On the Asymptotic Behavior for Singularities of the Powers of Mean Curvature Flow

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Curvature Flows

- (Huisken 84') Mean Curvature Flow

$$\frac{d}{dt}F(\cdot,t) = -H(\cdot,t)\nu(\cdot,t),$$
$$F(\cdot,0) = F_0(\cdot),$$

Curvature Flows

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- (Ben Chow 85') Flow by the *n*th root of the Gauss Curvature
- (Ben Chow 87') Flow by the square root of the scalar curvature
- (Andrews 94') considered a general class of such evolution equations

Speed has other positive degrees are more difficult,

- (Tso and Chow 85') K^α contract to point.
- (Andrews 00') K^α contract to point, homothetic for $\alpha \in (1/(n+2), 1/n]$
- (Andrews 96')convex, contract to a point, ellipsoids as unique limite.
- (Urbas 98')noncompact solutions evolve by homothetically expanding or translating.

In the case of curves in the plane, curve-shortening flow,

- (Gage and Hamilton 86') convex curves contract to round points
- (Grayson 87') compact embedded curve eventually becomes convex
- (Gage 93') anisotropic analogues of CSF in the convex case
- (Chou and Zhu 98') extend to complete embedded curve

Mean Curvature Flow

- Closed initial hypersurfaces, solution exists on a $[0, T)$, If $T < \infty$, the curvature becomes unbounded as $t \to T$.

Singular behavior as $t \to T$, Consider **Rescaled limit**

- **(type I)** If $\sup (T - t) |A|^2$ is uniformly bounded , we have selfsimilar, homothetically shrinking solution of the flow which is completely classified in the case of positive mean curvature (Huisken 90').
- **(type II)** If $\sup (T - t) |A|^2$ is unbounded , we have "eternal solution" . In the convex case, only translating soliton (Hamilton 95').
- (Huisken and Sinestrari 99') studied singularities in the mean convex case.

H^k-Flow

M^n compact manifold without boundary,
$F(\cdot, t) : M^n \times [0, T) \to R^{n+1}$. F_0 convex. $F(\cdot, t)$ solution to the initial value problem

$$\frac{dF}{dt}(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t),$$
$$F(\cdot, 0) = F_0(\cdot)$$

where H is the mean curvature and $\nu(\cdot, t)$ is the outer unit normal at $F(\cdot, t)$, $k > 0$.

This problem has been considered Andrews (94'), Huisken and Polden(96'), and Schulze (05') ... Schulze called it as H^k-**flow**.

Schulze proved following Theorem :

Theorem

F_0 a smooth immersion, $H(F_0) > 0$. There exists unique, smooth solution on finite time interval $[0, T)$.

In the case that,
i) F_0 strictly convex for $0 < k < 1$,
ii) F_0 weakly convex for $k \geq 1$,

then $F(\cdot, t)$ are strictly convex for all $t > 0$ and contract to a point as $t \to T$

Our result

Theorem

F_0 a smooth immersion, strictly convex for $0 < k < 1$, weakly convex for $k \geq 1$. After rescaling:

$$\widetilde{F}(x,\tau) = (F(x,t) - F(x,T))\left[(k+1)(T-t)\right]^{-\frac{1}{k+1}},$$

where $\tau = -\frac{1}{(k+1)}\log\left(\frac{T-t}{T}\right) \in [0,+\infty)$
the limiting hypersurface $\widetilde{F}_\infty$, *satisfies*

$$\widetilde{H}^{\frac{k+1}{2}} + \left|\widetilde{F}\right|^{\frac{k-1}{2}} \langle \widetilde{F}, \overrightarrow{n} \rangle = 0$$

where $\overrightarrow{n}$ *inner normal vector and* $\widetilde{H}$ *mean curvature of* $\widetilde{F}_\infty$.

Evolution Equations

Lemma

i) $\frac{\partial}{\partial t} g_{ij} = -2H^k h_{ij}$

ii) $\frac{\partial}{\partial t}\nu = kH^{k-1}\nabla H$

iii) $\frac{\partial}{\partial t} h_{ij} = kH^{k-1}\Delta h_{ij} + k(k-1)H^{k-2}\nabla_i H \nabla_j H - (k+1)H^k h_{jl} g^{lm} h_{mi}$
$+kH^{k-1}|A|^2 h_{ij}$

iv) $\frac{\partial}{\partial t} h_j^i = kH^{k-1}\Delta h_j^i + k(k-1)H^{k-2}\nabla^i H \nabla_j H - (k-1)\, H^k h_l^i h_j^l$
$+kH^{k-1}\, |A|^2\, h_j^i$

