Intersection Numbers on Moduli Spaces of Curves

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July 7, 2007

Topics

- Our explicit formula of n-point functions for intersection numbers on moduli spaces of curves, which provide a new way to compute correlation functions of 2d Gravity.
- New identities of intersection numbers. In particular, we found a new approach to Faber's intersection number conjecture.
- Our recursion formula of intersection numbers with arbitrary κ and ψ classes, which provides a new way to compute higher Weil-Petersson volumes.
- A proof of a conjecture of Itzykson and Zuber concerning denominators of intersection numbers.

My Special Thanks

I am very grateful to my advisor Prof. Kefeng Liu who introduced me into this fascinating subject. I am also indebted to him for insightful guidance, inspiring suggestions and numerous supports both in mathematics and life.

Moduli Spaces of Curves and Its Compactification

Fine moduli space of curves over $\mathbb{C}$ fail to exist due to automorphisms of curves (Riemann surfaces).

Constructions $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$

- The quotient of Teichmüller spaces by the action of the mapping class group.
- Geometric invariant theory (Gieseker, Mumford, etc).
- Deligne-Mumford-Knudsen compactification.

The coarse moduli space $\mathcal{M}_{g,n}$ is a dense open subset of its compactification $\overline{\mathcal{M}}_{g,n}$, which is projective. It's more natural and popular to treat $\overline{\mathcal{M}}_{g,n}$ as a Deligne-Mumford stack.

Mumford (1974' Fields medalist) defined the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ on moduli spaces of curves in 1983.

It's more reasonable to consider the tautological subring $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$. Since $A^*(\overline{\mathcal{M}}_{g,n})$ is too large and $R^*(\overline{\mathcal{M}}_{g,n})$ contains all geometrically natural classes.

Some tautological classes on $\overline{\mathcal{M}}_{g,n}$

- ψ_i the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the ith marked point.
- $\lambda_i = c_i(\mathbb{E})$ the ith Chern class of the Hodge bundle $\mathbb{E}$.
- κ classes originally defined by Miller-Morita-Mumford on $\overline{\mathcal{M}}_g$ and generalized to $\overline{\mathcal{M}}_{g,n}$ by Arbarello-Cornalba.

Netherlands School of Algebraic Geometry

There are many great Dutch mathematicians in history:

W. van Snell (1591-1626), C. Huygens (1629-1695), D.J. Korteweg (1848-1941), G. de Vries (1866-1934), L.E.J. Brouwer (1881-1966), B.L. van der Waerden (1903-1996), Edsger W. Dijkstra (1930-2002), etc.

In recent years, there are many world-renowned Dutch mathematicians working in algebraic geometry:

R. Dijkgraaf, C. Faber, G. van der Geer, A.J. de Jong, H. Lenstra, E. Looijenga, F. Oort, J. Steenbrink, A. Van de Ven, E. Verlinde, H. Verlinde, etc.

Witten's Conjecture and Kontsevich's Theorem

Correlation functions of 2D gravity

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \sum_{j=1}^{n} d_j = 3g - 3 + n.$$

The celebrated Witten's conjecture asserts that the generating function of these intersection numbers are governed by KdV hierarchy. Kontsevich (1998' Fields medalist) proved Witten's conjecture in 1992.

Witten's pioneering work revolutionized this field and motivated a surge of subsequent developments: Gromov-Witten theory, Faber's conjecture, Virasoro conjecture...

Reformulation of Witten's Conjecture

The following equivalent formulation of Witten's conjecture is due to Dijkgraaf-E.Verlinde-H.Verlinde.

