Recounting Dyons in $\mathcal{N} = 4$ String Theory

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Abstract

A recently discovered relation between 4D and 5D black holes is used to derive weighted BPS black hole degeneracies for 4D $\mathcal{N} = 4$ string theory from the well-known 5D degeneracies. They are found to be given by the Fourier coefficients of the unique weight 10 automorphic form of the modular group $Sp(2,\mathbb{Z})$. This result agrees exactly with a conjecture made some years ago by Dijkgraaf, Verlinde and Verlinde.

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A general D0-D2-D4-D6 black hole in a 4D IIA string compactification has an M-theory lift to a 5D black hole configuration in a multi-Taub-NUT geometry. This observation was used in [1] to derive a simple relation between 5D and 4D BPS black hole degeneracies. For the case of $K3 \times T^2$ compactification, corresponding to $\mathcal{N} = 4$ string theory, the relevant 5D black holes were found in [2,3] and the degeneracies are well known. In this paper we translate this into an exact expression for the 4D degeneracies, which turn out to be Fourier expansion coefficients of a well-studied weight 10 automorphic form $\Phi$ of the modular group of a genus 2 Riemman surface [4,5].

Almost a decade ago an inspired conjecture was made [3] by Dijkgraaf, Verlinde and Verlinde for the 4D degeneracies of $\mathcal{N} = 4$ black holes, and this was shown to pass several consistency checks. We will see that our analysis precisely confirms their old conjecture.

$\mathcal{N} = 4$ string theory in four dimensions can be obtained from IIA compactification on $K3 \times T^2$. The duality group is conjectured to be

$$SL(2, \mathbb{Z}) \times SO(6, 22; \mathbb{Z}).$$

The first factor may be described as an electromagnetic S-duality which acts on electric charges $q_{eA}$ and magnetic charges $q_{mA}$, $\Lambda = 0, \ldots, 27$ transforming in the 28 of the second factor. For the electric objects, we may take

$$q_{e} = (q_0; q_A; q_{23}; q_i),$$

where $q_0$ is D0-charge, $q_A, A = 1, \ldots, 22$ is $K3$-wrapped D2 charge, $q_{23}$ is $K3$-wrapped D4 charge, and $q_i, i = 24, \ldots, 27$ are momentum and winding modes of $K3 \times S^1$-wrapped NS5 branes. The magnetic objects are 24 types of D-branes which wrap $T^2 \times (K3$ cycle) and 4 types of F-string $T^2$ momentum/winding modes.

Now consider a black hole corresponding to a bound state of a single D6 brane with D0 charge $q_0$, $K3$-wrapped D2 charge $q_A$, and $T^2$-wrapped D2 charge $q_{23}$:

$$q_{m} = (1; q^A = 0; q_{23}; q^i = 0), \quad q_{e} = (q_0; q_A; q_{23} = 0; q_i = 0)$$

The duality invariant charge combinations are

$$\frac{1}{2} q_{e}^2 = \frac{1}{2} C^{AB} q_{A} q_{B}, \quad \frac{1}{2} q_{m}^2 = q_{23}^2, \quad q_{e} \cdot q_{m} = q_{0}$$

where $C^{AB}$ is the intersection matrix on $H^2(K3; \mathbb{Z})$. 

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By lifting this to M-theory on Taub-NUT, it was argued in [1] that the BPS states of this system are the same as those of a 5D black hole in a $K3 \times T^2$ compactification, with $T^2$-wrapped M2 charge $\frac{1}{2} q_m^2$, $K3$-wrapped M2 charge $q_A$ and angular momentum $J_L = q_0/2$. We now use one of the compactification circles to interpret the configuration as IIA on $K3 \times S^1$ with $\frac{1}{2} q_m^2$ F-strings winding $S^1$ and $q_A$ D2-branes. T-dualizing the $S^1$ yields $q_A$ D3-branes carrying momentum $\frac{1}{2} q_m^2$. This is then U-dual to a $Q_1$ D1 branes and $Q_5$ D5 branes on $K3 \times S^1$ with

$$N \equiv Q_1 Q_5 = \frac{1}{2} q_e^2 + 1$$

angular momentum

$$J_L = \frac{1}{2} q_e \cdot q_m$$

and left-moving momentum along the $S^1$:

$$L_0 = \frac{1}{2} q_m^2.$$ 

Hence, with the above relations between parameters, according to [1] the 4D degeneracy of states with charges (3) and 5D degeneracies are related by

$$d_4(1; 0; q_m^2; 0; q_0; q_A; 0; 0) = d_5 \left( q_m^2, q_A; \frac{q_0}{2} \right).$$

(8)

Since the degeneracies are U-dual we may also write

$$d_4(q_m^2, q_e^2, q_e \cdot q_m) = d_5(L_0, N, J_L) = d_5 \left( \frac{1}{2} q_m^2, \frac{1}{2} q_e^2 + 1, \frac{1}{2} q_e \cdot q_m \right).$$

(9)

Here and elsewhere in this paper by “degeneracies,” in a slight abuse of language, we mean the number of bosons minus the number of fermions of a given charge, and the center-of-mass multiplet is factored out.

