{ or } -1\).
Manifold with boundary – an example

Consider a smooth metric $g$ on $B = \{ x \in \mathbb{R}^N : |x| < 1 \}, N \geq 3$. Let $\Delta_g, R_g, \nu_g, h_g$ denote, respectively the Laplace-Beltrami operator, the scalar curvature of $(B, g)$, the outward normal to $\partial B = S^{N-1}$ with respect to $g$ and the mean curvature of $(S^{N-1}, g)$. Given two smooth functions $R'$ and $h'$, we will consider the existence of positive solutions $u \in H^1(B)$ of

$$
\begin{aligned}
-4 \frac{N-1}{N-2} \Delta_g u + R_g u &= R' u^{\frac{N+2}{N-2}} \text{ in } B, \\
\frac{2}{N-2} \partial_{\nu_g} u + h_g u &= h' u^{\frac{N}{N-2}} \text{ on } \partial B.
\end{aligned}
$$

(1.2)

It is well known that such a solution is $C^\infty$ provided $g, R'$ and $h'$ are. If $u > 0$ is a smooth solution of (1.2) then $g' = u^{\frac{4}{N-2}} g$ is a metric, conformally equivalent to $g$, such that $R'$ and $h'$ are, respectively, the scalar curvature of $(B, g')$ and the mean curvature of $(S^{N-1}, g')$. 
Suppose $M = S^N, g = g_0$, if we allow $\bar{R}$ to be general positive function, then (1.3) becomes

$$-\Delta_{g_0} u + \frac{N(N-2)}{4} u = \frac{N - 2}{4(N - 1)} \bar{R} u^{\frac{N+2}{N-2}}, \quad u > 0, \text{ in } S^N \quad (1.3)$$

Making the stereographic projection $\pi = \pi_N$, and setting $K_0(x) = K(\pi^{-1}x)$, we get the following

$$\begin{cases}
-\Delta u = K_0(x) u^{2^* - 1}, & x \in \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N,
\end{cases} \quad (1.4)$$

where $2^* = \frac{2N}{N-2}, N \geq 3$.

There have been many works on (1.4).

For example, SCHEON, SCHEON and YAU, CHANG and YANG, BAHRI and CORON, etc.
The main difficulty is that suppose there exists a sequence of approximating solutions, how can we get a subsequence which converges strongly in a suitable space?

Usually we use the variational method. Let $H$ be a Hilbert space and $I \in C^1(H, \mathbb{R})$. We can use many ways (constrained minimization, mountain pass lemma, Galerkin method, etc) to obtain Palais-Smale sequence $\{u_k\}$, that is $\{u_k\}$ satisfies

$$I(u_k) \to c, I'(u_k) \to 0$$

The Palais-Smale sequence is the corresponding sequence of approximating solutions.

If we can always choose a strongly convergent subsequence from any given Palais-Smale sequence, then $I$ is said to satisfy Palais-Smale condition. Unfortunately, for most problem with critical non-linearity, the corresponding functional does not satisfy Palais-Smale condition.
2. Problem, Previous Results and Our Result

Let \( N \geq 3 \), \( 2^* = \frac{2N}{N-2} \), and \( \Omega \) be an open bounded domain in \( \mathbb{R}^N \). We consider the following elliptic problem:

\[
\begin{align*}
-\text{div}(a(x)Du) &= Q(x)|u|^{2^*-2}u + \lambda u & x &\in \Omega, \\
&= 0 & x &\in \partial\Omega,
\end{align*}
\]

(2.1)

where \( a, Q \in C^4(\bar{\Omega}) \), \( a(x) \geq a_0 > 0 \), \( Q(x) \geq Q_0 > 0 \), and \( \lambda > 0 \) is a positive constant.

The functional corresponding to (2.1) is

\[
I(u) = \frac{1}{2} \int_{\Omega} (a(x)|Du|^2 - \lambda u^2) dx - \frac{1}{2^*} \int_{\Omega} Q(x)|u|^{2^*} dx, \quad u \in H^1_0(\Omega).
\]

(2.2)

Since the embedding of \( H^1_0(\Omega) \) into \( L^{2^*}(\Omega) \) is not compact, the functional \( I(u) \) does not satisfy the Palais-Smale condition (PS condition for short). This loss of compactness creates a lot of difficulties when variational method is used to obtain the existence result for (2.1).
The case of constant coefficients
\[ a(x) \equiv 1, \; Q(x) \equiv 1. \]
Let \( \lambda_1 \) denote the first eigenvalue of \(-\Delta\) in \( H^1_0(\Omega) \).

**Pohozaev** (1965): (2.1) has no nontrivial solution if \( N \geq 4 \) and \( \lambda < 0 \).

**Brezis-Nirenberg** (1983):
If \( N \geq 4 \) and \( \lambda \in (0, \lambda_1) \), then (2.1) has a positive solution.

**Capozzi, Fortunato and Palmieri** (1985), **Ambrosetti** and **Struwe** (1986)
Existence of nontrivial solutions were obtained when \( N \geq 4 \).

**Cerami, Solimini** and **Struwe** (1986)
The existence of solutions of changing sign was studied when \( N \geq 7 \).

**Fortunato** and **Jannelli** (1987)
For any real positive parameter \( \lambda \) and for all bounded domains \( \Omega \), which have suitable symmetry properties, (2.1) has infinitely many solutions when \( N \geq 4 \). When \( N = 3 \) it is shown that the number of solutions increases with \( \lambda \).

Let \( S \) be the best Sobolev constant.

The fact that for \( c < \frac{1}{N}S^\frac{N}{2} \), \( I(u) \) satisfies \((PS)_c\) condition were established and used in all the mentioned papers.
Recently, Devillanova and Solimini (2003) proved that (2.1) has infinitely many solutions if $N \geq 7$ and $\lambda > 0$.

**Framework of Devillanova and Solimini**

First consider the existence of infinitely many solutions of the following perturbed problem:

\[ \begin{cases} -\Delta u = |u|^{2^*-2-\varepsilon} u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]

(2.3)

where $\varepsilon > 0$ is a small constant.

By the result of Ambrosetti - Rabinowitz, for fixed $\varepsilon$, (2.3) has solutions $u_{\varepsilon, k}, k = 1, 2, \ldots$.

The strong convergence of $u_{\varepsilon, k}$ as $\varepsilon \to 0$ was proved by show that $|u_{\varepsilon, k}|$ has a uniform bound.

The procedure of obtaining the uniform bound consists of two steps.

