August 2008

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Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves

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Abstract

This is a continuation of our study of the foundations of D-branes from the viewpoint of Grothendieck in the region of the related Wilson's theory-space where "branes" are still branes. In this work, we focus on D-strings and construct the moduli stack of morphisms from Azumaya prestable curves C^{Az} with a fundamental module $\mathcal{E}$ to a fixed target Y of a given combinatorial type. Such a morphism gives a prototype for a Wick-rotated D-string of B-type on Y, following the Polchinski-Grothendieck Ansatz, and this stack serves as a ground toward a mathematical theory of topological D-string world-sheet instantons.

Key words: Polchinski-Grothendieck Ansatz, D-string of B-type, Azumaya prestable curve, Chan-Paton sheaf, fundamental module, morphism, surrogate, combinatorial type, moduli stack. D0-brane smearing, D-string world-sheet instanton.

MSC number 2000: 14A22, 81T30; 14D20, 14A20, 81T75.

Acknowledgements. We thank Andrew Strominger and Cumrun Vafa for topic course/lecture/conversations that continue to influence our understanding. S.L. thanks in addition Xi Yin for communicating issues on D-brane moduli from physics point of view. C.-H.L. thanks in addition Liang Kong for lectures/explanations/insights of D-branes from boundary conformal field theory aspect and Eric Sharpe for communicating to us a long itemized explanation of stringy/brany issues/themes/insights that both so generously share with and guide us; Miao Li and Edoardo Sernesi for communication on issues in project; Wenxuan Lu for insightful discussions; Nathan Berkovits, Frederik Denef, Sergei Gukov, Kentaro Hori, and Yongbin Ruan for related talks/discussions/conversations; Howard Georgi, Michael Hopkins, Pedram Safari, A.S., Katrin Wehrheim, Lauren Williams for topic courses fall 2007- spring 2008; Ann Willman for inspiring words and hospitality and Ling-Miao Chou for moral support. The project is supported by NSF grants DMS-9803347 and DMS-0074329.

Chien-Hao Liu dedicates this work to Ling-Miao Chou
for her tremendous love
throughout his preparatory brewing decade on D-branes.

0. Introduction and outline.

Introduction.

In [L-Y2], we addressed the foundation of D-branes in the region of the related Wilson's theory-space where "branes" are still branes from Grothendieck's point of view at Polchinski ([Po1], [Po2]) and Witten ([Wi]). In the current work, we continue this line of study with focus on D-strings and lay down part of the foundations to address D-string world-sheet instantons.

After highlighting in Sec. 1 from [L-Y2] the Polchinski-Grothendieck Ansatz and how it leads to a realization of D-branes as morphisms from Azumaya schemes with a fundamental module to an open-string target-space(-time), we give a self-contained re-do of this for D-strings on a commutative target-space in Sec. 2 and bring out the moduli stack $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ of morphisms from Azumaya prestable curves with a fundamental module to a fixed target space Y of combinatorial type $(g,r,\chi\,|\,\beta)$. The observation that the related Azumaya-type noncommutative-geometric setting in the problem (Sec. 2.1) can be recast back to a purely commutative-geometric setting makes the related moduli problem more accessible (Sec. 2.2). We prove then a boundedness property for family of morphisms in question using this observation (Sec. 2.3) and give another presentation of such morphisms when the target is a projective space (Sec. 2.4).

In Sec. 3, we discuss further the moduli stack $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ as a preparation for other parts in the project. A few terse remarks are given in Sec. 4 to relate the current work to technical themes in separate works.

Convention.

- Mathematicians are referred to [Po1], [Po2], and [Jo] for basics of D-branes when they are still branes, [Do] and [Sh] for an insight of how coherent sheaves and complexes thereof come to play as supersymmetric solitonic D-branes of B-type, [As] for a review of geometric phase of D-branes from open-string target-space(-time) aspect, and [H-H-P] for D-branes of B-type from the $2d$-gauged-linear-sigma-model-with-boundary point of view.

- All schemes are Noetherian and of finite type over $\mathbb{C}$. Similarly for their morphisms and products.

Outline.

1 Polchinski-Grothendieck Ansatz, Azumaya prestable curves, supersymmetric D-strings of B-type, and surrogates.

We introduce in this section Azumaya prestable curves with a fundamental module and review the most relevant points in [L-Y2] along the way. See ibidem for details and references.

From Polchinski to Grothendieck: Noncommutative structure on D-branes.

A *D-brane* (i.e. *Dirichlet membrane*) is meant to be a boundary condition for open strings in whatever form it may take, depending on where we are in the related Wilson's theory-space. A realization of D-branes that is most related to the current work is an embedding $f : X \to Y$ of a manifold X into the open-string target space-time Y with the end-points of open strings being required to lie in $f(X)$. This sets up a 2-dimensional Dirichlet boundary-value problem from the field theory on the world-sheet of open strings. Oscillations of open strings with end-points in $f(X)$ then create various fields on X, whose dynamics is governed by open string theory. This is parallel to the mechanism that oscillations of closed strings create fields in space-time Y, whose dynamics is governed by closed string theory. Cf. Figure 1-1.

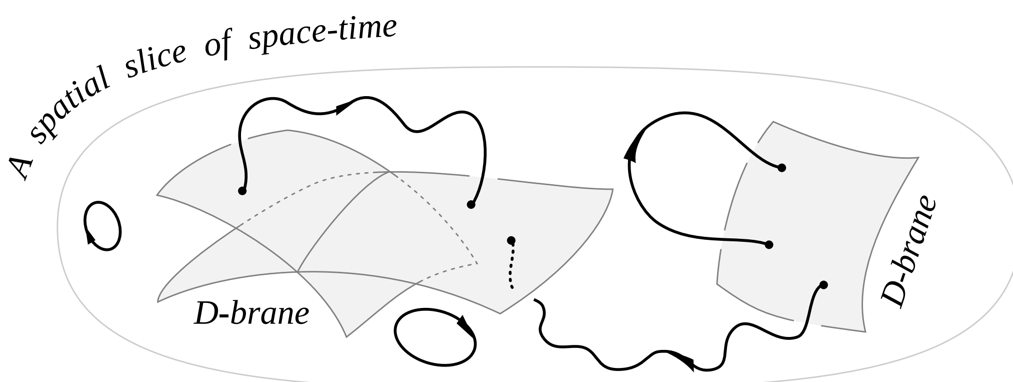


FIGURE 1-1. D-branes as boundary conditions for open strings in space-time. This gives rise to interactions of D-brane world-volumes with both open strings and closed strings. Properties of D-branes, including the quantum field theory on their world-volume and deformations of such, are governed by open and closed strings via this interaction. Both oriented open (resp. closed) strings and a D-brane configuration are shown.

In this setting, f is realized in local coordinates as a tuple of $\mathbb{R}$-valued scalar fields on X, describing the fluctuations/shapes of the D-brane after modding out the longitudinal redundancy along X. When there are r-many coincident D-branes with each of which being described by the same f, open string theory dictates that the components of the tuple that are transverse to $f(X)$ are enhanced to $M_r(\mathbb{R})$-valued. The meaning of this noncommutative enhancement of such scalar fields on coincident D-branes is somewhat mysterious from the aspect of space-time itself. (See the work of Polchinski, e.g. [Po1] and [Po2], and Witten [Wi] for more details; and [L-Y2: references] for a short list of literatures.)

However, when a formal noncommutative extension of Grothendieck's local contravariant equivalence of (function rings, morphisms) and (geometries, morphisms) is applied to the above picture, the mathematical meaning/content of this enhancement becomes the following statement: ([L-Y2: Introduction and Sec. 2.2]), cf. Figure 1-2.

Polchinski-Grothendieck Ansatz [D-brane: noncommutativity]. *A D-brane (or D-brane world-volume) X carries an Azumaya-type noncommutative structure locally associated to a function ring of the form $M_r(R)$ for some $r \in \mathbb{Z}_{\geq 1}$ and ring R. Here, $M_r(R)$ is the $r \times r$ matrix-ring over R.*

An additional statement hidden in this Ansatz that follows from mathematical naturality is that

- *fields on X are local sections of sheaves $\mathcal{F}$ of modules of the structure sheaf $\mathcal{O}_X^{nc}$ of X associated to the above noncommutative structure.*

Furthermore, this noncommutative structure on D-branes (or D-brane world-volumes) is more fundamental than that of space-time in the sense that,

- *from Grothendieck's equivalence, the noncommutative structure of space-time, if any, can be detected by a D-brane only when the D-brane probe itself is noncommutative.*

When the closed-string-created B-field on the open-string target space-time Y is turned off, R in the Ansatz is commutative. This is the case we will be considering throughout the work.

When D-branes are taken as fundamental objects as strings, we no longer want to think of their properties as derived from open strings. Rather, D-branes should have their own intrinsic nature in discard of open strings. Only that when D-branes co-exist with open strings in space-time, their nature has to be compatible/consistent with the originally-open-string-induced properties thereon. It is in this sense that we think of a D-brane world-volume as an Azumaya-type noncommutative space, following the Ansatz, on which other additional compatible structures – in particular, a Chan-Paton module – are defined.

Azumaya prestable curves and the world-sheet of SUSY D-strings of B-type.

The Polchinski-Grothendieck Ansatz for D-branes applies to both nonsupersymmetric and supersymmetric D-branes, and to both D-branes of A-type and D-branes of B-type in the latter case. After Wick-rotation, the world-volume of D-branes of B-type are holomorphic objects and can be studied in the realm of algebraic geometry. In the picture of Polchinski highlighted in the previous theme, the end-points of open strings serve as the source of gauge fields on X. The latter correspond to Chan-Paton bundles-with-a-connection on X. In particular, with the Ansatz and a Kobayashi-Hitchin type correspondence/Donaldson-Uhlenbeck-Yau type theorem in mind, a mathematical object that can serve as a prototype for the (Wick-rotated) world-sheet of a supersymmetric D-string of B-type is then given by

Definition 1.1. [Azumaya prestable curve with a fundamental module]. (Cf. [L-Y2: Definition 1.1.1, Definition 2.2.3].) An *Azumaya prestable curve* (over $\mathbb{C}$) *with a fundamental module* consists of the triple $(C, \mathcal{O}_C^{Az}, \mathcal{E})$, where C is a nodal curve, whose structure sheaf is denoted by $\mathcal{O}_C$; $\mathcal{O}_C^{Az}$ is a sheaf of noncommutative $\mathcal{O}_C$-algebra, and $\mathcal{E}$ is a (left) $\mathcal{O}_C^{Az}$-module such that, as $\mathcal{O}_C$-modules, $\mathcal{E}$ is locally-free on C and $\mathcal{O}_C^{Az} = \mathcal{E}nd\,\mathcal{E}$ $(:= \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{E}))$. The pair $(C, \mathcal{O}_C^{Az}) =: C^{Az}$ is called an *Azumaya prestable curve* and $\mathcal{E}$ a *fundamental module* on C^{Az}.

Let r be the rank of $\mathcal{E}$ as an $\mathcal{O}_C$-module. Then, here, $(C, \mathcal{O}_C^{Az})$ takes the role of the world-sheet of r-stacked D-strings of B-type and $\mathcal{E}$ a Chan-Paton module thereon. When the notion of morphisms from $(C^{Az}, \mathcal{E})$ to a space-time Y is correctly defined (cf. the next theme), the open-string-induced Higgsing/un-Higgsing behavior of D-strings on Y can be correctly reproduced via deformations of such morphisms. See also Definition 2.1.6 and Remark 2.1.7 in Sec. 2.1. This is what justifies the setting of [L-Y2] in the end as a beginning step to understand D-"branes" and their moduli – a topic that reveals different characters in different regions of the related Wilson's

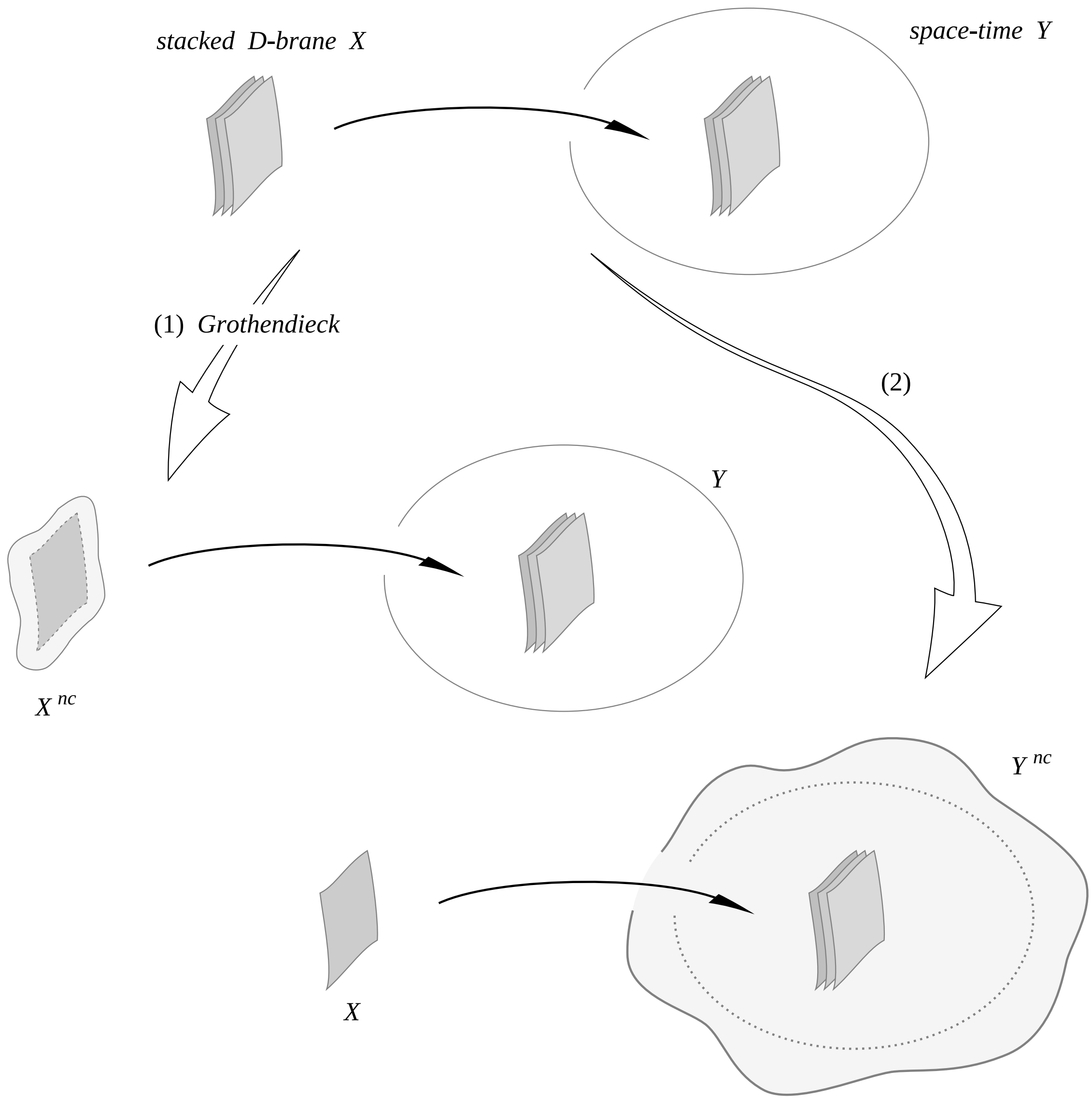


FIGURE 1-2. Two dual/counter aspects on noncommutativity related to coincident/stacked D-branes: (1) noncommutativity of D-brane world-volume as its fundamental/intrinsic nature versus (2) noncommutativity of space-time as probed by stacked D-branes. Aspect (1) leads to the *Polchinski-Grothendieck Ansatz* and is more fundamental from Grothendieck's viewpoint of contravariant equivalence of the category of local geometries and the category of function rings.

theory-space and remains overall very mysterious/challenging on the mathematical side despite its appearance in the string-theory literature [D-L-P] of Dai-Leigh-Polchinski and [Lei] of Leigh already in year 1989.

The geometry of Azumaya schemes revealed via morphisms therefrom: Surrogates.

