Counting Dyons in $\mathcal{N} = 8$ String Theory

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Abstract
A recently discovered relation between 4D and 5D black holes is used to derive exact (weighted) BPS black hole degeneracies for 4D $\mathcal{N} = 8$ string theory from the exactly known 5D degeneracies. A direct 4D microscopic derivation in terms of weighted 4D D-brane bound state degeneracies is sketched and found to agree.

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1. Introduction

In this paper, we deduce an exact formula for the modified elliptic genus of string theory in four dimensions with $\mathcal{N} = 8$ supersymmetry. The modified elliptic genus, as we review below, provides a weighted count of BPS states of $\mathcal{N} = 8$ string theory. We derive a formula for it using a recently proposed exact relation between 4D and 5D BPS degeneracies, together with the known degeneracies in 5D. In addition we sketch a direct microscopic counting of D0-D2-D4 bound states which gives the same result. Our hope is that this example will provide a useful laboratory for testing the string theory relations recently proposed in e.g. [3].

Some years ago an explicit formula for the elliptic genus for BPS states in 4D $\mathcal{N} = 4$ theories was presciently conjectured [4]. This formula was recently derived using the 4D-5D connection in [5]. The present work is an extension of [5] to 4D $\mathcal{N} = 8$ theories. Previous work in this direction includes [6,7,8].

In the next section we review the 5D index defined and computed in [2]. In section 3 we use the 4D-5D connection to derive the 4D index. In section 4 we sketch how this expression should follow (for one element of the U-duality class of black holes) from a microscopic analysis.

2. Review of the 5D modified elliptic genus

In this section, we want to summarize the work of reference [2] on counting the microstates of 1/8 BPS black holes in five dimensions. These can be realized in string theory as the usual D1-D5-momentum system of type IIB on $T^4 \times S^1$, with $Q_1$ D1-branes, $Q_5$ D5-branes and integral $S^1$ momentum $n$. The reason that microstate counting of this system is more difficult than for $K3$ compactification is because the usual supersymmetric index that counts these microstates, the orbifold elliptic genus of $Hilb^k(K3)$ with $k = Q_1 Q_5$, vanishes when $K3$ is replaced with $T^4$. In [2], this difficulty was overcome by defining
(and then computing) a new supersymmetric index $\mathcal{E}_2$, closely related with the elliptic genus, which is nonvanishing for $T^4$. We will refer to this new supersymmetric index as the modified elliptic genus of $\text{Hilb}^k(T^4)$. It is defined to be

$$
\mathcal{E}_2^{(k)} = \text{Tr} \left[ (-1)^{2 J_L^3 - 2 J_R^3} 2 (J_R^3)^2 q^{L_0} \bar{T}_0 y^{2 J_L^3} \right] \tag{2.1}
$$

where the trace is over states of the sigma model with target space $\text{Hilb}^k(T^4)$. Here $J_L^3$ and $J_R^3$ are the left and right half-integral $U(1)$ charges of the CFT, and they are identified with generators of $\text{SO}(4)$ rotations of the transverse $R^4$. The $S^1$ momentum is $n = L_0 - \bar{T}_0$. The usual elliptic genus is given by the same formula but without the $2 (J_R^3)^2$ factor; it is these two insertions of $J_R^3$ that make $\mathcal{E}_2$ nonvanishing for $T^4$.

As for $K3$, here it is convenient to define a generating function for the modified elliptic genus:

$$
\mathcal{E}_2 = \sum_{k \geq 1} p^k \mathcal{E}_2^{(k)} \tag{2.2}
$$

In [2], this was shown to be given by the following sum

$$
\mathcal{E}_2(p, q, y) = \sum_{s, k, n, \ell} s(p^k q^n y^\ell) s^c(nk, \ell) \tag{2.3}
$$

with the sum running over $s, k \geq 1, n \geq 0, \ell \in \mathbb{Z}$. Note that the $\bar{q}$ dependence has dropped out – only the $\bar{T}_0 = 0$ states contribute to the modified elliptic genus. Of course, the index must have this property in order to count BPS states, since the BPS condition is equivalent to requiring $\bar{T}_0 = 0$.

