Regularity of Refinable Functions with Exponentially Decaying Masks *

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Abstract

The smoothness property of refinable functions is an important issue in all multiresolution analysis and has a strong impact on applications of wavelets to image processing, geometric and numerical solutions of elliptic partial differential equations. The purpose of this paper is to characterize the smoothness properties of refinable functions with exponentially decaying masks and an isotropic dilation matrix by analyzing the spectral properties of associated transfer operators. The main results of this paper are really extensions of some results in [5], [14] and [25].

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1 Introduction

A homogeneous refinement equation with mask a and a general dilation matrix M is the functional equation of the form

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\, \phi(Mx-\alpha), \quad x \in \mathbb{R}^s, \tag{1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is an exponentially decaying sequence on $\mathbb{Z}^s$ called refinement mask, and M is an $s\times s$ integer matrix with $m = |\det M|$ such that $\lim_{n\to\infty} M^{-n} = 0$. The solution of (1.1) is called refinable function, and the matrix M is called a dilation matrix. It is well known that refinement equations play an important role in wavelet analysis and computer graphics. Most useful wavelets in applications are generated from refinable functions.

Great efforts have been spent on the study of refinable functions when mask a is a finitely supported sequence (see, e.g., [1, 5, 6, 10, 14, 26, 29] and many references therein). However, in some

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cases, one need to study the refinement equations with infinitely supported masks. For example, in engineering, infinitely supported masks are needed [9], the masks of various types of fractional splines in [33] are infinitely supported. In particular, due to some desirable properties, infinitely supported masks with exponential decay are of interest in the area of digital signal processing in electrical engineering (see e.g., [2, 3, 7, 12, 28]). To study Riesz bases of wavelets generated from multiresolution analysis, Han and Jia [11] investigated the L_2-solution of refinement equation (1.1) with exponentially decaying masks and a general dilation matrix. Li and Yang [24] studied existence of L_2-solution of vector refinement equation with exponentially decaying masks and a general dilation matrix. In the binary case ($s = 1, M = (2)$) and mask a is an exponentially decaying sequence, Cohen and Daubechies [3] studied the regularity of refinable functions by analyzing the spectral properties of transfer operators, Han [9] also investigated the smoothness of refinable function. In the case $s > 1$ and $M = 2I_{s\times s}$, Lorentz and Oswald [25] investigated the smoothness properties of refinable functions for the study of hierarchical bases in Sobolev spaces. In the case $s > 1$ and M is a general dilation matrix, Han and Jia [11] provided a sufficient condition to characterize the smoothness properties of refinable functions with exponentially decaying mask and a general dilation matrix.

Our goal is to investigate the smoothness of solutions of refinement equation (1.1) by studying the spectral properties of associated transfer operators, and a formula for the critical Sobolev exponent is obtained, which extends some main results in [5],[14] and [25] to the general setting. Our characterizations were inspired by the work of Cohen and Daubechies [3], Jia [14], Cohen, Gröchenig, and Villemoes [5] and Lorentz and Oswald [25].

Before proceeding further, we introduce some notations. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ denote the set of nonnegative integers. An element $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$ is called a multi-index. For $j = 1, \dots, s$, let e_j be the jth coordinate unit vector in $\mathbb{R}^s$. The norm in $\mathbb{R}^s$ is defined by

$$|y| := |y_1| + \cdots + |y_s|, \qquad y = (y_1, \cdots, y_s) \in \mathbb{R}^s.$$

We denote by $\ell(\mathbb{Z}^s)$ the linear space of all (complex valued) sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequence on $\mathbb{Z}^s$. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j a := a - a(\cdot - e_j), a \in \mathbb{Z}^s$. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$, and D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

As usual, for $1 \le p \le \infty$, we denote by $L_p(\mathbb{R}^s)$ the Banach space of all (complex-valued) functions f such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p dx \right)^{1/p} \qquad for \quad 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}^s$. We observe that $L_2(\mathbb{R}^s)$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle := \int_{\mathbb{R}^s} f(x)\overline{g(x)} dx.$$

Analogously, let $\ell_p(\mathbb{Z}^s)(1 \le p \le \infty)$ be the Banach space of all complex-valued sequence $a = (a(\alpha))_{\alpha\in\mathbb{Z}^s}$ such that $\|a\|_p < \infty$, where

$$\|a\|_p := \left(\sum_{\alpha\in\mathbb{Z}^s} |a(\alpha)|^p \right)^{1/p} \qquad for \quad 1 \le p < \infty,$$

and $\|a\|_\infty$ is the supremum of a on $\mathbb{Z}^s$.

Let b and c be two sequences on $\mathbb{Z}^s$, the convolution of b with c is defined by

$$b * c(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b(\beta)c(\alpha - \beta), \quad \alpha \in \mathbb{Z}^s$$

such that the sum is convergent. By the discrete version of Young's inequality, if $b \in \ell_1$ and $c \in \ell_p$ $(1 \le p \le \infty)$, then $b * c \in \ell_p$, and

$$\|b * c\|_p \le \|b\|_1 \|c\|_p. \tag{1.2}$$

The Fourier transform of a function f in $L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^s,$$

where $\xi \cdot x$ is the inner product of two vectors ξ and x in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to functions in $L_2(\mathbb{R}^s)$ and tempered distribution. Similarly, the Fourier series of a sequence $c \in \ell_1(\mathbb{Z}^s)$ is defined by

$$\hat{c}(\xi) := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s.$$

Evidently, $\hat{c}$ is a 2π-periodic continuous function on $\mathbb{R}^s$. In particular, $\hat{c}$ is a trigonometric polynomial if c is finitely supported. We call $\hat{c}$, the symbol of c.

In order to clarify the concept of exponential decay, we introduce the weighted space E_μ as follows. Suppose c is a (complex-valued) summable sequence on $\mathbb{Z}^s$. For $\mu \ge 0$, define

$$\|\hat{c}\|_\mu := \sum_{\alpha \in \mathbb{Z}^s} |c(\alpha)| e^{\mu|\alpha|}. \tag{1.3}$$

Let E_μ denote the Banach space of all 2π-periodic functions $\hat{c}$ on $\mathbb{R}^s$ such that $\|\hat{c}\|_\mu < \infty$.