Generalization of Grothendieck's theory of schemes to the noncommutative case turns out to be a very demanding task. To bypass this, we take "function rings" as more fundamental than a true "space with points and a topology". However, to address the notion of "gluing local charts on a space", one needs a notion of "localizations of a ring". This can be done through the notion of *Gabriel filters* $\mathfrak{F}$ on a ring. With applications to D-branes in mind, we restrict ourselves to a special class of filters that are associated to multiplicatively closed subsets in the center $Z(R)$ of rings R. This gives the notion of *central localizations* of a ring. Thus, we define contravariantly a "*space*" as an equivalence class $[\mathcal{S}]$ of gluing systems $\mathcal{S}$ of rings and a *morphism* from *Space* $[\mathcal{S}]$ to *Space* $[\mathcal{R}]$ as an equivalence class of systems of 3-step ring-system-morphisms

$$\mathcal{R} \xrightarrow{\Phi'} \mathcal{S}'' \precsim \mathcal{S}' \xrightarrow{\Phi} \mathcal{S}$$

with $\mathcal{S}''$ being a refinement of $\mathcal{S}'$ via central localizations.

In defining a morphism *Space* $[\mathcal{S}] \to$ *Space* $[\mathcal{R}]$ as such, we have in mind

(1) in realizing a morphism from *Space* $_1$ to *Space* $_2$ as a system of morphisms on local charts, the charts on the domain *Space* $_1$ in general needs to be refined;

(2) the composition of morphisms *Space* $_1 \to$ *Space* $_2$, *Space* $_2 \to$ *Space* $_3$ should be a morphism *Space* $_1 \to$ *Space* $_3$.

This would be redundant in the commutative case as in that case, the 3-step ring-system-morphism diagram $\mathcal{R} \xrightarrow{\Phi'} \mathcal{S}'' \precsim \mathcal{S}' \xrightarrow{\Phi} \mathcal{S}$ can always be reduced to a 2-step diagram $\mathcal{R} \xrightarrow{\Phi} \mathcal{S}''' \precsim \mathcal{S}$. This reduction no longer holds in general for the noncommutative case in our setting. Thus, the composability of morphisms enforces us to allow a morphism to be defined via a medium system $\mathcal{S}'$. This is how the notion of *surrogates* (cf. *Space* $[\mathcal{S}']$ in the above 3-step diagram) of *Space* $[\mathcal{S}]$ is enforced to occur through the notion of morphisms. It is *a compensation for the insufficiency of refinements via central localizations.* It turns out that *surrogates of Space* $[\mathcal{S}]$ *serve also to reveal the subtle geometry hidden in Space* $[\mathcal{S}]$.

In particular, when these constructions are applied to the gluing system of rings associated to an Azumaya-type noncommutative space $X^{nc} := (X, \mathcal{O}_X, \mathcal{O}_X^{nc})$, where $(X, \mathcal{O}_X)$ is a commutative (Noetherian) scheme and $\mathcal{O}_X^{nc}$ is a coherent sheaf of noncommutative $\mathcal{O}_X$-algebras that contains $\mathcal{O}_X$ as $1 \cdot \mathcal{O}_X$ in its center $\mathcal{Z}(\mathcal{O}_X^{nc})$, one can/should think of X^{nc} as $\mathcal{O}_X^{nc}$ together with the system $L_{\mathcal{O}_X^{nc}}$ of surrogates of X^{nc} given by sub-$\mathcal{O}_X$-algebra pairs:

$$L_{\mathcal{O}_X^{nc}} \;=\; \left\{ (\mathcal{A}, \mathcal{A}^{nc}) \;\middle|\; \begin{array}{l} \mathcal{O}_X \subset \mathcal{A} \subset \mathcal{A}^{nc} \subset \mathcal{O}_X^{nc}\,; \\ \mathcal{A},\, \mathcal{A}^{nc}:\ \text{sub-}\mathcal{O}_X\text{-algebras}\,;\ \mathcal{A} \subset \mathcal{Z}(\mathcal{A}^{nc}) \end{array} \right\}.$$

Cf. [L-Y2: Sec. 1.1]. It is this system of surrogates of X^{nc} that reveals the very rich geometry hidden in X^{nc}, allowing us to understand X^{nc} without having to directly deal with the technical issue of a functorial construction of $\mathbf{Spec}\,(\mathcal{O}_X^{nc})$ as a topological space. Furthermore, any morphism $X'^{nc} := (\mathbf{Spec}\,\mathcal{A}, \mathcal{A}, \mathcal{A}^{nc}) \to Y^{nc}$ from a surrogate X'^{nc} of X^{nc} should be thought of as defining a morphism $X^{nc} \to Y^{nc}$ from X^{nc} itself. See [L-Y2: Example 1.1.8 and Sec. 4.1] for the simplest example, where X^{nc} = the Azumaya point $(Spec\,\mathbb{C}, \mathbb{C}, M_r(\mathbb{C}))$, cf. Figure 1-3.

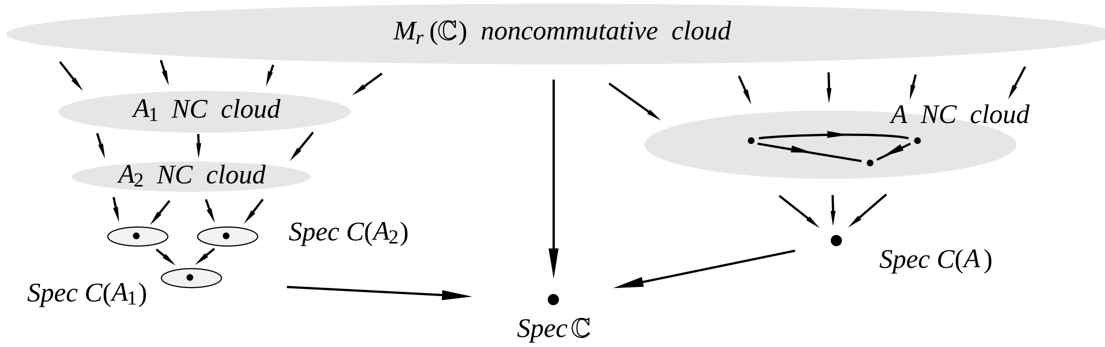


FIGURE 1-3. The very rich geometry of an Azumaya scheme is revealed by its surrogates. Indicated here is the geometry of an Azumaya point $pt^{Az} := (\mathit{Spec}\,\mathbb{C}, \mathbb{C}, M_r(\mathbb{C}))$. Here, A_i are $\mathbb{C}$-subalgebras of $M_r(\mathbb{C})$ and $C(A_i)$ is the center of A_i with

$$\begin{array}{ccccccc} M_r(\mathbb{C}) & \supset & A_1 & \supset & A_2 & \supset & \cdots \\ \cup & & \cup & & \cup & & \\ \mathbb{C}\cdot\mathbf{1} & \subset & C(A_1) & \subset & C(A_2) & \subset & \cdots\,. \end{array}$$

From Grothendieck's contravariant equivalence of local geometries and function rings, an inclusion pair $R_1 \hookrightarrow R_2$ of $\mathbb{C}$-algebras is equivalent to a dominant morphism $\mathit{Space}\,R_2 \to \mathit{Space}\,R_1$. Such a morphism is indicated by a set of $\to$'s in the figure. Since, while $M_r(\mathbb{C})$ contains only one prime ideal, namely $(\mathbf{0})$, a $\mathbb{C}$-subalgebra $A \subset M_r(\mathbb{C})$ can have more than one prime ideals, it follows that one should think of pt^{Az} as containing many secret points hidden in its surrogates. By smearing Azumaya points along a scheme, one sees also the rich Azumaya geometry in general dimensions.

D-branes in string theory as morphisms from Azumaya-type nc-space.

Coherent sheaves on $X^{nc} = (X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ are defined to be left $\mathcal{O}_X^{nc}$-modules that are coherent as $\mathcal{O}_X$-modules. The notion of push-forward under a morphism can also be defined.

Once these foundations are laid down, a (Wick-rotated/Euclidean) D-brane of B-type in an open-string target space Y is defined to be a morphism φ from an Azumaya-type noncommutative space $X^{nc} = (X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ to Y with the Chan-Paton sheaf the push-forward $\varphi_*\mathcal{E}$ of a fundamental $\mathcal{O}_X^{nc}$-module $\mathcal{E}$ on X.

A good match happens: *The notion of morphisms defined via surrogates*, which follows from the above mathematical reasonings that attempt to extend Grothendieck's language of (commutative) schemes to the noncommutative case, *is exactly what is needed to model/reproduce correctly the Higgsing/un-Higgsing behavior of D-branes under deformations.* It is this miracle that justifies the prototype formulation of supersymmetric D-branes of B-type in superstring theory in [L-Y2]. (See [L-Y2: Sec. 4.1] for the simplest example: D0-branes on the complex line.) Cf. Remark 2.1.7, Figure 2-1-1, and Remark 2.1.16.

For D-branes of A-type and nonsupersymmetric D-branes, the philosophy is the same but the language will be different.

When the target Y is a commutative scheme, the general formulation of [L-Y2: Sec. 1] – in particular, morphisms and the system of relevant surrogates – can be simplified/streamlined. This is what we will do in Sec. 2. Such simplification enables us to access the moduli space/stack of topological D-strings in Y along the line of the Polchinski-Grothendieck Ansatz.

2 Morphisms from Azumaya prestable curves with fundamental modules to a projective variety.

Basic definitions, objects, and properties concerning morphisms from Azumaya prestable curves to a projective variety Y are given in Sec. 2.1. This follows [L-Y2] and gives a natural noncommutative extension of the parallel notions in commutative algebraic geometry (e.g. [Ha]). In the passing, we incorporate its use in the description of (Euclidean) D1-branes of B-type in superstring theory and bring forth a moduli problem the project is devoted to. A commutative recast of the noncommutative setting is given in Sec. 2.2. This recast is technically important and it renders the moduli problem more accessible. Some boundedness properties on the family of morphisms from a bounded family of Azumaya prestable curves with a fundamental module to Y of a fixed combinatorial type are given in Sec. 2.3. A discussion on presentations of morphisms from an Azumaya prestable curve to a projective space is given in Sec. 2.4.

2.1 Morphisms, their associated surrogate, and a prototype description of topological D-strings.

Surrogates, morphisms, and D-strings of B-type.

Definition 2.1.1. [(commutative) surrogate]. (Cf. [L-Y2: Definition 1.1.1 and Definition/Example 1.1.2].) Let $\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az}$ be a commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az}$. Then $C_{\mathcal{A}} := \mathbf{Spec}\,\mathcal{A}$ is called a (commutative) *surrogate* of $C^{Az} := (C, \mathcal{O}_C^{Az})$.

One should think of $C_{\mathcal{A}}$ as a finite scheme over and dominating C that is itself canonically dominated by C^{Az}. An affine cover of $C_{\mathcal{A}}$ corresponds to a gluing system of algebras from central localizations. Following this, the notion of morphisms from C^{Az} to Y, as an equivalence class of gluing systems of ring-homomorphisms with respect to covers, can be phrased as

Definition 2.1.2. [morphism]. (Cf. [L-Y2: Definition 1.1.1].) A *morphism* from C^{Az} to Y, in notation $\varphi : C^{Az} \to Y$, is an equivalence class of pairs

$$(\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az}\,,\ f : C_{\mathcal{A}} := \mathbf{Spec}\,\mathcal{A} \to Y)\,,$$

where

(1) $\mathcal{A}$ is a commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az}$;

(2) $f : C_{\mathcal{A}} \to Y$ is a morphism of (commutative) schemes;

(3) two such pairs $(\mathcal{O}_C \subset \mathcal{A}_1 \subset \mathcal{O}_C^{Az}\,,\ f_1 : C_{\mathcal{A}_1} \to Y)$ and $(\mathcal{O}_C \subset \mathcal{A}_2 \subset \mathcal{O}_C^{Az}\,,\ f_2 : C_{\mathcal{A}_2} \to Y)$ are equivalent, in notation

$$(\mathcal{O}_C \subset \mathcal{A}_1 \subset \mathcal{O}_C^{Az}\,,\ f_1 : C_{\mathcal{A}_1} \to Y)\ \sim\ (\mathcal{O}_C \subset \mathcal{A}_2 \subset \mathcal{O}_C^{Az}\,,\ f_2 : C_{\mathcal{A}_2} \to Y)\,,$$

if there exists a third pair $(\mathcal{O}_C \subset \mathcal{A}_3 \subset \mathcal{O}_C^{Az}\,,\ f_3 : C_{\mathcal{A}_3} \to Y)$ such that $\mathcal{A}_3 \subset \mathcal{A}_i$ and that the induced diagram

$$\begin{array}{ccc} C_{\mathcal{A}_i} & \searrow^{f_i} & \\ \downarrow & & \\ C_{\mathcal{A}_3} & \xrightarrow{f_3} & Y \end{array}$$

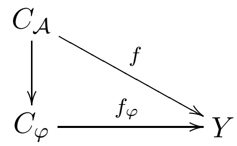
commutes, for $i = 1, 2$.

To improve clearness, we denote the set of pairs associated to φ by the bold-faced $\boldsymbol{\varphi}$.

Definition 2.1.3. [associated surrogate, canonical presentation, and image]. Let

$$\mathcal{A}_\varphi \;=\; \cap_{(\mathcal{O}_C\subset\mathcal{A}\subset\mathcal{O}^{Az},\, f:C_\mathcal{A}\to Y)\in\boldsymbol{\varphi}}\,\mathcal{A}\,.$$

Then $\mathcal{O}_C \subset \mathcal{A}_\varphi \subset \mathcal{O}_C^{Az}$ and there exists a unique $f_\varphi : C_\varphi := \mathbf{Spec}\,\mathcal{A}_\varphi \to Y$ such that the induced diagram



commutes, for all $(\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az}, f : C_\mathcal{A} \to Y) \in \boldsymbol{\varphi}$. We shall call the pair

$$(\mathcal{O}_C \subset \mathcal{A}_\varphi \subset \mathcal{O}_C^{Az}\;,\; f_\varphi : C_\varphi := \mathbf{Spec}\,\mathcal{A}_\varphi \to Y)\,,$$

which is canonically associated to φ, *the (canonical) presentation* for φ. The scheme C_φ, which dominates C, is called the *surrogate of* C^{Az} *associated to* φ. We will denote the built-in morphism $C_\varphi \to C$ by π_φ. The subscheme $f_\varphi(C_\varphi)$ of Y is called the *image* of C^{Az} under φ and will be denoted $Im\,\varphi$ or $\varphi(C^{Az})$ interchangeably.

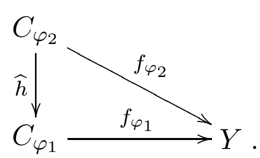
Remark 2.1.4. [*minimal property of* C_φ]*.* By construction,

- *there exists no* $\mathcal{O}_C$*-subalgebra* $\mathcal{O}_C \subset \mathcal{A}' \subset \mathcal{A}_\varphi$ *such that* f_φ *factors as the composition of morphisms* $C_\varphi \to \mathbf{Spec}\,\mathcal{A}' \to Y$.

We will call this feature the *minimal property* of the surrogate C_φ of C^{Az} associated to φ.

Definition 2.1.5. [isomorphism between morphisms]. Two morphisms $\varphi_1 : (C_1^{Az}, \mathcal{E}_1) \to Y$ and $\varphi_2 : (C_2^{Az}, \mathcal{E}_2) \to Y$ from Azumaya prestable curves with a fundamental module to Y are said to be *isomorphic* if there exists an isomorphism $h : C_1 \xrightarrow{\sim} C_2$ with a lifting $\widetilde{h} : \mathcal{E}_1 \xrightarrow{\sim} h^*\mathcal{E}_2$ such that

- $\widetilde{h} : \mathcal{A}_{\varphi_1} \xrightarrow{\sim} h^*\mathcal{A}_{\varphi_2}$,
- the following diagram commutes



Here, we denote the induced isomorphism $\mathcal{O}_{C_1}^{Az} \xrightarrow{\sim} h^*\mathcal{O}_{C_2}^{Az}$ of $\mathcal{O}_{C_1}$-algebras (or $\mathcal{A}_1 \xrightarrow{\sim} h^*\mathcal{A}_2$ of their respective $\mathcal{O}_{C_\bullet}$-subalgebras in question) via $\widetilde{h} : \mathcal{E}_1 \xrightarrow{\sim} h^*\mathcal{E}_2$ still by $\widetilde{h}$ and $\widehat{h} : C_{\varphi_2} \xrightarrow{\sim} C_{\varphi_1}$ is the scheme-isomorphism associated to $\widetilde{h} : \mathcal{A}_{\varphi_1} \xrightarrow{\sim} h^*\mathcal{A}_{\varphi_2}$.

