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K-THEORY FROM SEN'S CONJECTURE

PLAN OF THE TALK :

1) REVIEW OF THE BASIC NOTION OF D-BRANE CHARGES AND OF THE CONSEQUENCES THAT QUANTUM EFFECTS HAVE ON THEIR CLASSIFICATIONS

2) D-BRANES INTERACTIONS AND LOWER DIMENSIONAL BRANE CHARGES : THE ROLE OF TACHYON CONDENSATION

3) GYSIN MAP AND K-THEORETICAL CLASSIFICATION OF D-BRANES.

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REFERENCES :

a) ORIGINAL PAPERS :

MINASIAN - MOORE : HEP-TH / 9710230

WITTEN : HEP-TH / 9810188

b) REVIEW :

EVSLIN : HEP-TH / 0610328

Working hypotheses:

1) Topology of space-time: $\mathbb{R}^{1,n} \times X^{9-n}$

X^{9-n} compact.

$n \geq 1 \Rightarrow$ to have net D-Brane charge

$n = 0 \Rightarrow$ fluxes have nowhere to go; branes come together with anti-branes

2) H-flux vanishes both as integer cohomology class and representative 3-form.

3) Compact branes without worldvolume singularities

Branes are electric sources for the RR fields of type II string spectrum, to which they minimally couple via the Wess-Zumino action:

$$S \supset \int_{W_y} C_{p+1} \qquad W_y \to \text{Dp-brane worldvolume}$$

Invariance under the gauge transformation: $C \to C + d\Lambda$ imposes that the branes worldvolumes are cycles of space-time.

Classical mechanics suggests us to allow charge-preserving homotopic movements, but, due to stringy interactions, we want a brane to undergo even processes like this:

Thus, we have to admit homological deformations as well.

In order to obtain a suitable definition of the charge: Branes are sources for violation of Bianchi identity of RR magnetic field strengths.

Dp - brane Y at a fixed instant of time:

$$dG_{8-p} = d * G_{p+2} = \delta^{(9-p)}(qY) = \widetilde{PD}_{\mathbb{R}^n \times X}(qY)$$

where $\widetilde{PD}$ is just a representative of the class defined by Poincaré duality. $n = 0 \Rightarrow Y$ is the 0-class in homology, as expected!

$n > 0 \Rightarrow$ for $i : H^{9-p}_{cpt}(\mathbb{R}^n \times X) \longrightarrow H^{9-p}(\mathbb{R}^n \times X)$

$$PD_{\mathbb{R}^n \times X}(qY) \in \mathrm{Ker}(i)$$

This is equivalent to say that the non-triviality of the p-brane is measured by fluxes at infinity of $(8-p)$-forms that cannot be regarded just as the restriction of forms defined on the whole $\mathbb{R}^n \times X$.

~~Intuitively~~ Therefore:

$$\int_{B^{9-p}} \widetilde{PD}_{\mathbb{R}^n \times X}(qY) = \int_{B^{9-p}} d\, G_{8-p} = \int_{L^{8-p}} G_{8-p} \qquad \partial B = L$$

$$\left.\begin{array}{ll} B^{9-p} \text{ intersects } & Y \\ L^{8-p} \text{ links } & Y \end{array}\right\} \text{ in } \mathbb{R}^n \times X$$

Changing representative for $Y \Rightarrow$ change homology class for L

Homology classes for L are indexed by $\ell = \#\,\text{linking} = \#\,\text{intersections}$

Thus:
$$\boxed{q = \frac{1}{\ell} \int_{L^{8-p}} G_{8-p}}$$

As far as supergravity is concerned, de Rham cohomology is the end of the story

D-brane charges lie in the ring $\boxed{H^{\cdot}(X, \mathbb{R}) \simeq H_{dR}(X)}$

But Quantum Mechanics introduces dramatic effects:

I) Dirac quantization condition:

Invariance of the path integral measure e^{iS} under large gauge transformations (l.g.t.): $C_{p+1} \longrightarrow C_{p+1} + \Phi_{p+1}$ $\quad d\Phi_{p+1} = 0$ imposes quantization of RR fluxes $/2\pi$.