Step 1: find a safe region, where solutions of (2.3) are uniformly bounded.

The main ingredient used to achieve this goal is the mean value theorem for the Laplacian operator.

Step 2: establish a local Pohozaev identity in a small ball, whose boundary lies entirely in the safe region, to reach a contradiction.
The case of non-constant coefficients

that is, either $a(x)$, or $Q(x)$ is not identically constant.

For such a case, it is even more difficult to obtain a sign-changing solution for (2.1), because $I(u)$ does not satisfy $(PS)_{c}$ condition for any $c$ larger than the smallest number $\min\left\{ \frac{(a(x)S)^{\frac{N}{2}}}{N(Q(x))^{\frac{N-2}{2}}} \mid x \in \Omega\right\}$, where the $(PS)$ condition fails. The aim of this talk is to show that (2.1) has infinitely many solutions if $N \geq 7$, $a(x)$ and $Q(x)$ satisfy some degenerate conditions near their critical points.
Consider the following perturbed problem:

\[
\begin{align*}
-\text{div}(a(x)Du) &= Q(x)|u|^{2^*-2-\varepsilon}u + \lambda u & \quad x \in \Omega, \\
 u &= 0 & \quad \text{on } \partial \Omega,
\end{align*}
\]  

where \( \varepsilon > 0 \) is a small constant.

The functional corresponding to (2.4) becomes

\[
I_\varepsilon(u) = \frac{1}{2} \int_\Omega (a(x)|Du|^2 - \lambda u^2) \, dx - \frac{1}{2^* - \varepsilon} \int_\Omega Q(x)|u|^{2^*-\varepsilon} \, dx, \quad u \in H^1_0(\Omega).
\]

Now \( I_\varepsilon(u) \) is an even functional and satisfies the (PS) condition. So from Ambrosetti-Rabinowitz, (2.4) has infinitely many solutions. More precisely, there are positive numbers \( c_{\varepsilon, l}, l = 1, 2, \ldots \), with \( c_{\varepsilon, l} \to +\infty \) as \( l \to +\infty \), and a solution \( u_{\varepsilon, l} \) for (2.4), satisfying

\[
I_\varepsilon(u_{\varepsilon, l}) = c_{\varepsilon, l}.
\]

Moreover, \( c_{\varepsilon, l} \to c_l < +\infty \) as \( \varepsilon \to 0 \).

**Question:**

For each fixed \( l \), as \( \varepsilon \to 0 \), does \( u_{\varepsilon, l} \) converges strongly in \( H^1_0(\Omega) \) to \( u_l \)?

If the answer is yes, then \( u_l \) is a solution of (2.1) with \( I(u_l) = c_l \).

If we can prove that under suitable assumptions on \( a \) and \( Q \), \( u_{\varepsilon, l} \) as \( \varepsilon \to 0 \), then This will imply that (2.1) has infinitely many solutions.
Define
\[ \Sigma(x) = \frac{a^{N/2}(x)}{Q(N-2)/2(x)}. \] (2.6)

Let \( C_\Sigma \) be the set of all the critical points of \( \Sigma(x) \). Let \( \langle x, y \rangle \) denote the inner product of \( x, y \in \mathbb{R}^N \).

\( (C_1) \) There is a constant \( \delta > 0 \), such that for any \( x \in \bar{\Omega} \) with \( d(x, C_\Sigma) \leq \delta \),
\[ \langle Da(x), DQ(x) \rangle \leq 0. \]

\( (C_2) \) For each \( x_0 \in C_\Sigma \), there is a number \( m(x_0) > 2 \) and \( q(x_0) > 2 \), such that for \( x \) near \( x_0 \),
\[ |D^j a(x)| \leq C |Da(x)|^{(m(x_0)-j)/(m(x_0)-1)}, \quad j = 2, 3; \]
\[ |D^j Q(x)| \leq C |DQ(x)|^{(q(x_0)-j)/(q(x_0)-1)}, \quad j = 2, 3. \]

\( (C_3) \) There is a small \( \tau > 0 \), such that for any \( y \in \Omega \) with \( d(y, \partial \Omega) < \tau, t \in (0, \tau] \),
\[ \langle \tilde{y} - x, Da(x) \rangle \geq 0, \quad \forall x \in B_t(y), \]
\[ \langle \tilde{y} - x, DQ(x) \rangle \leq 0, \quad \forall x \in B_t(y), \]
where \( \tilde{y} = y + 2tn \), and \( n \) is the outward unit normal of \( \partial \Omega \) at \( \tilde{y}, |\tilde{y} - y| = d(y, \partial \Omega) \).
Remarks on the conditions \((C_2)\) and \((C_3)\).

Remark 2.1. If \(a(x) \equiv 1\), then \(\Sigma\) and \(Q(x)\) has the same critical points. Condition \((C_2)\) implies that any critical point of \(Q(x)\) must be degenerate. On the other hand, \((C_3)\) implies that \(Q(x)\) must be non-increasing near the boundary at the direction of the outward normal of \(\partial \Omega\). Note that if \(Q \equiv 1\), \((C_2)\) and \((C_3)\) hold. So Theorem 2.6 is a generalization of the result of Devillanova and Solimini.

Remark 2.2. If \(Q \equiv 1\), then \(\Sigma\) and \(a(x)\) has the same critical points. Condition \((C_2)\) implies that any critical point of \(a(x)\) must be degenerate, while \((C_3)\) implies that \(a(x)\) must be non-decreasing near the boundary at the direction of the outward normal of \(\partial \Omega\).

Remark 2.3. By \((C_3)\), it is easy to check that any critical point of \(\Sigma\) must be a critical point for \(a(x)\), and \(Q(x)\).

Remark 2.4. If \(\frac{\partial a(x)}{\partial n} > 0\) and \(\frac{\partial Q(x)}{\partial n} < 0\) for any \(x \in \partial \Omega\), where \(n\) is the outward unit normal of \(\partial \Omega\) at \(x\), then \((C_3)\) holds.