It was furthermore shown in [2] that the integers $\hat{c}(nm, \ell)$ are the coefficients in the following Fourier expansion

$$
Z(q, y) \equiv -\eta(q)^{-6} \vartheta_1(y|q)^2 = \sum_{n, \ell} \hat{c}(n, \ell) q^n y^\ell \tag{2.4}
$$

where $\eta(q)$ is the usual Dedekind eta function, and $\vartheta_1(y|q)$ is defined by the product formula

$$
\vartheta_1(y|q) = i(y^{1/2} - y^{-1/2}) q^{1/8} \prod_{n=1}^\infty (1 - q^n)(1 - y q^n)(1 - y^{-1} q^n) \tag{2.5}
$$

1 A free sigma model on $R^4 \times T^4$ is factored out here, and our definition differs by a factor of 2 from [2].
Finally, it was observed in [2] that \( \hat{c}(n, \ell) \) actually only depends on a single combination of parameters \( 4n - \ell^2 \):

\[
\hat{c}(n, \ell) = \hat{c}(4n - \ell^2) \tag{2.6}
\]

Using (2.6) in (2.3) yields

\[
\mathcal{E}_2(p, q, y) = \sum_{s,k,n,\ell} s(p^k q^n y^\ell)^s \hat{c}(4nk - \ell^2) \tag{2.7}
\]

When \( (k, n, \ell) \) are coprime, \( \hat{c}(4nk - \ell^2) \) counts BPS black holes with \( k = Q_1 Q_5, S^1 \) momentum \( n \) and spin \( J^3_L = \frac{\ell}{2}, \) multiplied by an overall \( (-)^\ell \) and summed over \( J^3_R \) weighted by \( 2(J^3_R)^2(-)^{2J^3_R} \):

\[
\hat{c}(4nk - \ell^2) \bigg|_{(k,n,\ell) \text{ coprime}} = (-)^\ell \sum_{J^3_R, \text{BPS states}} 2(J^3_R)^2(-)^{2J^3_R} \tag{2.8}
\]

When they are not coprime, the black hole can fragment, and the situation is more complicated due to multiple contributions in \( \mathcal{E}_2 \) [3]. In this paper we will always avoid this complication by choosing coprime charges.

We should note that \( Z(q, y) \) is also the modified elliptic genus of \( T^4 \), i.e.

\[
\mathcal{E}_2^{(1)} = \sum_{n,\ell} \hat{c}(n, \ell) q^n y^\ell = Z(q, y). \tag{2.9}
\]

This corresponds to the coprime D1-D5 system with \( k = 1 = Q_1 = Q_5 \). By writing

\[
Z(q, y) = \sum_m \hat{c}(4m) q^m \sum_k q^{k^2} y^{2k} + \sum_m \hat{c}(4m - 1) q^m \sum_k q^{k^2+k} y^{2k+1} \tag{2.10}
\]

and using (2.4) along with the standard Fourier expansion of the theta function

\[
\vartheta_1(y|q) = i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \tag{2.11}
\]

one can reorganize the generating functions for \( \hat{c} \) as

\[
\sum_m \hat{c}(4m) q^m = -q^{3/2} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 + m},
\]

\[
\sum_m \hat{c}(4m - 1) q^m = q^{3/2} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2}. \tag{2.12}
\]

These expressions will analyzed microscopically below in section 4.
3. The 4D modified elliptic genus

In this section we use the conjecture of \cite{1} to transform the 5D degeneracies into 4D ones. The fact that the \( \hat{c} \) coefficients depend only on the combination \( 4nk - \ell^2 \) is very encouraging, for the following reason. We expect the 1/8 BPS 5D degeneracies to be related to degeneracies of 1/8 BPS black holes in 4D, and in 4D U-duality implies \cite{9} that the black hole entropy must depend on the unique quartic invariant of \( E_{7,7} \), the so-called Cremmer-Julia invariant \cite{10}. In an \( \mathcal{N} = 4 \) language, this invariant takes the form

\[
J = q_e^2 q_m^2 - (q_e \cdot q_m)^2
\]  

(3.1)

where \( q_e \) and \( q_m \) are the electric and magnetic charge vectors for \( \mathcal{N} = 4 \) BPS states. (See e.g. \cite{5} for details on the notation.) This is precisely the dependence of \( \hat{c} \) on \( n, m, \ell \), provided we identify

\[
k = \frac{1}{2} q_e^2, \quad n = \frac{1}{2} q_m^2, \quad \ell = q_e \cdot q_m.
\]  

(3.2)

Note that from the purely 5D point of view, there was no obvious reason that \( \hat{c} \) should depend only on the combination \( 4nk - \ell^2 \) as there is no 5D U-duality which mixes spins with charges.