v) $\frac{\partial}{\partial t} H = kH^{k-1}\Delta H + k(k-1)H^{k-2}|\nabla H|^2 + |A|^2 H^k$

vi) $\frac{\partial}{\partial t} < F, \nu >= kH^{k-1}\Delta < F, \nu > -(k+1)H^k + kH^{k-1}|A|^2 < F, \nu > .$

vii) $\frac{\partial}{\partial t}|A|^2 = kH^{k-1}\Delta|A|^2 + 2k(k-1)H^{k-2}h_{lm}\nabla_i H \nabla_j H g^{il} g^{jm}$
$-2kH^{k-1}|\nabla A|^2 - 2(k-1)H^k C + 2kH^{k-1}|A|^4$

where $C = trA^3$.

By the strong maximum principle and the evolution equation of H, we know the $H > 0$ on $M^n \times [0, T)$.
Schulze have proved

Lemma

F_0 be strictly convex and $k > 0$.
Then M_t are strictly convex for all $t \in (0, T)$.
and $\kappa_{\min}(t)$ is monotonically increasing.

For $k \geq 1$, weakly convex hypersurfaces,

Lemma

F_0 weakly convex with $H(F_0) \geq \delta > 0$, and $k > 1$.
Then M_t is strictly convex for all $t \in (0, T)$.

Blow up rate

As we know, the curvature become unbounded when t tend to the T, here we give a lower bound for the blow up rate of the curvature.

Proposition

If the solution $F(\cdot,t)$ *of the flow* (1.1) *is convex and converges to a point when* $t \to T$ *and* $T < +\infty$, *then there exists a constant* $C(k,n)$ *such that*

$$\max_{F(\cdot,t)} |A|^2 \geq \frac{C(k,n)}{(T-t)^{2/(k+1)}}. \tag{3.1}$$

Classify and Rescale

We call a point $P \in R^{n+1}$ to be a **singularity** if there is $x \in M^n$ such that
(i) $F(x,t) \to P$ as $t \to T$, and
(ii) $|A(x,t)|$ becomes unbounded as t tends to T.

We call the flow is of **Type I**, if there is a constant C_0 such that

$$\max_{F(\cdot,t)} |A|^2 \le \frac{C_0}{(T-t)^{2/(k+1)}} \tag{3.2}$$

for all $t \in [0, T)$. Otherwise it is called to be of **Type II**.

Here we concentrate on the case of Type I. In this case we rescale the flow by setting

$$\widetilde{F}(x,\tau) = (F(x,t) - F(x,T))\left[(k+1)(T-t)\right]^{-\frac{1}{k+1}},$$

where $\tau = -\frac{1}{(k+1)}\log\left(\frac{T-t}{T}\right) \in [0,+\infty)$.

Then we can get

$$\widetilde{g}_{ij} = \left[(k+1)(T-t)\right]^{-\frac{2}{k+1}} g_{ij},$$
$$\widetilde{h}_{ij} = \left[(k+1)(T-t)\right]^{-\frac{1}{k+1}} h_{ij},$$
$$\widetilde{H} = \left[(k+1)(T-t)\right]^{\frac{1}{k+1}} H,$$
$$\left|\widetilde{A}\right|^2_{\widetilde{g}} = \left[(k+1)(T-t)\right]^{\frac{2}{k+1}} |A|^2_g .$$

and

$$\frac{\partial t}{\partial \tau} = \left[(k+1)(T-t)\right].$$

Substitute the above relations into the original equation we obtain the equation for the rescaled H^k-flow.

$$\frac{\partial \widetilde{F}(x,\tau)}{\partial \tau} = \widetilde{F}(x,\tau) - \widetilde{H}^k(\cdot,\tau)\widetilde{\nu}(\cdot,\tau) \tag{3.4}$$

In much the same way, we can get the corresponding evolution equation for H^k-Flow.

Gradient Estimate

In this section, we will show all higher derivatives of the second fundamental form $\widetilde{A}$ are bounded. We discuss H^k-flow at first. We have

Proposition

$F_0(M^n)$ strictly convex. If the norm of the second fundamental form of the solution $F(\cdot, t)$ is uniformly bounded on $M^n \times [0, T]$, that is

$$|A|^2(x,t) \leq C_0 \quad \textit{for } (x,t) \in M^n \times [0,T],$$

then $|\nabla A|^2$ is bounded also.

Proof.

We consider the function

$$G(x,t) = \left(1 + \frac{1}{2}|\nabla B|^2\right) e^{\phi\left(|B|^2\right)}$$

where ϕ is some smooth function to be defined later.