Definition 2.1.6. [Chan-Paton module]. Given a morphism $\varphi : C^{Az} = (C, \mathcal{O}_C^{Az}) \to Y$ with its canonical presentation $(\mathcal{O}_C \subset \mathcal{A}_\varphi \subset \mathcal{O}_C^{Az}\;,\; f_\varphi : C_\varphi \to Y)$. Let $\mathcal{E}$ be a fundamental $\mathcal{O}_C^{Az}$-module on C^{Az}. Then $\mathcal{E}$ is automatically a $\mathcal{O}_{C_\varphi}$-module[1], in notation, ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$. Define the *push-forward* $\varphi_*\mathcal{E}$ of $\mathcal{E}$ to Y under φ by $f_{\varphi\,*}({}_{\mathcal{O}_{C_\varphi}}\mathcal{E})$. It is a coherent $\mathcal{O}_Y$-module supported on $Im\,\varphi = f_\varphi(C_\varphi)$. We will call it also the *Chan-Paton module on* $\varphi(C^{Az})$ *associated to* $\mathcal{E}$ *under* φ.

[1]By construction, it is automatically a left $\mathcal{O}_{C_\varphi}$-module. We then set the right $\mathcal{O}_{C_\varphi}$-action on $\mathcal{E}$ the same as the left $\mathcal{O}_{C_\varphi}$-action.

Remark 2.1.7. [*topological D-string*]. The morphism $\varphi:(C^{Az},\mathcal{E})\to Y$ serves as our prototype-definition for the notion of *topological D-strings on* Y in superstring theory, cf. [L-Y2: Definition 2.2.3]. When Y is a target-space for open superstrings, as φ varies, the endomorphism sheaf $\mathcal{E}nd_{\mathcal{O}_{C_\varphi}}(_{\mathcal{O}_{C_\varphi}}\mathcal{E})$ on C_φ also varies. This $\mathcal{O}_{C_\varphi}$-module carries the information of the gauge group on the D-brane as observed/detected by open strings in Y. This is how the Higgsing/un-Higgsing behavior of the gauge theory on the Wick-rotated D-string world-sheet is revealed in the current setting through deformations of morphisms from Azumaya curves to Y. The case of D0-branes ([L-Y: Sec. 4]) is indicated in Figure 2-1-1. (See also the last theme of this subsection.)

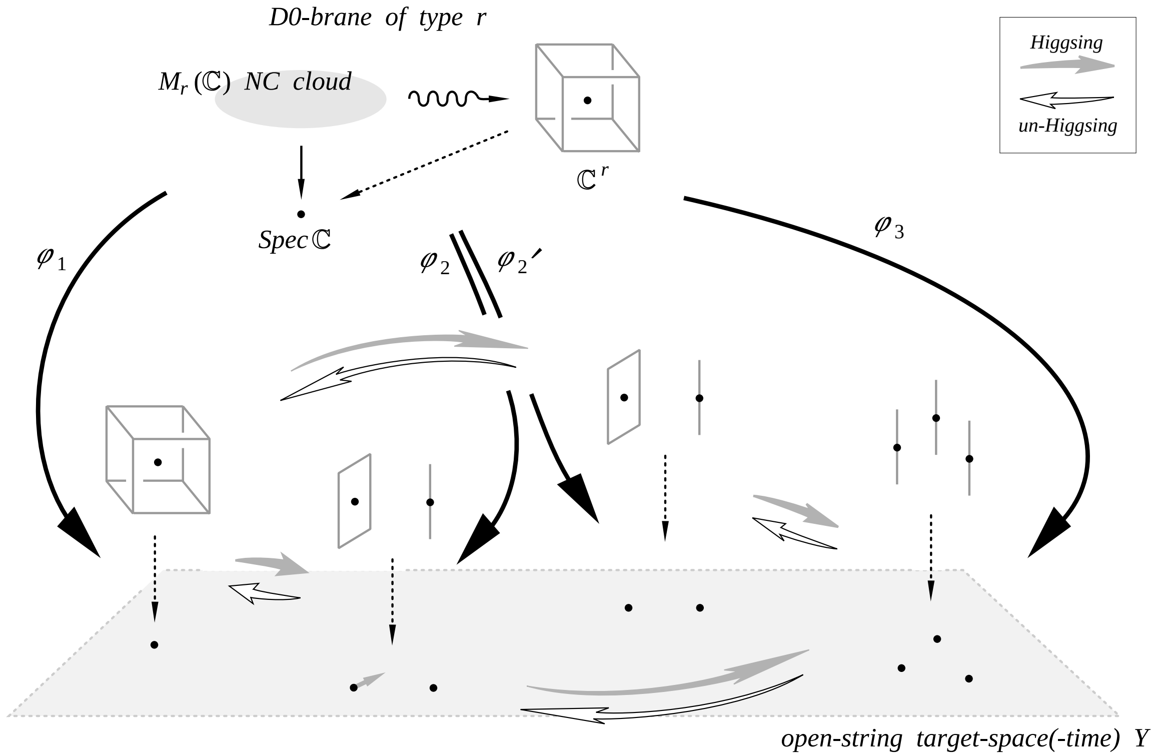


FIGURE 2-1-1. The Higgsing and un-Higgsing of D-branes in Y via deformations of morphisms φ from $(X^{Az},\mathcal{E})$ to Y. Here, a module over a scheme is indicated by $\dashrightarrow$.

Basic properties of the surrogate C_φ associated to $\varphi:C^{Az}\to Y$.

The following lemma realizes the surrogate C_φ of C^{Az} associated to $\varphi:C^{Az}\to Y$ as the graph of a multi-valued morphism from C to Y; it is a direct consequence of the minimal property of C_φ, cf. Remark 2.1.4:

Lemma/Definition 2.1.8. [**C_φ as graph of morphism**]. *With notations from above, C_φ is canonically isomorphic to a subscheme Γ_φ of $C\times Y$ such that $\pi_\varphi:C_\varphi\to C$ and $f_\varphi:C_\varphi\to Y$ are given respectively by $pr_1:\Gamma_\varphi\to C$ and $pr_2:\Gamma_\varphi\to Y$, where pr_1 and pr_2 are the projection maps from $C\times Y$ to its factors C and Y respectively. In particular, C_φ is projective.* We shall call $\Gamma_\varphi\subset C\times Y$ the *graph* of $\varphi:C^{Az}\to Y$.

Lemma 2.1.9. [**no embedded point**]. *A surrogate* **Spec** $\mathcal{A}$ *of* C^{Az}*, in particular* C_φ*, does not have embedded points.*

Proof. This follows from the fact that $\mathcal{A}$ is an $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az}$ and the latter is locally-free – and hence torsion-free – as an $\mathcal{O}_C$-module.

$\square$

Thus, C_φ can be visualized as a twisting-around of not necessarily connected and in general non-reduced 0-dimensional schemes over C that is flat over the smooth locus of C.

In more detail[2], let $\nu : \widetilde{C} \to C$ be the normalization of C and

$$0 \longrightarrow (\nu^*\mathcal{A}_\varphi)_{torsion} \stackrel{\imath}{\longrightarrow} \nu^*\mathcal{A}_\varphi \stackrel{\jmath}{\longrightarrow} \overline{\nu^*\mathcal{A}_\varphi} \longrightarrow 0\,,$$

where $(\nu^*\mathcal{A}_\varphi)_{torsion}$ is the torsion subsheaf of $\nu^*\mathcal{A}_\varphi$, and

$$\mathcal{A}_\varphi \stackrel{k}{\hookrightarrow} \nu_*\overline{\nu^*\mathcal{A}_\varphi}$$

be the canonical exact sequence (resp. inclusion) of $\mathcal{O}_{\widetilde{C}}$- (resp. $\mathcal{O}_C$-)modules. As $\mathcal{A}_\varphi$ is an $\mathcal{O}_C$-algebra, both $\nu^*\mathcal{A}_\varphi$ and $\overline{\nu^*\mathcal{A}_\varphi}$ are $\mathcal{O}_{\widetilde{C}}$-algebra, with $\mathbf{Spec}\,\nu^*\mathcal{A}_\varphi = \widetilde{C}\times_C C_\varphi$, $\imath$ an inclusion of an ideal sheaf on $\widetilde{C}\times_C C_\varphi$, and $\jmath$ is the $\mathcal{O}_{\widetilde{C}}$-algebra quotient that corresponds to the closed subscheme $\widehat{C_\varphi} := \mathbf{Spec}\,\overline{\nu^*\mathcal{A}_\varphi}$ of $\widetilde{C}\times_C C_\varphi$ obtained by removing all the embedded points from the latter. By construction, $\widehat{C_\varphi}$ is flat over $\widetilde{C}$. The $\mathcal{O}_C$-algebra inclusion k then implies that the morphism $\widehat{\nu} : \widehat{C_\varphi} \to C_\varphi$ from the composition $\widehat{C_\varphi} \hookrightarrow \widetilde{C}\times_C C_\varphi \to C_\varphi$ is surjective. In summary,

$$\begin{array}{ccc} \widehat{C_\varphi} & \stackrel{\widehat{\nu}}{\longrightarrow} & C_\varphi \\ {\scriptstyle \widehat{\pi_\varphi}}\downarrow{\scriptstyle \text{flat}} & & \downarrow{\scriptstyle \pi_\varphi} \\ \widetilde{C} & \stackrel{\nu}{\longrightarrow} & C \end{array}$$

where all the arrows/morphisms are surjective. Furthermore, as $\mathcal{A}_\varphi$ embeds in a locally-free $\mathcal{O}_C$-module, for q a node of C, let $\nu^{-1}(q) = \{q_-, q_+\}$; then,

$$(\widehat{\nu}(\widehat{\pi_\varphi}^{-1}(q_-)))_{\mathrm{red}} \;=\; (\pi_\varphi^{-1}(q))_{\mathrm{red}} \;=\; (\widehat{\nu}(\widehat{\pi_\varphi}^{-1}(q_+)))_{\mathrm{red}}\,.$$

Cf. Figure 2-1-2.

Finally, we compare C_φ in this picture with the related twisting-around of (commutative) surrogates of the Azumaya $\mathbb{C}$-point $\mathit{Space}\,M_r(\mathbb{C}) = (\mathit{Spec}\,\mathbb{C}, M_r(\mathbb{C}))$ over C. The inclusion $\mathcal{A} \hookrightarrow \mathcal{O}_C^{Az}$ induces a $\mathbb{C}$-algebra homomorphism $\mathcal{A}\otimes_{\mathcal{O}_C}\kappa_x \stackrel{\jmath_x}{\to} \mathcal{O}_C^{Az}\otimes_{\mathcal{O}_C}\kappa_x \simeq M_k(\mathbb{C})$, where κ_x is the residue field of $x \in C$.

Lemma 2.1.10. [kernel of $\jmath_x$]. *$Ker\,\jmath_x$ is contained in the nilradical $Nil(\mathcal{A}\otimes_{\mathcal{O}_C}\kappa_x)$ of $\mathcal{A}\otimes_{\mathcal{O}_C}\kappa_x$ and is non-zero only for finitely many x on C.*

Proof. Let $\mathcal{E}$ be a fundamental module on C^{Az} with $\mathcal{O}_C^{Az} = \mathcal{E}nd\,\mathcal{E}$. As $\mathcal{E} = \pi_{\varphi *}({}_{\mathcal{O}_{C_\varphi}}\mathcal{E})$ is locally-free of rank r, ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ on C_φ/C forms a flat family of 0-dimensional sheaves of length r over C. As $\mathcal{A}_\varphi$ is now a subsheaf of $\mathcal{E}nd\,\mathcal{E}$, $\mathit{Supp}\,(({}_{\mathcal{O}_{C_\varphi}}\mathcal{E})|_{\pi_\varphi^{-1}(x)}) = \pi_\varphi^{-1}(x)$ over a dense open subset of C. This implies that $\mathit{Supp}\,(({}_{\mathcal{O}_{C_\varphi}}\mathcal{E})|_{\pi_\varphi^{-1}(x)}) \subset \pi_\varphi^{-1}(x)$ with $(\mathit{Supp}\,(({}_{\mathcal{O}_{C_\varphi}}\mathcal{E})|_{\pi_\varphi^{-1}(x)}))_{\mathrm{red}} = (\pi_\varphi^{-1}(x))_{\mathrm{red}}$ for all $x \in C$. The lemma now follows.

$\square$

[2]This applies to general surrogates. Here we focus on C_φ.

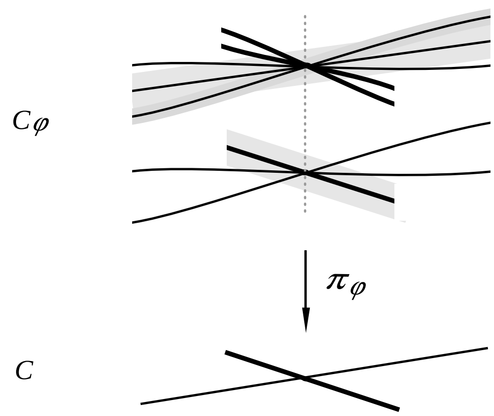


FIGURE 2-1-2. While C_φ is in general flat only over C_{smooth}, the reduced structure of C_φ over the two branches around a node of C has to coincide over that node.

The current theme helps us to understand part of the geometry behind the notion of "combinatorial type" of φ in the next theme. Lemma 2.1.10 indicates a general non-flat "lower-semicontinuous-like" dropping behavior of surrogates in a family, cf. Remark 2.1.16.

The combinatorial type of a morphism $\varphi : (C^{Az}, \mathcal{E}) \to Y$.

To define a notion of "combinatorial type" that fits our moduli problem, generalizes the situations in several related moduli problems from commutative geometry, and captures the features of D-branes in superstring theory, we have to bring in fundamental modules $\mathcal{E}$, i.e. Chan-Paton modules, as well. The triple (g, r, χ), where g is the (arithmetic) genus of C and r and χ are respectively the rank and the Euler characteristic of $\mathcal{E}$, is the standard combinatorial type data for the domain data $(C^{Az}, \mathcal{E})$. It remains to define the notion of "image curve class" for $\varphi : (C^{Az}, \mathcal{E}) \to Y$ in any of $A_1(Y)$, $N_1(Y)$, and $H_2(Y;\mathbb{Z})$.

Definition 2.1.11. [generic length]. Let C' be an irreducible component of C_φ. For a closed point of $x' \in C'$, define $l(x') = \mathit{length}((\mathcal{O}_{C_\varphi}\mathcal{E})|_{\widehat{x'}})$ $(= H^0((\mathcal{O}_{C_\varphi}\mathcal{E})|_{\widehat{x'}}))$, where $\widehat{x'}$ is the connected component of $\pi_\varphi^{-1}(\pi_\varphi(x'))$ that contains x'. There exists an open dense subset $U' \subset C'$ such that l is constant on the set of all closed points in U'. Define the *generic length* of $\mathcal{O}_{C_\varphi}\mathcal{E}$ *on* C' to be this constant.

Cf. Figure 2-1-3.

Definition 2.1.12. [image curve class]. Given a morphism $\varphi : (C^{Az}, \mathcal{E}) \to Y$, let $C_\varphi = \cup_i C'_i$ be the decomposition of C_φ by irreducible components and l_i be the generic length of $\mathcal{O}_{C_\varphi}\mathcal{E}$ on C'_i. Then the *image curve class*, in notation $\varphi_*[C]$, in $A_1(Y)$ (similarly, in $N_1(Y)$ or $H_2(Y;\mathbb{Z})$) is defined by

$$\varphi_*[C] \;:=\; \sum_i l_i \cdot (f_\varphi)_*[(C'_i)_{\mathrm{red}}] \;\in\; A_1(Y)\,.$$

Definition 2.1.13. [combinatorial type]. The tuple from the above discussion, defined by $(g, r, \chi\,|\,\beta) := (g(C), \mathit{rank}(\mathcal{E}), \chi(\mathcal{E})\,|\,\varphi_*[C])$, is called the *combinatorial type* of $\varphi : (C^{Az}, \mathcal{E}) \to Y$.