DVV recursion formula

$$\langle \tilde\tau_k \prod_{j=1}^{n} \tilde\tau_{d_j} \rangle_g = \sum_{j=1}^{n} (2d_j+1) \langle \tilde\tau_{d_1} \cdots \tilde\tau_{d_j+k-1} \cdots \tilde\tau_{d_n} \rangle_g$$

$$+ \frac{1}{2} \sum_{r+s=k-2} \langle \tilde\tau_r \tilde\tau_s \prod_{j=1}^{n} \tilde\tau_{d_j} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{r+s=k-2 \\ \underline{n}=I \coprod J}} \langle \tilde\tau_r \prod_{i\in I} \tilde\tau_{d_i} \rangle_{g'} \langle \tilde\tau_s \prod_{i\in J} \tilde\tau_{d_i} \rangle_{g-g'}$$

where $\langle \tilde\tau_{d_1} \cdots \tilde\tau_{d_n} \rangle_g = \left[\prod_{i=1}^{n} (2d_i+1)!! \right] \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$

Compute Intersection Numbers

Witten-Kontsevich's recursion formula is the only feasible way known before to calculate all intersection numbers on moduli spaces of curves, which baffled mathematicians for many years.

Different Proofs of Witten's conjecture

- Kontsevich' combinatorial model for the intersection numbers and matrix integral.
- Via ELSV fomula that relates intersection numbers with Hurwitz numbers (Okounkov-Pandharipande, Kazarian-Lando, Chen-Li-Liu).
- Mirzakhani's recursion formula of Weil-Petersson volumes.
- Kim-Liu's proof by localization on moduli spaces of relative stable morphisms and an asymptotic analysis.

Definition

We call the following generating function

$$F(x_1,\dots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

the n-point function.

Since n-point functions encoded all information of intersection numbers on moduli spaces of curves, a great deal of effort has been devoted to find explicit expressions of n-point function by many leading experts, including R. Dijkgraaf, D. Zagier, A. Okounkov, E. Brézin and S. Hikami.

Two and Three-Point Functions

One-point function $F(x) = \frac{1}{x^2}\exp(\frac{x^3}{24})$ (due to Witten)
R. Dijkgraaf's two-point function (1993)

$$F(x,y) = \frac{1}{x+y}\exp\left(\frac{x^3+y^3}{24}\right)\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k.$$

Don Zagier's three-point function (1996)

$$F(x,y,z) = \exp\left(\frac{x^3+y^3+z^3}{24}\right)\sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!}\frac{(\Delta/8)^s}{(r+s+1)!},$$

where S_r and Δ are the homogeneous symmetric polynomials

$$S_r(x,y,z) = \frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{2(x+y+z)},$$

$$\Delta(x,y,z) = (x+y)(y+z)(z+x) = \frac{(x+y+z)^3}{3} - \frac{x^3+y^3+z^3}{3}.$$

Okounkov's Analytic Formula of N-Point Functions

In 2001, A. Okounkov (2006' Fields medalist) obtained an analytic expression of the n-point functions using n-dimensional error-function-type integrals. The key ingredient in Okounkov's formula is the following function

$$\mathcal{E}(x_1,\ldots,x_n)=\frac{1}{2^n\pi^{n/2}}\frac{\exp\left(\frac{1}{12}\sum x_i^3\right)}{\prod\sqrt{x_i}}\times$$
$$\int_{s_i\geq 0} ds\,\exp\left(-\sum_{i=1}^{n}\frac{(s_i-s_{i+1})^2}{4x_i}-\sum_{i=1}^{n}\frac{s_i+s_{i+1}}{2}x_i\right)$$

Okounkov's proof borrows ingenious ideas from the theory of random matrices and random permutations.

Our Explicit Formula of N-Point Functions

Let $G(x_1, \ldots, x_n) = \exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right) \cdot F(x_1, \ldots, x_n)$ and $n \geq 2$.

$$G(x_1, \ldots, x_n) = \sum_{r,s\geq 0} \frac{(2r+n-3)!!P_r(x_1, \ldots, x_n)\Delta(x_1, \ldots, x_n)^s}{4^s(2r+2s+n-1)!!}$$

where P_r and Δ are homogeneous symmetric polynomials

$$\Delta = \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$P_r = \left(\frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I \coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 G(x_I) G(x_J)\right)_{3r+n-3}$$

$$= \frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I \coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 \sum_{r'=0}^{r} G_{r'}(x_I) G_{r-r'}(x_J).$$

Recovering Two and Three-Point Functions

Since $P_0(x,y) = \frac{1}{x+y}$, $P_r(x,y) = 0$ for $r > 0$ and

$$P_r(x,y,z) = \frac{r!}{2^r(2r+1)!} \cdot \frac{(xy)^r(x+y)^{r+1} + (\text{cyclic permutations})}{x+y+z},$$

we easily recover Dijkgraaf's two-point function and Zagier's three-point function obtained more than ten years ago.

