Good Geometry on the Curve Moduli

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The results are contained in

- *Canonical Metrics on the Moduli Spaces of Riemann Surfaces I*. JDG 68.
- *Canonical Metrics on the Moduli Spaces of Riemann Surfaces II*. JDG 69.
- *Good Geometry of the Moduli Spaces of Curves*. Preprint.
- *Good Metrics on the Moduli Spaces of Riemann Surfaces I*. Preprint.
- *Good Metrics on the Moduli Spaces of Riemann Surfaces II*. Preprint.

Moduli spaces and Teichmüller spaces of Riemann surfaces have been studied for many years, since Riemann.

They have appeared in many subjects of mathematics, from geometry, topology, algebraic geometry to number theory. They have also appeared in theoretical physics like string theory: many computations of path integrals are reduced to integrals of Chern classes on such moduli spaces.

The Teichmüller space $\mathcal{T}_{g,m}$ of Riemann surfaces of genus g with m punctures (such that $n = 3g - 3 + m > 0$) can be holomorphically embedded into $\mathbb{C}^n$. The moduli space $\mathcal{M}_{g,m}$ is a complex orbifold, as a quotient of $\mathcal{T}_{g,m}$ by mapping class group $\mathrm{Mod}_{g,m}$.

$\mathcal{M}_{g,m}$ has several natural compactifications such as the Baily-Borel-Satake compactification and the Deligne-Mumford compactification. In this talk we will use the DM compactification.

All of the following results hold for $\mathcal{M}_{g,m}$ and $\mathcal{T}_{g,m}$. To simplify the notation, we state the results for $\mathcal{M}_g$ and $\mathcal{T}_g$.

The topology of Teichmüller space is trivial. However, the moduli space and its compactification have highly nontrivial topology, and have been actively studied from many point of views in mathematics and physics.

For example, Harris-Mumford and Harris showed the moduli space is general type when the genus $g \geq 24$. On the other hand, there are famous counterexamples showed the moduli space is uniruled in low genus.

The moduli space has also played important role in physics. For examples, the Witten conjecture, proved by Kontsevich, states that the intersection numbers of the ψ-classes are governed by the KdV hierarchy. Other elegant proofs are given by Kim-Liu, Mirzakhani and Okounkov-Pandhripande.

Mariño-Vafa formula, proved by Liu-Liu-Zhou, gives a closed formula for the generating series of triple Hodge integrals of all genera and all possible marked points, in terms of Chern-Simons knot invariants. Many other conjectures related to Hodge integrals can be deduced from Mariño-Vafa formula by taking various limits.

Gromov-Witten theory can be viewed as a natural extension of the moduli space theory. In fact, Gromov-Witten theory for the DM moduli space is not well understood.

The geometry of the Teichmüller spaces and moduli spaces of Riemann surfaces also have very rich structures. There are many very famous classical metrics on the Teichmüller and the moduli spaces:

1. Finsler Metrics: (complete)

- Teichmüller metric;
- Kobayashi metric;
- Carathéodory metric.

2. Kähler Metrics:

- Weil-Petersson (WP) metric (incomplete);
- Kähler-Einstein metric;
- McMullen metric;
- Induced Bergman metric;
- Asymptotic Poincaré metric.

3. New Kähler Metrics:

- Ricci metric;
- Perturbed Ricci metric.

The last six Kähler metrics are complete.

Our project is to study the geometry of the Teichmüller and the moduli spaces. More precisely to understand the various metrics on these spaces, and more importantly, to introduce new metrics with good properties and to find their applications in algebraic geometry and physics.

The key point is the understanding of the **Ricci** and the **perturbed Ricci** metrics: two new complete Kähler metrics. Their curvature and asymptotic behavior, are studied in great details, and are very well understood.

As an easy corollary we have proved all of the above complete metrics are equivalent. Also we proved that the new metrics and the Kähler-Einstein metrics have (strongly) bounded geometry in Teichmüller spaces. Here by a metric with strongly bounded geometry we mean a complete metric whose curvature and its derivatives are bounded and whose injectivity radius is bounded from below.

From these we have good understanding of the Kähler-Einstein metric on both the moduli and the Teichmüller spaces, and find interesting applications to geometry.

The slope stability of the logarithmic cotangent bundle of the DM moduli spaces, Chern number inequality and other properties will follow.

The perturbed Ricci metric that we introduced has bounded negative holomorphic sectional and Ricci curvatures, bounded geometry and Poincaré growth. So this new metric has practically all interesting properties: close to be the best, except for the non-positivity of the bisetional curvature.

Goodness

The Weil-Petersson, Ricci and perturbed Ricci metrics are good in the sense of Mumford: Chern-Weil theory hold, study of L^2 cohomology.