Examples:
Suppose \(\forall x_0 \in C_{\Sigma}\), there are \(\theta > 2, \hat{\theta} > 2\) and constants \(q_{x_0, 1} > 0, q_{x_0, 2}, a_{x_0, 1} > 0, a_{x_0, 2}\) such that for \(x\) near \(x_0\),

\[
Q(x) = q_{x_0, 1} - q_{x_0, 2}|x - x_0|^\theta + q(x), \quad q(x) = o(|x - x_0|^\theta),
\]

\[
a(x) = a_{x_0, 1} + a_{x_0, 2}|x - x_0|^{\hat{\theta}} + \hat{a}(x), \quad \hat{a}(x) = o(|x - x_0|^{\hat{\theta}})
\]

with \(q(x)\) and \(\hat{a}(x)\) satisfy certain conditions of degenerate at \(x_0\), in particular \(q(x) \equiv 0, \hat{a}(x) \equiv 0\). Then \(C_1, C_2\) are satisfied.
The norm of $H^1_0(\Omega)$

$$\|u\| = \left( \int_{\Omega} |Du|^2 \, dx \right)^{\frac{1}{2}};$$

The norm of $L^p(\Omega) (1 \leq p < \infty)$

$$\|u\|_p = \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}}.$$

The main result is the following:

**Theorem 2.5.** Suppose that $N \geq 7$ and $(C_1)-(C_3)$ hold with

$$\inf_{x_0 \in S}(m(x_0), q(x_0)) > \frac{2(N-2)}{N-4}, \lambda > 0.$$ 

Then for any sequence $u_n$, which is a solution of (2.4) with $\varepsilon = \varepsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant independent of $n$, $u_n$ converges strongly in $H^1_0(\Omega)$ as $n \to +\infty$.

A direct consequence of Theorem 2.5 is the following multiplicity result.

**Theorem 2.6.** Suppose that $N \geq 7$ and $(C_1)-(C_3)$ hold with

$$\inf_{x_0 \in S}(m(x_0), q(x_0)) > \frac{2(N-2)}{N-4}, \lambda > 0.$$ 

Then (2.1) has infinitely many solutions.
Difference with previous work

It appears the techniques used in Devillanova and Solimini can not be applied to study (2.1).

1. It seems that result similar to the mean value theorem for the Laplacian operator is still unknown for elliptic operator in divergent form.

2. Under our assumptions on $a$ and $Q$, we can not obtain a contradiction by applying the local Pohozaev identity in a small ball whose boundary lies entirely in the region where the solutions are uniformly bounded.

To overcome these difficulties, in our paper, we will use the frozen coefficients technique, together with the mean value theorem for elliptic operator of constant coefficient, to obtain the desired local $L^1$ estimate for solutions of (2.4). And then we estimate the solutions of (2.4) in a carefully defined safe region for problem (2.4), where a local Pohozaev identity will be used to reach a contradiction. It is worthwhile to point out that in the safe region we choose for (2.4), the solutions are not uniformly bounded.
Framework of proof

Suppose $u_n$ is a solution of the perturbed problem with $\varepsilon = \varepsilon_n$ and $\|u_n\| \leq C$ for $C > 0$ independent of $n$. Suppose $u_n$ blow-up at some point. Then we have $x_n$ and $\sigma_n \to \infty$ such that $u_n$ behavior like $\sigma_n^{N-2} U(\sigma_n(x - x_n))$ near $x_n$.

Let $\alpha > \frac{1}{2}$, $B_n = B_{\sigma_n\alpha}(x_n) \cap \Omega$. We have a local Phozaev identity

$$
\lambda \int_{B_n} u_n^2 + \left( \frac{N}{2^* - \varepsilon_n} - \frac{N - 2}{2} \right) \int_{B_n} Q(x)|u_n|^{2^* - \varepsilon_n}
$$

$$
- \frac{1}{2} \int_{B_n} \langle Da(x), x - x_0 \rangle |Du_n|^2 + \frac{1}{2^* - \varepsilon_n} \int_{B_n} \langle DQ(x), x - x_0 \rangle |u_n|^{2^* - \varepsilon_n}
$$

$$
= \int_{\partial B_n} \left( a(x) \langle Du_n, x - x_0 \rangle + \frac{N - 2}{2} a(x) u_n \right) D_i u_n n_i
$$

$$
- \int_{\partial B_n} \left( \frac{1}{2} a(x) |Du_n|^2 - \frac{1}{2} \lambda u_n^2 - \frac{1}{2^* - \varepsilon_n} Q(x)|u_n|^{2-\varepsilon_n} \right) \langle n, x - x_0 \rangle,
$$

which deduces from $2^* - \varepsilon_n - \frac{N-2}{2} > 0$

$$
\lambda \int_{B_n} |u_n|^2 \, dx
$$

$$
\leq \int_{\partial B_n} \left( a(x) \langle Du_n, x - x_0 \rangle + \frac{N - 2}{2} a(x) u_n \right) D_i u_n n_i
$$

$$
- \int_{\partial B_n} \left( \frac{1}{2} a(x) |Du_n|^2 - \frac{1}{2} \lambda u_n^2 - \frac{1}{2^* - \varepsilon_n} Q(x)|u_n|^{2-\varepsilon_n} \right) \langle n, x - x_0 \rangle.
$$

We try to obtain a contradiction by estimating each term in the above inequality.
3. Some Integral Estimates

For any $\sigma > 0$ and $y \in \mathbb{R}^N$, we define

$$\rho_{y,\sigma}(u) = \sigma \frac{N}{2} u(\sigma(\cdot - y)), \quad u \in H_0^1(\Omega).$$

First, we need the following decomposition result for the solutions of (2.4).

**Proposition 3.1.** Suppose $N \geq 3$. Let $u_n$ be a solution of (2.4) with $\varepsilon = \varepsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant $C$. Then

(i) $u_n$ can be decomposed as

$$u_n = u_0 + \sum_{j=1}^{k} \rho_{x_n,j,\sigma_{n,j}}(U_j) + \omega_n. \quad (3.1)$$

where $\omega_n \to 0$ in $H^1(\Omega)$, $u_0$ is a solution for (2.1). For $j = 1, \cdots, k$, $x_{n,j} \in \Omega$, $\sigma_{n,j} d(x_{n,j}, \partial \Omega) \to +\infty$, $x_{n,j} \to x_j \in \bar{\Omega}$, as $n \to \infty$, and $U_j$ is a solution of

$$\begin{cases}
-a(x_j) \Delta u = b_j Q(x_j) |u|^{2^* - 2} u, & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases} \quad (3.2)$$

for some $b_j \in (0, 1]$.

(ii) For $i, j = 1, \cdots, k$, if $i \neq j$, then, as $n \to \infty$,

$$\frac{\sigma_{n,j}}{\sigma_{n,i}} + \frac{\sigma_{n,i}}{\sigma_{n,j}} + \sigma_{n,j} \sigma_{n,i} |x_{n,i} - x_{n,j}|^2 \to \infty. \quad (3.3)$$
To prove the strong convergence of \( u_n \) in \( H^1(\Omega) \), we just need to show that the bubbles \( \rho_{x_{n,j},\sigma_{n,j}}(U_j) \) will not appear in the decomposition of \( u_n \).