Weighted L_p spaces are defined as follows. Suppose f is a (complex-valued) measurable function on $\mathbb{R}^s$. For $\mu \ge 0$ and $1 \le p < \infty$, define

$$\|f\|_{p,\mu} := \left(\int_{\mathbb{R}^s} |f(x) e^{\mu|x|}|^p dx \right)^{1/p}. \tag{1.4}$$

For $\mu \ge 0$, let $\|f\|_{\infty,\mu}$ be the essential supremum of the function $|f(x)|e^{\mu|x|}$ on $\mathbb{R}^s$. We use $L_{p,\mu}$ to denote the Banach space of all measure functions f such that $\|f\|_{p,\mu} < \infty$.

In this paper, we always assume that mask a is an exponentially decaying sequence, i.e., there exists a $\mu > 0$, such that $\hat{a} \in E_\mu$. Denote continuous function

$$H(\xi) := \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha\cdot\xi}, \ \xi \in \mathbb{R}^s. \tag{1.5}$$

To investigate the smoothness of a refinable function associated with exponentially decaying mask a and dilation matrix M, following [3], [15] and [25], we shall also assume that

$$H(\xi) = p(\xi) q(\xi), \quad \text{with} \quad H(0) = 1, \tag{1.6}$$

where $p(\xi)$ is a trigonometric polynomial, $q(\xi)$ is a 2π-periodic C^∞ function with exponentially decaying Fourier coefficients, and $q(\xi) \neq 0$, $\forall \xi \in \mathbb{R}^s$. The particular case $q(\xi) \equiv 1$ covers the case of finite supported masks.

Let M be a fixed matrix with $m = |\det M|$. Then the coset space $\mathbb{Z}^s/M^T\mathbb{Z}^s$ consists of m elements, where M^T denotes the transpose of M. Let $\omega_k + M^T\mathbb{Z}^s, k = 0, 1, \ldots, m-1$ be the m distinct elements of $\mathbb{Z}^s/M^T\mathbb{Z}^s$ with $\omega_0 = 0$. We denote $\Omega := \{\omega_k, k = 0, 1, ..., m-1\}$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\omega + M^T\varepsilon$, where $\omega \in \Omega$ and $\varepsilon \in \mathbb{Z}^s$. Analogously, we denote by Γ a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$.

We say that mask a satisfies the basic sum rule if

$$\sum_{\alpha\in\mathbb{Z}^s} a(\gamma + M\alpha) = \sum_{\alpha\in\mathbb{Z}^s} a(M\alpha) \quad \forall \gamma \in \Gamma.$$

Suppose M is an $s \times s$ matrix with its entries in $\mathbb{C}$. We say that M is isotropic if M is similar to a diagonal matrix diag $\{\lambda_1, ..., \lambda_s\}$ with $|\lambda_1| = \cdots = |\lambda_s|$.

For $1 \leq p \leq \infty$, denote by $\mathcal{L}_p(\mathbb{R}^s)$ the linear space of all (complex-valued) functions f such that $|f|_p < \infty$, where

$$|f|_p := \left(\int_{[0,1)^s} (\sum_{\alpha\in\mathbb{Z}^s} |f(\cdot - \alpha)|)^p dx\right)^{1/p} \qquad for \quad 1 \leq p < \infty,$$

and $|f|_\infty$ is the essential supremum of $\sum_{\alpha\in\mathbb{Z}^s} |f(\cdot-\alpha)|$ on $[0,1)^s$. Equipped with the norm $|\cdot|_p$, $\mathcal{L}_p(\mathbb{R}^s)$ becomes a Banach space. There are several important subspaces of $\mathcal{L}_p$. For instance, if $f \in L_p$ is compactly supported, then $f \in \mathcal{L}_p (1 \leq p \leq \infty)$. Also, a function $f \in L_{p,\mu}$ with $\mu > 0$ is in $\mathcal{L}_p (1 \leq p \leq \infty)$. See [16] for more discussions of $\mathcal{L}_p$ spaces.

The concept of stability plays an important role in the study of the smoothness properties of refinable functions. Let $\phi \in L_p(\mathbb{R}^s)(1 \leq p \leq \infty)$. We say that the shifts of ϕ are ℓ_p-stable if there there exist positive constants A_p and B_p such that for all sequences $a \in \ell_p(\mathbb{Z}^s)$,

$$A_p\|a\|_p \leq \left\|\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(\cdot - \alpha)\right\|_p \leq B_p\|a\|_p. \tag{1.7}$$

See [16], [19] for more detail about ℓ_p-stability. It was proved by Jia and Micchelli in [16] that a function $\phi \in \mathcal{L}_p(\mathbb{R}^s)$ has ℓ_p-stable integer translates if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that

$$\hat{\phi}(\xi + 2\pi\beta) \neq 0.$$

For $\nu \geq 0$, denote by $H^\nu(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2(1 + |\xi|^2)^\nu d\xi < \infty.$$

The critical Sobolev exponent of a function $f \in L_2(\mathbb{R}^s)$ is defined by

$$s_f := \sup\{\nu : f \in H^\nu(\mathbb{R}^s)\}.$$

Sobolev spaces are closely related to Lipschitz Spaces, which is defined in terms of the modulus of continuity. For $y \in \mathbb{R}^s$, the difference operator is defined by

$$\nabla_y f := f - f(\cdot - y),$$

where f is a function defined on $\mathbb{R}^s$. The modulus of continuity of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega(f,h)_p := \sup_{|y|\leq h} \|\nabla_y f\|_p, \qquad h \geq 0.$$

Let k be a positive integer. The kth modulus of continuity of f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f,h)_p := \sup_{|y|\leq h} \|\nabla_y^k f\|_p, \qquad h \geq 0.$$

For $1 \leq p \leq \infty$, $0 < \nu \leq 1$ and a function $f \in L_p(\mathbb{R}^s)$, we say f belongs to the Lipschitz Space $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ if there exists a positive constant C independent of h such that

$$w(f,h)_p \leq Ch^\nu \qquad \forall\, h > 0.$$

For $\nu > 0$, we write $\nu = r + \eta$, where r is an integer and $0 < \eta \leq 1$. We say that f belongs to the Lipschitz Space $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ if $D^\mu f \in \mathrm{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu| = r$. For $\nu > 0$, let k be an integer greater than ν. The generalized Lipschitz Space $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$ consists of all functions $f \in L_p(\mathbb{R}^s)$ such that

$$\omega_k(f,h)_p \leq Ch^\nu \qquad \forall\, h > 0,$$

where C is a positive constant independent of h. If $\nu > 0$ is not an integer, then

$$\mathrm{Lip}(\nu, L_p(\mathbb{R}^s)) = \mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s)), \qquad 1 \leq p \leq \infty.$$

See [8], [32] and [14] for more detail about Lipschitz Spaces.

