On Projectively Flat Finsler Metrics

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Finsler metrics

Let M be a manifold. A function $F = F(x, y)$ on TM is called a Finsler Metric on M if it has the following properties:
(a)$F(x, y)$ is a C^∞ on TM_0;
(b)$F(x, y)$ is a Minkowski norm on T_xM for any $x \in M$.

(α, β)-metrics

Finsler metrics under our consideration are special (α, β)-metric, it is expressed in the following form

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form.
$\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \leq b \leq b_0.$$

It is known that F is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_0$ for any $x \in M$.
Let G^i and G^i_α denote the spray coefficients of F and α, respectively, give by

$$\begin{aligned} G^i &= \frac{g^{il}}{4}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}, \\ G^i_\alpha &= \frac{a^{il}}{4}\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l}\}. \end{aligned}$$

where $(g_{ij}) := (\frac{1}{2}[F^2]_{y^i y^j})$ and $(a^{ij}) := (a_{ij})^{-1}$.

we have the following
Lemma 1(Chern and Shen) The geodesic coefficients G^i are related to G^i_α by

$$\begin{aligned} G^i &= G^i_\alpha + \alpha Q s^i_0 + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} \\ &\quad + H\{-2Q\alpha s_0 + r_{00}\}\{b^i - s\frac{y^i}{\alpha}\} \end{aligned}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}$$

$$J := \frac{\phi'(\phi - s\phi')}{2\phi((\phi - S\phi') + (b^2 - s^2)\phi'')}$$

$$H := \frac{\phi''}{2((\phi - S\phi') + (b^2 - s^2)\phi'')}$$

where $s = \frac{\beta}{\alpha}$, and $b := \|\beta_x\|_\alpha$.

$$s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \qquad r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i})$$

$$s^i_j = a^{ik}s_{kj}, \qquad s^i_0 = s^i_j y^j, \qquad s_0 = b_i s^i_0, \qquad r_0 = r_{ij}y^i y^j.$$

Projectively flat

The Hilbert's Fourth Problem is to characterize the (not-necessarily-reversible) distance functions on an open subset in R^n such that straight lines are shortest paths. Distance functions induced by a Finsler metrics are regarded as smooth ones. Thus the Hilbert's Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in R^n whose geodesics are straight lines. Finsler metrics on an open domain in R^n with this property are said to be *projectively flat*.

G.Hamel first found a simple system of PDE's to characterize projectively flat Finsler metrics on an open subset in R^n. That is, a Finsler metric $F = F(x, y)$ on an open subset in R^n is projectively flat if and only if it satisfies the following partial differential equations

$$F_{x^k y^i} y^k = F_{x^i}$$

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metrics on an open domain in R^n. This problem is very difficult for general Finsler metrics.

A natural problem is to study and characterize all (α, β)-metrics which are projectively flat. In general, this is very complicated. The fist step for us is to study some special (α, β)-metrics. we have following

Lemma 2(Shen and Yildirim) An (α, β)-metric $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$, is projectively flat on an open subset $U \subset R^n$ if and only if

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0$$

where $y_m = a_{mi}y^i$

A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed.

Z. Shen and G. Civi Yildirim have proved that $F = \frac{(\alpha+\beta)^2}{\alpha}$ is projectively flat if and only if

$$\begin{aligned} b_{i|j} &= \tau\{(1+2b^2)a_{ij} - 3b_i b_j\}, \\ G^i_\alpha &= \theta y^i - \tau\alpha^2 b^i, \end{aligned}$$

where $b = \sqrt{a_{ij}(x)b^i(x)b^j(x)}$, $\tau = \tau(x)$ is a scalar function and $\theta = t_i(x)y^i$ is a 1-form.

P. Sennarath and G. Thornley have given an equation in local coordinates that characterizes projectively flat Finsler metrics in the form $F = \frac{\alpha^2+\beta^2}{\alpha}$.

These are some special forms of (α, β)-metric.

Exponential Finsler Metric

we consider a special (α, β)-metric in the following form:

$$F = \alpha \exp(\frac{\beta}{\alpha}) + \epsilon\beta$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form on M, ϵ is a constant. Let $b_0 > 0$ be the largest number such that

$$1 - s + b^2 - s^2 > 0, \quad |s| \leq b < b_0.$$

where b_0 depends on ϵ such that $\phi(s) = exp(s) + \epsilon s > 0$.

Lemma 3 If $F = \alpha \exp(\frac{\beta}{\alpha}) + \epsilon\beta$ is a Finsler metric, then $b_0 \leq 1$.