Among all the bubbles \( \rho_{x_{n,j},\sigma_{n,j}}(U_j) \), we can choose a bubble, such that this bubble has the slowest concentration rate. That is, the corresponding \( \sigma_{n,i} \) is the lowest order infinity among all the \( \sigma_{n,j} \) appearing in the bubbles. For simplicity, we denote \( \sigma_n \) the slowest concentration rate and \( x_n \) the corresponding concentration point.

Let \( w_n(x) = |u_n(x)| \) in \( \Omega \); \( w_n(x) = 0 \) in \( \mathbb{R}^N \setminus \Omega \). We have

**Lemma 3.2.** Let \( D \) be any bounded domain in \( \mathbb{R}^N \). Then for any \( \phi \in H^1(\mathbb{R}^N) \) with \( \phi \geq 0 \),

\[
\int_D a(x)Dw_nD\phi \, dx - \int_{\partial D} a(x)\langle Dw_n, n \rangle \phi \leq A \int_D (w_n^{2s-1} + 1) \phi \, dx,
\]

where \( A > 0 \) is a large constant, \( n \) is the outward unit normal of \( \partial D \). In particular,

\[
\int_{\mathbb{R}^N} a(x)Dw_nD\phi \, dx \leq A \int_{\mathbb{R}^N} (w_n^{2s-1} + 1) \phi \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N), \ \phi \geq 0.
\]
To analyze the asymptotic behaviors of the solutions of (2.4), we need some integral estimates for $u_n$.

For any $p_2 < 2^* < p_1$, $\alpha > 0$ and $\sigma > 0$, we consider the following relation:

$$\begin{cases}
\|u_1\|_{p_1} \leq \alpha, \\
\|u_2\|_{p_2} \leq \alpha \sigma^{\frac{N}{2^*}} \frac{N}{p_2}.
\end{cases} \quad (3.6)$$

Define

$$\|u\|_{p_1,p_2,\sigma} = \inf \{\alpha > 0 : \text{there are } u_1 \text{ and } u_2, \text{ such that } (3.6) \text{ holds and } |u| \leq u_1 + u_2\}.$$ 

The main result of this section is the following proposition.

**Proposition 3.3.** Let $w_n$ be a weak solution of (3.5). For any $p_2 < 2^* < p_1 < +\infty$, there is a constant $C$, depending on $p_1$ and $p_2$, such that

$$\|w_n\|_{p_1,p_2,\sigma_n} \leq C.$$ 

**Proof.** By using the estimates in Appendix A, we can prove Proposition 3.3 in exactly the same way as in the paper of Devillanova-Solimini. \qed
4. **Estimates on Safe Regions**

Let $B_t(y)$ be the open ball centered at $y$ with radius $t > 0$ and Denote $B_t^*(y) = B_{\sqrt{a(y)t}}(y)$.

Let $\alpha \in \left(\frac{1}{2}, \frac{N-4}{N-2}\right)$ be a constant, which is to be determined later. Since the number of the bubbles of $u_n$ is finite, we may always find a constant $\bar{C} > 0$, independent of $n$, such that the region

$$A_1^n = \left( \overline{B_{(\bar{C}+5)\sigma_n^{-\alpha}}(x_n)} \setminus B_{\bar{C}\sigma_n^{-\alpha}}(x_n) \right) \cap \Omega,$$

does not contain any concentration point of $u_n$ for any $n$. We call this region a safe region for $u_n$.

Let

$$A_2^n = \left( \overline{B_{(\bar{C}+4)\sigma_n^{-\alpha}}(x_n)} \setminus B_{(\bar{C}+1)\sigma_n^{-\alpha}}(x_n) \right) \cap \Omega.$$

In this section, we will prove the following result.

**Proposition 4.1.** Let $w_n$ be a weak solution of $(3.5)$. Then, there is a constant $C > 0$, independent of $n$, such that

$$w_n(x) \leq C\sigma_n^{(N-2)(\alpha-\frac{1}{2})}, \quad \forall \ x \in A_2^n.$$

To prove Proposition 4.1, we need the following lemma.

**Lemma 4.2.** Suppose that $w_n$ satisfies $(3.5)$. Then, there is a constant $C > 0$, independent of $n$, such that

$$\frac{1}{r^{N-1}} \int_{\partial B_r^*(x_n)} w_n \, dS \leq C\sigma_n^{(N-2)(\alpha-\frac{1}{2})},$$

for all $r \in [\bar{C}\sigma_n^{-\alpha}, (\bar{C}+5)\sigma_n^{-\alpha}]$.

**Proof.** From $\int_{B_1(x_n)} w_n \, dx \leq C$, we can find a $r_n \in \left[\frac{1}{2}, 1\right]$, such that

$$\frac{1}{r_n^{N-1}} \int_{\partial B_{r_n}(x_n)} w_n \, dS \leq C.$$
We make use of the following formula: For any $C^2$ function $v$, we have

$$\int_{r_n}^{r} \frac{d}{dt}\left( \frac{1}{t^{N-1}} \int_{B_t(x_n)} v dx \right) dt = \int_{r_n}^{r} \frac{1}{\omega_N t^{N-1}} \int_{B_t(x_n)} \Delta v dx \, dt,$$

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. As a result, there are constants $C_1 > 0$, such that

$$\int_{r_n}^{r} \frac{d}{dt}\left( \frac{1}{t^{N-1}} \int_{B_t^+(x_n)} v dx \right) dt = \int_{r_n}^{r} \frac{C_1}{\omega_N t^{N-1}} \int_{B_t^+(x_n)} a(x_n) \Delta v dx \, dt. \tag{4.1}$$