Negativity

The Weil-Petersson metric is dual Nakano negative: vanishing theorems of L^2 cohomology, infinitesimal rigidity.

Fix an orientable surface Σ of genus $g \geq 2$.

Uniformization Theorem

Each Riemann surface of genus $g \geq 2$ can be viewed as a quotient of the hyperbolic plane $\mathbb{H}$ by a Fuchsian group. Thus there is a unique Poincaré metric, or the hyperbolic metric on Σ.

The group $Diff^+(\Sigma)$ of orientation preserving diffeomorphisms acts on the space $\mathcal{C}$ of all complex structures on Σ by pull-back.

Teichmüller Space

$$\mathcal{T}_g = \mathcal{C}/Diff_0^+(\Sigma)$$

where $Diff_0^+(\Sigma)$ is the set of orientation preserving diffeomorphisms which are isotopic to identity.

Moduli Space

$$\mathcal{M}_g = \mathcal{C}/Diff^+(\Sigma) = \mathcal{T}_g/Mod(\Sigma)$$

is the quotient of the Teichmüller space by the mapping class group where

$$Mod\,(\Sigma) = Diff^+(\Sigma)/Diff_0^+(\Sigma).$$

Dimension

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3.$$

$\mathcal{T}_g$ is a pseudoconvex domain in $\mathbb{C}^{3g-3}$: Bers' embedding theorem. $\mathcal{M}_g$ is a complex orbifold, it can be compactified to a projective orbifold by adding normal crossing divisors D consisting of stable nodal curves, called the Deligne-Mumford compactification, or DM moduli.

Tangent and Cotangent Space

By the deformation theory of Kodaira-Spencer and the Hodge theory, for any point $X \in \mathcal{M}_g$,

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X)$ is the space of holomorphic quadratic differentials on X.

For $\mu \in HB(X)$ and $\phi \in Q(X)$, the duality between $T_X\mathcal{M}_g$ and $T_X^*\mathcal{M}_g$ is

$$[\mu : \phi] = \int_X \mu\phi.$$

Teichmüller metric is the L^1 norm and the WP metric is the L^2 norm. Alternatively, let
$\pi : \mathfrak{X} \to \mathcal{M}_g$ be the universal curve and let $\omega_{\mathfrak{X}/\mathcal{M}_g}$ be the relative dualizing sheaf. Then

$$\omega_{WP} = \pi_* \left(c_1 \left(\omega_{\mathfrak{X}/\mathcal{M}_g} \right)^2 \right).$$

Curvature

Let $\mathfrak{X}$ be the total space over the $\mathcal{M}_g$ and π be the projection. Pick $s \in \mathcal{M}_g$, let $\pi^{-1}(s) = X_s$. Let $s_1, \cdots, s_n$ be local holomorphic coordinates on $\mathcal{M}_g$ and let z be local holomorphic coordinate on X_s.

The Kodaira-Spencer map is

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\overline{z} \in HB(X_s).$$

The Weil-Petersson metric is

$$h_{i\bar{j}} = \int_{X_s} A_i \overline{A}_j \; dv$$

where $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\overline{z}$ is the volume form of the KE metric λ on X_s.

By the work of Royden, Siu and Schumacher, let

$$a_i = -\lambda^{-1}\partial_{s_i}\partial_{\overline{z}} \log \lambda.$$

Then

$$A_i = \partial_{\overline{z}} a_i.$$

Let η be a relative $(1,1)$ form on $\mathfrak{X}$. Then

$$\frac{\partial}{\partial s_i}\int_{X_s} \eta = \int_{X_s} L_{v_i}\eta$$

where

$$v_i = \frac{\partial}{\partial s_i} + a_i \frac{\partial}{\partial z}$$

is called the harmonic lift of $\frac{\partial}{\partial s_i}$. In the following, we let

$$f_{i\overline{j}} = A_i\overline{A}_j \text{ and } e_{i\overline{j}} = T(f_{i\overline{j}}).$$

Here $T = (\Box + 1)^{-1}$ with $\Box = -\lambda^{-1}\partial_z\partial_{\overline{z}}$, is the Green operator. The functions $f_{i\overline{j}}$ and $e_{i\overline{j}}$ will be the building blocks of the curvature formula.