Following the discussion in [14], we have

$$\sup\{\nu : f \in H^\nu(\mathbb{R}^s)\} = \sup\{\nu : f \in \mathrm{Lip}(\nu, L_2(\mathbb{R}^s))\} = \sup\{\nu : f \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))\}.$$

2 Some auxiliary results

In this section, we shall introduce some auxiliary results. In the following, let $\mathcal{T} := [-\pi, \pi]^s$. We denote by $C(\mathcal{T})$ the space of all continuous 2π-period functions on $\mathbb{R}^s$ and by $L_\infty(\mathcal{T})$ the space of all 2π-period measurable functions f on $\mathcal{T}$ such that $\|f\|_{L_\infty(\mathcal{T})} < \infty$, where $\|f\|_{L_\infty(\mathcal{T})}$ denotes the essential supremum of $|f|$ on $\mathcal{T}$.

Following [6] and [25], it is easy to obtain Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *Let a be an exponentially decaying sequence and $H(\xi)$ be given by* (1.5) *with $H(0) = 1$, then the infinite product*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} H((M^T)^{-j}\xi) \tag{2.1}$$

converges absolutely and uniformly on any compact subsets of $\mathbb{R}^s$ to a continuous function.

Lemma 2.2. *For $\mu > 0$, let b be an exponentially decaying sequence with $\hat{b} \in E_\mu$ and $\hat{b}(0) = 1$, then for all positive integers N, there exist trigonometric polynomials*

$$\hat{b}_N(\xi) = \sum_{|\alpha| \le N} b_N(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s,$$

such that

$$\|\hat{b}(\xi) - \hat{b}_N(\xi)\|_{L_\infty(\mathcal{T})} \le Ce^{-\mu N}, \quad and \quad \hat{b}_N(0) = 1,$$

where C is a positive constant independent of N and $|\alpha| = |\alpha_1| + \cdots + |\alpha_s|$ with $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{Z}^s$.

Proof. Let $\hat{b}_N(\xi)$ be defined by

$$\hat{b}_N(\xi) = \sum_{|\alpha| \le N} b(\alpha) e^{-i\alpha\cdot\xi} + \sum_{|\alpha| > N} b(\alpha).$$

Then we have $\hat{b}_N(0) = 1$ and

$$\|\hat{b}(\xi) - \hat{b}_N(\xi)\|_\infty \le \sum_{|\alpha| > N} 2|b(\alpha)| \le Ce^{-\mu N},$$

where C is independent of N. □

Transfer operator is an useful tool for the study of refinable functions (see e.g., [3, 5, 6, 17] and [31]). For a given 2π-periodic function $u(\xi)$, the transition operator T_u acts on 2π-periodic functions according to

$$T_u f(\xi) = \sum_{\omega_i \in \Omega} u((M^T)^{-1}(\xi + 2\pi\omega_i)) f((M^T)^{-1}(\xi + 2\pi\omega_i)). \tag{2.2}$$

We fix $u(\xi) = |H(\xi)|^2$, where $H(\xi)$ is defined by (1.5) and $H(\xi) \in E_\mu$ for some $\mu > 0$. It is easy to see that $u(\xi) \in E_\mu$. In terms of the Fourier coefficients u_α of $u(\xi) = \sum_{\alpha\in\mathbb{Z}^s} u_\alpha e^{-i\alpha\cdot\xi}$, (2.2) can also be rewritten as

$$(T_u f)_\alpha = \frac{1}{(2\pi)^s} \int_{[0,2\pi)^s} T_u f(\xi) e^{i\alpha\cdot\xi} d\xi = m \sum_{\beta\in\mathbb{Z}^s} u_{M\alpha-\beta} f_\beta, \tag{2.3}$$

where $f(\xi) = \sum_{\beta\in\mathbb{Z}^s} f_\beta e^{-i\beta\cdot\xi}$.

It follows from [11] that

$$\|T_u f\|_\mu = \sum_{\alpha\in\mathbb{Z}^s} |(T_u f)_\alpha| e^{\mu|\alpha|} = m \sum_{\alpha\in\mathbb{Z}^s} |\sum_{\beta\in\mathbb{Z}^s} u_{M\alpha-\beta} f_\beta| e^{\mu|\alpha|} \le m\|u\|_\mu \|f\|_\mu.$$

Therefore, T_u is a bounded linear operator on E_μ. Furthermore, T_u is also a compact operator on E_u.

Let $V = \big\{\hat{v} \in E_\mu : \quad \sum_{\alpha\in\mathbb{Z}^s} |v(\alpha)||\alpha| < \infty \quad and \quad \sum_{\alpha\in\mathbb{Z}^s} v(\alpha) = 0\big\}$.

Theorem 2.3. *Let T_u be given by (2.2) with u being exponential decay. For $L \in \mathbb{N}$, define $z_L = [\sin^2(\frac{\xi_1}{2}) + \cdots + \sin^2(\frac{\xi_s}{2})]^L$. Then*

$$\overline{\lim}_{k\to\infty}\|T_u^k z_L\|_{L_\infty(\mathcal{T})}^{1/k} \leq \overline{\lim}_{k\to\infty}\|T_u^k z_L\|_\mu^{1/k} \leq \rho(T_u|_V), \tag{2.4}$$

where $\rho(T_u|_V)$ denotes the spectral radius of the restriction of T_u to the subspace V of E_μ.

Proof. It is easy to check that $z_L \in V$. Hence, for $k \geq 1$, we have

$$\|T_u^k z_L\|_{L_\infty(\mathcal{T})} \leq \|T_u^k z_L\|_\mu \leq \|T_u^k|_V\|_\mu \|z_L\|_\mu.$$

Since

$$\lim_{k\to\infty} \|T_u^k|_V\|_\mu^{1/k} = \rho(T_u|_V),$$

we obtain that

$$\overline{\lim}_{k\to\infty}\|T_u^k z_L\|_{L_\infty(\mathcal{T})}^{1/k} \leq \overline{\lim}_{k\to\infty}\|T_u^k z_L\|_\mu^{1/k} \leq \rho(T_u|_V).$$

□

Lemma 2.4. *[21, 5] Let T_u be given by (2.2) with u being exponential decay. For any $f(\xi), g(\xi) \in C(\mathcal{T})$, and any positive integer n, we have the following identity*

$$\int_{\mathcal{T}} T_u^n f(\xi)\overline{g(\xi)}d\xi = \int_{(M^T)^n\mathcal{T}} [\prod_{k=1}^n u((M^T)^{-k}\xi)] f((M^T)^{-n}\xi)\overline{g(\xi)}d\xi.$$

To characterize the smoothness of refinable functions with masks having exponential decay, we need the following Theorem 2.5, which was established by Lorentz and Oswald in [25] for the special case $M = 2I_{s\times s}$.