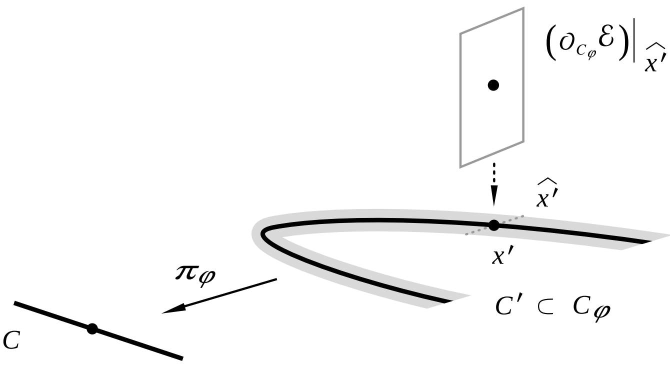


FIGURE 2-1-3. The length function $l(\bullet)$ of ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ on $(C_\varphi)_{\mathbb{C}}$. Here, a module over a scheme is indicated by $\dashrightarrow$.

Remark 2.1.14. (1) The notion of image curve class for a D-string world-sheet under a morphism has to go through the length function in Definition 2.1.11 due to the complication that in general a coherent $\mathcal{O}_{X'}$-module $\mathcal{E}'$ on a nonreduced scheme X' may not admit an open dense subscheme over which $\mathcal{E}'$ is locally free. Such situation can indeed happen for ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ on C_φ.

(2) When ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ is locally free on an open dense subset of C_φ, then the rank of such restrictions define the rank r_i of ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ on each irreducible component C_i' of C_φ. In this case, the generic length l_i of ${}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ on an irreducible component C_i' of C_φ is given by $m_i \cdot r_i$, where m_i is the multiplicity of C_i' in the sense of [Fu], and $\varphi_*[C] = \sum_i m_i r_i \cdot (f_\varphi)_*[(C_i')_{\mathrm{red}}]$. This form fits into superstring-theory setting directly.

(3) For morphisms to $\mathbb{P}^k$ such that Item (2) above applies, let d_i be the degree of $f_\varphi|_{C_i'}$. Then $\varphi_*[C]$ has degree $\sum_i r_i d_i = rd$ for some $d \in \mathbb{Z}_{\geq 0}$. Here, $r = \mathit{rank}(\mathcal{E})$. This illustrates that the notion of image curve class of D-string world-sheet in a target Y involves not only the 1-cycle that supports the D-string world-sheet but also the Chan-Paton module on the D-string world-sheet.

Morphisms over a (commutative) base scheme/$\mathbb{C}$.

The notion of a morphism from an Azumaya prestable curve to Y, as given in Definition 2.1.2, works for a general Azumaya scheme $X^{Az} := (X, \mathcal{O}_X^{Az})$, where $X = (X, \mathcal{O}_X)$ is a (commutative, Noetherian) scheme/$\mathbb{C}$ and $\mathcal{O}_X^{Az} = \mathcal{E}nd(\mathcal{E}) := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ for a locally-free coherent $\mathcal{O}_X$-module $\mathcal{E}$ on X. One can thus extend the notion of morphism to that of "morphism over a base scheme" as follows:

Definition 2.1.15. [morphism over a base and its pull-back]. Let S be a (commutative Noetherian) scheme over $\mathbb{C}$. A *family of morphisms over* S from Azumaya prestable curves with a fundamental module to Y, in notation $\Phi := \Phi_S : (\mathcal{C}_S^{Az}, \mathcal{E}_S)/S \to Y$, consists of the following data:

(1) A flat family $(\mathcal{C}_S^{Az}, \mathcal{E}_S)/S := (\mathcal{C}_S, \mathcal{O}_{\mathcal{C}_S}^{Az}, \mathcal{E}_S)/S$, where $\mathcal{O}_{\mathcal{C}_S}^{Az} = \mathcal{E}nd_{\mathcal{O}_{\mathcal{C}_S}}(\mathcal{E}_S)$, of Azumaya prestable curves with a fundamental module..

(2) An equivalence class of pairs

$$(\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A} \subset \mathcal{O}_{\mathcal{C}_S}^{Az}\,,\ f : \mathcal{C}_{\mathcal{A}} := \mathbf{Spec}\,\mathcal{A} \to Y)\,,$$

where

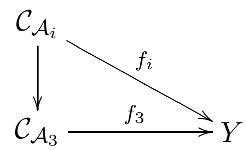
(2.1) $\mathcal{A}$ is a commutative $\mathcal{O}_{\mathcal{C}_S}$-subalgebra of $\mathcal{O}_{\mathcal{C}_S}^{Az}$;

(2.2) $f : \mathcal{C}_{\mathcal{A}} \to Y$ is a morphism of (commutative) schemes;

(2.3) two such pairs $(\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A}_1 \subset \mathcal{O}_{\mathcal{C}_S}^{Az}\,,\ f_1 : \mathcal{C}_{\mathcal{A}_1} \to Y)$ and $(\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A}_2 \subset \mathcal{O}_{\mathcal{C}_S}^{Az}\,,\ f_2 : \mathcal{C}_{\mathcal{A}_2} \to Y)$ are equivalent, in notation

$$(\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A}_1 \subset \mathcal{O}_{\mathcal{C}_S}^{Az}\,,\ f_1 : \mathcal{C}_{\mathcal{A}_1} \to Y)\ \sim\ (\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A}_2 \subset \mathcal{O}_{\mathcal{C}_S}^{Az}\,,\ f_2 : \mathcal{C}_{\mathcal{A}_2} \to Y)\,,$$

if there exists a third pair $(\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A}_3 \subset \mathcal{O}_{\mathcal{C}_S}^{Az}\,,\ f_3 : \mathcal{C}_{\mathcal{A}_3} \to Y)$ such that $\mathcal{A}_3 \subset \mathcal{A}_i$ and that the induced diagram



commutes, for $i = 1, 2$.

Let $h : T \to S$ be a morphism and $\mathcal{C}_T = h^*\mathcal{C}_S := T \times_S \mathcal{C}_S$. Define the *pull-back* of Φ to T to be the morphism $\Phi_T := h^*\Phi_S : (h^*\mathcal{C}_S^{Az}, h^*\mathcal{E}_S)/T \to Y$ that corresponds to the equivalence class of pairs $(\mathcal{O}_{\mathcal{C}_T} \subset \mathcal{A}' \subset \mathcal{O}_{\mathcal{C}_T}^{Az}\,,\ f' : \mathcal{C}_{\mathcal{A}'} := \mathbf{Spec}\,\mathcal{A}' \to Y)$ that contains the pair

$$(\mathcal{O}_{\mathcal{C}_T} \subset \overline{h^*\mathcal{A}_\Phi} \subset \mathcal{O}_{\mathcal{C}_T}^{Az}\,,\ \overline{h^*f_\Phi} : \mathbf{Spec}\,\overline{h^*\mathcal{A}_\Phi} \to Y)\,,$$

where

- $\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{A}_\Phi \subset \mathcal{O}_{\mathcal{C}_S}^{Az}$ gives the surrogate $\mathcal{C}_\Phi$ of $\mathcal{C}_S^{Az}$ associated to Φ, it goes with a morphism $f_\Phi : \mathcal{C}_\Phi \to Y$,

- $\overline{h^*\mathcal{A}_\Phi}$ is the image $\mathcal{O}_{\mathcal{C}_T}$-subalgebra of the $\mathcal{O}_{\mathcal{C}_T}$-algebra homomorphism $h^*\mathcal{A}_\Phi \to \mathcal{O}_{\mathcal{C}_T}^{Az} = h^*\mathcal{O}_{\mathcal{C}_S}^{Az}$, it contains $\mathcal{O}_{\mathcal{C}_T}$ canonically as a $\mathcal{O}_{\mathcal{C}_T}$-subalgebra,

- $\overline{h^*f_\Phi} : \mathbf{Spec}\,\overline{h^*\mathcal{A}_\Phi} \to Y$ is the composition of the canonical $\mathbf{Spec}\,\overline{h^*\mathcal{A}_\Phi} \to h^*\mathcal{C}_\Phi \stackrel{h^*f_\Phi}{\to} Y$.

When $\iota_s : \mathit{Spec}\,\mathbb{C} \to S$ corresponds to a closed point $s \in S$, the pull-back $\Phi_s := \iota_s^*\Phi$ is called the *fiber* of Φ at $s \in S$.

$\Phi : (\mathcal{C}_S^{Az}, \mathcal{E}_S)/S \to Y$ is said to be a *family of morphisms* over S *of combinatorial type* $(g, r, \chi\,|\,\beta)$ if, in addition, Φ_s is of combinatorial type $(g, r, \chi\,|\,\beta)$ for every closed point $s \in S$.

Remark 2.1.16. [*surrogates in family and the significance of Chan-Paton modules*]. Caution that, while both $\mathcal{O}_{\mathcal{C}_S}$ and $\mathcal{O}_{\mathcal{C}_S}^{Az}$ are flat over S, the $\mathcal{O}_{\mathcal{C}_S}$-module $\mathcal{A}_\Phi$ on $\mathcal{C}_S$ that gives the family $\mathcal{C}_\Phi$ of surrogates associated to Φ in general is *not* flat over S. *Nor* does the surrogate pass from one to another directly via the usual pull-back under base change. This is an indication that the pure moduli problem of morphisms from Azumaya prestable curves to Y is a technically worse problem than the moduli problem of D-strings. In the latter, the extra data of Chan-Paton modules on D-branes enlarges the moduli problem while taming it significantly. This will become clearer from the point of view of the next subsection.

2.2 Azumaya without Azumaya, morphisms without morphisms.

Recasting into commutative algebro-geometric setting.

A morphism from an Azumaya prestable curve with a fundamental module to Y under our setting can be redescribed/recast effectively and faithfully in terms of commutative geometric data. Two equivalent such data are given below.

Lemma 2.2.1. [Azumaya without Azumaya, morphisms without morphisms]. *Given a prestable curve C, a morphism $\varphi : (C^{Az}, \mathcal{E}) \to Y$ from an Azumaya curve C^{Az} over C with a fundamental module $\mathcal{E}$ of rank r is given by a coherent $\mathcal{O}_{C\times Y}$-module $\widetilde{\mathcal{E}}$ on $(C\times Y)/C$ of relative length r. The correspondence is functorial and bijective.*

Proof. Given φ, it follows from Sec. 2.1 that one can canonically associated to it an $\mathcal{O}_{\Gamma_\varphi}$-module $\widetilde{\mathcal{E}} := {}_{\mathcal{O}_{C_\varphi}}\mathcal{E}$ via the canonical isomorphism $C_\varphi \simeq \Gamma_\varphi \subset C\times Y$. As an $\mathcal{O}_{C\times Y}$-module on $(C\times Y)/C$, $\widetilde{\mathcal{E}}$ satisfies the stated properties.

Conversely, given $\widetilde{\mathcal{E}}$ on $(C \times Y)/C$ as stated, let $\Gamma = \mathit{Supp}\,\widetilde{\mathcal{E}}$, $\mathcal{E} := pr_{1*}\widetilde{\mathcal{E}}$, and $\mathcal{O}_C^{Az} := \mathcal{E}nd\,\mathcal{E}$. Then the tautological action of $\mathcal{O}_\Gamma$ on $\widetilde{\mathcal{E}}$ gives rise to an action of $\mathcal{O}_\Gamma$ on $\mathcal{E}$. This realizes $\mathcal{O}_\Gamma$ canonically as a commutative $\mathcal{O}_C$-sub-algebra $\mathcal{A}$ in $\mathcal{O}_C^{Az}$. The morphism $\Gamma \to Y$ from the restriction of $pr_2 : C \times Y \to Y$ gives rise to a morphism $f : C_\mathcal{A} := \mathbf{Spec}\,\mathcal{A} \to Y$ that satisfies the minimal property in Remark 2.1.4. Thus, $\widetilde{\mathcal{E}}$ corresponds canonically to $\varphi : (C^{Az}, \mathcal{E}) \to Y$ given by $\mathcal{E}$ and the pair $(\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az} := \mathcal{E}nd\,\mathcal{E}\,,\, f : C_\mathcal{A} \to Y)$ as defined.

The functoriality and bijectivity of the correspondence follow from the fact that the above two-way correspondence is canonical.

□

Cf. Figure 2-2-1. It is worth emphasizing that $\widetilde{\mathcal{E}}$ on $C \times Y$ in the above lemma is flat over C and, hence, its push-forward $pr_{1*}\widetilde{\mathcal{E}}$ to C is locally free.

Remark 2.2.2. [*AwoA & MwoM in general*]. Replacing the prestable curve C in Lemma 2.2.1 by a scheme X gives a canonical description of a morphism $\varphi : (X^{Az}, \mathcal{E}) \to Y$ in terms of a coherent $\mathcal{O}_{X\times Y}$-module $\widetilde{\mathcal{E}}$ of relative length r on $(X \times Y)/X$.

Lemma 2.2.1 and Remark 2.2.2 allow us to bring the language and techniques in purely commutative algebraic geometry into our study. In this way we are tossed back to the category of commutative geometry.

Applying Lemma 2.2.1 and Remark 2.2.2 to the case of D0-branes (cf. [Va] and [L-Y2: Sec. 4]), one then realizes the moduli stack of D0-branes of type r on Y as the algebraic stack of 0-dimensional, length r, torsion sheaves on Y; see Sec. 3.1 for more discussions. It follows immediately that

Corollary 2.2.3. [topological D-string as a smearing of D0-branes]. *A morphism $\varphi : (C^{Az}, \mathcal{E}) \to Y$ corresponds canonically to a morphism from C to the stack of D0-branes on Y; and vice versa.*

Replacing C in the corollary by a scheme X realizes Euclidean D-branes of B-type as morphisms from X to the stack of D0-branes on Y. This gives a precise realization of general Dp-branes of B-type as "smearing" of D0-branes along a scheme/cycle, cf. [L-Y2: the paragraph before Remark 2.2.5] and Figure 3-1-1.

Definition 2.1.12 of image curve class β of a D-string world-sheet under a morphism follows the feature of D-branes in superstring theory. The following lemma shows that it is indeed a "correct" definition for the current prototype mathematical formulation of D-strings in Definition 2.1.6 and Remark 2.1.7 along the Polchinski-Grothendieck Ansatz:

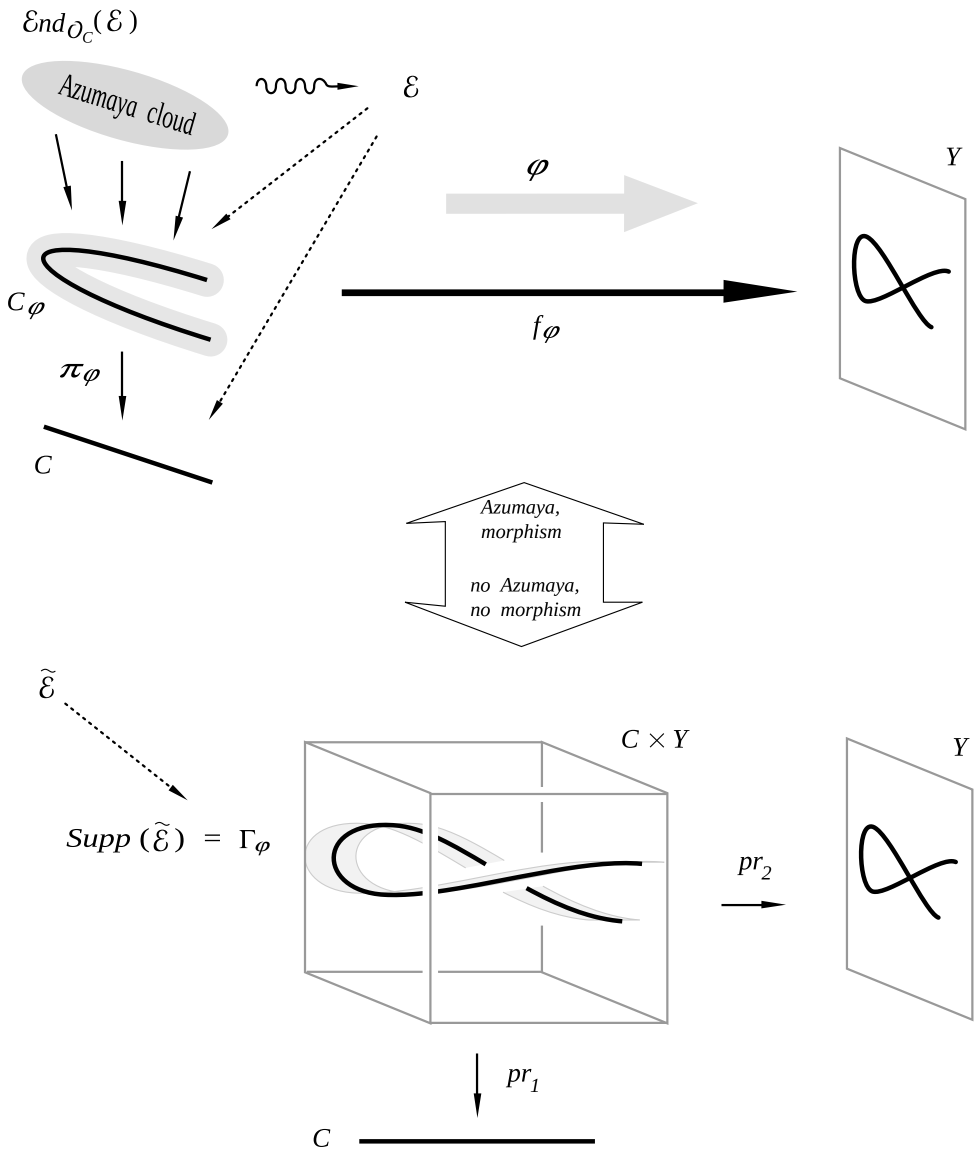


FIGURE 2-2-1. The correspondence between $\{$ morphisms $\varphi : (C^{Az}, \mathcal{E}) \to Y \}$ and $\{$ torsion $\mathcal{O}_{C\times Y}$-modules $\widetilde{\mathcal{E}}$ on $C \times Y$ that are flat over C with relative dimension $0\}$. Here, only Y is fixed and the correspondence is at the level of related moduli stacks. In the figure, a module over a scheme is indicated by a dash-arrow $\dashrightarrow$.