Since $w_n$ does not belong to $C^2$, we need to modify it. Anyway we can obtain

$$\frac{1}{r^{N-1}} \int_{\partial B_t^+(x_n)} w_n \, dS = \frac{1}{r^{N-1}} \int_{\partial B_t^+(x_n)} w_n \, dS + \int_{r_n}^{r} \frac{C_1}{\omega_N t^{N-1}} \int_{\partial B_t^+(x_n)} (-a(x_n) D_i w_n n_i) \, dS \, dt, \tag{4.2}$$
which, together with Lemma 3.2, gives

\[
\frac{1}{r^{N-1}} \int_{\partial B_r^*(x_n)} w_n \, dS = \frac{1}{r_n^{N-1}} \int_{\partial B_{r_n}^*(x_n)} w_n \, dS + \int_r^{r_n} \frac{C_1}{\omega_N t^{N-1}} \int_{\partial B^*_t(x_n)} (-a(x) D_i w_n n_i) \, dS \, dt \\
+ \int_r^{r_n} \frac{C_1}{\omega_N t^{N-1}} \int_{\partial B^*_t(x_n)} (a(x) - a(x_n)) D_i w_n n_i \, dS \, dt \leq C + C \int_r^{r_n} \frac{1}{t^{N-1}} \int_{B_t^*(x_n)} (w_n^{2*} - 1) \, dx \, dt \\
+ C \int_r^{r_n} \frac{1}{t^{N-2}} \int_{\partial B_t^*(x_n)} |Dw_n| \, dS \, dt = C + C \int_r^{r_n} \frac{1}{t^{N-1}} \int_{B_t^*(x_n)} (w_n^{2*} - 1) \, dx \, dt \\
+ C \int_r^{r_n} \frac{1}{t^{N-2}} \int_{\partial B_t^*(x_n)} |Du_n| \, dS \, dt.
\] (4.3)

By the Sobolev embedding theorem,

\[
\int_{\partial B_t^*(x_n)} |Du_n| \, dS \leq C \int_{B_t^*(x_n)} |D^2 u_n| + \frac{C}{t^2} \int_{B_t^*(x_n)} |u_n|.
\] (4.4)
Let \( \bar{\theta} > 0 \) be a fixed small constant and let \( q = 1 + \bar{\theta} > 1 \). By the \( L^q \) estimate for the elliptic equation, we have

\[
\int_{B_t^*(x_n)} |D^2u_n| \leq t^{N(1 - \frac{1}{q})} \left( \int_{B_t^*(x_n)} |D^2u_n|^q \right)^{1/q} 
\]

\[
\leq Ct^{N(1 - \frac{1}{q})} \left( \int_{B_{2t}^*(x_n)} (|u_n|^q + (|u_n|^{2^* - 1} + 1)^q) \right)^{1/q} 
\]

\[
\leq Ct^{N(1 - \frac{1}{q})} \left( \int_{B_{2t}^*(x_n)} (|u_n|^{(2^* - 1)q} + 1) \right)^{1/q} . 
\]

Combining (4.3), (4.4) and (4.5), we are led to

\[
\frac{1}{r^{N-1}} \int_{\partial B_t^*(x_n)} w_n dS 
\]

\[
\leq C + C \int_r^{r_n} \frac{1}{t^{N-1}} \int_{B_t^*(x_n)} w_n^{2^* - 1} dx \, dt 
\]

\[
+ C \int_r^{r_n} \frac{t^{N(1 - 1/q)}}{t^{N-2}} \left( \int_{B_{2t}^*(x_n)} |u_n|^{(2^* - 1)q} dx \right)^{1/q} dt 
\]

\[
+ C \int_r^{r_n} \frac{1}{t^{N}} \int_{B_{2t}^*(x_n)} |u_n| dx \, dt. 
\]

We can use Proposition 3.3 to estimate each term in the right hand side of (4.6) and finish our proof of Lemma 4.2.

\[ \square \]

**Proof of Proposition 4.1.** It follows from Lemma 4.2 that for any \( y \in \mathcal{A}_n^2 \), we have

\[
\frac{1}{\sigma_{\alpha n}^n} \int_{B_{C\sigma_{\alpha n}^{-\alpha}(y)}^*} |w_n| \leq C \sigma_n^{(N-2)(\alpha - \frac{1}{2})}. 
\]
Let
\[ v_n(x) = w_n(C\sigma_n^{-\alpha} x + y), \quad x \in \Omega_n, \]
where \( \Omega_n = \{ x : C\sigma_n^{-\alpha} x + y \in \Omega \} \). Then \( v_n \) satisfies \( \forall \, \eta \in H^1_{\text{loc}}(\mathbb{R}^N) \), \( \eta \geq 0 \)
\[
\int_{\mathbb{R}^N} a(C\sigma_n^{-\alpha} x + y) Dv_n D\eta \leq C\sigma_n^{-2\alpha} \int_{\mathbb{R}^N} (|v_n|^{2^* - 2} + \lambda) v_n \eta.
\]

Since \( B^*_{C\sigma_n^{-\alpha}}(y), y \in A^2_n \), does not contain any concentration point of \( u_n \), we can deduce
\[
\int_{B_1(0)} |\sigma_n^{-2\alpha} (|v_n|^{2^* - 2} + \lambda)|^{\frac{N}{2}} dx \leq C \int_{B^*_{C\sigma_n^{-\alpha}}(y)} |u_n|^{2^*} dx + C\sigma_n^{-\alpha N} \to 0
\]
as \( n \to +\infty \). Thus, by Moser iteration, we obtain
\[
\|v_n\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C \left( \int_{B_1(0)} |v_n| dx + 1 \right) = C \left( \frac{1}{\sigma_n^{\alpha N}} \int_{B^*_{C\sigma_n^{-\alpha}}(y)} |w_n| dx + 1 \right)
\]
\[
\leq C\sigma_n^{(N-2)\left(\frac{1}{2} - \frac{1}{2}\right)}.
\]
\[ \square \]
Let
\[ A^3_n = \left( B^*_s(\bar{C}+3)\sigma_n^{-\alpha}(x_n) \setminus B^*_s(\bar{C}+2)\sigma_n^{-\alpha}(x_n) \right) \cap \Omega. \]

**Proposition 4.3.** We have
\[
\int_{A^3_n} |Du_n|^2 \, dx \leq C \int_{A^3_n} (|u_n|^2 + 1) \, dx + C\sigma_n^{2\alpha} \int_{A^2_n} |u_n|^2 \, dx. \tag{4.7}
\]
In particular,
\[
\int_{A^3_n} |Du_n|^2 \, dx \leq C\sigma_n^{-(N-2)(1-\alpha)} . \tag{4.8}
\]

**Corollary 4.4.** There exists \( t_n \in [\bar{C} + 2, \bar{C} + 3] \), such that
\[
\int_{\partial B^*_{t_n\sigma_n^{-\alpha}(x_n)}} |Du_n|^2 \, dS \leq C\sigma_n^{-(N-2)(1-\alpha)+\alpha}. \tag{4.9}
\]
5. Local Pohozaev Identities and Location of Concentration Point

In this section, we will first find two identities by applying the general Pucci and Serrin identity. We then use one of these two identities to locate the concentration points. The other one will be used to prove the main result in the next section.