Theorem 2.5. *Let $u(\xi) = |H(\xi)|^2 := P(\xi)Q(\xi)$, where $H(\xi)$ is defined by (1.5) and $P(\xi) = |p(\xi)|^2, Q(\xi) = |q(\xi)|^2$. Suppose $Q_N(\xi)$ is a trigonometric polynomial sequence of approximations of $Q(\xi)$ as in Lemma 2.2 such that $\|Q(\xi) - Q_N(\xi)\|_{L_\infty(\mathcal{T})} \leq Ce^{-\mu N}$. Let $u_N(\xi) = P(\xi)Q_N(\xi)$. Then*

$$\rho_L = \lim_{N\to\infty} \rho(T_{u_N}|_{V_{u_N,Z_L}}) \tag{2.5}$$

and

$$\rho = \lim_{L\to\infty} \rho_L \tag{2.6}$$

both exist and do not depend on the specific choice of the sequence $Q_N(\xi)$, where V_{u_N,Z_L} is defined by

$$V_{u_N,Z_L} = span\left\{T_{u_N}^n z_L : \quad n \geq 0\right\}.$$

Proof. Since $Q(\xi)$ is a positive 2π-periodic continuous function, there exists a positive constant c, such that $|Q(\xi)| \geq c > 0$. Consequently, for sufficiently large N, we have

$$\max(|1 - \frac{Q_N(\xi)}{Q(\xi)}|, |1 - \frac{Q(\xi)}{Q_N(\xi)}|) \leq C^* e^{-\mu N}, \tag{2.7}$$

where C^* is a positive constant independent of N.

Therefore,

$$(1 - C^* e^{-\mu N}) Q_N(\xi) \le Q(\xi) \le (1 + C^* e^{-\mu N}) Q_N(\xi),$$

and

$$(1 - C^* e^{-\mu N}) u_N(\xi) \le u(\xi) \le (1 + C^* e^{-\mu N}) u_N(\xi).$$

By Lemma2.4, we have

$$(1 - C^* e^{-\mu N}) \|T_{u_N}^k z_L\|_{L_\infty(\mathcal{T})}^{1/k} \le \|T_u^k z_L\|_{L_\infty(\mathcal{T})}^{1/k} \le (1 + C^* e^{-\mu N}) \|T_{u_N}^k z_L\|_{L_\infty(\mathcal{T})}^{1/k}. \tag{2.8}$$

Since u_N is a trigonometric polynomial, by Lemma 2.4 of [10], we have that V_{u_N,Z_L} is finite dimensional and

$$\lim_{k\to\infty} \|T_{u_N}^k z_L\|_{L_\infty(\mathcal{T})}^{1/k} = \rho(T_{u_N}|_{V_{u_N,Z_L}}).$$

Let $k \to \infty$ in (2.8), by Theorem 2.3, we have that

$$(1 - C^* e^{-\mu N}) \rho(T_{u_N}|_{V_{u_N,Z_L}}) \le \overline{\lim}_{k\to\infty} \|T_u^k z_L\|_{L_\infty(\mathcal{T})}^{1/k} \le (1 + C^* e^{-\mu N}) \rho(T_{u_N}|_{V_{u_N,Z_L}}). \tag{2.9}$$

Then, let $N \to \infty$ in (2.9), we obtain that

$$\rho_L := \lim_{N\to\infty} \rho(T_{u_N}|_{Z_L}) = \overline{\lim}_{k\to\infty} \|T_u^k z_L\|_{L_\infty(\mathcal{T})}^{\frac{1}{k}} \tag{2.10}$$

exists.

If $\tilde{u}_N(\xi)$ is another sequence of approximations of $u(\xi)$, we can easily obtain

$$(1 - C^* e^{-\mu N}) u_N(\xi) \le \tilde{u}_N(\xi) \le (1 + C^* e^{-\mu N}) u_N(\xi).$$

Thus,

$$(1 - C^* e^{-\mu N})^k \|T_{u_N}^k z_L\|_{L_\infty(\mathcal{T})} \le \|T_{\tilde{u}_N}^k z_L\|_{L_\infty(\mathcal{T})} \le (1 + C^* e^{-\mu N})^k \|T_{u_N}^k z_L\|_{L_\infty(\mathcal{T})}.$$

This together with (2.10) imply that ρ_L is independent of the specific choice of the sequence u_N. Since $0 \le z_{L+1} \le s z_L$, (2.10) yields

$$\rho = \lim_{L\to\infty} \rho_L \le \rho_{L+1} \le \rho_L \cdots \le \rho_0. \tag{2.11}$$

The proof of Theorem 2.5 is complete. □

3 Characterization of Smoothness

In this section, we give some characterizations of the smoothness property of refinable functions with mask a having exponential decay and an isotropic dilation matrix M.

When mask a is finitely supported, $H(\xi) = \frac{1}{m} \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) e^{-i\alpha\cdot\xi}$ is a trigonometric polynomial. Define S_c as the largest nonnegative integer such that

$$D^\mu H(2\pi(M^T)^{-1}\omega) = 0, \quad \forall\, \omega \in \Omega\backslash\{0\}, \quad \textit{and} \quad |\mu| \le S_c - 1.$$

In the following, L denotes an integer large than S_c.

It follows from [5] that $V_{|H|^2,Z_L} = span\{T^k_{|H|^2} z_L, k \geq 0, k \in \mathbb{N}_0\}$ is finite dimensional and is an invariant subspace of $T_{|H|^2}$.

Cohen, Gröchenig, and Villemoes [5] gave a characterization of the smoothness of refinable functions based on the spectral radius of $T_{|H|^2}$ restricted to $V_{|H|^2,Z_L}$ when mask a is finitely supported. Their results were stated as following:

Theorem 3.1. [*Cohen et.al*] *Let $\phi \in L_2(\mathbb{R}^s)$ be a compactly supported solution to* (1.1) *with mask a being finitely supported and M being an isotropic dilation matrix. Suppose the shifts of ϕ are stable. Let ρ be the spectral radius of $T_{|H|^2}$ restricted to $V_{|H|^2,Z_L}$. Then the Sobolev exponent s_ϕ satisfies*

$$s_\phi = -\frac{s}{2}\log_m \rho, \tag{3.1}$$

and $\phi \in H^s$ if and only if $s < s_\phi$, where $m = |\det M|$.