Curvature Formula of the WP Metric

By the work of Wolpert, Siu and Schumacher, the curvature of the Weil-Petersson metric is

$$R_{i\bar{j}k\bar{l}} = -\int_{X_s} (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}})\, dv.$$

- The sign of the curvature of the WP metric can be seen directly.
- The precise upper bound $-\frac{1}{2\pi(g-1)}$ of the holomorphic sectional curvature and the Ricci curvature of the WP metric can be obtained by the spectrum decomposition of the operator $(\Box + 1)$.
- The curvature of the WP metric is not bounded from below. But *surprisingly* the Ricci and the perturbed Ricci metrics have bounded (negative) curvatures.
- The WP metric is incomplete.

Observation

The Ricci curvature of the Weil-Petersson metric is bounded above by a negative constant, one can use the negative Ricci curvature of the WP metric to define a new metric.
We call this metric the **Ricci metric**

$$\tau_{i\bar{j}} = -Ric(\omega_{WP})_{i\bar{j}}.$$

We proved the Ricci metric is complete, Poincaré growth, and has bounded geometry.
We perturbed the Ricci metric with a large constant multiple of the WP metric to define the **perturbed Ricci metric**

$$\omega_{\tilde{\tau}} = \omega_{\tau} + C\,\omega_{WP}.$$

We proved that the perturbed Ricci metric is complete, Poincaré growth and has bounded negative holomorphic sectional and Ricci curvatures, and bounded geometry.

Curvature Formula of the Ricci Metric

$$
\begin{aligned}
\widetilde{R}_{i\bar{j}k\bar{l}} = & - h^{\alpha\bar{\beta}} \left\{ \sigma_1\sigma_2 \int_{X_s} T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\bar{\beta}})\ dv \right\} \\
& - h^{\alpha\bar{\beta}} \left\{ \sigma_1\sigma_2 \int_{X_s} T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\ dv \right\} \\
& - h^{\alpha\bar{\beta}} \left\{ \sigma_1 \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\bar{\beta}}\ dv \right\} \\
& + \tau^{p\bar{q}} h^{\alpha\bar{\beta}} h^{\gamma\bar{\delta}} \left\{ \sigma_1 \int_{X_s} \xi_k(e_{i\bar{q}})e_{\alpha\bar{\beta}}\ dv \right\} \times \\
& \left\{ \widetilde{\sigma}_1 \int_{X_s} \overline{\xi}_l(e_{p\bar{j}})e_{\gamma\bar{\delta}})\ dv \right\} \\
& + \tau_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}}.
\end{aligned}
$$

Here σ_1 is the symmetrization of indices i, k, α.
σ_2 is the symmetrization of indices j, β.
$\tilde{\sigma}_1$ is the symmetrization of indices j, l, δ.
ξ_k and $Q_{k\bar{l}}$ are combinations of the Maass operators and the Green operators.

The curvature formula has 85 terms, since it contains fourth order derivatives of the WP metric. The curvature formula of the perturbed Ricci metric even has more. It is too complicated to see the sign. We work out its asymptotic near the boundary.

Selected Applications of These Metrics

- Royden proved that

$$\text{Teichmüller metric} = \text{Kobayashi metric.}$$

  This implies that the isometry group of $\mathcal{T}_g$ is exactly the mapping class group.

- Ahlfors: the WP metric is Kähler, the holomorphic sectional curvature is negative.
- Masur: WP metric is incomplete.

Wolpert studied WP metric in great details, found many important applications in topology(relation to Thurston's work) and algebraic geometry(relation to Mumford's work).

Each family of stable curves induces a holomorphic maps into the moduli space.

A version of Schwarz lemma that I proved, gave very sharp geometric height inequalities in algebraic geometry. Corollaries include:

- Kodaira surface X has strict Chern number inequality:

$$c_1(X)^2 < 3c_2(X).$$

- Beauville conjecture: the number of singular fibers for a non-isotrivial family of semi-stable curves over $\mathbb{P}^1$ is at least 5.

Geometric Height Inequalities, by K. Liu, MRL 1996.

McMullen proved that the moduli spaces of Riemann surfaces are Kähler hyperbolic, by using his metric ω_M which he obtained by perturbing the WP metric.

This means ω_M has bounded geometry and the Kähler form on the Teichmüller space is of the form $d\alpha$ with α bounded one form. Corollaries include:

- The lowest eigenvalue of the Laplacian on the Teichmüller space is positive.
- Only middle dimensional L^2 cohomology is non-zero on the Teichmüller space.

Theorem

All complete metrics on $\mathcal{T}_g$ and $\mathcal{M}_g$ are equivalent. Furthermore, the Carathéodory metric, Kobayashi metric, Bergman metric and KE metric are equivalent on general homogeneous holomorphic regular manifolds.

Subsequently, S.K. Yeung published a weaker version of this theorem.

Theorem

The Ricci, perturbed Ricci and Kähler-Einstein metrics are complete, have (strongly) bounded geometry and Poincaré growth. The holomorphic sectional and Ricci curvatures of the perturbed Ricci metric are negatively pinched.

