Projectively Flat Metrics

Benling Li
Department of Mathematics, Zhejiang University
lblmath@163.com

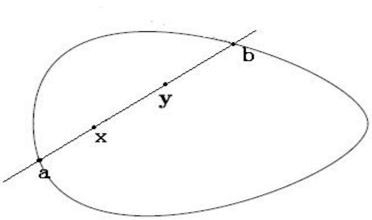
June 30, 2007

Hilbert Metric

Hilbert geometry(1894) (C, d_c):

$$d_C(x,y) = \frac{1}{2}\ln[a,x,y,b] = \frac{1}{2}\ln\frac{\|y-a\|\|x-b\|}{\|x-a\|\|y-b\|}, \quad \forall x,y \in C$$

where $C \in \mathbb{R}^n$ is a convex domain.



C centrally symmetric, (C, d_c) Minkowski geometry ;
C ellipsoid, (C, d_c) the Lobachevskii geometry;
$C = B^n(1)$, the Klein model of hyperbolic geometry

$$\cosh d_{B^n(1)}(x, y) = \frac{1 - xy}{\sqrt{(1 - |x|^2)}\sqrt{(1 - |y|^2)}}.$$

Hilbert's Fourth problem

Hilbert's Fourth problem is the problem of determining all metrizations of C for which the geodesics are straight lines.

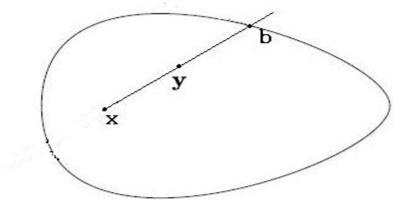
Such metric is also called projectively flat metric.

A differentiable metric is Finsler metric. The regular solution of Hilbert's Fourth problem is just projectively flat Finsler metric.
Example: Hilbert metric, Funk metric.

Funk metric

Funk metric on C:

$$d_f(x,y) = \ln \frac{\|x-b\|}{\|y-b\|}.$$



Hilbert metric is the symmetrization of Funk metric:

$$d_C(x,y) = \frac{1}{2}\Big[d_f(x,y) + [d_f(y,x)\Big]$$

Funk metric is non-reversable and projectively flat.
Hilbert metric is reversable and projectively flat.

History

- Euclidean Geometry
- Riemannnian Geometry (B. Riemann 1854)
- (P. Finsler 1918). In Finsler's Ph.D thesis he studied the variation problem in Finsler geometry.
- In 1926, Berwald introduced the notion of connection into Finsler geometry. Then in Finsler geometry there is the definition of curvature.
- In 1928, J. Douglas(First Fields medalist 1936), projectively geometry, Douglas curvature.
- In 1934, E. Cartan, generalized regular geometry.
- In 1941, S.S. Chern got interested in Finsler geometry.
- In 1941, L. Berwald died in Nazi concentration camp. In 1947-1949, some papers of Berwald is published in Ann. Math. by his students.

Riemannian metrics

Riemannian metrics: (M, g)

$$g = g_{ij} dx^i \otimes dx^j, \quad (g_{ij}) > 0.$$

Christoffel symbols of the second kind:

$$\gamma^i_{jk} := \frac{1}{2} g^{is} \Big(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \Big)$$

Curvature tensor:

$$R_i{}^j{}_{kl} := \frac{\partial \gamma^j_{il}}{\partial x^k} - \frac{\partial \gamma^j_{ik}}{\partial x^l} + \gamma^h_{il} \gamma^j_{hk} - \gamma^h_{ik} \gamma^j_{hl}$$

$$R_{ijkl} = g_{jh} R_i{}^h{}_{kl}$$

Sectional Curvature

Sectional curvature:

$$K(x, U, V) := \frac{V^j (U^i R_{ijkl} U^l) V^k}{g(U, U) g(V, V) - [g(U, V)]^2}.$$

$$R_{ijkl} = K(g_{jk} g_{il} - g_{ik} g_{jl})$$

By Cartan's classification, there are only three space forms.