Lemma 2.2.4. [invariance of combinatorial type under flat deformation]. *Let $\mathcal{C}_S/S$ be a flat family of prestable curves over (a connected) S, $\widetilde{\mathcal{E}}_S$ be an $\mathcal{O}_{\mathcal{C}_S\times Y}$-module of relative length r on $(\mathcal{C}_S\times Y)/\mathcal{C}_S$, (in particular, $\widetilde{\mathcal{E}}_S$ on $\mathcal{C}_S\times Y$ is flat over S), and $\varphi_s:(C_s^{Az},\mathcal{E}_s)\to Y$ be the morphism, with a fundamental module, associated to $\widetilde{\mathcal{E}}_s$ for a closed point $s\in S$. Then, the combinatorial type $(g,r,\chi\,|\,\beta)$ of φ_s is constant over S.*

Proof. Let $\mathcal{O}_Y(1)$ be the fixed polarization on Y and take $\mathcal{O}_{C\times Y}(1)$ to be $\mathcal{O}_C(1)\boxtimes\mathcal{O}_Y(1)$. Let $\widetilde{\mathcal{E}}$ be an $\mathcal{O}_{C\times Y}$-module of relative length r on $(C\times Y)/C$. Then, $Supp\,\widetilde{\mathcal{E}}$ is 1-dimensional and, hence, there exists an $\widetilde{\mathcal{E}}$-regular hyperplane $H=pr_1^*H_C+pr_2^*H_Y\in|\mathcal{O}_{C\times Y}(1)|$ on $C\times Y$ with H_C (resp. H_Y) being a hyperplane in C (resp. Y) with respect to $\mathcal{O}_C(1)$ (resp. $\mathcal{O}_Y(1)$) such that $(pr_1^*H_C\cap pr_2^*H_Y)\cap Supp\,(\widetilde{\mathcal{E}})$ is empty. Since the morphism $Supp\,(\widetilde{\mathcal{E}})\to C$ from the restriction of the projection map $pr_1:C\times Y\to C$ is affine, one has the values of Hilbert polynomials $P(\widetilde{\mathcal{E}},\bullet)$ of $\widetilde{\mathcal{E}}$ at 0 and 1 given respectively by

$$P(\widetilde{\mathcal{E}},0)\;=\;\chi(\widetilde{\mathcal{E}})\;=\;\chi(pr_{1*}\widetilde{\mathcal{E}})\;=\;\chi$$

and

$$P(\widetilde{\mathcal{E}},1)\;=\;\chi(\widetilde{\mathcal{E}}(1))\;=\;\chi\left(\widetilde{\mathcal{E}}|_H\right)\binom{1}{1}+\chi_0\;=\;(r\deg(C)+H_Y\cdot\beta)+\chi\,.$$

This implies that

$$P(\widetilde{\mathcal{E}},m)\;=\;(r\deg(C)+H_Y\cdot\beta)m+\chi\,.$$

In particular, it is completely determined by the combinatorial type $(g,r,\chi\,|\,\beta)$ of the associated $\varphi:(C^{Az},\mathcal{E})\to Y$ and the polarization chosen on C and Y respectively. The lemma follows.

□

Remark 2.2.5. [*Euler characteristic vs. degree*]. Note that for a locally free sheaf $\mathcal{E}$ of rank r on a prestable curve C of (arithmetic) genus g with a fixed polarization $\mathcal{O}_C(1)$, its Hilbert polynomial $P(\mathcal{E},m)$ is given by

$$P(\mathcal{E},m)\;=\;r\deg(C)m+\deg(\mathcal{E})+r(1-g)\,.$$

This follows from the Riemann-Roch Theorem for smooth curves, additivity of Hilbert polynomials with respect to an short exact sequence, and the consideration of the canonical short exact sequence of $\mathcal{O}_C$-modules $0\to\mathcal{E}\to\nu_*\nu^*\mathcal{E}\to\mathcal{T}\to 0$, where $\nu:C'\to C$ is the normalization of C and note that $\mathcal{T}$ is a torsion sheaf supported on the set of nodes of C with the fiber dimension at each node equal to r. Thus, in our problem, χ in $(g,r,\chi\,|\,\beta)$ can be replaced by the degree of fundamental modules.

Isomorphisms.

Definition 2.2.6. [isomorphism]. An $\mathcal{O}_{C_1\times Y}$-module $\widetilde{\mathcal{E}}_1$ of relative length r on $(C_1\times Y)/C_1$ and an $\mathcal{O}_{C_2\times Y}$-module $\widetilde{\mathcal{E}}_2$ of relative length r on $(C_2\times Y)/C_2$ are said to be *isomorphic* if there exists a pair $(h,\widetilde{h})$, where $h:C_1\simeq C_2$ and $\widetilde{h}:\widetilde{\mathcal{E}}_1\simeq h^*\widetilde{\mathcal{E}}_2$ are isomorphisms of schemes and coherent sheaves respectively. Here, we denote the h-induced isomorphisms $C_1\times Y\simeq C_2\times Y$ between the products still by h.

The following lemma follows from the correspondence between the noncommutative and the commutative picture of morphisms, given in the proof of Lemma 2.2.1, and a definition-tracing:

Lemma 2.2.7. [isomorphism: noncommutative vs. commutative picture]. *An $\mathcal{O}_{C_1\times Y}$-module $\widetilde{\mathcal{E}}_1$ of relative length r on $(C_1\times Y)/C_1$ and an $\mathcal{O}_{C_2\times Y}$-module $\widetilde{\mathcal{E}}_2$ of relative length r on $(C_2\times Y)/C_2$ are isomorphic if and only if the corresponding morphisms $\varphi_1:(C_1,\mathcal{E}_1)\to Y$ and $\varphi_2:(C_2,\mathcal{E}_2)\to Y$ are isomorphic in the sense of Definition 2.1.5.*

It follows from the immediate generalization of Lemma 2.2.1 to families and Lemma 2.2.7 that:

Proposition 2.2.8. [stack of morphisms as stack of sheaves on products]. *The stack of morphisms from Azumaya prestable curves with a fundamental module to Y is canonically isomorphic to the stack of torsion sheaves of modules on the product of prestable curves with Y of finite relative length over the prestable curves.*

We will denote the stack of morphisms from Azumaya prestable curves with a fundamental module to Y of combinatorial type $(g,r,\chi\,|\,\beta)$ by $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$. This is an algebraic stack over $\mathbb{C}$, but only locally of finite-type. We will describe an atlas of $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ in Sec. 3.2.

2.3 Boundedness of morphisms.

Proposition 2.3.1. [boundedness of morphisms]. *Let $(\mathcal{C}_S^{Az},\mathcal{E}_S)/S$ be a family of Azumaya prestable curves with a fundamental module of type (g,r,χ) over a base scheme S (of finite type). Then the set $\{\varphi_\bullet\}_\bullet$ of morphisms from fibers of $(\mathcal{C}_S^{Az},\mathcal{E}_S)/S$ to Y of type $(g,r,\chi\,|\,\beta)$ is bounded.*

Proof. Fix a relative ample line bundle $\mathcal{O}_{\mathcal{C}_S^{Az}/S}(1)$ on $\mathcal{C}_S^{Az}/S$ and an ample line bundle $\mathcal{O}_Y(1)$ on Y with the associated hyperplane class H_Y. Let $\mathcal{O}_{\mathcal{C}_S^{Az}\times Y}(1):=\mathcal{O}_{\mathcal{C}_S^{Az}/S}(1)\boxtimes\mathcal{O}_Y(1)$. A morphism φ from $(C_s^{Az},\mathcal{E}_s)$ to Y, where s is a closed point of S, of type of $(g,r,\chi\,|\,\beta)$ is given by an $\mathcal{O}_{\mathcal{C}_s^{Az}\times Y}$-module $\widetilde{\mathcal{E}}_s$ such that $\widetilde{\mathcal{E}}_s$ is of relative length r over C_s^{Az} and has the Hilbert polynomial $P(\widetilde{\mathcal{E}}_s,m)=(r\deg(C)+H_Y\cdot\beta)m+\chi$ and that $pr_{1*}(\widetilde{\mathcal{E}}_s)\simeq\mathcal{E}_s$. As $Supp(\widetilde{\mathcal{E}}_s)$ is affine over C_s, once fixing the isomorphism $pr_{1*}(\widetilde{\mathcal{E}}_s)\simeq\mathcal{E}_s$, the tautological morphism $pr_1^*(\mathcal{E}_s)\to\widetilde{\mathcal{E}}_s\to 0$ of $\mathcal{O}_{C_s\times Y}$-modules is surjective. Since $\mathcal{E}_S/S$ is given, it follows that the set of all such $\widetilde{\mathcal{E}}_s$ on $C_s\times Y$ is bounded, cf. [Gr] and [H-L: Lemma 1.7.6]. The proposition now follows from Lemma 2.2.1. □

Continuing the notation in the above proof, as the set of $\widetilde{\mathcal{E}}_s$ is bounded, so is the set of their (scheme-theoretic) support $Supp(\widetilde{\mathcal{E}}_s)$, described by the ideal sheaf $Ker(\mathcal{O}_{C_s\times Y}\to\mathcal{E}nd(\widetilde{\mathcal{E}}_s))$. A bound of $\chi(Supp(\widetilde{\mathcal{E}}_s))$, hence a bound of the (arithmetic) genus $1-\chi$ of the surrogate associated to morphisms of type $(g,r,\chi\,|\,\beta)$ from fibers of $(\mathcal{C}_S^{Az},\mathcal{E}_S)/S$ to Y, can be expressed in terms of Hilbert polynomials and the Mumford-Castelnuovo regularity of certain sheaves involved as follows. Let

$$m_0\;:=\;\max\left\{reg(\mathcal{F})\;:\;\begin{array}{l}\mathcal{F}\;=\;\mathcal{O}_{C_s\times Y}\;\text{ or }\;Ker(\mathcal{O}_{C_s\times Y}\to\mathcal{E}nd(\widetilde{\mathcal{E}}_s))\\ \text{from the above bounded family}\end{array}\right\}.$$

Here, $reg(\mathcal{F})$ is the Mumford-Castelnuovo regularity for an $\mathcal{O}_{C_s\times Y}$-module $\mathcal{F}$ with respect to $\mathcal{O}_{C_s\times Y}(1)$. Since $reg(\,\bullet\,)$ here is taken over a bounded family of modules, m_0 is finite in $\mathbb{Z}_{\geq 0}$. It follows that the Hilbert polynomials satisfy inequalities

$$0\;\leq\;P(Supp(\widetilde{\mathcal{E}}_s),m)\;\leq\;P(\mathcal{O}_{C_s\times Y},m)\quad\text{for all }\;m\geq m_0\,.$$

Note that $P(\mathcal{O}_{C_s\times Y},m)$ is a polynomial $P(m)$ in m, independent of $s\in S$. It follows that

$$P(Supp(\widetilde{\mathcal{E}}_s),m)\;=\;\alpha(Supp(\widetilde{\mathcal{E}}_s))\,m\;+\;\chi(Supp(\widetilde{\mathcal{E}}_s))$$

with

$$1 \ \le\ \alpha(Supp\,(\widetilde{\mathcal{E}}_s))\ \le\ r\deg(C)+H_Y\cdot\beta$$

and

$$-(r\deg(C)+H_Y\cdot\beta)\,m\ \le\ \chi(Supp\,(\widetilde{\mathcal{E}}_s))\ \le\ P(m)-m \quad \text{for all}\ \ m\ge m_0\,.$$

In particular,

$$1-P(m_0)+m_0\ \le\ g(C_\varphi)\ \le\ 1\,+\,(r\deg(C)+H_Y\cdot\beta)\,m_0\,.$$

2.4 Morphisms from an Azumaya prestable curve to $\mathbb{P}^k$.

Morphisms to a projective space $\mathbb{P}^k$ deserves some attention. In the commutative case, there are three presentations of a morphism from a scheme X to a projective space: (1) as a morphism between schemes, (2) as a quotient $\mathcal{O}_X^{k+1}\to\mathcal{L}\to 0$ to a line bundle, and (3) as a $(k+1)$-tuple of rational functions on X. Let $Y=\mathbb{P}^k$, then Sec. 2.1 gives the presentation for morphisms from Azumaya prestable curves to $\mathbb{P}^k$ from Aspect (1). In the current subsection, we discuss the presentation of such morphisms from Aspect (3).[3]

We formulate first a format of localizations that is akin to the problem and then define and study morphisms from Azumaya curves to $\mathbb{P}^k$. This subsection is a consequence of [L-Y2]. Details that follow ibidem immediately are omitted.

Localizations and covers of commutative $\mathcal{O}_C$-subalgebras of $\mathcal{O}_C^{Az}$.

Let $\mathcal{O}_C\subset\mathcal{A}\subset\mathcal{O}_C^{Az}$ be a commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az}$. Let $\mathcal{K}_C$ (resp. $\mathcal{K}_{C_\mathcal{A}}$) be the sheaf of total quotient rings of $\mathcal{O}_C$ (resp. $\mathcal{O}_{C_\mathcal{A}}$), cf. [Ha]. An element h of $\Gamma(\mathcal{K}_{C_\mathcal{A}})$ is called a *rational function*[4] on $C_\mathcal{A}$. As the tautological morphism $\pi^\mathcal{A}: C_\mathcal{A}\to C$ is finite, $\mathcal{K}_{C_\mathcal{A}}\simeq\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C$ canonically as $\mathcal{A}$-algebras.

Notation 2.4.1. The element in $\Gamma(\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C)$ associated to $h\in\Gamma(\mathcal{K}_{C_\mathcal{A}})$ is denoted by m_h and the element in $\Gamma(\mathcal{K}_{C_\mathcal{A}})$ associated to $m\in\Gamma(\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C)$ is denoted by h_m.

The notion of function rings of local charts and their localizations and that of open covers of $C_\mathcal{A}$ in commutative algebraic geometry can be re-phrased in terms of localizations of $\mathcal{O}_C$-subalgebras of $\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C$, as follows.

Recall that, for two commuting idempotents e and e' in the matrix algebra $M_r(\mathbb{C})$ over $\mathbb{C}$, we say that e' is *subordinate to* e if there exists an idempotent $e''\in M_r(\mathbb{C})$ that commutes with both e and e' such that $e=e'+e''$. In notation, $e'\preceq e$ or $e\succeq e'$.