By Lemma1, we get

$$
\begin{aligned}
Q &= \frac{\alpha(1+\epsilon\exp(-\frac{\beta}{\alpha}))}{\alpha-\beta}, \\
J &= \frac{(\exp(\frac{\beta}{\alpha})+\epsilon)(\alpha-\beta)\alpha^2}{2(\exp(\frac{\beta}{\alpha})\alpha+\epsilon\beta))((1+b^2)\alpha^2-\beta\alpha-\beta^2)}, \\
H &= \frac{\alpha^2}{2((1+b^2\alpha^2)-\alpha\beta-\beta^2)},
\end{aligned}
$$

remark $1+\epsilon\exp(-\frac{\beta}{\alpha})\neq 0$. If $1+\epsilon\exp(-\frac{\beta}{\alpha})=0$, then $\phi(s)=\exp(s)+\epsilon s$ is a constant, thus F is a Riemannian metric.

Lemma 4 If $(a_{ml}\alpha^2-y_m y_l)G^m_\alpha=0$, then α is locally projectively flat.

Result 1

Theorem 1 Let $F = \alpha \exp(\frac{\beta}{\alpha}) + \epsilon\beta$ be a Finsler metric on a manifold M. F is locally projectively flat if and only if the following conditions holds
(a) β is parallel with respect to α,
(b) α is locally projectively flat. That is, α is of constant curvature.
Proof: If F is projective flat, by lemma 2, we have

$$\begin{aligned}
&2(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)(a_{ml}\alpha^2-y_my_l)G^m_\alpha \\
&+2\alpha^4(1+\epsilon\exp(-\frac{\beta}{\alpha}))((1+b^2)\alpha^2-\alpha\beta-\beta^2)s_{l0} \\
&+\alpha^3(-2\alpha^2(1+\epsilon\exp(-\frac{\beta}{\alpha}))s_0+(\alpha-\beta)r_{oo})(b_l\alpha-sy_l)=0. \qquad (1)
\end{aligned}$$

Case 1: Assume that $\epsilon \neq 0$. Contract (1) with b^l yields

$$\begin{aligned}
&2((1+b^2)\alpha^2-\alpha\beta-\beta^2)(b_l\alpha^2-y_m\beta)G^m_\alpha \\
&+\alpha^2(b^2\alpha^2-\beta^2)r_{oo}+2\alpha^5(1+\epsilon\exp(-\frac{\beta}{\alpha}))s_0=0. \qquad (2)
\end{aligned}$$

replace y to $-y$, we get

$$
\begin{aligned}
&2((1+b^2)\alpha^2+\alpha\beta-\beta^2)(b_l\alpha^2-y_m\beta)G^m_\alpha \\
&+\alpha^2(b^2\alpha^2-\beta^2)r_{oo}-2\alpha^5(1+\epsilon\exp(\frac{\beta}{\alpha}))s_0=0.
\end{aligned} \tag{3}
$$

(2)-(3), we get

$$
2\beta(b_l\alpha^2-y_m\beta)G^m_\alpha=\alpha^4 s_0(2+\epsilon\exp(\frac{\beta}{\alpha})+\epsilon\exp(-\frac{\beta}{\alpha})) \tag{4}
$$

Use Taylor expansion of $\exp(\frac{\beta}{\alpha})$, we can find that the left of (4) is a integral expression in y and the right of (4) is a fractional expression in y, we get

$$
s_0=0, \qquad (b_l\alpha^2-y_m\beta)G^m_\alpha=0. \tag{5}
$$

Substituting (5) back into (3), we get $r_{00}=0$. Thus (1) became

$$
(\alpha-\beta)(a_{ml}\alpha^2-y_m y_l)G^m_\alpha+\alpha^4(1+\epsilon\exp(-\frac{\beta}{\alpha}))s_{l0}=0. \tag{6}
$$

With the same discussion, we get

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0, \qquad s_{l0} = 0.$$

By lemma 4, α is locally projective flat.
By $s_{l0} = 0$ and $r_{00} = 0$, we get $b_{i|j} = 0$, thus β is parallel with respect α.