We also point out that similar characterization was also established by Jia in [14] with a different method. Several other researchers have considered the regularity of refinable functions in high dimensions (see [4, 18, 20, 21, 22, 23, 27, 30] and [31]).

We are in a position to establish the following characterization of smoothness of a refinable function in terms of the refinement mask.

Theorem 3.2. *Let M be an isotropic dilation matrix with $|detM| = m$ and ϕ be a L_2-solution of refinement equation* (1.1). *Assume that $H(\xi)$ satisfies* (1.6). *Let ρ be defined by* (2.6). *Then $s_\phi \geq -\frac{s}{2}\log_m \rho$. Moreover, if the shifts of ϕ are stable, then $s_\phi = -\frac{s}{2}\log_m \rho$.*

Proof. Set $\mathcal{T} = [-\pi, \pi]^s, G = \mathcal{T}\backslash(M^T)^{-1}\mathcal{T}$, then $\mathbb{R}^s = \mathcal{T} \cup \bigcup_{n=1}^\infty (M^T)^n G$. For $\xi \in (M^T)^n\mathcal{T}$, it is easy to see that

$$|\hat{\phi}(\xi)|^2 \leq C_1 \prod_{k=1}^n |H((M^T)^{-k}\xi)|^2 \tag{3.2}$$

with $C_1 = \max_{\xi\in\mathcal{T}} |\hat{\phi}(\xi)|^2$. For any fixed $L \in \mathbb{N}$, since $(M^T)^{-1}\mathcal{T}$ contains an open ball centered at the origin, we have

$$z_L(\xi) \geq C_2 > 0 \qquad for \quad \xi \in G. \tag{3.3}$$

It follows from Lemma 2.4,

$$\int_{(M^T)^n G} [\prod_{k=1}^n |H((M^T)^{-k}\xi)|^2] z_L((M^T)^{-n}\xi) d\xi = \langle T^n_{|H|^2} z_L, 1\rangle \leq \|T^n_{|H|^2} z_L\|_{L_\infty(\mathcal{T})}. \tag{3.4}$$

Thus, combine (3.2), (3.3), and (3.4), we have

$$\int_{(M^T)^n G} |\hat{\phi}(\xi)|^2 (1+|\xi|^2)^\gamma \leq C_3 m^{\frac{2\gamma n}{s}} \int_{(M^T)^n G} |\hat{\phi}(\xi)|^2 d\xi \leq C_4 m^{\frac{2\gamma n}{s}} \|T^n_{|H|^2} z_L\|_{L_\infty(\mathcal{T})}. \tag{3.5}$$

Note that,

$$\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2 (1+|\xi|^2)^\gamma d\xi = \int_{\mathcal{T}} |\hat{\phi}(\xi)|^2 (1+|\xi|^2)^\gamma d\xi + \sum_{n=1}^\infty \int_{(M^T)^n G} |\hat{\phi}(\xi)|^2 (1+|\xi|^2)^\gamma d\xi. \tag{3.6}$$

Since $|\hat{\phi}(\xi)|^2(1+|\xi|^2)^\gamma$ is continuous on $\mathcal{T}$, the first integer on the right-hand side of (3.6) is a finite constant C_5. This together with (3.5) imply that

$$\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1+|\xi|^2)^\gamma d\xi \le C_5 + C_4 \sum_{n=1}^{\infty} m^{\frac{2\gamma n}{s}} \|T^n_{|H|^2} z_L\|_{L_\infty(\mathcal{T})}.$$

By Theorem 2.5, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$, such that for large enough L,

$$\|T^n_{|H|^2} z_L\|_{L_\infty(\mathcal{T})} \le C(\varepsilon)(\rho+\varepsilon)^n, \qquad n \in \mathbb{N}.$$

Consequently,

$$\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1+|\xi|^2)^\gamma d\xi \le C_5 + C_4 C(\varepsilon) \sum_{n=1}^{\infty} m^{\frac{2\gamma n}{s}} (\rho+\varepsilon)^n.$$

When $\gamma < -\frac{s}{2}\log_m(\rho+\varepsilon)$,

$$\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1+|\xi|^2)^\gamma d\xi < \infty.$$

Since ε can be chosen arbitrary small, we conclude that $s_\phi \ge -\frac{s}{2}\log_m \rho$ and the first part of the theorem follows. In all these estimates, the positive constants C_j are independent of n.

To prove the second part, for $H(\xi) = p(\xi)q(\xi)$, we can choose a sequence of trigonometric polynomials $q_N(\xi)$ of approximations of $q(\xi)$ as in Lemma 2.2. Since the shifts of ϕ are stable, it follows from Theorem 4.1 of [11] that the cascade algorithm associated mask a and dilation matrix M converges in $L_2(\mathbb{R}^s)$. Therefore, a satisfies the basic sum rule and $\rho(T_{|H|^2}|_V) < 1$. Let

$$H_N(\xi) = p(\xi)q_N(\xi),$$

then $H_N(\xi)$ generate a sequence of refinable functions $\phi_N(\xi)$ with compact support. Since a satisfies the basic sum rule if and only if $H(2\pi(M^T)^{-1}\omega) = 0, \forall\, \omega \in \Omega\backslash\{0\}$. It follows that the masks a_N associated with ϕ_N also satisfy the basic sum rule for N large enough. By the choice of H_N, we have that

$$\lim_{N\to\infty} T_{|H_N|^2} \to T_{|H|^2}$$

in the E_μ-norm, which implies that

$$\rho(T_{|H_N|^2}|_V) < 1,$$

for N large enough. Then, it follows from Theorem 4.2 of [11] that the cascade algorithms associated with masks a_N and dilation matrix M converge in L_2, which implies that $\phi_N \in L_2$.

Since

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} H(\frac{\xi}{2^j}) = \prod_{j=1}^{\infty} p(\frac{\xi}{2^j}) \prod_{j=1}^{\infty} q(\frac{\xi}{2^j}),$$

and

$$\hat{\phi}_N(\xi) = \prod_{j=1}^{\infty} H_N(\frac{\xi}{2^j}) = \prod_{j=1}^{\infty} p(\frac{\xi}{2^j}) \prod_{j=1}^{\infty} q_N(\frac{\xi}{2^j}).$$

If the shifts of ϕ are stable, we obtain that there exist positive constants A and B such that

$$A \le \sum_{k\in\mathbb{Z}^s} |\hat{\phi}(\xi+2\pi k)|^2 \le B, \quad \forall \xi \in \mathbb{R}^s.$$

Thus, for any $\xi \in \mathbb{R}^s$, there exists a $\alpha \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\pi\alpha) \neq 0$, which implies that $\prod_{j=1}^{\infty} p(\frac{\xi+2\pi\alpha}{2^j}) \neq 0$. Therefore, $\hat{\phi}_N(\xi + 2\pi\alpha) \neq 0$ for N large enough. We obtain that the shifts of ϕ_N are also stable for sufficiently large N.