Let us now derive the identification (3.2) from the dictionary of \cite{1}, beginning from the IIB spinning 5D D1-D5-n black hole of the previous section. First we T-dual on \( S^1 \) to obtain a black hole with spin \( \frac{\ell}{2} \), F-string winding \( n \), \( Q_1 \) D0-branes, and \( Q_5 \) D4-branes. Now T-dual so that there are \( Q_1 + Q_5 \) D2 branes with intersection number \( Q_1 Q_5 = k \) on the \( T^4 \). Next we compactify on a single center Taub-NUT, whose asymptotic circle we identify as the the new M-theory circle. The result is three orthogonal sets of \( (n, Q_1, Q_5) \) D2-branes on \( T^6 \), \( \ell \) D0-branes, and one D6-brane. For IIA D-brane configurations with D0, D2, D4, D6 charges \( (q_0, q_{ij}, p^{ij}, p^0) \), where \( i = 1, \ldots 6 \) runs over the \( T^6 \) cycle and \( p^{ij} = -p^{ji} \), \( q_{ij} = -q_{ji} \) \( \mathcal{J} \) reduces to

\[
\mathcal{J} = \frac{1}{12} (q_0 \epsilon_{ijklmn} p^{ij} p^{kl} p^{mn} + p^0 \epsilon_{ijklmn} q_{ij} q_{kl} q_{mn})
\]

\[
- p^{ij} q_{jk} p^{kl} q_{li} + \frac{1}{4} p^{ij} q_{ij} p^{kl} q_{kl} - (p^0 q_0)^2 + \frac{1}{2} p^0 q_0 p^{ij} q_{ij}.
\]  

(3.3)

\[\text{See e.g. } \cite{11}, \text{ equation (66), and take } p^0 = p_{87}, \ p_{8i} = 0, \text{ etc. Our definition of } \mathcal{J} \text{ differs from that of } \cite{11} \text{ by a sign.}\]
For our D0-D2-D6 configuration, we can pick a basis of cycles without loss of generality such that the nonzero charges are

\[ p^0 = 0, \quad q_0 = \ell, \quad q_{12} = -q_{21} = n, \quad q_{34} = -q_{43} = Q_1, \quad q_{56} = -q_{65} = Q_5 \]  

(3.4)

Then (3.3) reduces to

\[ J = 4nk - \ell^2, \]  

(3.5)

which, as stated above, is exactly the argument of (2.3).

According to [1] the weighted degeneracy of the 4D black hole resulting from U-duality and Taub-NUT compactification equals that of the original 5D black hole, when \( J^3_R \) in (2.8) is identified with the generator \( J^3 \) of \( \mathbb{R}^3 \) rotations in 4D. Note that, since \( J \) is odd if and only if \( \ell \) is, we may trade \((-\ell)^J\) for \((-\ell)^J\) in (2.8). Therefore, for fixed coprime charges, the weighted 4D BPS degeneracy depends only on the the Cremmer-Julia invariant and is given by

\[ \sum_{J^3, BPS \ states} 2(J^3)^2(-)^2J^3 = (-)^J \hat{c}(J). \]  

(3.6)

Note that, although this formula for the 4D BPS degeneracy was derived assuming a specific D6-D2-D0 configuration, it applies to all D-brane configurations by U-duality.

As a first check on this conjecture, we note that for large charges \( \hat{c}(J) \sim e^{\pi \sqrt{J}} \). From the supergravity solutions \( \text{Area} = 4\pi \sqrt{J} \), so there is agreement with the Bekenstein-Hawking entropy.

As an example, let’s consider the modified elliptic genus for the D4-D0 black hole on \( T^6 \), in which we fix the D4 charges and sum over D0 charge \( q_0 \). Consider the \( T^6 \) of the form \( T^2 \times T^2 \times T^2 \) with \( \alpha_1, \alpha_2, \alpha_3 \) being the three 2-cycles associated with the \( T^2 \)’s. Let \( A^1, A^2, A^3 \) be the dual 4-cycles. We shall consider the D4-brane wrapped on the cycle \([P] = A^1 + A^2 + A^3\). Its triple self-intersection number is \( D = P \cdot P \cdot P = 6 \). From (3.3) we have

\[ J = 4q_0. \]  

(3.7)

We then have

\[ \mathcal{E}_2(q) = \sum_{q_0 \in \mathbb{Z}} \hat{c}(4q_0)q^{q_0} = -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2+m}. \]  