Algebraic-geometric consequences

Theorem

The log cotangent bundle $T^*_{\overline{\mathcal{M}}_g}(\log D)$ of the DM moduli of stable curves is stable with respect to its canonical polarization.

Corollary

Orbifold Chern number inequality.

Basic Ideas of Proof

Equivalence: Schwarz Lemma and asymptotic analysis.
Example: $\omega_{\tilde{\tau}} \sim \omega_{KE}$: the perturbed Ricci metric $\omega_{\tilde{\tau}}$ has negatively pinched holomorphic sectional curvature and the KE metric has constant Ricci curvature -1. Apply the versions of Schwarz lemma

$$id : (\mathcal{M}_g, \omega_{\tilde{\tau}}) \to (\mathcal{M}_g, \omega_{KE})$$

we get

$$\omega_{KE} \leq c_1 \omega_{\tilde{\tau}}.$$

Conversely, since

$$id : (\mathcal{M}_g, \omega_{KE}) \to (\mathcal{M}_g, \omega_{\tilde{\tau}})$$

we get

$$\omega_{\tilde{\tau}}^{3g-3} \leq c_2 \omega_{KE}^{3g-3}.$$

These imply $\omega_{\tilde{\tau}} \sim \omega_{KE}$.

Asymptotic

- Deligne-Mumford Compactification: For a Riemann surface X, a point $p \in X$ is a node if there is a neighborhood of p which is isomorphic to the germ

$$\{(u, v) \mid uv = 0, |u| < 1, |v| < 1\} \subset \mathbb{C}^2.$$

A Riemann surface with nodes is called a nodal surface.
A nodal Riemann surface is stable if each connected component of the surface subtracting the nodes has negative Euler characteristic. In this case, each connected component has a complete hyperbolic metric.
The union of $\mathcal{M}_g$ and moduli of stable nodal curves of genus g is the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, the DM moduli.
$D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a divisor of normal crossings.

- Principle: To compute the asymptotic of the Ricci metric and its curvature, we work on surfaces near the boundary of $\mathcal{M}_g$. The geometry of these surfaces localize on the pinching collars.
- Model degeneration: Earle-Marden, Deligne-Mumford, Wolpert: Consider the variety

$$V = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\} \subset \mathbb{C}^3$$

and the projection $\Pi : V \to \Delta$ given by

$$\Pi(z, w, t) = t$$

where Δ is the unit disk.
If $t \in \Delta$ with $t \neq 0$, then the fiber $\Pi^{-1}(t) \subset V$ is an annulus (collar).
If $t = 0$, then the fiber $\Pi^{-1}(t) \subset V$ is two transverse disks $|z| < 1$ and $|w| < 1$.
This is the local model of degeneration of Riemann surfaces.

Methods

- Find the harmonic Beltrami differentials A_i.
- Find the KE metric on the collars.
- Estimate the Green function of $(\Box + 1)^{-1}$.
- Estimate the norms and error terms.

We use elliptic estimates to control the error terms causing by the transition of the plumbing coordinate to the rotationally symmetric coordinates to deal with the first two problems.
We then construct approximation solutions on the local model, single out the leading terms and then carefully estimate the error terms one by one.

Asymptotic in pinching coordinates

Theorem

Let $(t_1, \cdots t_m, s_{m+1}, \cdots s_n)$ be the pinching coordinates. Then WP metric h has the asymptotic:

(1) $h_{i\bar{i}} = \frac{1}{2} \frac{u_i^3}{|t_i|^2}(1 + O(u_0))$ *for* $1 \le i \le m$;

(2) $h_{i\bar{j}} = O(\frac{u_i^3 u_j^3}{|t_i t_j|})$ *if* $1 \le i, j \le m$ *and* $i \ne j$;

(3) $h_{i\bar{j}} = O(1)$ *if* $m + 1 \le i, j \le n$;

(4) $h_{i\bar{j}} = O(\frac{u_i^3}{|t_i|})$ *if* $i \le m < j$.

Here $u_i = \frac{l_i}{2\pi}$, $l_i \approx -\frac{2\pi^2}{\log|t_i|}$ *and* $u_0 = \sum u_i + \sum |s_j|$.

Theorem

The Ricci metric τ has the asymptotic:

(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *if* $i \le m$;

(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i+u_j)\right)$ *if* $i,j \le m$ *and* $i \neq j$;

(3) $\tau_{i\bar{j}} = O\left(\frac{u_i^2}{|t_i|}\right)$ *if* $i \le m < j$;

(4) $\tau_{i\bar{j}} = O(1)$ *if* $i,j \ge m+1$.