Assume $\phi \in H^{\gamma}(\mathbb{R}^s)$ for some $\gamma > -\frac{s}{2}\log_m \rho$, then

$$\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1+|\xi|^2)^{\gamma} d\xi < \infty.$$

Fix any $\tilde{\gamma} \in (-\frac{s}{2}\log_m \rho, \gamma)$, we claim that $\phi_N \in H^{\tilde{\gamma}}(\mathbb{R}^s)$ for N large enough.

For sufficiently large N, we have $1 + C^* e^{-\mu N} < m^{\frac{2(\gamma-\tilde{\gamma})}{s}}$, where C^* is the same as in (2.7). Set $\Omega_k = (M^T)^k \mathcal{T} \backslash (M^T)^{(k-1)}\mathcal{T}$, it is easy to see that $\mathbb{R}^s = (\bigcup_{k=1}^{\infty} \Omega_k) \bigcup \mathcal{T}$. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^s} (1+|\xi|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi)|^2 d\xi \\
&= \int_{\mathcal{T}} \sum_{\alpha\in\mathbb{Z}^s} (1+|\xi+2\pi\alpha|^2)^{\tilde{\gamma}} \left(\prod_{k=1}^{\infty} |H_N((M^T)^{-k}(\xi+2\pi\alpha))|^2 \right) d\xi \\
&\le C \sum_{k=1}^{\infty} m^{\frac{2\tilde{\gamma}k}{s}} \int_{\mathcal{T}} \sum_{\alpha\in\Omega_k} \left(\prod_{l=1}^{k} |H_N((M^T)^{-l}(\xi+2\pi\alpha))|^2 \right) d\xi + \int_{\mathcal{T}} \sum_{\alpha\in\mathcal{T}} (1+|\xi+2\pi\alpha|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi+2\pi\alpha)|^2 d\xi.
\end{aligned}$$

Note that $|\hat{\phi}_N(\xi+2\pi\alpha)|^2$ is continuous on $\mathcal{T}$. Hence, there exists a positive constant B_1 such that $\int_{\mathcal{T}} \sum_{\alpha\in\mathcal{T}} (1+|\xi+2\pi\alpha|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi+2\pi\alpha)|^2 d\xi \le B_1$.

Consequently, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^s} (1+|\xi|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi)|^2 d\xi \\
&\le C \sum_{k=1}^{\infty} m^{\frac{2\gamma k}{s}} (1+C^* e^{-\mu N})^{-k} \int_{\mathcal{T}} \sum_{\alpha\in\Omega_k} \left(\prod_{l=1}^{k} |H_N((M^T)^{-l}(\xi+2\pi\alpha))|^2 \right) d\xi + B_1 \\
&\le C \sum_{k=1}^{\infty} m^{\frac{2\gamma k}{s}} \int_{\mathcal{T}} \sum_{\alpha\in\Omega_k} [\prod_{l=1}^{k} |H((M^T)^{-l}(\xi+2\pi\alpha))|^2] d\xi + B_1 \\
&= C \int_{\mathcal{T}} \sum_{k=1}^{\infty} m^{\frac{2\gamma k}{s}} \sum_{\alpha\in\Omega_k} [\prod_{l=1}^{k} |H((M^T)^{-l}(\xi+2\pi\alpha))|^2] d\xi + B_1 \\
&\le \tilde{C} \int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2 (1+|\xi|^2)^{\gamma} d\xi + B_1 < \infty.
\end{aligned}$$

Therefore, $\phi_N \in H^{\tilde{\gamma}}(\mathbb{R}^s)$ for N large enough. Since the shifts of ϕ_N are stable, it follows from Theorem 3.1 that $s_{\phi_N} = -\frac{s}{2}\log_m \rho(T_{u_N}, V_{u_N,Z_L})$ for any $L \ge L_0$. Therefore, $s_{\phi_N} \ge \tilde{\gamma} > -\frac{s}{2}\log_m \rho$ for N large enough. By the definition of ρ, this is impossible. Hence, $s_\phi \le -\frac{s}{2}\log_m \rho$. Combined with the proof of the first part, we complete the proof of Theorem 3.2.

□

Remark 3.3. *Theorem 3.2 characterizes the optimal smoothness of a refinable function with mask a having exponential decay and an isotropic dilation matrix M, which extends Theorem* 3.1 *to the* *case that mask a is infinitely supported. Theorem 3.2 was also established in [25] for the case $M = 2I_{s\times s}$.*

In some cases, when mask a is exponential decay, the solution of equation (1.1) belongs to $\mathcal{L}_p(\mathbb{R}^s)$ for $1 \le p \le \infty$. For example, Han [9] characterized the existence of $L_{2,\mu}$-solution of equation (1.1) with a being exponential decay and $M = 2$ for the case $s = 1$, and Jia [15] also gave a characterization of the existence of the solution of refinement equation in $L_{p,\mu}(1 \le p \le \infty)$ by considering the convergence of the corresponding cascade algorithm associated with mask a being exponential decay. Hogan [13] investigated some properties of refinement equation (1.1) under the assumptions that solution $\phi \in \mathcal{L}_p(\mathbb{R}^s)$ for $1 \le p \le \infty$. In this section, we will characterize the smoothness of solution of refinement equation (1.1) in $\mathcal{L}_p(\mathbb{R}^s)$ for $1 \le p \le \infty$. The following Theorems 3.4 and 3.5 give some characterizations of the smoothness of refinable functions in $\mathcal{L}_2(\mathbb{R}^s)$ and in $\mathcal{L}_p(\mathbb{R}^s)$ for $1 \le p \le \infty$, respectively. Our characterizations are based on a discrete version of Young's inequality.

Theorem 3.4. *Suppose $\nu > 0$ and k is a positive integer. Let M be an isotropic dilation matrix with $m = |\det M|$. Suppose $\phi \in \mathcal{L}_2(\mathbb{R}^s)$ is the normalized solution of (1.1) with mask a being exponential decay. For $n = 1, 2, \ldots$, let a_n be given by*

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)a_{n-1}(\beta), \quad \alpha \in \mathbb{Z}^s, \tag{3.7}$$

where a_0 is the δ sequence given by $\delta(0) = 1$ and $\delta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s\backslash\{0\}$.
If there exists a constant $C > 0$ such that

$$\|\nabla_j^k a_n\|_2 \le C m^{(1/2-\nu/s)n} \qquad \forall n \in \mathbb{N} \quad \text{and} \quad j = 1, ..., s, \tag{3.8}$$

then $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$. Conversely, if $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$, and the shifts of ϕ are stable, then (3.8) holds for $k > \nu$.