Our n-point functions are derived from Witten's KdV equation. Although our derivation of n-point functions is hard, the verification is purely combinatorial and straightforward.

Proof of Our N-Point Functions

We need to check that

$$E(x_1, \dots, x_n) := (\sum_{j=1}^{n} x_j) \cdot G(x_1, \dots, x_n)$$

satisfies the differential equation

$$2x_1 \sum_{j=1}^{n} x_j \cdot \frac{\partial}{\partial x_1} E(x_1, \dots, x_n) + \left(x_1 + \frac{x_1^3}{4} \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} x_j - \frac{x_1}{4} (\sum_{j=1}^{n} x_j)^3 \right) E(x_1, \dots, x_n)$$
$$= \frac{x_1}{2} \sum_{\underline{n}=I \coprod J} \left((\sum_{i \in J} x_i)^2 + 2(\sum_{i \in I} x_i) \cdot (\sum_{i \in J} x_i) \right) E(x_I) E(x_J).$$

and the initial value condition (i.e. the string equation)

$$G(x_1, \dots, x_n, 0) = (\sum_{j=1}^{n} x_j) \cdot G(x_1, \dots, x_n).$$

New Way to Compute Intersection Numbers

Our formula of n-point functions provides an elementary algorithm to calculate all intersection numbers on moduli spaces of curves other than the celebrated Witten-Kontsevich's recursion formula.

There is another slightly different formula of n-point functions whose coefficients look a little simpler (When $n = 3$, this has also been obtained by Zagier). For $n \geq 2$,

$$F(x_1, \dots, x_n) = \exp \frac{(\sum_{j=1}^n x_j)^3}{24} \sum_{r,s \geq 0} \frac{P_r(x_1, \dots, x_n) \Delta(x_1, \dots, x_n)^s}{(-8)^s (2r + 2s + n - 1) s!}$$

Faber's Conjecture on Tautological Rings

In 1993, Carel Faber proposed his remarkable conjectures about the structure of tautological ring $\mathcal{R}^*(\mathcal{M}_g)$. Faber's conjecture motivated a tremendous progress toward understanding of the topology of moduli spaces of curves.

Many leading experts have made important contributions to Faber's conjecture and its generalizations.

C. Faber, E. Getzler, I. Goulden, R. Hain, E. Ionel, D. Jackson, S. Morita, E. Looijenga, R. Pandharipande, R. Vakil, etc.

Faber's Intersection Number Conjecture

An important part (the only quantitative part) of Faber's conjecture is the famous Faber's intersection number conjecture. There are three approaches.

1. The $\lambda_g \lambda_{g-1}$ theorem by the work of Getzler and Pandharipande conditional to degree 0 Virasoro conjecture for $\mathbb{P}^2$, a proof of which is announced by Givental in 2001.
2. Goulden, Jackson and Vakil give a proof of the following relation when $n \leq 3$

$$\pi_*(\psi_1^{d_1+1} \dots \psi_n^{d_n+1}) = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)! \prod_{j=1}^{n}(2d_j+1)!!} \kappa_{g-2},$$

3. Our new identities coming from study of n-point functions.

Our Approach to Faber's Intersection Number Conjecture

From the coefficients of the special n-point functions $G(y, -y, x_1, \ldots, x_n)$, we proved

$$\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{j=1}^{n}\tau_{d_j}\rangle_{g-1}=\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!}\cdot\frac{2}{\prod_{j=1}^{n}(2d_j-1)!!}$$

Faber's intersection number conjecture is equivalent to

$$\langle\tau_{d_1}\cdots\tau_{d_n}\tau_{2g}\rangle_g=\sum_{j=1}^{n}\langle\tau_{d_1}\cdots\tau_{d_{j-1}}\tau_{d_j+2g-1}\tau_{d_{j+1}}\cdots\tau_{d_n}\rangle_g$$
$$-\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$