Finally we derive the curvature asymptotic:

Theorem

The holomorphic sectional curvature of the Ricci metric τ satisfies

$$\widetilde{R}_{i\bar{i}i\bar{i}} = -\frac{3u_i^4}{8\pi^4|t_i|^4}(1+O(u_0)) > 0 \quad \text{if} \quad i \le m$$

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \quad \text{if} \quad i > m.$$

To prove that the holomorphic sectional curvature of the perturbed Ricci metric

$$\omega_{\tilde{\tau}} = \omega_\tau + C\,\omega_{_{WP}}$$

is negatively pinched, we notice that it remains negative in the degeneration directions when C varies and is dominated by the curvature of the Ricci metric.

When C large, the holomorphic sectional curvature of $\tilde{\tau}$ can be made negative in the interior and in the non-degeneration directions near boundary from the negativity of the holomorphic sectional curvature of the WP metric.

The estimates of the bisectional curvature and the Ricci curvature of these new metrics are long and complicated computations.

The lower bound of the injectivity radius of the Ricci and perturbed Ricci metrics and the KE metric on the Teichmüller space is obtained by using Bers embedding theorem, minimal surface theory and the boundedness of the curvature of these metrics.

Bounded Geometry of the KE Metric

The first step is to perturb the Ricci metric by using the Kähler-Ricci flow

$$\begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -(R_{i\bar{j}} + g_{i\bar{j}}) \\ g(0) = \tau \end{cases}$$

to avoid complicated computations of the covariant derivatives of the curvature of the Ricci metric.

For $t > 0$ small, let $h = g(t)$ and let g be the KE metric. We have

- h is equivalent to the initial metric τ and thus is equivalent to the KE metric.
- The curvature and its covariant derivatives of h are bounded.

Then we consider the Monge-Amperé equation

$$\log\det(h_{i\bar{j}}+u_{i\bar{j}})-\log\det(h_{i\bar{j}})=u+F$$

where $\partial\bar{\partial}u=\omega_g-\omega_h$ and $\partial\bar{\partial}F=Ric(h)+\omega_h$.

- Equivalences: $\partial\bar{\partial}u$ has C^0 bound.
- The strong bounded geometry of h implies $\partial\bar{\partial}F$ has C^k bounds for $k\geq 0$.

We need C^k bounds of u for $k\geq 2$. Let

$$S=g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}u_{;i\bar{q}k}u_{;\bar{j}p\bar{l}}$$

$$V=g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}\left(u_{;i\bar{q}k\bar{n}}u_{;\bar{j}p\bar{l}m}+u_{;i\bar{n}kp}u_{;\bar{j}m\bar{l}\bar{q}}\right)$$

where the covariant derivatives of u were taken with respect to the metric h.

C^3 estimate implies S is bounded.
Let $f = (S + \kappa)V$ where κ is a large constant. By using the generalized maximum principle, the inequality

$$\Delta^{'} f \geq Cf^2 + (\text{ lower order terms })$$

implies f is bounded and thus V is bounded. So the curvature of the KE metric are bounded. Same method can be used to derive boundedness of higher derivatives of the curvature.

A recent work of D. Wu on the complete asymptotic expansion of the KE metric on a quasi-projective manifold (assuming $K + [D] > 0$) may give a different proof of the boundedness of the curvature of the KE metric.

Stability of the Log Cotangent Bundle $\overline{E}$

The proof of the stability needs the detailed understanding of the boundary behaviors of the KE metric to control the convergence of the integrals of the degrees.

- As a current, $\omega_{_{KE}}$ is closed and represent the first Chern class of $\overline{E}$.

$$[\omega_{_{KE}}] = c_1(\overline{E}).$$

- The singular metric $g^*_{_{KE}}$ on $\overline{E}$ induced by the KE metric defines the degree of $\overline{E}$.

$$\deg(\overline{E}) = \int_{\mathcal{M}_g} \omega^n_{_{KE}}.$$

- The degree of any proper holomorphic sub-bundle F of $\overline{E}$ can be defined using $g^*_{KE} \mid_F$.

$$\deg(F) = \int_{\mathcal{M}_g} -\partial\overline{\partial} \log \det \left(g^*_{KE} \mid_F \right) \wedge \omega^{n-1}_{KE}.$$

Also needed is a basic non-splitting property of the mapping class group and its subgroups of finite index.

Goodness and Negativity

Now I will discuss the goodness of the Weil-Petersson metric, the Ricci and the perturbed Ricci metrics in the sense of Mumford, and their applications in understanding the geometry of moduli spaces.