Proof. Since $\phi \in \mathcal{L}_2(\mathbb{R}^s)$ is the normalized solution of (1.1), it is easy to check that

$$\phi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\phi(M^n \cdot -\alpha). \tag{3.9}$$

Therefore,

$$\nabla_{M^{-n}e_j}^k \phi = \sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n(\alpha)\phi(M^n \cdot -\alpha). \tag{3.10}$$

By (1.2), we obtain that

$$\begin{aligned}
\|\nabla_{M^{-n}e_j}^k \phi\|_2 &= \left(\int_{\mathbb{R}^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n \phi(M^n x - \alpha)|^2 dx\right)^{\frac{1}{2}} \\
&= m^{-\frac{n}{2}} \left(\sum_{\beta\in\mathbb{Z}^s} \int_{\beta+[0,1]^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n(\alpha)\phi(x-\alpha)|^2 dx\right)^{\frac{1}{2}} \\
&= m^{-\frac{n}{2}} \left(\sum_{\beta\in\mathbb{Z}^s} \int_{[0,1]^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n(\alpha)\phi(x-\beta-\alpha)|^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$= m^{-\frac{n}{2}} \left(\int_{[0,1]^s} \sum_{\beta \in \mathbb{Z}^s} | \sum_{\alpha \in \mathbb{Z}^s} \nabla_j^k a_n(\alpha) \phi(x - \beta - \alpha)|^2 dx \right)^{\frac{1}{2}}$$

$$\leq m^{-\frac{n}{2}} \left(\int_{[0,1]^s} \|\nabla_j^k a_n\|_2^2 (\sum_{\beta \in \mathbb{Z}^s} |\phi(x - \beta)|)^2 dx \right)^{\frac{1}{2}}$$

$$\leq m^{-\frac{n}{2}} \|\nabla_j^k a_n\|_2^2 \, |\phi|_{\mathcal{L}_2[0,1]^s} \, .$$

This in connection with (3.8) tell us that

$$\|\nabla^k_{M^{-n}e_j} \phi\|_2 \leq C m^{-\frac{\nu n}{s}} \, |\phi|_{\mathcal{L}_2[0,1]^s} \, . \tag{3.11}$$

It follows from Theorem 2.1 in [14] that (3.11) is equivalent to

$$\|\nabla^k_y \phi\|_2 \leq C|y|^\nu \qquad \forall y \in \mathbb{R}^s. \tag{3.12}$$

This shows $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$, as desired.

Conversely, since the shifts of ϕ are stable. It follows from (1.7) and (3.10) that

$$m^{-\frac{n}{2}} \|\nabla_j^k a_n\|_2 \leq C_2 \|\nabla^k_{M^{-n}e_j} \phi\|_2, \tag{3.13}$$

where C_2 is a constant independent of n and j. If $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$, then for $k > \nu$

$$\|\nabla^k_{M^{-n}e_j} \phi\|_2 \leq C m^{-\frac{\nu n}{s}}. \tag{3.14}$$

Therefore, (3.8) follows from (3.13) and (3.14) immediately. □

Theorem 3.5. *Let $\nu > 0$ and k be a positive integer. Suppose $M = qI_{s\times s}$ and $q \geq 2$ is an integer. Let ϕ be the normalized solution of* (1.1) *with mask a being exponential decay and $\phi \in \mathcal{L}_p(\mathbb{R}^s)$ for $1 \leq p \leq \infty$. For $n = 1, 2, \ldots$, let a_n be given by* (3.7)*. If there exists a constant $C > 0$ such that*

$$\|\nabla_j^k a_n\|_p \leq C q^{(s/p - \nu)n} \qquad \forall n \in \mathbb{N} \quad \textit{and} \quad j = 1, ..., s, \tag{3.15}$$

then $\phi \in Lip^*(\nu, L_p(\mathbb{R}^s))$. Conversely, if $\phi \in Lip^*(\nu, L_p(\mathbb{R}^s))$, and the shifts of ϕ are ℓ_p-stable, then (3.15) *holds for $k > \nu$.*

Proof. Since ϕ is the normalized solution of (1.1), we have that

$$\nabla^k_{q^{-n}e_j} \phi = \sum_{\alpha \in \mathbb{Z}^s} \nabla_j^k a_n(\alpha) \phi(q^n \cdot - \alpha). \tag{3.16}$$

By virtue of (1.2), we have

$$
\begin{aligned}
\|\nabla^k_{q^{-n}e_j}\phi\|_p &= \left(\int_{\mathbb{R}^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla^k_j a_n \phi(q^n x-\alpha)|^p dx\right)^{\frac{1}{p}} \\
&= q^{-\frac{ns}{p}} \left(\sum_{\beta\in\mathbb{Z}^s} \int_{\beta+[0,1]^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla^k_j a_n(\alpha)\phi(x-\alpha)|^p dx\right)^{\frac{1}{p}} \\
&= q^{-\frac{ns}{p}} \left(\sum_{\beta\in\mathbb{Z}^s} \int_{[0,1]^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla^k_j a_n(\alpha)\phi(x-\beta-\alpha)|^p dx\right)^{\frac{1}{p}} \\
&= q^{-\frac{ns}{p}} \left(\int_{[0,1]^s} \sum_{\beta\in\mathbb{Z}^s} |\sum_{\alpha\in\mathbb{Z}^s} \nabla^k_j a_n(\alpha)\phi(x-\beta-\alpha)|^p dx\right)^{\frac{1}{p}} \\
&\le q^{-\frac{ns}{p}} \left(\int_{[0,1]^s} \|\nabla^k_j a_n\|_p^p (\sum_{\beta\in\mathbb{Z}^s} |\phi(x-\beta)|)^p dx\right)^{\frac{1}{p}} \\
&\le q^{-\frac{ns}{p}} \|\nabla^k_j a_n\|_p \, |\phi|_{\mathcal{L}_p[0,1]^s}\,.
\end{aligned}
$$

By (3.15), we obtain that

$$\|\nabla^k_{q^{-n}e_j}\phi\|_p \le Cq^{-\nu n}\, |\phi|_{\mathcal{L}_p[0,1]^s}\,. \tag{3.17}$$

From Theorem 3.1 in [23], we know that (3.17) is equivalent to

$$\|\nabla^k_y\phi\|_p \le C|y|^\nu \qquad \forall y\in\mathbb{R}^s. \tag{3.18}$$

This shows $\phi\in Lip^*(\nu, L_p(\mathbb{R}^s))$, as desired.