Let $C=\cup_i C_i$ be the decomposition of C into irreducible components. A *pseudo-section* (resp. *local pseudo-section*) s of an $\mathcal{O}_C$-module $\mathcal{F}$ is by definition a tuple $(s_i)_i$, where s_i is a section (resp. local section) of $\mathcal{F}|_{C_i}$.

Definition 2.4.2. [admissible idempotent pseudo-section]. An *admissible idempotent pseudo-section* of $\mathcal{O}_C^{Az}$ is a tuple $e:=(e_i)_i$, where

(1) e_i is an idempotent section of $\mathcal{O}_{C_i}^{Az}:=\mathcal{O}_C^{Az}\big|_{C_i}$ of constant rank as an endomorphism of $\mathcal{E}|_{C_i}$;

[3]It would be nice to have a presentation of morphisms from Azumaya curves to $\mathbb{P}^k$ in terms of Aspect (2) as well. In some sense, a fundamental module $\mathcal{E}$ of C^{Az} may be thought of as a line bundle on $\mathcal{C}^{Az}$.

[4]We do not distinguish two rational functions h_1 and h_2 on a scheme X whose domains of definition $V_{h_1}\subset X$ and $V_{h_2}\subset X$ share a common open dense subset $V\subset V_{h_1}$ and $V\subset V_{h_2}$. Similarly for rational sections of an $\mathcal{O}_X$-module.

(2) for each node $p \in C_i \cap C_j$, $e_i(p)$ and $e_j(p)$ commute and either $e_i(p) \preceq e_j(p)$ or $e_i(p) \succeq e_j(p)$.

The *addition* $e_1 + e_2$ and *multiplication* $e_1 e_2$ of two commuting idempotent pseudo-sections $e_1 = (e_{1,i})_i$ and $e_2 = (e_{2,i})_i$ are defined by $(e_{1,i} + e_{2,i})_i$ and $(e_{1,i}e_{2,i})_i$ respectively. Similarly, the multiplications me and em of a local section m of $\mathcal{O}_C^{Az}$ with e are defined component by component. Both me and em are local pseudo-sections of $\mathcal{O}_C^{Az}$. The same notions are defined for any $\mathcal{O}_C$-subalgebra $\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az}$.

Definition 2.4.3. [sheaf of algebra-subsets]. (Cf. [L-Y2: Definition 3.2.1].) (Continuing the notations from above.) Let $e = (e_i)_i$ be an admissible idempotent pseudo-section of $\mathcal{O}_C^{Az}$. Then e specifies a sheaf ${\mathcal{O}_C^{Az}}_{(e)}$ of $\mathcal{O}_C$-submodules of $\mathcal{O}_C^{Az}$ that consists of local sections m of $\mathcal{O}_C^{Az}$ that satisfy $me = em = m$. The restriction $\left({\mathcal{O}_C^{Az}}_{(e)}, e\right)\Big|_{C_i}$ is a sheaf of $\mathcal{O}_{C_i}$-algebra-subsets of $\mathcal{O}_C^{Az}|_{C_i}$ with the section of identities given by e_i. With a slight abuse of terminology, we shall call ${\mathcal{O}_C^{Az}}_{(e)}$ the *sheaf of $\mathcal{O}_C$-algebra-subsets of $\mathcal{O}_C^{Az}$ determined by e.*

Cf. Figure 2-4-1.

FIGURE 2-4-1. The meaning of e: e selects for each surrogate $C_{\mathcal{A}}$ of C^{Az} a subcurve $C_{\mathcal{A},e}$ that is maximal with respect to the property that $\mathbf{Spec}\,(\mathcal{A} \cap \mathcal{O}_{C(e)}^{Az})$ is dominated by it as $\mathbb{C}$-schemes. ($C_{\mathcal{A},e}$ can be empty.)

Example 2.4.4. $[{\mathcal{O}_C^{Az}}_{(e)}]$**.** If $e_i = 0$ for some C_i, then ${\mathcal{O}_C^{Az}}_{(e)}\Big|_{C_i}$ is the zero $\mathcal{O}_{C_i}$-algebra-subsets of $\mathcal{O}_C^{Az}|_{C_i}$. For $e = (e_i)_i$ associated to $1 \in \Gamma(\mathcal{O}_C^{Az})$, ${\mathcal{O}_C^{Az}}_{(e)} = \mathcal{O}_C^{Az}$.

Lemma 2.4.5. [characteristic function vs. admissible idempotent pseudo-section]. *Let $\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az}$ be a commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az}$ and $C_{\mathcal{A}} := \mathbf{Spec}\,\mathcal{A} = C_{\mathcal{A},1} \cup C_{\mathcal{A},2}$ be a decomposition of $C_{\mathcal{A}}$ by two disjoint sets of irreducible components. Let χ be the characteristic pseudo-function on $C_{\mathcal{A}}$ with support $C_{\mathcal{A},1}$ (i..e it takes value 1 on $C_{\mathcal{A},1}$ and 0 on $C_{\mathcal{A}} - C_{\mathcal{A},1}$). Then χ corresponds to a unique admissible idempotent pseudo-section e_χ of $\mathcal{A}$ such that for any rational function h on $C_{\mathcal{A}}$ with $h|_{C_{\mathcal{A},2}} = 0$, $m_h e_\chi \,(= e_\chi m_h) \, = m_h$.*

Let $\mathcal{O}_C \subset \mathcal{A} \subset \mathcal{O}_C^{Az}$ be a commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az}$ and h be a rational function on $C_{\mathcal{A}} := \mathbf{Spec}\,\mathcal{A}$. Let χ_h be the characteristic pseudo-function on $C_{\mathcal{A}}$ with support the union of the irreducible components of $C_{\mathcal{A}}$ to which the restriction of h is not nilpotent.

Lemma 2.4.6. [inverse of rational section and open set]. *Let e_{χ_h} be the admissible idempotent pseudo-section of $\mathcal{A}$ associated to χ_h and $\mathcal{A}_{(e_{\chi_h})} := \mathcal{A} \cap {\mathcal{O}_C^{Az}}_{(e_{\chi_h})}$, which is an $\mathcal{O}_C$-algebra-subset of $\mathcal{A}$ with the section of identities e_{χ_h}. Then, $m_h e_{\chi_h}$ is invertible in $\Gamma(\mathcal{A}_{(e_{\chi_h})} \otimes_{\mathcal{O}_C}$*

$\mathcal{K}_C$). *It follows that the open subcurve of* $C_{\mathcal{A}}$ *defined as the regular locus of* $1/h$ *in the support of* χ_h *corresponds to the localization* $\mathcal{A}_{e_{\chi_h}}[(m_h e_{\chi_h})^{-1}]$ *of the* $\mathcal{O}_C$*-algebra-subset* $\mathcal{A}_{e_{\chi_h}}$ *in* $\mathcal{A}_{(e_{\chi_h})}\otimes_{\mathcal{O}_C}\mathcal{K}_C$.

Given now a commutative $\mathcal{O}_C$-subalgebra $(\mathcal{O}_C\subset)\,\mathcal{A}\subset\mathcal{O}_C^{Az}$, a collection $\{\,(\mathcal{A}_\alpha\,,\,e_\alpha)\,\}_{\alpha\in I}$ of sheaves of $\mathcal{O}_C$-algebra-subsets of $\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C$, where e_α is an admissible idempotent pseudo-section of $\mathcal{A}$ that serves as the identity section of $\mathcal{A}_\alpha$ for $\alpha\in I$, that satisfy the following conditions:

(1) fix a well-ordering of the index set I, then

$$1\;=\;\sum_\alpha e_\alpha\;-\;\sum_{\alpha_1<\alpha_2}e_{\alpha_1}e_{\alpha_2}\;+\;\sum_{\alpha_1<\alpha_2<\alpha_3}e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}\;\pm\;\cdots\;+\;(-1)^{|I|+1}\sum_{\alpha_1<\cdots<\alpha_{|I|}}e_{\alpha_1}\,\cdots\,e_{\alpha_{|I|}}\,;$$

(2) $(\mathcal{O}_C\langle\mathcal{A}_{\alpha_1}\cdots\mathcal{A}_{\alpha_l}\rangle\,,\,e_{\alpha_1}\cdots e_{\alpha_l})$ is a sheaf of $\mathcal{O}_C$-algebra-subsets in $\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C$, for all $\{\alpha_1\,,\cdots,\alpha_l\}\subset I$;

(3) for $I_0\subset I$, let

$$\begin{aligned}e_{I_0}&=\prod_{\alpha\in I_0}e_\alpha\,,\\ \hat{e}_{I_0}&=1-\sum_{\alpha'}e_{(\alpha')}+\sum_{\alpha'_1<\alpha'_2}e_{(\alpha'_1)}e_{(\alpha'_2)}+\cdots\\ &\qquad+(-1)^l\sum_{\alpha'_1<\cdots<\alpha'_l}e_{(\alpha'_1)}\cdots e_{(\alpha'_l)}+\cdots+(-1)^{|I|-|I_0|}e_{(\alpha'_1)}\cdots e_{(\alpha'_{|I|-|I_0|})}\,,\end{aligned}$$

where $|I|$, $|I_0|$ are the cardinality of I, $|I_0|$ respectively and all $\alpha'_\bullet\in I-I_0$; then,

$$\mathcal{A}\;=\;\sum_{I_0\subset I}\,\bigcap_{\alpha\in I_0}\mathcal{A}_\alpha e_{I_0}\hat{e}_{I_0}$$

in $\mathcal{A}\otimes_{\mathcal{O}_C}\mathcal{K}_C$;

give rise to an open cover $\{U_\alpha\}_\alpha$ of $C_{\mathcal{A}}:=\mathbf{Spec}\,\mathcal{A}$ with $U_\alpha:=\mathbf{Spec}\,\mathcal{A}_\alpha$.

Morphisms from C^{Az} to $\mathbb{P}^k$.

Let Y be the projective space over $\mathbb{C}$ with a fixed standard affine atlas:

$$Y\;=\;\mathbb{P}^k\;=\;\mathit{Proj}\,\mathbb{C}[y_0,y_1,\cdots,y_k]\;=\;\cup_{i=0}^k V_i\;=\;\cup_{i=0}^k\mathit{Spec}\,\mathbb{C}[\tfrac{y_0}{y_i}\,,\,\cdots\,,\,\tfrac{y_k}{y_i}]\,.$$

Here $y_\bullet/y_i$ are treated as formal variables with $y_i/y_i=$ the identity 1 of the ring $\mathbb{C}[\frac{y_0}{y_i}\,,\,\cdots\,,\,\frac{y_k}{y_i}]$; the gluing $V_i\supset V_{ij}:=V_i\cap V_j\overset{\sim}{\leftarrow}V_{ji}:=V_j\cap V_i\subset V_j$ of local affine charts is given by

$$\begin{array}{ccccccc}\mathbb{C}[\frac{y_0}{y_i}\,,\,\cdots\,,\,\frac{y_k}{y_i}] & \hookrightarrow & \dfrac{\mathbb{C}[\frac{y_0}{y_i},\cdots,\frac{y_k}{y_i},\frac{y_i}{y_j}]}{\left(\frac{y_j}{y_i}\cdot\frac{y_i}{y_j}-1\right)} & \overset{\sim}{\longrightarrow} & \dfrac{\mathbb{C}[\frac{y_0}{y_j},\cdots,\frac{y_k}{y_j},\frac{y_j}{y_i}]}{\left(\frac{y_i}{y_j}\cdot\frac{y_j}{y_i}-1\right)} & \hookleftarrow & \mathbb{C}[\frac{y_0}{y_j}\,,\,\cdots\,,\,\frac{y_k}{y_j}]\\ & & \frac{y_\bullet}{y_i} & \longmapsto & \frac{y_\bullet}{y_j}\cdot\frac{y_j}{y_i} & & \\ & & \frac{y_i}{y_j} & \longmapsto & \frac{y_i}{y_j} & & .\end{array}$$

Then, a morphism $\varphi: C^{Az} \to \mathbb{P}^k$ is presented by the following $(k+1)$-tuple of globally-generated $\mathcal{O}_C$-algebra-subsets of $\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C$ from an underlying gluing system of ring-homomorphisms:

$$\big\{\big(\,\mathcal{A}_{(i)} \;:=\; \mathcal{O}_C\langle m_{(i),j}\,|\, j\in\{0,\,\cdots,\,k\}-\{i\}\,\rangle\;,\; e_{(i)}\,\big)\big\}_{i=0}^k\;,$$

where (we set $m_{(i),i}=e_{(i)}$ by convention)

- $\{m_{(i),j}\,|\, i,j=0,\,\ldots\,,k\}$ is a commuting set of elements in $\Gamma(\mathcal{O}_C^{Az}\otimes_{\mathcal{O}_C}\mathcal{K}_C)$,
- $e_{(i)}$, $i=0,\,\ldots\,,k$, are admissible idempotent pseudo-section of $\mathcal{O}_C^{Az}$,

that satisfies the following gluing conditions that match with those for the fixed atlas $\{V_i\}_{i=0}^k$ on $\mathbb{P}^k$:

(1) $1=\sum_i e_{(i)} - \sum_{i_1<i_2} e_{(i_1)}e_{(i_2)} + \cdots$
$\qquad + (-1)^{\bullet-1}\sum_{i_1<\cdots<i_\bullet} e_{(i_1)}\,\cdots\, e_{(i_\bullet)} + \cdots + (-1)^k e_{(0)}\,\cdots\, e_{(k)}$;

(2) $(m_{(j),j}m_{(i),j})\,(m_{(j),j}m_{(j),i}) \;=\; m_{(i),i}m_{(j),j}\;,\;\; i,j=0,\,\ldots\,,k$;

(3) $m_{(j),j}m_{(i),\bullet} \;=\; m_{(j),\bullet}\cdot(m_{(j),j}m_{(i),j})\,,\;\; i,j,\bullet=0,\,\ldots\,,k$;

(4) for $I\subset\{0,\,\cdots,\,k\}$, let

$$\begin{aligned} e_I &= \prod_{i\in I} e_i\,,\\ \hat{e}_I &= 1-\sum_j e_{(j)} + \sum_{j_1<j_2} e_{(j_1)}e_{(j_2)} + \cdots \\ &\qquad + (-1)^l\sum_{j_1<\cdots<j_l} e_{(j_1)}\,\cdots\, e_{(j_l)} + \cdots + (-1)^{k+1-|I|}e_{(j_1)}\,\cdots\, e_{(j_{k+1-|I|})}\,,\end{aligned}$$

where $|I|$ is the cardinality of I and all $j_\bullet\in\{0,\,\cdots,\,k\}-I$; then, the $\mathcal{O}_C$-subalgebra

$$\mathcal{A} \;:=\; \sum_{I\subset\{0,\,\cdots,\,k\}}\,\bigcap_{i\in I}\mathcal{A}_i e_I\hat{e}_I$$

of $\mathcal{O}_C^{Az}\otimes_{\mathcal{O}_C}\mathcal{K}_C$ lies in $\mathcal{O}_C^{Az}$.

Note that Condition (1) – Condition (3) resemble the smearing of D0-branes along C (cf. [L-Y2: Sec. 4.4]) while the additional Condition (4) says that $C_\varphi := \mathbf{Spec}\,\mathcal{A}$ is dominated by the Azumaya noncommutative space *Space* $\mathcal{O}_C^{Az}$, should the latter be constructed functorially. It implies also that C_φ is proper over C. (These properties from Condition (4) are automatic in the case of D0-branes).

In terms of this $(k+1)$-tuple of $\mathcal{O}_C$-algebra-sets, any gluing system of ring-homomorphisms behind φ follows from[5]

$$\left\{\begin{array}{cccccc} \varphi_{(i)}^\sharp & : & \mathbb{C}[\frac{y_0}{y_i},\,\cdots,\,\frac{y_k}{y_i}] & \longrightarrow & \Gamma(\mathcal{A}_{(i)}) \\ & & 1 & \longmapsto & e_{(i)} \\ & & \frac{y_j}{y_i} & \longmapsto & m_{(i),j} \end{array}\right\}_{i=0}^k\,.$$

The corresponding $\varphi_{(i)}:\mathbf{Spec}\,\mathcal{A}_{(i)}\to V_i$, $i=0,\,\ldots\,,k$, glue to $f_\varphi: C_\varphi=\mathbf{Spec}\,\mathcal{A}\to\mathbb{P}^k$ that represents φ.