By consider the derivative of G at the maximum point, we have,

$$|\nabla B|^2 \leq C$$

for some constant C depending only on n, k

Notice that $\nabla_l h^i_j = -h^i_p \left(\nabla_l b^p_q\right) h^q_j$, we have

$$|\nabla A|^2 \leq |A|^4 |\nabla B|^2 \leq C$$

Next we consider the **rescaled flow**. In the case of **Type I**, we have

Proposition

For each $m \geq 0$, there exists a constant $C(m)$ depending only on m, n, C_0, k and the initial hypersurface such that

$$\left|\widetilde{\nabla}^m \widetilde{A}\right|^2 \leq C(m)$$

on $M^n \times [1, +\infty)$.

Corollary

For each sequence $\tau_j \to +\infty$, there is a subsequence τ_{j_k} such that $\widetilde{F}(\cdot, \tau_{j_k})$converge smoothly to an immersed nonempty limiting hypersurface $\widetilde{F}_\infty$.

Monotonicity Formula

Theorem

$\widetilde{F}(x,\tau)$ solution of rescaled H^k-flow, then

$$\frac{d}{d\tau}\int_{\widetilde{F}(x,\tau)}\widetilde{\rho}\, d\widetilde{\mu}_\tau \le -\int_{\widetilde{F}(x,\tau)}\widetilde{\rho}|\widetilde{F}|^{k-1}|\langle\widetilde{F},\overrightarrow{n}\rangle+\sigma\widetilde{H}|^2 d\widetilde{\mu}_\tau$$

where $\widetilde{\rho}\left(\widetilde{F}\right)=\exp\left(-\frac{1}{k+1}\left|\widetilde{F}\right|^{k+1}\right)$, *and* $\sigma=\widetilde{H}^{\frac{k-1}{2}}/\left|\widetilde{F}\right|^{\frac{k-1}{2}}$. *Here* $\overrightarrow{n}$ *is the inner normal vector of the rescaled surface, and* $\widetilde{H}>0$.

Thus from the previous Corollary we know that every limit hypersurfaces $\widetilde{F}_\infty$ satisfying the equation

$$\langle \overrightarrow{\widetilde{F}}, \overrightarrow{n} \rangle + \sigma \widetilde{H} = 0$$

i.e.

$$\widetilde{H}^{\frac{k+1}{2}} + \left| \overrightarrow{\widetilde{F}} \right|^{\frac{k-1}{2}} \langle \overrightarrow{\widetilde{F}}, \overrightarrow{n} \rangle = 0 \tag{5.1}$$

Therefore we have

Theorem

Each limiting hypersurface $\widetilde{F}_\infty$ as obtained in Corollary 4.1 satisfies the equation (5.1).

Remark

- *Mean curvature flow, we can deduce an elliptic equation from the above equation of the limiting hypersurface. By maximum principle, only sphere.*
- *however, the same approach seem doesn't work for the H^k-flow.*
- *The paper of K-S Chou and X-J Wang suggest that there may be infinitely many solutions.*

Type II singulalities

In this section we will discuss the **Type II** singularities. We will prove the following

Theorem

Assume $F_0 : M^n \to R^{n+1}$ $(n \geq 2)$ *is compact and convex (as in Theorem S). Then the flow will not develop type II singularity.*

First by standard method of blowup argument,we dilate the solution F $t \in [0, T)$ into F_i such that

$$\max_{M^n} H_i(\cdot, \tau) \leq 1 + \varepsilon$$

Proposition

Assume $F_0 : M^n \to R^{n+1}$ $(n \geq 2)$ is compact and convex (as in Theorem S), and type II .
Then a sequence of the rescaled flow F_i converges smoothly on every compact set to F_∞ defined for all $\tau \in (-\infty, +\infty)$. Moreover, $0 < H_\infty \leq 1$ everywhere and is equal to 1 at least at one point.

Next we need to classify all such solutions.
we need a result of Jie Wang,

Proposition

Any strictly convex solution $F(\cdot, t)$, $t \in (-\infty, +\infty)$, to the H^k-flow for $k > 0$, where the mean curvature assume its maximum value at a point in space-time must be a strictly convex translating soliton.

Now let us consider a translating soliton $F(\cdot, t)$ translates in the direction of a constant vector e_{n+1}. We can write

$$e_{n+1} = V - H^k \nu$$

where V is the tangential part of e_{n+1}. From this we get

$$\langle e_{n+1}, \nu \rangle = -H^k.$$

Since we have shown that $F(\cdot, t)$ is convex for each t, then $\langle e_{n+1}, \nu \rangle < 0$. Then the image of the Gauss map of $F(\cdot, t)$, $\nu(F(\cdot, t))$, is located in a semi-sphere. Noncompact, contradict to the fact that $F(\cdot, t)$ is compact.

Thanks!