(3.8)

according to (2.12).
A straightforward generalization of this example is the D4-D2-D0 system, where we wrap $(q_1, q_2, q_3)$ D2 branes on the 2-cycles $(\alpha_1, \alpha_2, \alpha_3)$. In this case, the Cremmer-Julia invariant becomes

$$J = 4(q_0 + q_1 q_2 + q_1 q_3 + q_2 q_3) - (q_1 + q_2 + q_3)^2$$

(3.9)

and the sum over $q_0$ produces

$$E_2(q) = \sum_{q_0 \in \mathbb{Z}} (-1)^J \hat{c}(J) q^{q_0} = \begin{cases} -q^{\frac{1}{2}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 + m - \frac{1}{4} J} & q_1 + q_2 + q_3 \text{ even} \\ -q^{\frac{1}{2}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 - \frac{1}{4} J - \frac{1}{4}} & q_1 + q_2 + q_3 \text{ odd} \end{cases}$$

(3.10)

where $\hat{J} = 4(q_1 q_2 + q_1 q_3 + q_2 q_3) - (q_1 + q_2 + q_3)^2$. Now let us turn to the 4D derivation of (3.8) and (3.10).

4. Microscopic derivation in 4D

In this section we sketch a derivation of (3.8) and (3.10) using a 4D microscopic analysis. The derivation is not complete because, as we will discuss below, we ignore some potential subtleties associated to the fact that $P$ is not simply connected. In principle it should be possible to close this gap. A microscopic description of $T^6$ black holes using the M-theory picture of wrapped fivebranes has been given in [8], adapting the description given in [12] for a general Calabi-Yau, in terms of a $(0, 4)$ 2D CFT living on the M-theory circle. For uniformity and simplicity of presentation, we here will use the IIA description in which fivebrane momenta around the M-theory circle become bound states of D0 branes to D4 branes.

As above (3.7) we examine the special case of the D4-D0 system wrapped on $[P] = A_1 + A_2 + A_3$. The D4-D0 system can be described in terms of the quantum mechanics of $q_0$ D0-branes living on the D4-brane world volume $P$. The D4-brane world volume $P$ is holomorphically embedded in the $T^6$. One can compute its Euler character, $\chi(P) = 6$. It follows from the Riemann-Roch formula that the only modulus of $P$ is the overall translation in $T^6$. Since $\chi(P) = 6$, $P$ has $4 + 2b_1$ 2-cycles. By the Lefschetz hyperplane theorem we have $b_1(P) = b_1(T^6) = 6$, and therefore $b_2(P) = 16$. All but one of the 2-cycles

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3 The dual line bundle $L_P$ of the divisor $P$ has only one holomorphic section. However as $T^6$ is not simply connected, the line bundle $L_P$ is not only determined by $c_1(L_P) = [P]$. In fact the translation of $T^6$ takes it to a different line bundle.
come from the intersection of $P$ with $\binom{6}{4} = 15$ 4-cycles in $T^6$. We will be mostly interested in 3 of these, denoted by $\tilde{\alpha}_i$, corresponding to intersections of $A^i$ with $P$. Turning on fluxes along these three 2-cycles corresponds to having charges of D2-branes wrapped on the $\alpha_i$’s. Their intersection numbers are

$$\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases} \quad (4.1)$$

There is, however, one extra 2-cycle in $P$, which we shall denote by $\beta$, that does not correspond to any cycle in the $T^6$.

One can show from the adjunction formula that $c_1(P)$ is Poincaré dual to $-(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)$. It then follows from Hirzebruch signature theorem that

$$\sigma(P) = -\frac{2}{3} \chi(P) + \frac{1}{3} \int_P c_1^2 = -2. \quad (4.2)$$

We conclude that the intersection form on $P$ is odd (and that $P$ is not a spin manifold). Essentially the unique way to extend (4.1) to an odd rank 4 unimodular quadratic form is to have an extra 2-cycle $\gamma$ with

$$\gamma \cdot \tilde{\alpha}_i = 1, \quad \gamma \cdot \gamma = 1. \quad (4.3)$$

Now if we choose $\beta = 2\gamma - \sum \tilde{\alpha}_i$, we have

$$\beta \cdot \tilde{\alpha}_i = 0, \quad \beta \cdot \beta = -2. \quad (4.4)$$

Note that $(\tilde{\alpha}_i, \beta)$ is not an integral basis for $H_2(P, \mathbb{Z})$, yet $\beta$ is the smallest 2-cycle that doesn’t intersect $\tilde{\alpha}_i$. The total intersection form on $P$ is the sum of this rank 4 form together with 6 copies of $\sigma_1$ coming from the 12 other 2-cycles in $P$.