New Identities of Intersection Numbers

For $K > 2g$, $\sum_{j=0}^{K}(-1)^j\langle\tau_{K-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_g = 0$

$$\langle\prod_{j=1}^{n}\tau_{d_j}\tau_K\rangle_g = \sum_{j=1}^{n}\langle\tau_{d_j+K-1}\prod_{i\neq j}\tau_{d_i}\rangle_g - \frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{K-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{K-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$

$$\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\frac{1}{\prod_{j=1}^{n}(2d_j-1)!!} + \sum_{j=1}^{n}\langle\tau_{d_j+2g-3}\tau_{d_{j+1}}\prod_{i\neq j}\tau_{d_i}\rangle_g$$
$$= \langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g-2}\rangle_g + \frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\rangle$$

Compute $\mathrm{ch}_{2g-3}(\mathbb{E})$ Integral

The following identity is still conjectural ($n=1$ is known).

$$\frac{(2g-2)!}{B_{2g-2}}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\mathrm{ch}_{2g-3}(\mathbb{E})$$

$$=\frac{2g-2}{|B_{2g-2}|}\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-1}\lambda_{g-2}-3\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-3}\lambda_g\right)$$

$$=\frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_{2g-4-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}$$

$$+\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\cdot\frac{1}{\prod_{j=1}^{n}(2d_j-1)!!}$$

Recursion formula of Higher Weil-Petersson Volumes

For $\kappa(\mathbf{b}) = \prod_{j\geq 1} \kappa_j^{b_j}$, we have

$$
\begin{aligned}
&(2d_1+1)!!\langle \kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g \\
&= \sum_{j=2}^{n} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i}\rangle_g \\
&+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i=2}^{n} \tau_{d_i}\rangle_{g-1} \\
&\quad + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I\coprod J=\{2,\dots,n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}} (2r+1)!!(2s+1)!! \\
&\qquad\qquad \times \langle \kappa(\mathbf{e})\tau_r \prod_{i\in I} \tau_{d_i}\rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i\in J} \tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

Values of Tautological Constants

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0, \qquad \mathbf{b} \neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}'\neq\mathbf{0}}} \frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b} \neq 0,$$

with the initial value $\alpha_{\mathbf{0}} = 1$.
Denote $\alpha(k, 0, 0, \dots)$ by α_k, we recover Mirzakhani's recursion formula with

$$\alpha_k = (-1)^{k-1}(2^{2k}-2)\frac{B_{2k}}{(2k-1)!!}.$$

Compatibility of κ and ψ classes

The proof of the previous recursion formula of higher Weil-Petersson volumes uses Witten-Kontsevich theorem and some combinatorial lemmas. We also found that recursion formula of ψ classes can always be generalized to include κ classes, which display compatibility between κ and ψ classes.

1. Generalization of the string equation

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = \sum_{j=1}^{n} \langle \tau_{d_j - 1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \rangle_g$$

2. Generalization of the dilaton equation

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g-2+n) \langle \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g$$

New Way to Compute Higher Weil-Petersson Volumes

We can use the previous recursion formula to compute any κ and ψ intersection numbers on $\overline{\mathcal{M}}_{g,n}$ recursively with the three initial values

$$\langle\tau_0\kappa_1\rangle_1=\frac{1}{24},\qquad \langle\tau_0^3\rangle_0=1,\qquad \langle\tau_1\rangle_1=\frac{1}{24}.$$

When $n=0$, i.e. for higher Weil-Petersson volumes of $\overline{\mathcal{M}}_g$, we may apply the following formula first

$$\langle\kappa(\mathbf{b})\rangle_g=\frac{1}{2g-2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\kappa(\mathbf{L}')\rangle_g.$$

Virasoro Constraints

The following is a generalization of Mulase and Safnuk's work.

$$V_k = -\frac{1}{2}\sum_{\mathbf{L}}(2(|\mathbf{L}|+k)+3)!!\gamma_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\frac{\partial}{\partial t_{|\mathbf{L}|+k+1}}$$
$$+\frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}}$$
$$+\frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}}+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{48},$$

where $k \geq -1$ and $\gamma_{\mathbf{L}}$ are defined by

$$\gamma_{\mathbf{L}} = \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}.$$

We have $V_k \exp(G) = 0$ for $k \geq -1$ and $[V_n, V_m] = (n-m)V_{n+m}$.