[5]Recall that a morphism from a (commutative) scheme X to $\mathbb{A}^k$ is specified by a k-tuple of elements in the function ring $R(X):=\Gamma(\mathcal{O}_X)$ of X.

Definition 2.4.7. [nondegenerate/degenerate with respect to standard atlas]. A morphism $\varphi : C^{Az} \to \mathbb{P}^k$ is said to be *nondegenerate with respect to* a standard affine atlas $\mathbb{P}^k = \cup_{i=0}^k V_i$, as above, for $\mathbb{P}^k$ if the pullback $f_\varphi^{-1}(H_i)$ of the hyperplane divisor $H_i := \mathbb{P}^k - V_i$ on $\mathbb{P}^k$ to C_φ is a divisor on C_φ and is supported on the smooth locus of $(C_\varphi)_{\mathrm{red}}$,[6] for $i = 0, \ldots, k$. φ is said to be *degenerate with respect to* $\mathbb{P}^k = \cup_0^k V_i$ if φ is not non-degenerate with respect to $\mathbb{P}^k = \cup_0^k V_i$.

In the above discussion, we fix a standard affine atlas on $\mathbb{P}^k$ and a morphism $\varphi : C^{Az} \to \mathbb{P}^k$ can be degenerate with respect to the atlas. However, given $\varphi : C^{Az} \to \mathbb{P}^k$, one can always choose an atlas, still denoted by $\mathbb{P}^k = \cup_0^k V_i$ with $V_i = \mathit{Spec}\,\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}]$, so that φ becomes nondegenerate with respect to $\mathbb{P}^k = \cup_0^k V_i$. In this case, the above presentation

$$\big\{ \big(\mathcal{A}_{(i)} := \mathcal{O}_C \langle m_{(i),j} \,|\, j \in \{0, \cdots, k\} - \{i\} \rangle \,,\, e_{(i)} \big) \big\}_{i=0}^k \,,$$

of φ has the additional property that:

- $e_{(i)} = 1 \in \Gamma(\mathcal{O}_C^{Az})$, for $i = 0, 1, \ldots, k$.

The above presentation of φ can then be simplified to a commuting $(k+1)$-tuple of rational sections of $\mathcal{O}_C^{Az}$

$$(m_{(0),0}, m_{(0),1}, \cdots, m_{(0),k}) \in (\Gamma(\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C))^{k+1}$$

that satisfies: ($m_{(0),0} = 1$ by convention)

(1) $m_{(0),j}$ is generically invertible, for $j = 1, \ldots, k$.

In this case, $m_{(0),j}$ is invertible in $\Gamma(\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C)$ and its inverse therein will be denoted by $m_{(0),j}^{-1}$ or $1/m_{(0),j}$.

(2) Let

$$\mathcal{A}_{(i)} = \mathcal{O}_C \langle m_{(0),0}/m_{(0),i}, m_{(0),1}/m_{(0),i}, \cdots, m_{(0),k}/m_{(0),i} \rangle \,.$$

Then, the $\mathcal{O}_C$-subalgebra $\mathcal{A} := \cap_{i=0}^k \mathcal{A}_{(i)}$ of $\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C$ lies in $\mathcal{O}_C^{Az} \subset \mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C$.

In terms of this k-tuple, any gluing system of ring-homomorphisms behind φ follows from

$$\left\{ \begin{array}{ccccc} \varphi_{(i)}^\sharp & : & \mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_k}{y_i}] & \longrightarrow & \Gamma(\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C) \\ & & 1 & \longmapsto & 1 \\ & & \frac{y_j}{y_i} & \longmapsto & m_{(0),j}/m_{(0),i} \end{array} \right\}_{i=0}^k .$$

The corresponding $\varphi_{(i)} : \mathbf{Spec}\,\mathcal{A}_{(i)} \to V_i$, $i = 0, \ldots, k$, glue to an $f_\varphi : C_\varphi = \mathbf{Spec}\,\mathcal{A} \to \mathbb{P}^k$ that represents φ.

Definition 2.4.8. [strongly nondegenerate with respect to standard atlas]. A morphism $\varphi : C^{Az} \to \mathbb{P}^k$ is said to be *strongly nondegenerate with respect to* a standard affine atlas $\mathbb{P}^k = \cup_{i=0}^k V_i$ for $\mathbb{P}^k$ if φ is nondegenerate with respect to $\{V_i\}_{i=0}^k$ and $f_\varphi(C_\varphi) \subset \mathbb{P}^k - \cup_{0 \le i < j \le k}(H_i \cap H_j)$.

Realize $(y_0, \cdots, y_k)$ as a $(k+1)$-tuple $\mathbf{t} := (t_0, \cdots t_k)$ of global sections of $\mathcal{O}_{\mathbb{P}^k}(1)$ with $t_i^{-1}(0) = H_i$. Then, $\mathbf{t}$ determines an embedding

$$\begin{array}{ccccc} \pi = (\pi_{(ij)})_{0 \le i < j \le k} & : & \mathbb{P}^k - \cup_{0 \le i < j \le k}(H_i \cap H_j) & \longrightarrow & \prod_{0 \le i < j \le k} \mathbb{P}^1_{(ij)} \simeq (\mathbb{P}^1)^{k(k+1)/2} \\ & & [y_0 : \cdots : y_k] & \longmapsto & (y_j/y_i)_{i<j} = (t_j/t_i)_{i<j} \end{array} ,$$

[6] Recall that C_φ does not have embedded points.

where $\mathbb{P}^1_{(i),j} \simeq \mathbb{P}^1$ with a fixed coordinate $\mathbb{C}\cup\{\infty\}$. When $\varphi : C^{Az} \to \mathbb{P}^k$ is strongly nondegenerate with respect to $\{V_i\}_{i=0}^k$, the presentation $(m_{(0),1}, \cdots, m_{(0),k}) \in (\Gamma(\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C))^k$ of φ with respect to the above data is characterized by (recall $m_{(0),0} = 1$ by convention)

(1) $m_{(0),j}$ is generically invertible and $m_{(0),j_1} m_{(0),j_2} = m_{(0),j_2} m_{(0),j_1}$, for $j, j_1, j_2 = 1, \ldots, k$;

(2) $\mathcal{O}_C\langle m_{(0),j}/m_{(0),i}\rangle \cap \mathcal{O}_C\langle m_{(0),i}/m_{(0),j}\rangle \subset \mathcal{O}_C^{Az}$.

It follows from the minimal property of C_φ that

$$\mathcal{A}_\varphi \;=\; \mathcal{O}_C\left\langle \mathcal{O}_C\langle m_{(0),j}/m_{(0),i}\rangle \cap \mathcal{O}_C\langle m_{(0),i}/m_{(0),j}\rangle \;:\; 0 \le i < j \le k \right\rangle$$

in such a presentation of φ.

Remark 2.4.9. [*geometry of commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C$*]. Let a be a non-nilpotent rational section of $\mathcal{O}_C^{Az}$ and $\mathcal{O}_C\langle a\rangle$ be the $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C$ generated by[7] a. Then, $C_a := \mathbf{Spec}\,(\mathcal{O}_C\langle a\rangle)$ is an open dense subscheme of a finite scheme $\overline{C}_a$ over C with a tautological rational function h_a on C_a (or equivalently on $\overline{C}_a$) corresponding to a. The Generalized Cayley-Hamilton Theorem for a linear operator on a finite-dimensional vector space implies that C_a is a subscheme of the spectral curve $\det(\lambda - a) = 0$ in $C \times \mathit{Spec}\,\mathbb{C}[\lambda]$ with the same reduced-scheme structure. This implies that C_a/C is canonically a closed subscheme of $C \times \mathbb{A}^1$. Note that C_a contains an open dense subset that is finite over an open subset of C. However, C_a itself in general may not be finite over every open subset of C. Cf. Figure 2-4-2. In general, let $\mathcal{B}$ be a commutative $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C$ that is globally generated by $a_1, \ldots, a_l \in \Gamma(\mathcal{O}_C^{Az} \otimes_{\mathcal{O}_C} \mathcal{K}_C)$; then $C_{\mathcal{B}} := \mathbf{Spec}\,\mathcal{B}$ is a closed subscheme of the fibered product $C_{a_1} \times_C \cdots \times_C C_{a_l}$ and hence is a closed subscheme of $C \times \mathbb{A}^l$. Using this, one obtains a second proof of Lemma/Definition 2.1.8.

FIGURE 2-4-2. The geometry of C_a.

3 The stack $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \beta)$ of morphisms with a fundamental module.

3.1 D0-branes revisited: the stack $\mathfrak{M}_r^{D0}(Y)$ of D0-branes of type r on Y.

The moduli stack $\mathfrak{M}_r^{D0}(Y)$ of D0-branes of type r on Y under the Polchinski-Grothendieck Ansatz is the moduli stack of morphisms from the (un-fixed) Azumaya point $\mathit{Space}\,(M_n(\mathbb{C}))$

[7]Recall that 1 is always included in $\mathcal{O}_C\langle\,\cdot\,\rangle$ by convention.

with the fundamental module $\mathbb{C}^r$ to Y. It follows from Lemma 2.2.1 and Remark 2.2.2 that this is precisely the stack of 0-dimensional, length r, torsion sheaves on Y. Such an $\mathcal{O}_Y$-module has the Hilbert polynomial the constant r, in disregard of the polarization $\mathcal{O}_Y(1)$ on Y chosen. Such a torsion sheaf is thus a quotient sheaf of $\mathcal{O}_Y \otimes \mathbb{C}^r$. It follows that $\mathfrak{M}_r^{D0}(Y)$ is a quotient stack of the Quot-scheme of $\mathcal{O}_Y \otimes \mathbb{C}^r$ with the constant Hilbert polynomial r:

$$\mathfrak{M}_r^{D0}(Y) \;=\; [\,Quot_r(\mathcal{O}_Y \otimes \mathbb{C}^r)/GL(r;\mathbb{C})]\,.$$

There are three substacks of $\mathfrak{M}_r^{D0}(Y)$ worth mentioning:

(1) $\mathfrak{M}_{r,Chow}^{D0}(Y)$ consists of objects $[\widetilde{\mathcal{E}}] \in \mathfrak{M}_r^{D0}(Y)$ such that $Supp\,\widetilde{\mathcal{E}}$ is reduced; it fibers over the stack $[Y^r/Sym_r]$ of 0-cycles of order r on Y.

(2) $\mathfrak{M}_{r,Hilb}^{D0}(Y)$ consists of objects $[\widetilde{\mathcal{E}}] \in \mathfrak{M}_r^{D0}(Y)$ such that $\mathcal{O}_{Supp\,\widetilde{\mathcal{E}}} \simeq \widetilde{\mathcal{E}}$; it fibers over the Hilbert scheme $Hilb_r(Y)$ of 0-dimensional subschemes of length r on Y.

(3) $\mathfrak{M}_{r,singleton}^{D0}(Y)$ consists of objects $[\widetilde{\mathcal{E}}] \in \mathfrak{M}_r^{D0}(Y)$ such that $(Supp\,(\widetilde{\mathcal{E}}))_{\mathrm{red}}$ is a singleton; it fibers over Y.

The role of these three substacks in superstring theory are respectively:

(1′) $\mathfrak{M}_{r,Chow}^{D0}(Y)$ is related to the classical moduli space of gas of r-many D0-branes on Y.

(2′) $\mathfrak{M}_{r,Hilb}^{D0}(Y)$ is related to the quantum moduli space of gas of r-many D0-branes on Y, when Y is a smooth projective surface,

(3′) $\mathfrak{M}_{r,singleton}^{D0}(Y)$ could contain various birational models for Y and, when Y is singular, partial smooth resolutions of Y. These partial resolutions are the candidate target-space structures revealed as the vacuum manifolds/varieties/spaces in the different phases of the quantum field theory on the r-stacked D0-brane probe to a singular Y.

See [Va] for Item (1′) and Item (2′) and [L-Y2: Sec. 2.1, Sec. 4.4, and quoted references].

It follows from Corollary 2.2.3 that one can construct $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ as a substack of the stack of morphisms from prestable curves of genus g to $\mathfrak{M}_r^{D0}(Y)$. Cf. Figure 3-1-1.

3.2 The stack $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ and its decorated atlas.

The forgetful functor that forgets sheaves of modules on products of prestable curves with Y give rise to a fibration/morphism:

$$\pi_{\mathfrak{M}_g}\;:\; \mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta) \;\longrightarrow\; \mathfrak{M}_g\,,$$

where $\mathfrak{M}_g$ is the stack of prestable curves of genus g. It is known that $\mathfrak{M}_g$ is algebraic (Artinian in this case), smooth of stacky dimension $3g-3$, but only locally of finite type. That $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ is algebraic (Artinian in this case) follows from the fact that both $\mathfrak{M}_g$ and the fiber stacks $\mathfrak{Coh}(C\times Y)$ of coherent $\mathcal{O}_{C\times Y}$-modules are algebraic. An atlas of $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ can be constructed as follows, combining the morphism $\pi_{\mathfrak{M}_g}$ and the existing construction of atlas for $\mathfrak{M}_g$ ([Kn]) and for $\mathfrak{Coh}(C\times Y)$ ([L-MB]) respectively.

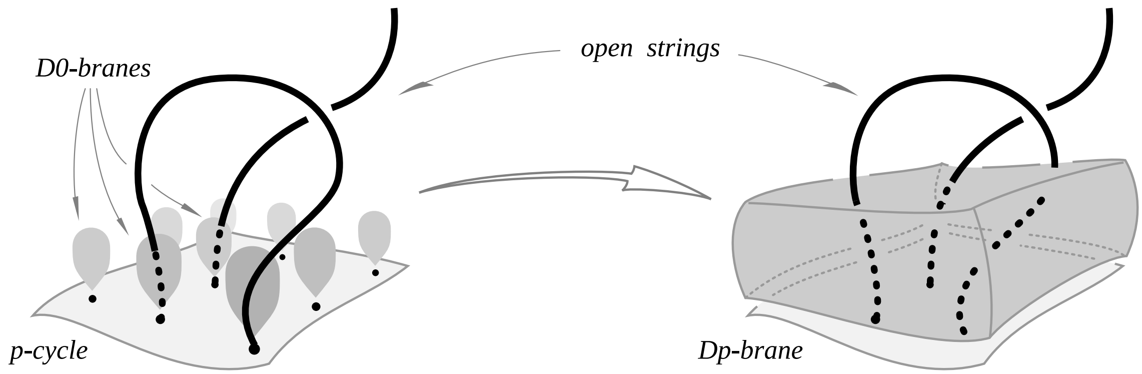


FIGURE 3-1-1. The original stringy operational definition of D-branes as objects in the target space(-time) Y of strings where end-points of open-strings can and have to stay suggests that smearing D0-branes along a (real) p-dimensional cycle/submanifold/distribution Z in Y renders Z a Dp-brane. Such a smearing in our case is realized as a morphism from a manifold/variety Z to the stack $\mathfrak{M}^{D0}(Y)$ of D0-branes on Y. In the figure, the Chan-Paton sheaf $\mathcal{E}$ that carries the index information on the end-points of open strings is indicated by a shaded cloud. Its endomorphism sheaf $\mathcal{E}nd\,\mathcal{E}$ carries the information of the gauge group of the quantum field theory on the D-brane world-volume.