Of course these microscopic BPS degeneracies $d_5$ of the D1-D5 system are well known [2,3]. Their main contribution comes from the coefficients in the Fourier expansion of the elliptic genus of Hilb$^N(K3)$:

$$\chi_N(\rho, \nu) = \sum_{L_0, J_L} d_5'(L_0, N, J_L)e^{2\pi i (L_0 \rho + 2J_L \nu)}$$

(10)

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1. One should keep in mind that $J_L$ is half the R-charge $F_L$ [3], and is hence takes values in $\frac{1}{2}\mathbb{Z}$.

2. Note that $d_n$ denotes fixed-charge degeneracies and does not involve a sum over U-duality orbits.
It is shown in [6] that the weighted sum of the elliptic genera has a product representation:

\[
\sum_{N \geq 0} \chi_N(\rho, \nu) e^{2\pi i N \sigma} = \frac{1}{\Phi'(\rho, \sigma, \nu)}
\]

where \(\Phi'\) is given by

\[
\Phi'(\rho, \sigma, \nu) = \prod_{k \geq 0, l > 0, m \in \mathbb{Z}} (1 - e^{2\pi i (k \rho + l \sigma + m \nu)}) c(4kl - m^2),
\]

with \(c(4k - m^2) = d'_5(k, 1, m)\) the elliptic genus coefficients for a single \(K3\) as given in [7].

Equation (11) is the generating function for BPS states of CF Ts on \(\text{Hilb}^N(K3)\) in the D5 worldvolume. However, it does not quite give the degeneracies needed in (9) because it leaves out the decoupled contribution from the elliptic genus of a single fivebrane. This remains even when \(N = 0\) and there are no D1 branes at all. (By U-duality, we are free to view the system as a single fivebrane and \(N\) D1 branes.) Using the U-dual relation of a \(K3\)-wrapped D5 brane to a fundamental heterotic string, the elliptic genus, not including the center of mass contribution, is [8,9]

\[
Z_0(\nu, \rho) = (e^{\pi i \nu} - e^{-\pi i \nu})^{-2} e^{-2\pi i \rho} \prod_{n \geq 1} (1 - e^{2\pi i (n \rho + \nu)})^{-2} (1 - e^{2\pi i (n \rho - \nu)})^{-2} (1 - e^{2\pi i n \rho})^{-20}.
\]

(13)

This shifts \(\Phi'\) to

\[
\frac{1}{\Phi'(\rho, \sigma, \nu)} \rightarrow \frac{Z_0(\nu, \rho)}{\Phi'(\rho, \sigma, \nu)} = e^{2\pi i \sigma} \Phi(\rho, \sigma, \nu)
\]

(14)

where \(\Phi(\rho, \sigma, \nu)\) has a product representation

\[
\Phi(\rho, \sigma, \nu) = e^{2\pi i (p + \sigma + \nu)} \prod_{(k, l, m) > 0} (1 - e^{2\pi i (k \rho + l \sigma + m \nu)}) c(4kl - m^2)
\]

(15)

where \((k, l, m) > 0\) means that \(k, l \geq 0, m \in \mathbb{Z}\) and in the case \(k = l = 0\), the product is only over \(m < 0\). \(\Phi(\rho, \sigma, \nu)\) is the unique automorphic form of weight 10 of the modular group \(Sp(2, \mathbb{Z})\) and was studied in [4]. The 5D BPS degeneracies are then the Fourier coefficients in

\[
\sum_{L_0, N, J_L} d_5(L_0, N, J_L) e^{2\pi i (L_0 \rho + (N-1) \sigma + 2J_L \nu)} = \frac{1}{\Phi(\rho, \sigma, \nu)}.
\]

(16)

Note \(c(-1) = 2, c(0) = 20,\) and \(c(n) = 0\) for \(n \leq -2.\)
Inserting the 4D-5D relation (3), (16) agrees exactly with the formula proposed in [5] for the microscopic degeneracy of BPS black holes of $\mathcal{N} = 4$ string theory.

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References