KdV Hierarchy

The Witten-Kontsevich theorem states that the generating function

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy.
Since Virasoro constraints uniquely determine the generating functions $G(\mathbf{s}, t_0, t_1, \ldots)$ and $F(t_0, t_1, \ldots)$, we have

$$G(\mathbf{s}, t_0, t_1, \ldots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \ldots),$$

where p_k are polynomials in $\mathbf{s}$ given by

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{||\mathbf{L}||-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of $\mathbf{s}$, $G(\mathbf{s}, \mathbf{t})$ is a τ-function for the KdV hierarchy.

Quantities Related to Singularities

Definition

$$D_{g,n} = lcm\left\{denom\left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\right) \middle| \sum_{i=1}^{n} d_i = 3g-3+n\right\}$$

$$\mathcal{D}_g = lcm\left\{denom\left(\int_{\overline{\mathcal{M}}_g} \kappa_{a_1}\cdots\kappa_{a_m}\right) \middle| \sum_{i=1}^{m} a_m = 3g-3\right\}, g \geq 2$$

A neighborhood of $\Sigma \in \overline{\mathcal{M}}_{g,n}$ is of the form $U/\mathrm{Aut}(\Sigma)$, where U is an open subset of $\mathbb{C}^{3g-3+n}$. The denominators of intersection numbers all come from these orbifold quotient singularities. As remarked by S. Lando, these constants should "reflect the geometry of some virtual smooth spaces".

Properties of Denominators

- $D_{g,n} \mid D_{g,n+1}$, $D_{g,n} \mid \mathcal{D}_g$. If $n \geq \lfloor \frac{g}{2} \rfloor + 1$, $D_{g,n} = \mathcal{D}_g$.
- $\mathrm{ord}(2, \mathcal{D}_g) = \mathrm{ord}(2, 24^g g!)$, $\mathrm{ord}(3, \mathcal{D}_g) = \mathrm{ord}(3, 24^g g!)$ and $\mathrm{ord}(p, \mathcal{D}_g) = \lfloor \frac{2g}{p-1} \rfloor$ for prime $p \geq 5$. (Exact values of $\mathcal{D}_g$)
- We order all Witten-Kontsevich τ-functions of given genus g

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prec \langle \tau_{k_1} \cdots \tau_{k_m} \rangle_g$$

if $n < m$ or $n = m$ and there exists some i, such that $d_j = k_j$ for $j < i$ and $d_i < k_i$.

If $5 \leq p \leq 2g + 1$ is a prime number, then the smallest tau function of genus g in the above lexicographical order that satisfies $\mathrm{ord}(p, denom\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g) = \lfloor \frac{2g}{p-1} \rfloor$ is

$$\langle \underbrace{\tau_{\frac{p-1}{2}} \cdots \tau_{\frac{p-1}{2}}}_{\lfloor \frac{2g}{p-1} \rfloor} \tau_d \rangle_g$$

An stronger form of Itzykson and Zuber's conjecture

We proved

Theorem

For $1 < g' \leq g$, the order of any automorphism group of a Riemann surface of genus g' divides $D_{g,3}$.

The following corollary is a conjecture raised by Itzykson and Zuber in 1992.

Corollary

For $1 < g' \leq g$, the order of any automorphism group of an algebraic curve of genus g' divides $\mathcal{D}_g$.

Hints From Denominators of Intersection Numbers

The n-point function should contain $n-1$ parameters. Of course some parameters may be coupled!

In fact, our first belief that Dijkgraaf's two-point function and Zagier's three-point function can be generalized comes from the well behavior of denominators of intersection numbers.

Thank You Very Much!