Take for simplicity Y lying in the intersection of two patches:

$$\int_Y C^{(+)}_{p+1} - \int_Y C^{(-)}_{p+1} = \int_Y \Phi_{p+1}$$

$C^{(\pm)}_{p+1}$ being two local potentials for G_{p+2}

$$= \int_{U^+} G_{p+2} - \left[- \int_{U^-} G_{p+2} \right] = \int_U G_{p+2}$$

$\partial U^+ = -\partial U^- = Y$

$U = U^+ \cup U^-$

$$\Rightarrow \quad \int_U G_{p+2} = 2\pi k \qquad k \in \mathbb{Z}$$

In other words, l.g.t. are topologically non-trivial transition functions for the gerbe whose ~~...~~ characteristic class is measured by G_{p+2} and on which C_{p+1} plays the role of a connection.

These gerbes are defined on $\mathbb{R}^{1,n} \times X \setminus W_Z$ where Z is the dual $(6-p)$-brane, because only there G_{p+2} are closed. Hence there is the problem of defining a local Lorentz invariant theory for the (self-dual) 3-branes.

So, brane charges lie in: $\boxed{H^p(X,\mathbb{Z}) = \underbrace{\mathbb{Z}^{b_p}}_{\text{free}} \oplus_i \underbrace{\mathbb{Z}_{p_i^{k_i}}}_{\text{torsion}}}$

II) Worldsheet global anomalies (Freed - Witten):

a) Branes wrapping cycles with $W_3(TY) + H|_Y \neq 0$ are anomalous.

b) Branes wrapping PD of $W_3 + H$ within the worldvolume of some other brane are unstable (MMS-instantons)

Thus, Homology is not any more a good classification.
In order to improve it, there exists a spectral sequence (Atiyah - Hirzebruch) whose first non-trivial differential d_3 has the following property:

a) F.W. anomalous cycles fall in $\mathrm{Ker}\, d_3 \Rightarrow$ cut

b) MMS-instantons fall in $\mathrm{Im}\, d_3 \Rightarrow$ quotiented out

Carrying out this process we end up with something closely related to K-theory (talk by F. FERRARI RUFFINO)

D-branes have two kinds of interactions:

a) Gauge interaction (gauge field configurations from open strings determine the gauge bundle).

b) Gravitational interaction (graviton configurations from closed strings determine the tangent bundle $TX|_Y$)

Minasian and Moore computed the <u>unique non-anomalous form</u> of these couplings:

$$S \supset \int_{W_Y} i^* C \wedge ch(E) \wedge e^{d/2} \wedge \frac{\sqrt{\hat{A}(TY)}}{\sqrt{\hat{A}(NY)}} \qquad \text{where}$$

$E \to$ gauge bundle on the worldvolume W_Y

$i : W_Y \to X \times \mathbb{R}$ embedding

$\hat{A} \to$ A-roof genus (function of Pontrjagin classes)

$d \to$ $Spin^c$ class of NY (the normal bundle of the brane)

Thus, a brane contains lower dimensional brane charges!!
The old charge is recovered as the rank of the gauge bundle on the brane: $i_\#(\mathrm{rk}\, E) = i_\#(ch_0(E))$
$i_\#$ being the Gysin map in cohomology $(PD_X\, i_*\, PD_Y)$.

Charge density must be a class of the entire space, but now dependent on further data than simply the cycle. How to get it? Here comes Sen's conjecture.

p-Brane (possibly equipped with a gauge bundle) $\Longleftrightarrow$

$\Longleftrightarrow$ a stack of sufficiently many coincident pairs $D(9-n)-\overline{D(9-n)}$ branes equipped with a suitable K-theory class.

The equivalence is due to the phenomenon of tachyon condensation

Simplest example (brane with trivial line bundle of codimension 2):

$Dp - \overline{Dp}$ system:

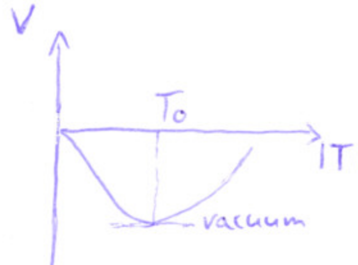
$\left.\begin{matrix} p-p \\ \bar{p}-\bar{p} \end{matrix}\right\} \xrightarrow{GSO}$ Maxwell supermultiplet $U(1) \times U(1)$ || $\left.\begin{matrix} p-\bar{p} \\ \bar{p}-p \end{matrix}\right\} \xrightarrow{GSO}$ Complex scalar field Tachyon $(1,-1)$

reversed GSO!

Tachyon means instability $\Longleftrightarrow$ $T=0$ is a maximum of the scalar potential computed after integration of all other massive modes.

$V(T)$ must be gauge invariant $\Rightarrow$ $V(|T|)$. It looks like:

$T_0 \neq 0$ minimum of V.