Case 2: $\epsilon = 0$, we get

$$\begin{aligned}
&2((1+b^2)\alpha^3(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 r_{00}(b_l\alpha^2 - \beta y_l) \\
&-2\alpha^5\beta s_{l0} - (2\alpha^4 s_0 + \alpha^2\beta r_{oo})(b_l\alpha^2 - \beta y_l) \\
&+(-2(2+b^2)\alpha^2\beta + 2\beta^3)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha \\
&+(2(1+b^2)\alpha^6 - 2\alpha^4\beta^2)s_{l0} = 0. \qquad (7)
\end{aligned}$$

Because α^{even} is a polynomial in y^i, then the coefficients of α and the coefficients of α^2 must be zero. We obtain

$$\begin{aligned}
&2((1+b^2)\alpha^3(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 r_{00}(b_l\alpha^2 - \beta y_l) \\
&= 2\alpha^5\beta s_{l0} \qquad (8) \\
&2(-(2+b^2)\alpha^2\beta + \beta^3)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha \\
&+2((1+b^2)\alpha^6 - \alpha^4\beta^2)s_{l0} \\
&= (2\alpha^4 s_0 + \alpha^2\beta r_{oo})(b_l\alpha^2 - \beta y_l). \qquad (9)
\end{aligned}$$

With the same discussion, we get

$$\beta(b_m\alpha^2 - y_m\beta)G^m_\alpha = \alpha^4 s_0. \tag{10}$$

Note that the polynomial α^4 is not divisible by β, Thus $(b_m\alpha^2 - y_m\beta)G^m_\alpha$ is divisible by α^4. Therefore, there is a scalar function $\tau = \tau(x)$ such that

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha = \tau(x)\alpha^4 \tag{11}$$

$$s_0 = \tau(x)\beta. \tag{12}$$

Contracting (8) with b^l yields

$$2((1+b^2)\alpha^2\beta(b_m\alpha^2 - y_m\beta)G^m_\alpha + \alpha^2\beta r_{00}(b^2\alpha^2 - \beta^2) = 2\alpha^4\beta^2 s_0 \tag{13}$$

Substituting (11) and (12) back into (13), we get

$$r_{00}(b^2\alpha^2 - \beta^2) = 2\alpha^2(\beta^2 - (1+b^2)\alpha^2)\tau(x). \tag{14}$$

Note that the polynomial $b^2\alpha^2 - \beta^2$ is not divisible by α^2 ,then r_{00} is divisible by α^2.therefore, there is a scalar function $\lambda = \lambda(x)$ such that

$$r_{00} = \lambda\alpha^2, \tag{15}$$

$$\lambda(b^2\alpha^2 - \beta^2) = 2(\beta^2 - (1 + b^2)\alpha^2)\tau(x). \tag{16}$$

Because the polynomial $b^2\alpha^2 - \beta^2$ is not divisible by $\beta^2 - (1 + b^2)\alpha^2$, then

$$\lambda = 0, \quad \tau(x) = 0 \tag{17}$$

thus we get $r_{00} = 0, s_0 = 0$.
By (8) and (9), we get

$$\begin{aligned}&(-2(2 + b^2)\alpha^2\beta + 2\beta^3)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha \\ &+(2(1 + b^2)\alpha^6 - 2\alpha^4\beta^2)s_{l0} = 0,\end{aligned}$$

$$2((1 + b^2)\alpha^2\beta(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha - 2\alpha^4\beta^2 s_{l0} = 0.$$

Because

$$\begin{vmatrix} -2(2+b^2)\alpha^2\beta+2\beta^3 & 2(1+b^2)\alpha^6-2\alpha^4\beta^2 \\ 2((1+b^2)\alpha^2\beta & -2\alpha^4\beta^2 \end{vmatrix} \neq 0$$

We get

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0, \qquad s_{l0} = 0.$$

By lemma 4, α is locally projective flat. By $s_{l0} = 0$ and $r_{00} = 0$, we get $b_{i|j} = 0$, thus β is parallel with respect α.

We say a Finsler metric on an open domain in R^n is *trivial*, if it satisfies the conclusion of theorem 1. Thus the above theorem tells us that in the class of exponential Finsler metrics, there is no non-trivial projectively flat metrics.

A theorem due to Douglas states that a Finsler metric F is projectively flat if and only if two special curvature tensors are zero. The first is the Douglas tensor. The second is the projective Weyl tensor for $n \geq 3$, and the Berwald-Weyl tensor for $n = 2$. It is known that the projective Weyl tensor vanishes if and only if the flag curvature of F have no dependence on the transverse edges (but can possibly depend on the position x and the flagpole y). If the Douglas tensor of F vanishes, we call F a *Douglas metric*.