- $(K = 0)$ $M = \mathbb{R}^n$,

$$ds^2 = \sum_{i=1}^{n} (dx^i)^2.$$

- $(K = c > 0)$ $M = S^n(\frac{1}{\sqrt{c}})$,

$$ds^2 = \frac{\sum_i (dx^i)^2}{\left(1 + \frac{4}{c}\sum_i (x^i)^2\right)^2}.$$

- $(K = c < 0)$ $M = \{x \in \mathbb{R}^n | \sum_i (x^i)^2 < -\frac{4}{c}\}$,

$$ds^2 = \frac{\sum_i (dx^i)^2}{\left(1 + \frac{4}{c}\sum_i (x^i)^2\right)^2}.$$

Projectively Flat Riemannian metrics

Geodesics: $c = c(t)$,

$$\frac{d^2x^i}{dt} + \gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

Projectively flat metrics: geodesics are lines.

Beltrami theorem

A Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature.
By Beltrami theorem we can express the three metrics by Cartan's classification as the following.

$$a_{ij}y^iy^j = \alpha_\mu^2 = \frac{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}{(1+\mu|x|^2)^2}, \quad y \in T_xB^n(r_\mu) \cong R^n,$$

where $r_\mu := 1/\sqrt{-\mu}$ if $\mu < 0$ and $r_\mu := +\infty$ if $\mu \geq 0$. Obviously,

$$a_{ij} = \frac{1}{1+\mu|x|^2}\left\{\delta_{ij} - \frac{\mu x^i x^j}{1+\mu|x|^2}\right\}.$$

Finsler metrics

Finsler metric: $F : TM \to R$ has the following properties

$$F \text{ is } C^\infty \text{ on } TM \setminus \{0\},$$

$$F(x, y) > 0, \quad y \neq 0,$$

$$F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0.$$

The fundamental form $\mathbf{g}_y(u, v) = g_{ij}(x, y)u^i v^j$,

$$g_{ij} = \left[\frac{1}{2}F^2\right]_{y^i y^j} > 0.$$

Projectively flat Finsler metrics

Geodesics:
$\ddot{x} + 2G^i(x, \dot{x}) = 0$, where $G^i = G^i(x, y)$ are given by

$$G^i = \frac{1}{4} g^{il} \Big\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \Big\}.$$

Projectively flat metrics: $G^i = P(x, y) y^i$.
Another equivalent condition of projectively flat Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset R^n$,

$$F_{x^k y^l} y^k - F_{x^l} = 0. \quad \textit{due to G.Hamel}$$

The Hilbert metric and Funk metric we mentioned previous are both projectively flat Finsler metrics.

Projectively flat Randers metric

Randers metric(1941):

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form.

Theorem(Matsumoto-Bacso, 1997) A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed.

Flag curvature

Riemann curvature: $\mathbf{R}_y(u) = R^i{}_k(x,y)u^k$,

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k \mathbf{b}_i} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Flag curvature:

$$\mathbf{K} = \mathbf{K}(P,y) = \frac{\mathbf{g}_y(\mathbf{R}_y(u), u)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,y) - [\mathbf{g}_y(y,u)]^2},$$

where $P = \mathrm{span}\{y,u\} \subset T_xM$.
For a Riemannian metric $F = \sqrt{g_{ij}(x)y^iy^j}$, $\mathbf{g}_y = \mathbf{g}$,

$$\mathbf{R}_y(u) = R(u,y)y.$$

$$\mathbf{g}(\mathbf{R}_y(u), u) = \mathbf{g}(R(u,y)y, u) = \mathbf{g}(R(y,u)u, y) = \mathbf{g}(\mathbf{R}_u(y), y).$$

$$\mathbf{K}(P,y) = \mathbf{K}(P) \qquad \text{(sectional curvature)}.$$

Beltrami Theorem in Finsler geometry

Is Beltrami theorem true in Finsler geometry? No.

- If a Finsler metric is projectively flat, then $K = K(x, y)$.

- $K = constant \not\Leftrightarrow G^i = Py^i$.

- How about the intersection part?

Projectively Flat Randers metric with constant flag curvature

Theorem (Z. Shen, 2003) A Randers metric $F = \alpha + \beta$ is locally projectively flat and with constant flag curvature K then $K \leq 0$. If $K = 0$, F is locally Minkowskian. If $K = -\frac{1}{4}$, F is a Funk metric which is given by

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} \pm \langle x, y\rangle}{1 - |x|^2} \pm \frac{\langle a, y\rangle}{1 + \langle a, x\rangle},$$

where $a \in R^n$ is a constant vector.