Conversely, since the shifts of ϕ are ℓ_p-stable. It follows from (1.7) and (3.16) that

$$q^{-\frac{ns}{p}}\|\nabla^k_j a_n\|_p \le C_2\|\nabla^k_{q^{-n}e_j}\phi\|_p, \tag{3.19}$$

where C_2 is a constant independent of n and j. If $\phi\in Lip^*(\nu, L_p(\mathbb{R}^s))$, then for $k>\nu$

$$\|\nabla^k_{q^{-n}e_j}\phi\|_p \le Cq^{-\nu n}. \tag{3.20}$$

Therefore, (3.15) follows from (3.19) and (3.20). The proof of the theorem is complete. □

Remark 3.6. *We remark that Theorem* 3.4 *was established by Jia in* [*14*] *for the case in which* ϕ *is a compactly supported* L_2*-solution of equation* (1.1) *and Theorem* 3.5 *was obtained by Li in* [*23*] *and Jia et.al in* [*18*] *for the case in which* ϕ *is a compactly supported* L_p*-solution of equation* (1.1) *for* $1\le p\le\infty$*.*

References

[1] A. S. Cavaretta, W. Dahmen, C. A. Micchelli, Stationay subdivision, Mem. Amer. Math. Soc. **93** (1991), 1-186.

[2] A. Cohen, Wavelets and Multiscale Signal Processing, Chapman & Hall, New York, 1995.

[3] A. Cohen, I. Daubechies, A new technique to estimate the regularity of refinable functions, Rev. Mat. Iberoamericana. **12** (1996), 527-591.

[4] A. Cohen, I. Daubechies, and G. Plonka, Regularity of refinable function vectors, J. Fourier Anal. Appl. **3** (1997), 295-324.

[5] A. Cohen, K. Gröchenig, L. F. Villemoes, Regularity of multivariate refinable functions, Constr. Approx. **15** (1999), 241-255.

[6] I. Daubechies, Ten lectures on wavelets, CBMS-NSF Conference, SIAM, Philadelphia, 1992.

[7] I. Daubechies, Y. Huang, A decay theorem for refinable functions, Appl. Math. Lett. **7** (1994), 1-4.

[8] R. A. Devore and G. G. Lorentz, Constuctive Approximation, Springer-Verlag, Berlin, 1993.

[9] B. Han, Refinable functions and cascade algorithms in weighted spaces with hölder continuous masks, SIAM J. Math. Anal. **41** (2008), 70-102.

[10] B. Han, R. Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, SIAM J. Math. Anal. **29** (1998), 1177-1199.

[11] B. Han, R. Q. Jia, Characterization of Riesz bases of wavelets generated from multiresolution analysis, Appl. Comput. Harmon. Anal. **23** (2007), 321-345.

[12] C. Herley, M. Vetterli, Wavelets and recursive filter banks, IEEE Tran. Signal Proc. **41** (1993), 2536-2556.

[13] T. A. Hogan, A note on matrix refinement equations, SIAM J. Math. Anal. **29**(1998), 849-854.

[14] R. Q. Jia, Characterization of smoothness of multivariate refinable functions in Sobolev spaces, Trans. Amer. Math. Soc. **351** (1999), 4089-4112.

[15] R. Q. Jia, Refinable Functions with Exponential Decay: An Approach via Cascade Algorithms, manuscript.

[16] R. Q. Jia, C. A. Micchelli, Using the refinement equations for the construction of pre-wavelets II: Powers of two, in Curves and Surfaces, P. J. Laurent, A. Laurent, A. Le Mehauté, and L. L. Schumaker (eds.), Academic Press, New York, 1991, 209-246.

[17] R. Q. Jia, Q. T. Jiang, Spectral analysis of the transition operator and its applications to smoothness analysis of wavelets, SIAM J. Matrix Anal. Appl. **24** (2003), 1071-1109.

[18] R. Q. Jia, S. D. Riemenschneider, D. X. Zhou, Smoothness of multiple refinable functions and multiple wavelets. SIAM J. Matrix Anal. Appl. **21** (1999), 1-28.

[19] R. Q. Jia, J. Z. Wang, Stability and linear independence asociated with wavelet decompositions, Proc. Amer. Math. Soc. **117** (1993), 1115-1124.

[20] Q. T. Jiang, On the regularity of matrix refinable functions, SIAM J. Math. Anal. **29** (1998), 1157-1176.

[21] Q. T. Jiang, Multivariate matrix refinable functions with arbitrary matrix dilation, Trans. Amer. Math. Soc. **351** (1999), 2407-2438.

[22] S. Li, Characterization of smoothness of multivariate refinable functions and convergence of cascade algorithms of nonhomogeneous refinement equations, Adv. Comput. Math. **20** (2004), 311-331.

[23] S. Li, Convergence rates of vector casscade algorithms in L_p, J. Approx. Theory. **137** (2005), 123-142.

[24] S. Li, J. B. Yang, Vector refinement equations with infinitely supported masks, J. Approx. Theory. **148** (2007), 158-176.

[25] R. Lorentz, P. Oswald, Criteria for hierarchical bases in Sobolev spaces, Appl. Comput. Harmon. Anal. **8** (2000), 32-85.

[26] B. Ma, Q. Sun, Compactly supported refinable distributions in Triebel-Lizorkin spaces and Besov spaces, J. Fourier Anal. Appl. **5** (1999), 87-104.

[27] C. A. Micchelli, T. Sauer, Regularity of multiwavelets, Adv. Comput. Math. **7**(1997), 455-545.

[28] A. Oppenheim, R. Schafer, Digital Signal Processing, Prentice Hall, New York, 1975.

[29] A. Ron, Z. Shen, The Sobolev regularity of refinable functions, J. Approx. Theory. **106** (2000), 185-225.

[30] A. Ron, Z. Shen, K.C.Toh, Computing the Sobolev regularity of refinable functions by Arnoldi method, SIAM J. Matrix Anal. Appl. **29** (2001), 57-76.

[31] Z. Shen, Refinable Function Vectors, SIAM J. Math. Anal. **29** (1998), 235-250.

[32] E. M. Stein, Singualr Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, New Jersey, 1997.

[33] M. Unser, T. Blu, Fractional splines and wavelets, SIAM Rev. **42** (2000), 43-67.