The question that WP metric is good or not has been open for many years, according to Wolpert. Corollaries include:

- Chern classes can be defined on the moduli spaces by using the Chern forms of the WP metric, the Ricci or the perturbed Ricci metrics; the L^2-index theory and fixed point formulas can be applied on the Teichmüller spaces.
- The log cotangent bundle is Nakano positive; vanishing theorems of L^2 cohomology; rigidity of the moduli spaces.

Goodness of Hermitian Metrics

For an Hermitian holomorphic vector bundle (F, g) over a closed complex manifold M, the Chern forms of g represent the Chern classes of F. However, this is no longer true if M is not closed since g may be singular.

- X: quasi-projective variety of $\dim_{\mathbb{C}} X = k$ by removing a divisor D of normal crossings from a closed smooth projective variety $\overline{X}$.
- $\overline{E}$: a holomorphic vector bundle of rank n over $\overline{X}$ and $E = \overline{E}\mid_X$.
- h: Hermitian metric on E which may be singular near D.

Mumford introduced conditions on the growth of h, its first and second derivatives near D such that the Chern forms of h, as currents, represent the Chern classes of $\overline{E}$.
We cover a neighborhood of $D \subset \overline{X}$ by finitely many polydiscs

$$\left\{ U_\alpha = \left(\Delta^k, (z_1, \cdots, z_k) \right) \right\}_{\alpha \in A}$$

such that $V_\alpha = U_\alpha \setminus D = (\Delta^*)^m \times \Delta^{k-m}$. Namely, $U_\alpha \cap D = \{z_1 \cdots z_m = 0\}$. We let $U = \bigcup_{\alpha \in A} U_\alpha$ and $V = \bigcup_{\alpha \in A} V_\alpha$. On each V_α we have the local Poincaré metric

$$\omega_{P,\alpha} = \frac{\sqrt{-1}}{2} \left(\sum_{i=1}^{m} \frac{1}{2|z_i|^2 \left(\log |z_i|\right)^2} dz_i \wedge d\overline{z}_i + \sum_{i=m+1}^{k} dz_i \wedge d\overline{z}_i \right).$$

Definition

Let η be a smooth local p-form defined on V_α.

- We say η has Poincaré growth if there is a constant $C_\alpha > 0$ depending on η such that

$$|\eta(t_1, \cdots, t_p)|^2 \leq C_\alpha \prod_{i=1}^{p} \|t_i\|^2_{\omega_{P,\alpha}}$$

  for any point $z \in V_\alpha$ and $t_1, \cdots, t_p \in T_z X$.
- η is good if both η and $d\eta$ have Poincaré growth.

Definition

An Hermitian metric h on E is good if for all $z \in V$, assuming $z \in V_\alpha$, and for all basis $(e_1, \cdots, e_n)$ of $\overline{E}$ over U_α, if we let $h_{i\bar{j}} = h(e_i, e_j)$, then

- $\left|h_{i\bar{j}}\right|, (\det h)^{-1} \leq C \left(\sum_{i=1}^m \log |z_i|\right)^{2n}$ for some $C > 0$.
- The local 1-forms $\left(\partial h \cdot h^{-1}\right)_{\alpha\gamma}$ are good on V_α. Namely the local connection and curvature forms of h have Poincaré growth.

Properties of Good Metrics

- The definition of Poincaré growth is independent of the choice of U_α or local coordinates on it.
- A form $\eta \in A^p(X)$ with Poincaré growth defines a p-current $[\eta]$ on $\overline{X}$. In fact we have

$$\int_X |\eta \wedge \xi| < \infty$$

  for any $\xi \in A^{k-p}(\overline{X})$.
- If both $\eta \in A^p(X)$ and $\xi \in A^q(X)$ have Poincaré growth, then $\eta \wedge \xi$ has Poincaré growth.
- For a good form $\eta \in A^p(X)$, we have $d[\eta] = [d\eta]$.

The importance of a good metric on E is that we can compute the Chern classes of $\overline{E}$ via the Chern forms of h as currents.

Mumford has proved:

Theorem

Given an Hermitian metric h on E, there is at most one extension $\overline{E}$ of E to $\overline{X}$ such that h is good.

Theorem

If h is a good metric on E, the Chern forms $c_i(E, h)$ are good forms. Furthermore, as currents, they represent the corresponding Chern classes $c_i(\overline{E}) \in H^{2i}(\overline{X}, \mathbb{C})$.

With the growth assumptions on the metric and its derivatives, we can integrate by part, so Chern-Weil theory still holds.

Good Metrics on Moduli Spaces

Now we consider the metrics induced by the Weil-Petersson metric, the Ricci and perturbed Ricci metrics on the logarithmic extension of the holomorphic tangent bundles over the moduli space of Riemann surfaces.