S. Bácsó and M. Matsumoto proved that a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is a closed 1-form.

M. Matsumoto obtained that for $n = dimM \geq 3$, $F = \frac{\alpha^2+\beta^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j} = \tau((1 + 2b^2)a_{ij} - 3b_i b_j)$$

where $\tau = \tau(x)$ is a scalar function.

Definition Let

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right) \tag{18}$$

It is easy to verify that $D := D^i_{jkl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_0. We say D the Douglas tensor.

It is know that the Douglas tensor is a projective invariant, namely, if two Finsler metrics F and $\overline{F}$ are projectively equivalent,

$$G^i = \overline{G}^i + Py^i, \tag{19}$$

where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\overline{F}$.

Theorem 2 Let $F = \alpha \exp(\frac{\beta}{\alpha}) + \epsilon\beta$ be a Finsler metric on a manifold M. Then the Douglas tensor of F vanishes if and only if β is parallel with respect to α.

Arctangent Finsler Metric

we consider a special (α, β)-metric in the following form:

$$F = \alpha + \epsilon\beta + \beta \arctan \frac{\beta}{\alpha}$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form on M, ϵ is a constant. Let $b_0 > 0$ be the largest number such that

$$\frac{1 - s^2 + 2b^2}{(1+s^2)^2} > 0, \quad |s| \leq b < b_0.$$

where b_0 depends on ϵ such that $\phi(s) = 1 + \epsilon s + s \arctan s > 0$.
By Lemma1, we get

$$\begin{aligned} Q &= \frac{(\epsilon + \arctan \frac{\beta}{\alpha})(\alpha^2 + \beta^2) + \alpha\beta}{\alpha^2}, \\ J &= \frac{\alpha[(\epsilon + \arctan \frac{\beta}{\alpha})(\alpha^2 + \beta^2) + \alpha\beta]}{2[(1+2b^2)\alpha^2 - \beta^2][\alpha + (\epsilon + \arctan \frac{\beta}{\alpha})\beta]}, \\ H &= \frac{\alpha^2}{(1+2b^2)\alpha^2 - \beta^2}, \end{aligned}$$

Result 2

Theorem 3 Let $F = \alpha + \epsilon\beta + \beta \arctan \frac{\beta}{\alpha}$ be a Finsler metric on a manifold M. F is locally projectively flat if and only if the following conditions holds
(a) $b_{i|j} = \tau[(1 + 2b^2)a_{ij} - b_i b_j]$,
(b) $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,
where $\tau = \tau(x)$ and $\theta = a_i(x)y^i$. In this case,

$$G^i = (\theta + \tau\chi\alpha)y^i,$$

where

$$\chi = \frac{\epsilon + \arctan s}{2[1 + (\epsilon + \arctan s)s]}, \qquad s = \frac{\beta}{\alpha}.$$

We say a Finsler metric on an open domain in R^n is *trivial*, if it satisfies the conclusion of theorem 3.
Theorem 4 Suppose that $F = \alpha + \epsilon\beta + \beta \arctan \frac{\beta}{\alpha}$ is projectively flat with constant flag curvature $K = \lambda$, then $\lambda = 0$.
Theorem 5 Let $F = \alpha + \epsilon\beta + \beta \arctan \frac{\beta}{\alpha}$ is projectively flat with zero flag curvature, then α is flat metric and β is parallel. In this case, F is locally Minkowshian.

Solutions

Theorem 6 Let $F = \alpha + \epsilon\beta + \beta \arctan\frac{\beta}{\alpha}$ be a Finsler metric, where ϵ is a constant. Let $\rho := \rho(h)$ and $h := h(x)$ be as follows:

$$\rho = -\ln(4C_2^2(\mu h^2 - 2\theta h - C_3)),$$
$$h = \frac{1}{\sqrt{1+\mu|x|^2}}\{C_1 + < a, x > + \frac{\theta|x|^2}{1+\sqrt{1+\mu|x|^2}}\},$$

where C_1, $C_2 > 0$, C_3, μ and θ are constants, and $a \in R^n$ is a constant vector. Define

$$\alpha := e^{\rho}\bar{\alpha}, \qquad \beta := C_2 e^{\frac{3}{2}\rho} h_0,$$

where

$$\bar{\alpha} := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - < x, y >^2)}}{1+\mu|x|^2} \qquad h_0 = h_{x^i} y^i.$$

Then $F = \alpha + \epsilon\beta + \beta \arctan\frac{\beta}{\alpha}$ is projectively flat.

Thank you very much!