Berwald's Metric

$K = 0$.

$$B = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}},$$

where $y \in T_x\mathrm{B}^n \approx R^n$.

The Funk metric Θ and Berwald's metric B are related and they can be expressed in the form

$$\Theta = \bar{\alpha} + \bar{\beta}, \qquad B = \frac{(\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}},$$

where

$$\bar{\alpha} := \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2}, \qquad \bar{\beta} := \frac{\langle x, y\rangle}{1 - |x|^2},$$

$$\tilde{\alpha} := \lambda\bar{\alpha}, \qquad \tilde{\beta} := \lambda\bar{\beta}, \qquad \lambda := \frac{1}{1 - |x|^2}.$$

(α, β)-metrics

An (α, β)-metric is expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. $\phi = \phi(s)$ is a C^∞ function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(0) = 1, \qquad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,$$

where $b = \|\beta_x\|_\alpha$.

(α,β)-metrics

An (α,β)-metric is expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. $\phi = \phi(s)$ is a C^∞ function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(0) = 1, \qquad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,$$

where $b = \|\beta_x\|_\alpha$.

Special (α,β)-metrics:

$\phi(s) = 1$, $F = \alpha$, Riemannian metric.

$\phi(s) = 1 + s$, $F = \alpha + \beta$, Randers metric.

$\phi(s) = 1/(1 - s)$, $F = \alpha^2/(\alpha - \beta)$, Matsumoto metric.

Projectively Flat (α,β)-Metrics

$$G^i = G^i_\alpha + \alpha Q {s^i}_0 + \alpha^{-1}\Theta\Big(-2\alpha Q s_0 + r_{00}\Big)y^i + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)b^i,$$

where G^i_α is the geodesic coefficient of α and $s_{ij} = b_{i|j} - b_{j|i}$, $r_{ij} = b_{i|j} + b_{j|i}$, $s_{i0} = s_{ij}y^j$, $s_0 = s_{i0}b^i$, $r_{00} = r_{ij}y^iy^j$ and

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'} \\ \Theta &= \frac{\phi - s\phi'}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \cdot \frac{\phi'}{\phi} - s\Psi \\ \Psi &= \frac{1}{2}\,\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

There are many projectively flat (α, β)-metrics which are trivial (β is parallel with respect to α, then α is projectively flat) such as Matsumoto metric, etc. However there also exist many nontrivial ones. Besides Randers metric, the following theorem gives another one.

Theorem(Shen-Yildirim) *Let $(\alpha+\beta)^2/\alpha$ be a Finsler metric on a manifold M. F is projectively flat if and only if*

(i) $b_{i|j} = \tau\{(1+2b^2)a_{ij} - 3b_ib_j\}$,

(ii) *the geodesic coefficients G^i_α of α are in the form:* $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,

where $\tau = \tau(x)$ is a scalar function and $\theta = a_i(x)y^i$ is a 1-form on M.

Theorem (Shen) *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n\geq 3)$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i\neq 0$. Suppose that β is not parallel with respect to α and F is not of Randers type. Then F is projectively flat on $\mathcal{U}$ if and only if the function $\phi=\phi(s)$ satisfies*

$$\Big\{1+(k_1+k_2s^2)s^2+k_3s^2\Big\}\phi''(s)=(k_1+k_2s^2)\Big\{\phi(s)-s\phi'(s)\Big\},$$

$$b_{i|j}=2\tau\Big\{(1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j\Big\},$$

$$G^i_\alpha=\xi y^i-\tau\Big(k_1\alpha^2+k_2\beta^2\Big)b^i,$$

where $\tau=\tau(x)$ is a scalar function on $\mathcal{U}$ and k_1,k_2 and k_3 are constants.

Projectively Flat (α,β)-Metrics with Constant Flag Curvature

In recent years, some special projectively flat (α,β)-metrics with constant flag curvature have been studied.

- Randers metric $F=\alpha+\beta$.

(2000 Bao-Robels-Shen) The classification of Randers metrics with constant flag curvature.
(Mo-Shen-Yang) The classification of projectively flat Randers metrics with constant flag curvature.