Our theorems hold for the moduli space of Riemann surfaces with punctures.

Let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces with $g \geq 2$ and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. Let $n = 3g - 3$ be the dimension of $\mathcal{M}_g$ and let $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ be the compactification divisor.

Let $\overline{E} = T^*_{\overline{\mathcal{M}}_g}(\log D)$ be the logarithmic cotangent bundle over $\overline{\mathcal{M}}_g$.

For any Kähler metric p on $\mathcal{M}_g$, let p^* be the induced metric on $\overline{E}$. We know that near the boundary $\{t_1 \cdots t_m = 0\}$,

$$\left(\frac{dt_1}{t_1}, \cdots, \frac{dt_m}{t_m}, dt_{m+1}, \cdots, dt_n \right)$$

is a local holomorphic frame of $\overline{E}$.

In these notations, near the boundary the log tangent bundle $F = T_{\overline{\mathcal{M}}_g}(-\log D)$ has local frame

$$\left\{ t_1 \frac{\partial}{\partial t_1}, \cdots, t_m \frac{\partial}{\partial t_m}, \frac{\partial}{\partial t_{m+1}}, \cdots, \frac{\partial}{\partial t_n} \right\}.$$

We have proved several results about the goodness of the metrics on moduli spaces. By very subtle analysis on the metric, connection and curvature tensors.

We first proved the following theorem:

Theorem

The metric h^* on the logarithmic cotangent bundle $\overline{E}$ over the DM moduli space induced by the Weil-Petersson metric is good in the sense of Mumford.

Based on the curvature formulae of the Ricci and perturbed Ricci metrics we derived before, we have proved the following theorem from much more detailed and harder analysis: estimates over 80 terms.

Theorem

The metrics on the log tangent bundle $T_{\overline{\mathcal{M}}_g}(-\log D)$ over the DM moduli space induced by the Ricci and perturbed Ricci metrics are good in the sense of Mumford.

A direct corollary is

Theorem

The Chern classes $c_k\left(T_{\overline{\mathcal{M}}_g}(-\log D)\right)$ *are represented by the Chern forms of the Weil-Petersson, Ricci and perturbed Ricci metrics.*

This in particular means we can use the explicit formulas of Chern forms of the Weil-Petersson metric derived by Wolpert to represent the classes, as well as those Chern forms of the Ricci and the perturbed Ricci metrics.

Dual Nakano Negativity of WP Metric

It was shown by Ahlfors, Royden and Wolpert that the Weil-Petersson metric have negative Riemannian sectional curvature.

Schumacher showed that the curvature of the WP metric is strongly negative in the sense of Siu.

In 2005, we showed that the curvature of the WP metric is dual Nakano negative.

Let (E^m, h) be a holomorphic vector bundle with a Hermitian metric over a Kähler manifold (M^n, g). The curvature of E is given by

$$P_{i\bar{j}\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} h_{i\bar{j}} + h^{p\bar{q}} \partial_\alpha h_{i\bar{q}} \partial_{\bar{\beta}} h_{p\bar{j}}.$$

(E,h) is Nakano positive if the curvature P defines a positive form on the bundle $E \otimes T_M$. Namely, $P_{i\bar{j}\alpha\overline{\beta}} C^{i\alpha} \overline{C^{j\beta}} > 0$ for all $n \times n$ complex matrix $C \neq 0$.

E is dual Nakano negative if the dual bundle (E^*, h^*) is Nakano positive. Our result is

Theorem

The Weil-Petersson metric on the tangent bundle $T_{\mathcal{M}_g}$ and on the log tangent bundle $T_{\overline{\mathcal{M}}_g}(-\log D)$ are dual Nakano negative.

To prove this theorem, we only need to show that $(T^*\mathcal{M}_g, h^*)$ is Nakano positive. Let $R_{i\bar{j}k\bar{l}}$ be the curvature of $T\mathcal{M}_g$ and $P_{i\bar{j}k\bar{l}}$ be the curvature of the cotangent bundle.

We first have $P_{m\bar{n}k\bar{l}} = -h^{i\bar{n}} h^{m\bar{j}} R_{i\bar{j}k\bar{l}}$.

If we let $a_{kj} = \sum_m h^{m\bar{j}} C^{mk}$, we then have

$$P_{m\bar{n}k\bar{l}} C^{mk} \overline{C^{nl}} = -\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} a_{ij} \overline{a_{lk}}.$$

Recall that at $X \in \mathcal{M}_g$ we have

$$R_{i\bar{j}k\bar{l}} = -\int_X \left(e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}} \right) dv.$$

By combining the above two formulae, to prove that the WP metric is Nakano negative is equivalent to show that

$$\int_X \left(e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}} \right) a_{ij} \overline{a_{lk}} \; dv > 0.$$

For simplicity, we assume that matrix $[a_{ij}]$ is invertible.