**Lemma 5.1.** Suppose that $u_\varepsilon$ is a solution of (2.4). Then for any bounded domain $B$ contained in $\Omega$,

$$
\lambda \int_B u_\varepsilon^2 + \left( \frac{N}{2^* - \varepsilon} - \frac{N-2}{2} \right) \int_B Q(x)|u_\varepsilon|^{2^* - \varepsilon}
$$

$$
- \frac{1}{2} \int_B \langle Da(x), x - x_0 \rangle |Du_\varepsilon|^2 + \frac{1}{2^* - \varepsilon} \int_B \langle DQ(x), x - x_0 \rangle |u_\varepsilon|^{2^* - \varepsilon}
$$

$$
= \int_{\partial B} \left( a(x) \langle Du_\varepsilon, x - x_0 \rangle + \frac{N-2}{2} a(x) u_\varepsilon \right) D_i u_\varepsilon n_i
$$

$$
- \int_{\partial B} \left( \frac{1}{2} a(x)|Du_\varepsilon|^2 - \frac{1}{2} \lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x)|u_\varepsilon|^{2^* - \varepsilon} \right) \langle n, x - x_0 \rangle
$$

where $n$ is the outward unit normal vector of $\partial B$.

**Lemma 5.2.** Suppose that $u_\varepsilon$ is a solution of (2.4). Then for any bounded domain $B$ contained in $\Omega$, $k = 1, \ldots, N$,

$$
\frac{1}{2} \int_B D_k a(x)|Du_\varepsilon|^2 - \frac{1}{2^* - \varepsilon} \int_B D_k Q(x)|u_\varepsilon|^{2^* - \varepsilon}
$$

$$
= \int_{\partial B} \left( \frac{1}{2} a(x)|Du_\varepsilon|^2 - \frac{1}{2} \lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x)|u_\varepsilon|^{2^* - \varepsilon} \right) n_k
$$

$$
- \int_{\partial B} \left( a(x) D_k u_\varepsilon + \frac{N-2}{2} u_\varepsilon a(x) - \frac{N-2}{2} a(x) \right) D_i u_\varepsilon n_i.
$$

where $n$ is the outward unit normal vector of $\partial B$. 

In the rest of this section and whole the next section, we will always assume that \( u_{\varepsilon_n} \) is a solution of (2.4) with \( \varepsilon = \varepsilon_n \) and \( x_n \) is the concentration point of \( u_{\varepsilon_n} \) chosen as in section 2. To simplify notation we will write \( \varepsilon_n \) and \( u_{\varepsilon_n} \) as \( \varepsilon \) and \( u_{\varepsilon} \) respectively. Now, we locate the limit position of the concentration points.

**Lemma 5.3.** Suppose that \( x_n \to x_0 \). Then \( x_0 \) is a critical point of \( \Sigma(x) \).

**Proof.** Taking \( B = B_{t_n\sigma_n^{-\alpha}(x_n)}^* \) in Lemma 5.1, then from Proposition 4.1 and Corollary 4.4 we obtain

\[
\frac{1}{2} \int_{B} D_k a(x) |Du_{\varepsilon}|^2 - \frac{1}{2^* - \varepsilon} \int_{B} D_k Q(x) |u_{\varepsilon}|^{2^*-\varepsilon}
= O \left( \int_{\partial B} (|Du_{\varepsilon}|^2 + |u_{\varepsilon}|^{2-\varepsilon} + 1) \right)
= O \left( \sigma_n^{-(N-2)+(N-1)\alpha} + \sigma_n^{-N+(N+1)\alpha+o(1)} + \sigma_n^{-(N-1)\alpha} \right)
= O \left( \sigma_n^{-(N-2)+(N-1)\alpha} \right).
\]

(5.1)

On the other hand,

\[
\frac{1}{2} \int_{B} D_k a(x) |Du_{\varepsilon}|^2 - \frac{1}{2^* - \varepsilon} \int_{B} D_k Q(x) |u_{\varepsilon}|^{2^*-\varepsilon}
= \frac{1}{2} \int_{B} D_k a(x_n) |Du_{\varepsilon}|^2 - \frac{1}{2^* - \varepsilon} \int_{B} D_k Q(x_n) |u_{\varepsilon}|^{2^*-\varepsilon}
+ O \left( |D^2 a(x_n)| \sigma_n^{-\alpha} + |D^3 a(x_n)| \sigma_n^{-2\alpha} + \sigma_n^{-3\alpha} \right)
+ O \left( |D^2 Q(x_n)| \sigma_n^{-\alpha} + |D^3 Q(x_n)| \sigma_n^{-2\alpha} + \sigma_n^{-3\alpha} \right).
\]

(5.2)
Moreover,

\[
\int_B D_k a(x_n) |Du_\varepsilon|^2 = \frac{D_k a(x_n)}{a(x_n)} \int_B a(x_n) |Du_\varepsilon|^2 \\
= \frac{D_k a(x_n)}{a(x)} \int_B a(x_n) |Du_\varepsilon|^2 + o(1) \\
= \frac{D_k a(x_n)}{a(x)} \left( \int_B (Q(x)|u_\varepsilon|^{2^*-\varepsilon} + \lambda u_\varepsilon) + \int_{\partial B} a(x) D_i u_\varepsilon n_i \right) + o(1) \\
= \frac{Q(x_n) D_k a(x_n)}{a(x)} \int_B |u_\varepsilon|^{2^*-\varepsilon} + o(1).
\]

(5.3)

Combining (5.1), (5.2) and (5.3), we are led to

\[
\left( \frac{1}{2} \frac{Q(x_n) D_k a(x_n)}{a(x)} - \frac{1}{2^*-\varepsilon} D_k Q(x_n) \right) \int_B |u_\varepsilon|^{2^*-\varepsilon} = o(1).
\]

(5.4)

Letting \( n \to +\infty \), we obtain

\[
Q(x_0) D_k a(x_0) - \frac{N-2}{N} a(x_0) D_k Q(x_0) = 0.
\]

So, the result follows. \( \square \)

Let \( \gamma = \min \left( \frac{m(x_0)-2}{m(x_0)-1}, \frac{q(x_0)-2}{q(x_0)-1} \right) \).