- (Shen-Yildirim) $F=(\alpha+\beta)^2/\alpha$.
- (Shen-Zhao) $F=\alpha+\epsilon\beta+2k\beta^2/\alpha-k^2\beta^4/(3\alpha^3)$.
- (Li) Matsumoto metric and $F=\alpha+\epsilon\beta+\frac{3}{2}\beta\arctan(\beta/\alpha)+\frac{\alpha\beta^2}{2(\alpha^2+\beta^2)}$.
- (Yu) $F=\alpha+\epsilon\beta+\arctan\left(\frac{\beta}{\alpha}\right)$.

Then it's natural to study all projectively flat (α,β)-metrics with constant flag curvature.

Classification

By the above two lemmas we obtain the following.

Theorem(Li-Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i \neq 0$. Then F is projectively flat with constant flag curvature K if and only if one of the following holds*

(i) *α is projectively flat and β is parallel with respect to α;*

(ii) *$F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ is projectively flat with constant flag curvature $K < 0$, where k and $\epsilon \neq 0$ are constants;*

(iii) *$F = (\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2 / \sqrt{\alpha^2 + k\beta^2}$ is projectively flat with $K = 0$, where k and $\epsilon \neq 0$ are constants.*

Classification

By the above two lemmas we obtain the following.

Theorem(Li-Shen) *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n\geq 3)$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ and $\beta=b_iy^i\neq 0$. Then F is projectively flat with constant flag curvature K if and only if one of the following holds*

(i) *α is projectively flat and β is parallel with respect to α;*

(ii) *$F=\sqrt{\alpha^2+k\beta^2}+\epsilon\beta$ is projectively flat with constant flag curvature $K<0$, where k and $\epsilon\neq 0$ are constants;*

(iii) *$F=(\sqrt{\alpha^2+k\beta^2}+\epsilon\beta)^2/\sqrt{\alpha^2+k\beta^2}$ is projectively flat with $K=0$, where k and $\epsilon\neq 0$ are constants.*

- It is a trivial fact that if F is trivial and the flag curvature $K=constant$, then it is either Riemannian $(K\neq 0)$ or locally Minkowskian $(K=0)$. (by S. Numata).

- The Finsler metric in (ii) is of Randers type, i.e.,

$$F = \bar{\alpha} + \bar{\beta},$$

where $\bar{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\bar{\beta} := \epsilon\beta$.
Shen proved that a Finsler metric in this form is projectively flat with constant flag curvature if and only if it is locally Minkowskian or it is locally isometric to a generalized Funk metric

$$F = c(\bar{\alpha} + \bar{\beta})$$

on the unit ball $B^n \subset R^n$, where $c > 0$ is a constant, and

$$\bar{\alpha} : = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} \tag{1}$$

$$\bar{\beta} : = \pm\left\{\frac{\langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\right\}, \tag{2}$$

where $a \in R^n$ is a constant vector.

- The Finsler metric in (iii) is in the form

$$F = (\tilde{\alpha} + \tilde{\beta})^2 / \tilde{\alpha},$$

where $\tilde{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\tilde{\beta} := \epsilon\beta$. It is proved (by Mo, Shen , Yang and Yildirim) that a non-Minkowkian metric $F = (\tilde{\alpha} + \tilde{\beta})^2 / \tilde{\alpha}$ is projectively flat with $K = 0$ if and only if it is, after scaling on x, locally isometric to a metric

$$F = c(\tilde{\alpha} + \tilde{\beta})^2 / \tilde{\alpha}$$

on the unit ball $B^n \subset R^n$, where $c = constant$, $\tilde{\alpha} = \lambda\bar{\alpha}$ and $\tilde{\beta} = \lambda\bar{\beta}$, where $\bar{\alpha}$ and $\bar{\beta}$ are given in (1) and (2), and

$$\lambda := \frac{(1 + \langle a, x\rangle)^2}{1 - |x|^2}.$$

Open Problem

Is there any metric $F = (\alpha + \beta)^2/\alpha$ of constant flag curvature which is not locally projectively flat?

• In the past ten years, R. Bryant has classified projectively flat Finsler metrics on S^n with constant curvature $K = 1$.
• In 2003, Shen have given the Taylor extensions at the origin $0 \in R^n$ for x-analytic projectively flat metrics $F = F(x, y)$ of constant flag curvature K.