Fix an $e \geq 3$. Then, for any stable curve with n marked points $(C; p_1, \cdots, p_n)$, the line bundle $(\omega_C(p_1 + \cdots + p_n))^{\otimes e}$, where ω_C is the dualizing sheaf of C, is very ample. Let

$$\begin{aligned} d &= e(2g-2+n) \;=\; deg\,((\omega_C(p_1+\cdots+p_n))^{\otimes e})\,, \\ P &= \text{Hilbert polynomial of } (\omega_C(p_1+\cdots+p_n))^{\otimes e}\;: \\ &\quad\; P(m) \;=\; dm + (1-g)\,, \\ N &= P(1)-1 \;=\; d-g\,. \end{aligned}$$

Let

$$H_{g,n} \;\subset\; Hilb_{\,P}(\mathbb{P}^N) \times (\mathbb{P}^N)^n$$

be the locally closed subscheme that parameterizes[8] tuples $([C], p_1, \cdots, p_n)$ where C is a subscheme of $\mathbb{P}^N$ of Hilbert polynomial P that is nodal, $p_i \in C$ for $i = 1 \ldots, n$, and $\mathcal{O}_{\mathbb{P}^N}(1)|_C \simeq (\omega_C(p_1 + \cdots p_n))^{\otimes e}$. Then, the morphism $\pi_{g,n} : H_{g,n} \to \mathfrak{M}_g$ defined by the tautological family $C_{g,n}$ of nodal curves over $H_{g,n}$ is smooth, with image an open substack of $\mathfrak{M}_g$ of finite type, and the morphism

$$\pi_g \;:=\; \amalg_n \pi_{g,n} \;:\; \amalg_n H_{g,n} \;\longrightarrow\; \mathfrak{M}_g$$

is a smooth epimorphism and defines an atlas on $\mathfrak{M}_g$ in the fppf topology.

Furthermore, by construction, the universal subscheme $C_{g,n}$ over $H_{g,n}$ is naturally endowed with a relative very ample line bundle, namely $\mathcal{O}_{C_{g,n}/H_{g,n}}(1) := \iota_{g,n}^*\mathcal{O}_{\mathbb{P}^N}(1)$, where $\iota_{g,n} : C_{g,n} \to \mathbb{P}^N$ is the tautological morphism. We will call $\{(H_{g,n}, \pi_{g,n}; \mathcal{O}_{C_{g,n}/H_{g,n}}(1))\}_n$ a *decorated atlas*[9] on $\mathfrak{M}_g$. By construction, $\mathcal{O}_{C_{g,n}/H_{g,n}}(1)$ has relative degree $d = e(2g-2+n)$.

[8]It is standard to write both $H_{g,n}$ here and $Quot^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}} \otimes \mathbb{C}^{N'})$ in the next more precisely as schemes that represent related moduli functors.

[9]It is conceptually instructive to think of the system $\{\mathcal{O}_{C_{g,n}/H_{g,n}}(1)\}_n$ as defining a *twisted relative (very) ample line bundle* on the universal curve $\mathfrak{C}_g$ over $\mathfrak{M}_g$ since there is no relative ample line bundle on the whole $\mathfrak{C}_g/\mathfrak{M}_g$. Similarly, for the system $\{\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(1)\}_{n;N',n'}$ in the next.

Fix an ample line bundle $\mathcal{O}_X(1)$ on X and let $H_Y \in A^1(X)$ be the hyperplane class associated to $\mathcal{O}_Y(1)$. Then, $\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(1) := \mathcal{O}_{C_{g,n}/H_{g,n}}(1) \boxtimes \mathcal{O}_X(1)$ is a relative ample line bundle on $(C_{g,n}\times Y)/H_{g,n}$. Define

$$\chi_{n'} \;:=\; \chi + n'\Big(\text{rel.deg}\,(\mathcal{O}_{C_{g,n}/H_{g,n}}(1)) + H_Y\cdot\beta\Big) \;=\; \chi + n'\Big(re(2g-2+n) + H_Y\cdot\beta\Big)\,.$$

Let $\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}\otimes\mathbb{C}^{N'})$ be the open subscheme of the Quot-scheme $\mathit{Quot}(\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}\otimes\mathbb{C}^{N'})$ that parameterizes isomorphism classes of quotient $\mathcal{O}_{C\times Y}$-modules $\mathcal{O}_{C\times Y}\otimes\mathbb{C}^{N'}\rightarrow\widetilde{\mathcal{E}}\rightarrow 0$ on fibers $C\times Y$ of $(C_{g,n}\times Y)/H_{g,n}$ such that

- $\widetilde{\mathcal{E}}$ is of relative length r on $(C\times Y)/C$;
- the Hilbert polynomial $P(\widetilde{\mathcal{E}},m) = \Big(re(2g-2+n)+H_Y\cdot\beta\Big)\,m+\chi_{n'}$;
- $H^i(C\times Y,\widetilde{\mathcal{E}})=0$, for $i>0$;
- the natural map $\mathbb{C}^{N'}\rightarrow H^0(C\times Y,\widetilde{\mathcal{E}})$ is an isomorphism.

Then, the universal quotient sheaf twisted by the pull-back of $\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(-n')$

$$\begin{aligned}\mathcal{O}_{\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})\times_{H_{g,n}}(C_{g,n}\times Y)}\otimes\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(-n')\otimes\mathbb{C}^{N'}&\\ \longrightarrow\;\widetilde{\mathcal{F}}\otimes\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(-n')\;\longrightarrow\;0&\end{aligned}$$

on $\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})\times_{H_{g,n}}(C_{g,n}\times Y)$, for $n'\geq 0$, defines a smooth morphism

$$\Theta_{(n;N',n')}\;:\;\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})\;\longrightarrow\;\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)\,.$$

The morphism from their union

$$\Theta\;:=\;\amalg_{n,N',n'\geq 0}\,\Theta_{(n;N',n')}\;:\;\amalg_{n,N',n'\geq 0}\,\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})\;\longrightarrow\;\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$$

is a smooth epimorphism and defines an atlas on $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ in the fppf topology. We will call

$$\Big\{\Big(\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})\,,\Theta_{(n;N',n')}\,;\,\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(1)\Big)\Big\}_{n;N',n'}$$

a *decorated atlas* on $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$, where $\mathcal{O}_{(C_{g,n}\times Y)/H_{g,n}}(1)$ is now regarded as a relative ample line bundle on $(\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})\times_{H_{g,n}}(C_{g,n}\times Y))/\mathit{Quot}^{\circ}_{(r,\chi_{n'})}(\mathcal{O}_{C_{g,n}\times Y}\otimes\mathbb{C}^{N'})$.

4 Remarks: D-strings as a master object for curves and D-string world-sheet instantons.

The moduli stack $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ will serve as a prototype moduli stack to address topological D-strings in this project. It has its own deformation-obstruction theory phrasable in three different aspects.

The content of this stack already renders it as a model for a master object for curves: Via various forgetful functors - applied to appropriate substacks and with perhaps additional stabilizations -, $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ can be related to the moduli stack $\overline{M}_g$ of stable curves and the moduli stack $\mathit{Bun}_{(g,r,\chi)}$ of bundles over prestable curves, and in the genus 0 case, taking surrogates associated to morphisms relates $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ to Hurwitz schemes. It is more technical to address how these standard moduli stacks for curves can be embedded back into $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$, after allowing reasonable ambiguities. These ring with an anticipation explained in [L-Y2: Introduction and Sec. 4.5], which says that the moduli space/stack of D-branes should serve as a universal/master moduli space/stack for commutative geometry from the aspect of the Wilson's theory-space picture of stringy dualities, cf. Diagram 4-1 (i.e. [L-Y1: Diagram A-1-1]) and Figure 4-2.

The notion of (semi-)stability of objects in the current moduli problem comes from three sources/origins naturally built into the problem. Choices of this notion define special fillable substacks of finite type in $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$. The stack $\mathfrak{M}_{Az(g,r,\chi)^f}(Y,\beta)$ is part of the ingredients toward a mathematical theory of topological D-string world-sheet instanton invariants of Y as a weighted counting of maps from Azumaya prestable curves with a fundamental module to Y, along the line of the Polchinski-Grothendieck Ansatz.

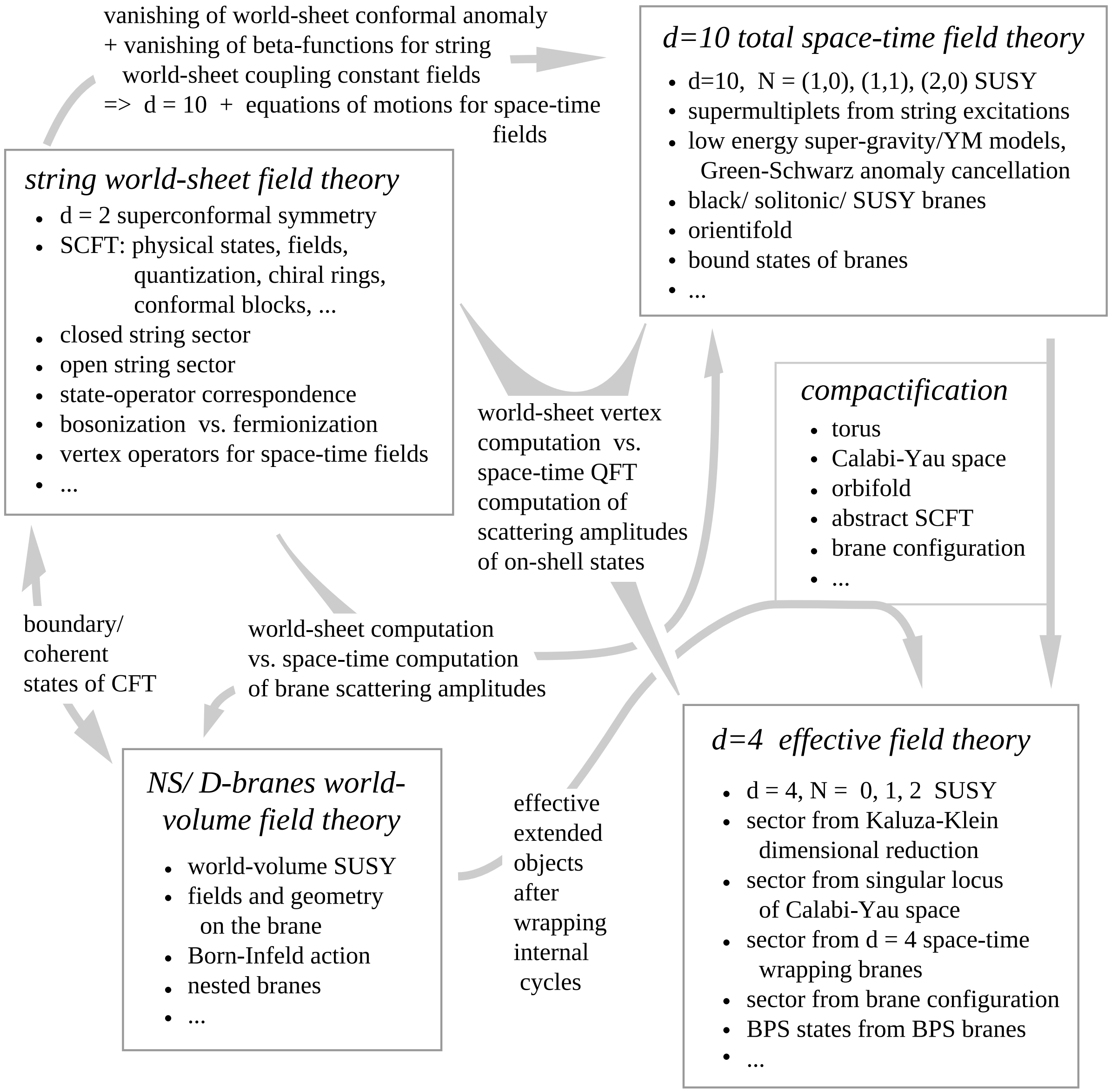


Diagram 4-1. ([L-Y1: Diagram A-1-1].) The *string world-sheet*, the *brane-world-volume*, the *total space-time*, and the 4-*dimensional effective field theory* aspect of a string theory. Scattering amplitudes of fields and D-branes in space-time computed via string world-sheet methods and via space-time field theory method have to match order by order. Each of the four aspects itself has a *Wilson's theory-space* associated to it, containing all the phases of the field theory associated to that aspect of the string theory. *Dualities* can be realized either as a local isomorphism or a coordinate change on the Wilson's theory-space $\mathcal{S}_{Wilson}$ that induces an isomorphism on the universal family of Hilbert spaces of states, ring of operators, and correlation functions of these operators over $\mathcal{S}_{Wilson}$. In the weak string coupling regime, strings are light and branes are heavy and hence string are regarded as more fundamental. In the strong string coupling regime, branes can become light and strings become heavy and should be no longer treated as the most/unique fundamental object in the theory.

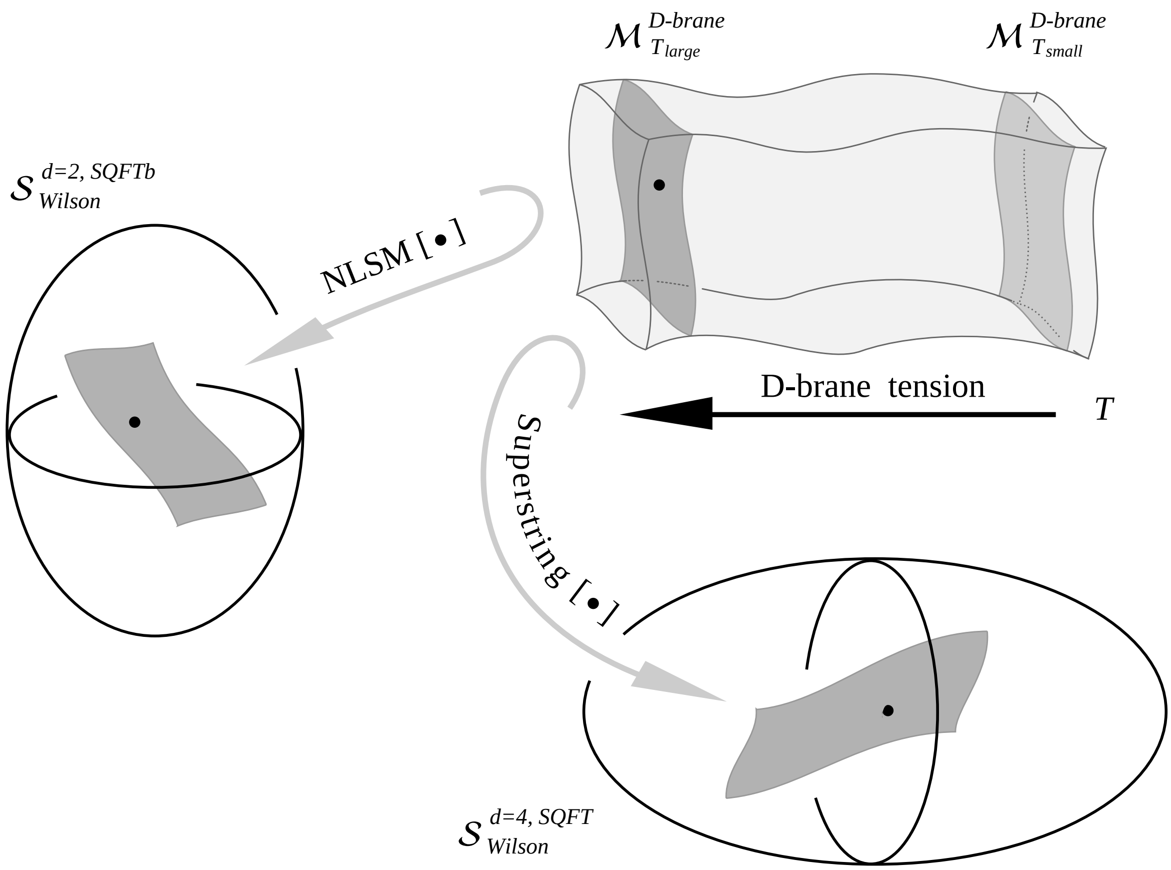


FIGURE 4-2. The moduli space of soft/fundamental D-branes and the hard/solitonic D-branes are connected by a continuum fibered over the brane-tension T-parameter line. The moduli space of solitonic D-branes is then embedded into the Wilson's theory-space of either $d = 4$ supersymmetric quantum field theories (SQFT) or $d = 2$ supersymmetric quantum field theories-with-boundary (SQFTb) via respectively the Kaluza-Klein compactification $Superstring[\,\bullet\,]$ of a $d = 10$ superstring model or taking nonlinear sigma model-with-boundary $NLSM[\,\bullet\,]$ on the internal geometry-with-D-branes of such a compactification. Stringy dualities that arise from D-brane mechanisms render then the moduli space of D-branes a master object that relates different standard moduli spaces from moduli problems in commutative geometry. The study of D0-branes in [L-Y2] and D-strings in the current work following the Polchinski-Grothendieck Ansatz is consistent with such a stringy picture/anticipation.

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