Fixing T_0 we don't fix the phase of T !!

Giving the vev T_0 to the Tachyon the $Dp-\overline{Dp}$ system will look like the vacuum everywhere except very close to the locus in which T vanishes. Being T complex, it can still have a non-trivial winding number around this locus depending on the topology of the unbroken gauge theory $\left(U(1)\times U(1) \xrightarrow{vev} U(1)_D\right)$

$$T \longrightarrow T e^{i(\vartheta_1 - \vartheta_2)}$$

$\vartheta_1 = \vartheta_2$ is the residual gauge symmetry.

What is now the $T=0$ locus in terms of bundles?
Put a line bundle $\mathcal{L}$ on the Dp and the trivial line bundle on $\overline{Dp}$
Since T belongs to the $p \leftrightarrow \bar{p}$ sector it is a section of $\mathcal{L} \otimes 1^{\vee} = \mathcal{L} \otimes 1 = \mathcal{L}$ that can be regarded also as a map $T : \mathcal{L} \to 1$.
Of course it can have zeros and the zero locus is precisely the Divisor of $\mathcal{L}$ inside the Dp worldvolume

$$\boxed{\mathrm{Div}\,\mathcal{L} \simeq \{T=0\}}$$

It's a codimension 2 locus and it behaves like a $D(p-2)$-brane with charge given by $i_{\#}[c_1(\mathcal{L})]$, after $Dp - \overline{Dp}$ have annihilated.
Provided we use a non-abelian gauge theory on the brane-brane system, the generalization to higher codimension is straightforward.

- How do we get a class on the entire space?
  Use Sen's conjecture with maximal ($D(9-n)$) branes!
- How can we keep track of a possible gauge bundle on the brane?
  Use Gysin map in K-theory! $i_! : K(Y) \to \tilde{K}(X)$

Since FW $\Rightarrow$ $W_3(TY) = W_3(NY) = 0$, the Gysin map is defined in the following way. Let U be a tubular neighborhood of Y:

$i_! = \psi^* \circ \varphi^* \circ T$ where

$T : K(Y) \simeq K(NY) \equiv \tilde{K}(NY^+)$ is the Thom isomorphism

$\varphi : U^+ \to NY^+$ is a diffeomorphism

$\psi : X \to U^+$ fixes U and sends $X \setminus U$ to $+$.

everything in the case of even codimension.
In the odd case we end up with $K^1(X)$

Ready to get the K-theory charge of a brane:
Rewrite S in this way (integral on $\mathbb{R}\times X$)

$$S \supset \int_{\mathbb{R}\times X} C \wedge i_{\#}\left(ch(E)\wedge e^{d/2}\wedge \frac{\sqrt{\hat{A}(TY)}}{\sqrt{\hat{A}(NY)}}\right) \underset{\text{Hirzebruch-Riemann-Roch theorem}}{=}$$

$$= \int_{\mathbb{R}\times X} C \wedge ch(i_! E)\wedge \sqrt{\hat{A}(TX)}$$

Being $\hat{A}(TX)$ independent of the brane $\Rightarrow$ $i_! E$ is the K-theory charge of the brane and $ch(i_! E)$ its differential form approximation

$$\boxed{Q_Y = i_! E \in \hat{K}(X)}$$

Physical relevance:
- It takes into account all the couplings of the brane.
- It is compatible with Sen's conjecture.
- Stable equivalence relation $\Leftrightarrow$ Total Tachyon condensation
  Adding the same bundle H on both $D(9-n)$ and $\overline{D(9-n)}$ will produce a Tachyon as section of $H\otimes H^\vee = n$ ($n = \mathrm{rk}\, H$), that is a global never zero section.
  So $\mathrm{Div}(n) = \emptyset$ $\Rightarrow$ vacuum everywhere $\Rightarrow$ charge doesn't change

Checks: first two non trivial charges:

$$ch_0(i_! E) = i_{\#}(\mathrm{rk}\, E)$$
$$ch_1(i_! E) = i_{\#}(c_1(E))$$

Shortcoming: Tachyon condensation is like an RG-flow
it relates trajectories (deformation in space-time)

$\Downarrow$

$\boxed{i_! E \text{ is a trajectory charge}}$ In general if we compute $i_{t!} E$ at a fixed instant, this will be not conserved in time.

AHSS gives instead really a charge conserved in time!

How to relate it to $i_! E$? $\Rightarrow$ Talk by F. FERRARI RUFFINO