**Proposition 5.4.** Suppose that \( x_n \to x_0 \in \Omega \). Suppose that (\( C_1 \)) and (\( C_2 \)) hold. Then

\[
|Da(x_n)| + |DQ(x_n)| = O \left( \sigma_n^{-(N-2)+(N-1)\alpha} + \sigma_n^{-\frac{\alpha}{1-\gamma}} + \sigma_n^{-3\alpha} \right).
\]
Proof. Combining (5.1) and (5.2), we obtain
\[ \frac{1}{2} D_k a(x_n) \int_B |Du| \, dx - \frac{1}{2^{*+\varepsilon}} D_k Q(x_n) \int_B |u| \, dx = O\left( \sigma^{-\alpha} + |D^2 a(x_n)|\sigma^{-2\alpha} + \sigma^{-3\alpha} \right) + O\left( |D^2 Q(x_n)|\sigma^{-\alpha} + |D^3 Q(x_n)|\sigma^{-2\alpha} + \sigma^{-3\alpha} \right). \]

(5.5)

It follows from \((C_1)\) that
\[ |D a(x_n)| + |D Q(x_n)| = O\left( \sigma^{-\alpha} + |D^2 a(x_n)|\sigma^{-2\alpha} + \sigma^{-3\alpha} \right) + O\left( |D^2 Q(x_n)|\sigma^{-\alpha} + |D^3 Q(x_n)|\sigma^{-2\alpha} + \sigma^{-3\alpha} \right). \]

(5.6)

Let \( \gamma_1 = \frac{m(x_0)-2}{m(x_0)-1} \) and \( \gamma_2 = \frac{q(x_0)-2}{q(x_0)-1} \). By \((C_2)\), (5.6) becomes
\[ |D a(x_n)| + |D Q(x_n)| = O\left( \sigma^{-\alpha} + |D a(x_n)|^{\gamma_1} \sigma^{-\alpha} + |D a(x_n)|^{2\gamma_1-1} \sigma^{-2\alpha} + \sigma^{-3\alpha} \right) + O\left( |D Q(x_n)|^{\gamma_2} \sigma^{-\alpha} + |D Q(x_n)|^{2\gamma_2-1} \sigma^{-2\alpha} + \sigma^{-3\alpha} \right), \]

from which, we obtain
\[ |D a(x_n)| + |D Q(x_n)| = O\left( \sigma^{-\alpha} + \sigma^{-\alpha/(1-\gamma_1)} + \sigma^{-\alpha/(1-\gamma_2)} + \sigma^{-3\alpha} \right) + O\left( \sigma^{-\alpha} + \sigma^{-\alpha/(1-\gamma)} + \sigma^{-3\alpha} \right). \]

□
6. Proof of the Main Result

In this section, we will prove Theorem 2.5.

Proof of Theorem 2.5. We have two different cases: (i) \( d(x_n, \partial \Omega) \leq \tau \); (ii) \( d(x_n, \partial \Omega) \geq \tau \), where \( \tau > 0 \) is the constant in \((C_3)\).

Let \( B_n = B_{t_n \sigma_n^{-\alpha}}(x_n) \cap \Omega \). By Lemma 5.1

\[
\begin{align*}
\lambda \int_{B_n} u_\varepsilon^2 + \left( \frac{N}{2^* - \varepsilon} - \frac{N-2}{2} \right) \int_{B_n} Q(x)|u_\varepsilon|^{2^*-\varepsilon} \\
- \frac{1}{2} \int_{B_n} \langle Da(x), x - x_0 \rangle |Du_\varepsilon|^2 + \frac{1}{2^*-\varepsilon} \int_{B_n} \langle DQ(x), x - x_0 \rangle |u_\varepsilon|^{2^*-\varepsilon} \\
= \int_{\partial B_n} \left( a(x)\langle Du_\varepsilon, x - x_0 \rangle + \frac{N-2}{2} a(x)u_\varepsilon \right) D_i u_\varepsilon n_i \\
- \int_{\partial B_n} \left( \frac{1}{2} a(x)|Du_\varepsilon|^2 - \frac{1}{2} \lambda u_\varepsilon^2 - \frac{1}{2^*-\varepsilon} Q(x)|u_\varepsilon|^{2^*-\varepsilon} \right) \langle n, x - x_0 \rangle,
\end{align*}
\]

(6.1)

where \( \nu \) is the outward normal to \( \partial B_n \). The point \( x_0 \) in (6.1) is chosen as follows. In case (i), we take \( x_0 = x_n + 2t_n \sigma_n^{-\alpha} \nu \), where \( \nu \) is the outward unit normal of \( \partial \Omega \) at \( \bar{x}_n |\bar{x}_n - x_n| = d(x_n, \partial \Omega) \). In case (ii), we take a point \( x_0 = x_n \).

We first consider case (i).

Since \( 2^* - \varepsilon < 2^* \), the second term in the left hand side of (6.1) is non-negative. By the choice of \( x_0 \) and \((C_3)\), the last two terms in the left hand side of (6.1) are non-negative as well. We thus obtain
from (6.1) that

\[
\lambda \int_{B_n} |u_n|^2 \, dx \\
\leq \int_{\partial B_n} \left( a(x)\langle Du_\varepsilon, x - x_0 \rangle + \frac{N - 2}{2} a(x) u_\varepsilon \right) D_i u_\varepsilon n_i \\
- \int_{\partial B_n} \left( \frac{1}{2} a(x) |Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x) |u_\varepsilon|^{2^* - \varepsilon} \right) \langle n, x - x_0 \rangle.
\]

(6.2)

Now we decompose \( \partial B_n \) into

\[
\partial B_n = \partial_i B_n \cup \partial_e B_n,
\]

where \( \partial_i B_n = \partial B_n \cap \Omega \) and \( \partial_e B_n = \partial B_n \cap \partial \Omega \).