Write $T = (\square + 1)^{-1}$ the Green operator. Recall $e_{i\bar{j}} = T\left(f_{i\bar{j}} \right)$ where $f_{i\bar{j}} = A_i \overline{A_j}$ and A_i is the harmonic representative of the Kodaira-Spencer class of $\frac{\partial}{\partial t_i}$.

Let $B_j = \sum_{i=1}^n a_{ij} A_i$. Then the inequality we need to prove is equivalent to

$$-\sum_{j,k} R(B_j, \overline{B_k}, A_k, \overline{A_j}) =$$

$$\sum_{j,k} \int_X \left(T\left(B_j \overline{A_j}\right) A_k \overline{B_k} + T\left(B_j \overline{B_k}\right) A_k \overline{A_j} \right) dv \geq 0.$$

Let $\mu = \sum_j B_j \overline{A_j}$. Then the first term in the above equation is

$$\sum_{j,k} \int_X T\left(B_j \overline{A_j}\right) A_k \overline{B_k}\, dv = \int_X T(\mu)\overline{\mu}\, dv \geq 0.$$

We then let $G(z, w)$ be the Green's function of the operator T.

Let

$$H(z, w) = \sum_j \overline{A_j}(z) B_j(w).$$

The second term is

$$\sum_{j,k} \int_X T\left(B_j \overline{B_k}\right) A_k \overline{A_j}\, dv =$$

$$= \int_X \int_X G(z, w) H(z, w) \overline{H}(z, w) dv(w) dv(z) \geq 0$$

where the last inequality follows from the fact that the Green's function G positive.

Application

As corollaries of goodness and the positivity or negativity of the metrics, first we directly obtain:

Theorem

The Chern classes of the log cotangent bundle of the moduli spaces of Riemann surfaces are positive.

We have several corollaries about cohomology groups of the moduli spaces:

Theorem

The Dolbeault cohomology of the log tangent bundle $T_{\overline{\mathcal{M}}_g}(-\log D)$ on $\overline{\mathcal{M}}_g$ computed via the singular WP metric g is isomorphic to the ordinary cohomology (or Cech cohomology) of the sheaf $T_{\overline{\mathcal{M}}_g}(-\log D)$.

Here we need the goodness of the metric g induced from the WP metric in a substantial way.

Saper proved that the L^2-cohomology of $\mathcal{M}_g$ of the WP metric h (with trivial bundle $\mathbb{C}$) is the same as the ordinary cohomology of $\overline{\mathcal{M}}_g$. Parallel to his result, we have

Theorem

$$H^*_{(2)}\left((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})\right) \cong H^*(\overline{\mathcal{M}}_g, F).$$

An important and direct application of the goodness of the WP metric and its dual Nakano negativity is the vanishing theorem of L^2-cohomology group.

Theorem

The L^2-cohomology groups

$$H^{0,q}_{(2)}\left((\mathcal{M}_g,\omega_\tau),\left(T_{\overline{\mathcal{M}}_g}(-\log D),\omega_{WP}\right)\right)=0$$

unless $q=n$. Here ω_τ is the Ricci metric.

We put the Ricci metric on the base manifold to avoid the incompleteness of the WP metric. This implies a result of Hacking

$$H^q(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D))=0,\ \ q\neq n.$$

To prove this, we first consider the Kodaira-Nakano identity

$$\Box_{\overline{\partial}}=\Box_\nabla+\sqrt{-1}\left[\nabla^2,\Lambda\right].$$

We then apply the dual Nakano negativity of the WP metric to get the vanishing theorem by using the goodness to deal with integration by part. There is no boundary term.

Remark

- As corollaries, we also have: the moduli space of Riemann surfaces is rigid: no holomorphic deformation.
- We are proving that the KE and Bergman metric are also good metrics and other applications to algebraic geometry and topology.

Idea of Proving Goodness

The proof of the goodness of the WP, Ricci and perturbed Ricci metrics requires very sharp estimates on the curvature and local connection forms of these metrics. We need:

- Different lifts of tangent vectors of $\mathcal{M}_g$ near the boundary divisor which are not harmonic.
- Balance between the use of the rs-coordinates and plumbing coordinates on pinching collars.
- Trace the dependence of error terms.

These give us control on the local connection forms.

- Sharp estimates on the full curvature tensor.

These estimates give us control on the curvature forms.