Now one can turn on gauge field flux on the D4-brane world volume along $\beta$, which does not correspond to any D2-brane charge. This flux nevertheless induces D0-brane charge. There is a subtlety in the quantization of this flux. As well known, the curvature of the D4-brane world volume induces an anomalous D0-brane charge $-\chi(P)/24 = -\frac{1}{4}$. In order that the total D0 charge be integral the flux along the cycle $\beta$ on the D4-brane must be half-integer, i.e. of the form $(m + \frac{1}{2})\beta$. The total induced D0-brane charge is $\Delta q_0 = -\frac{1}{2}(m + \frac{1}{2})^2 \beta \cdot \beta - \frac{1}{4} = m^2 + m$, which is indeed an integer$^4$.

$^4$ In the M-theory picture the anomalous D0 charge is the left-moving zero point energy $-\frac{c_L}{24} = -\frac{1}{4}$, and the 2-cycle fluxes correspond to momentum zero modes of scalars on a Narain lattice.
We ignore here the facts arising from nonzero $b_1(P)$ that there is a moduli space of flat connections as well as overall $T^6$ translations which must be quantized. These factors are treated in the language of the 2D CFT in \[8\]. They are found to lead to extra degrees of freedom which are however eliminated by extra gauge constraints. A complete microscopic derivation, not given here, would have to show that a careful accounting of these factors give a trivial correction to our result.

It is now straightforward to reproduce (3.8). Each D0-D4 bound state is in a hypermultiplet which contributes minus one to $Tr[2(J^3)^2(-)^{2J^3}]$. Counting the number of ways of distributing $n$ D0-branes among the $\chi(P) = 6$ ground states of the supersymmetric quantum mechanics, and then summing over $n$, gives the factor of $q^{-6} = \prod_{k=1}^{\infty} (1 - q^k)^{-6}$ in (3.8). Including finally the sum over fluxes on $\beta$, we precisely reproduce the degeneracy (3.8).

Let us now consider the more general case of D4-D2-D0 system. Again we shall assume $(p^1, p^2, p^3) = (1, 1, 1)$. The D2-brane charges are labelled by $(q_1, q_2, q_3)$. The bound state is described by the D4-brane with D2-brane dissolved in its world volume. We end up with the gauge flux

$$F = (m + 1/2)\beta + \sum_{i=1}^{3} q_i \delta_i, \quad \delta_i \cdot \tilde{\alpha}_j = \delta_{ij}. \quad (4.5)$$

In above expression $\delta_i$ is defined up to a shift of an integer multiple of $\beta$. Since we are summing over $m$, this ambiguity is irrelevant. We can choose $\delta_i = \gamma_i - \tilde{\alpha}_i$. The total induced D0-brane charge is then

$$\Delta q_0 = -\int \frac{1}{2} F^2 - \frac{1}{4} = (m + 1/2)^2 + (m + 1/2) \sum q_i + \frac{1}{2} \sum q_i^2 - \frac{1}{4} \quad (4.6)$$

where $D^{AB}$ is the inverse matrix of $D_{AB} = D_{ABC}p^C$,

$$D^{AB} q_A q_B = 3(2q_1 q_2 + 2q_2 q_3 + 2q_3 q_1 - q_1^2 - q_2^2 - q_3^2). \quad (4.7)$$

Note that $\frac{1}{3} D^{AB} q_A q_B = 0 \mod 4$ if $\sum q_i$ is even, and $\frac{1}{3} D^{AB} q_A q_B = -1 \mod 4$ if $\sum q_i$ is odd. Therefore $\Delta q_0$ is always an integer, as expected. The Cremmer-Julia invariant is in this case

$$J = 4 \left( q_0 + \frac{1}{12} D^{AB} q_A q_B \right). \quad (4.8)$$
The counting of D0-brane states as before gives the generating function

\[
\sum_{q_0} (-)^J c(J) q^{q_0} = -\prod_{k=1}^\infty (1-q^k)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2+m-1/2} D^{AB} q_A q_B
\]

(4.9)

in the case \( \sum q_i \in 2\mathbb{Z} \) and \( J \equiv 0 \mod 4 \), and

\[
\sum_{q_0} (-)^J c(J) q^{q_0} = -\prod_{k=1}^\infty (1-q^k)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2-1/2} D^{AB} q_A q_B - \frac{1}{4}
\]

(4.10)

in the case \( \sum q_i \in 2\mathbb{Z} + 1 \) and \( J \equiv -1 \mod 4 \). These are precisely the degeneracies (3.10) we derived from 5D earlier!

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References