Noting \( u_n = 0 \) on \( \partial \Omega \), we find

\[
\int_{\partial_e B_n} \left( a(x)\langle Du_\varepsilon, x - x_0 \rangle + \frac{N - 2}{2} a(x) u_\varepsilon \right) D_i u_\varepsilon n_i \\
- \int_{\partial_e B_n} \left( \frac{1}{2} a(x) |Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x) |u_\varepsilon|^{2 - \varepsilon} \right) \langle n, x - x_0 \rangle \\
= \frac{1}{2} \int_{\partial_e B_n} a(x) |Du_n|^2 \langle n, x - x_0 \rangle \leq 0.
\]

(6.3)
So, we can rewrite (6.2) as

$$
\lambda \int_{B_n} |u_n|^2 \, dx
\leq \int_{\partial_i B_n} \left( a(x) \langle Du_\varepsilon, x - x_0 \rangle + \frac{N - 2}{2} a(x) u_\varepsilon \right) D_i u_\varepsilon n_i
- \int_{\partial_i B_n} \left( \frac{1}{2} a(x) |Du_\varepsilon|^2 - \frac{1}{2} \lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x)|u_\varepsilon|^{2-\varepsilon} \right) \langle n, x - x_0 \rangle.
$$

(6.4)

Now we consider case (ii).
By Proposition 5.4 and \((C_2)\), we obtain

$$
\left| \int_{B_n} \langle Da(x), x - x_0 \rangle |Du_\varepsilon|^2 \right| + \left| \int_{B_n} \langle DQ(x), x - x_0 \rangle |u_\varepsilon|^{2^*-\varepsilon} \right|
= O \left( |Da(x_n)| \sigma_n^{-\alpha} + |D^2 a(x_n)| \sigma_n^{-2\alpha} + |D^3 a(x_n)| \sigma_n^{-3\alpha} + \sigma_n^{-4\alpha} \right)
+ O \left( |DQ(x_n)| \sigma_n^{-\alpha} + |D^2 Q(x_n)| \sigma_n^{-2\alpha} + |D^3 Q(x_n)| \sigma_n^{-3\alpha} + \sigma_n^{-4\alpha} \right)
= O \left( \sigma_n^{-\frac{2-\gamma}{1-\gamma} \alpha} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha} \right).
$$

(6.5)
As a result, we can rewrite (6.2) as

\[
\lambda \int_{B_n} |u_n|^2 \, dx
\leq \int_{\partial_i B_n} \left( a(x)(Du_\varepsilon, x-x_0) + \frac{N-2}{2} a(x)u_\varepsilon \right) D_i u_\varepsilon n_i \\
- \int_{\partial_i B_n} \left( \frac{1}{2} a(x) |Du_\varepsilon|^2 - \frac{1}{2} \lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x) |u_\varepsilon|^{2-\varepsilon} \right) \langle n, x - x_0 \rangle \\
+ O \left( \sigma_n^{-\frac{2-\gamma}{1-\gamma} \alpha} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha} \right).
\] (6.6)

So, we have proved that in both cases, we always have

\[
\lambda \int_{B_n} |u_n|^2 \, dx
\leq \int_{\partial_i B_n} \left( a(x)(Du_\varepsilon, x-x_0) + \frac{N-2}{2} a(x)u_\varepsilon \right) D_i u_\varepsilon n_i \\
- \int_{\partial_i B_n} \left( \frac{1}{2} a(x) |Du_\varepsilon|^2 - \frac{1}{2} \lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x) |u_\varepsilon|^{2-\varepsilon} \right) \langle n, x - x_0 \rangle \\
+ O \left( \sigma_n^{-\frac{2-\gamma}{1-\gamma} \alpha} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha} \right).
\] (6.7)
Using Corollary 4.4, noting that $|x - x_0| \leq C\sigma_n^{-\alpha}$ for $x \in \partial_i B_n$, we see

$$\int_{\partial_i B_n} \left( a(x) \langle Du_\varepsilon, x - x_0 \rangle + \frac{N-2}{2} a(x) u_\varepsilon \right) D_i u_\varepsilon n_i$$

$$- \int_{\partial_i B_n} \left( \frac{1}{2} a(x) |Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x) |u_\varepsilon|^{2^* - \varepsilon} \right) \langle n, x - x_0 \rangle$$

$$\leq C\sigma_n^{-\alpha} \int_{\partial_i B_n} (|u_n|^{2^* - \varepsilon} + u_n^2 + |Du_n|^2) d\sigma + C \int_{\partial_i B_n} |Du_n| |u_n| d\sigma,$$

$$\leq O\left(\sigma_n^{-N(1-\alpha)+o(1)} + \sigma_n^{-(N-2)+(N-4)\alpha} + \sigma_n^{-(N-2)(1-\alpha)}\right)$$

$$= O\left(\sigma_n^{-(N-2)(1-\alpha)}\right),$$

(6.8)

which, together with (6.7), implies

$$\int_{B_n} |u_n|^2 \, dx \leq + O\left(\sigma_n^{-\frac{2-\gamma}{1-\gamma}} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha}\right).$$

(6.9)

Since

$$\int_{B_n} |u_n|^2 \, dx \geq c'\sigma_n^{-2},$$

for some $c' > 0$, we obtain from (6.9),

$$\sigma_n^{-2} \leq C\left(\sigma_n^{-\frac{2-\gamma}{1-\gamma}} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha}\right).$$

(6.10)

Choose $\alpha = \frac{N-4}{N-2} - \theta$, where $\theta > 0$ is a small constant. Then $\alpha > \frac{1}{2}$, since $N \geq 7$. So, we have

$$4\alpha > 2, \quad (N-2)(1-\alpha) > 2.$$
Moreover, from
\[ \gamma \geq \min_{x_0 \in S} \left( \min \left( \frac{m(x_0) - 2}{m(x_0) - 1}, \frac{q(x_0) - 2}{q(x_0) - 1} \right) \right), \]
and
\[ \min_{x_0 \in S} \left( \min(m(x_0), q(x_0)) \right) > \frac{2(N - 2)}{N - 4}, \]
we deduce
\[ \gamma > \frac{4}{N}. \]
So,
\[ \frac{2 - \gamma}{1 - \gamma} \alpha > 2. \]
Thus, (6.10) is a contradiction. So, we have proved Theorem 2.5.

\( \square \)

**Proof of Theorem 2.6.** As pointed out in the introduction, by the result of Ambrosetti and Rabinowitz [1], for any given positive integer \( k \) and \( \varepsilon > 0 \) small, (2.4) has a solution \( u_{k, \varepsilon} \) such that \( I_\varepsilon(u_{k, \varepsilon}) = c_{k, \varepsilon} \) and \( c_{k, \varepsilon} \rightarrow +\infty \) as \( k \rightarrow +\infty \) independent of \( \varepsilon \) small. From Theorem 2.5 we can choose subsequence \( \{u_{k, \varepsilon_n}\} \) such that \( u_{k, \varepsilon_n} \rightarrow u_k \) strongly in \( H_0^1(\Omega) \) for some \( u_k \) and \( c_{k, \varepsilon_n} \rightarrow c_k \). \( u_k \) is a solution of (2.1) and \( I_0(u_k) = c_k \). Since \( c_k \) goes to \( \infty \) we get infinitely many solutions.

\( \square \)
REFERENCES


