ISOMETRIC EMBEDDING OF POSITIVE DISKS R^3

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Given a smooth Riemannian manifolds (M^n, g) can we find a map

$$\phi : M^n \mapsto R^q \text{ such that } \phi^* h = g$$

where h is the standard metric in R^q

This is a classical problem. A famous fundamental result is due to Nash.

• Weyl problem :(S^2, g) with $K(g) > 0$, always admits an analytic isometric embedding in R^3 by Weyl-Lewy if $g \in C^\omega$ and a smooth one by Nirenberg-pogorelov if $g \in C^\infty$

•For $K(g) \geq 0$, the results on existence are due to Guan-Li and also to Zuily-Hong independently.

By a positive disk we mean the closed unit disk $\bar{D}$ equipped with a positive curvature metric g, denoted by $(\bar{D}, g)$

Heinze, Jour of Anal.Math, (1966)

Pogorelov, Extrinsic Geometry of convex surf. (1973)

A very important counter example is due to Gromov and Rokhlin

• There is an analytic positive disk not admitting any C^2 isometric immersion in R^3.

To my knowledge, it always admits a smooth isometric embedding provide that

$$k_g \geq 0 \text{ (pogorelov)or } \int K < 4\pi$$

Recently the study in boundary value problems for isometric embedding has attracted much attention of mathematicians

there are two kinds of (BVP).

•Dirichlet Problem(Pogorelov problem);

•Neumann problem.

Neumann Problem

Given $(\bar{D}, g)$ and a function $h \in C^{\infty}(\partial D, R^1)$

$$\vec{r} : \bar{D} \mapsto R^3$$

such that

$$d^2\vec{r} = g, \text{ and } H(\vec{r}) = h \text{ on } \partial D$$

A necessary condition

$$h \geq \sqrt{K} \text{ on } \partial D$$

Assume:

A_0: for the given $(\bar{D}, g)$ there is a smooth isometric immersion $\vec{r}_0$ in R^3

Theorem 1(Hong, Asian J. of Math., 2001) Let A_0 be satisfied for the given $(\bar{D}, g)$ and let $h > \sqrt{K}$ on ∂D. Then for $\forall n \in 0, 1, 2, \ldots$ and for arbitrary $n+1$ points $p_0 \in \partial D, p_1, p_2, \ldots, p_n \in D$, (BVP) always admits two and only two solutions $\vec{r} \in C^\infty$ satisfying

$$Index(\vec{r}) = n$$

$$H(p_k) = H_0(p_k), k = 1, 2, \ldots n$$

and at p_0 the principal direction parallel to ∂D provided that

$$\frac{h}{\sqrt{K}} - 1 > 4 \max \left[\frac{H_0}{\sqrt{K}} - 1 \right] \text{ on } \partial D$$

where H_0 is the mean curvature of $\vec{r}_0$ $(\bar{D}, g)$

Always existence

If $(\bar{D}, g)$ is of constant curvature and $\sqrt{K} < h \in C^\infty(\partial D)$, then for each nonnegative integer n and Then for $\forall n \in 0, 1, 2, ...$ and for arbitrary $n+1$ points $p_0 \in \partial D, p_1, p_2, ..., p_n \in D$, (BVP) always admits two and only two solutions satisfying

$$Index(\vec{r}) = n$$

$$p_k, k = 1, 2, ...n \text{ are umbilic points}$$

$$\text{and at } p_0 \text{ the principal direction parallel to } \partial D$$

Nonexitence

For some $G(r) \in C^\infty([0,1])$,

$$g = dr^2 + G^2(r)d\theta^2$$

$$G(0) = 0, G_r(0) = 1, G > 0$$

$$\text{and moreover,} G_r(1) > -1.$$

Then $(\bar{D}, g)$ has such a smooth isom. embe.

$$\vec{r}_0 : x = G(r)\cos\theta, y = G(r)\sin\theta,$$

$$z = -\int_r^1 \sqrt{1 - G_r^2} dr$$

Denote its mean curvature by $H_0 = H_0(r)$. If $H_0(1) > \sqrt{K(1)}$, then for arbitrary $h \in C^\infty(\partial D)$ satisfying

$$\sqrt{K(1)} \leq h < H_0(1) \implies$$

(NP) has no any C^2 solution.

Dircichlet problem(Pogorelov Problem)

Given a smooth complete surface Σ in R^3 and $(\bar{D}, g)$, to find a map $\vec{r}: \bar{D} \longrightarrow$ into R^3 such that

$$d^2\vec{r} = g, \text{ and}$$

$$\vec{r}(\partial D) \subset \Sigma \text{on } \partial D$$

Σ is the plane $z = 0$

DP : To find an isometric embedding $\vec{r} = (x, y, z)$ of the given positive disk $(\bar{D}, g)$,such that $z(\partial D) = 0$

$$det(\nabla_{ij} z) = K \det(g_{ij})(1 - |\nabla z|^2) \text{ in } D$$

with

$$z = 0 \text{ on } \partial D \text{ and } |\nabla z| < 1 \text{ on } \bar{D}$$

Theorem(Pogorelov) (DP) always admits a solution $\vec{r} \in C^{\infty}(D) \cap C^{0.1}(\bar{D})$ provided that the geodesic curvature of ∂D with respect to the metric g is nonnegative.

- **Under what cond, $\exists$ a solu in $C^{\infty}(\bar{D})$?**

Theorem 2 (Hong, Chin.Ann of Math., 1999) If $k_g > 0$ on ∂D, (DP) always admits a unique solution in $C^{\infty}(\bar{D})$ if one of the following assumptions is satisfied (1) $K > 0$ on $\bar{D}$, (2) $K > 0$ in D and $K = 0 \neq |dK|$ on ∂D.

Necessary condition:

$$k^2 = k_g^2 + k_n^2 \text{ on } \partial D$$

$$\int |k_g| < \int k = 2\pi$$

Proposition(Hong, ICCM 1998)
There is a smooth positive disk not admitting any C^2 solution to the above boundary value problem

Remark too many changes of the sign of k_g might make the problem unsolvable !

$$\text{Case } a : k_g > 0 \text{ and Case } b : k_g < 0$$

- **Does $(\bar{D}, g)$ in case b satisfying necessary condition always admit a global smooth solution ?**

The situation is very complicated!Indeed,

- **(Hong Proceeding of ICCM)**There is a convex surface which is a smooth isometric embedding of a positive disk in case b satisfying (NC) but not infinitesimally rigid

If $(\bar{D}, g)$ is in Case b and if there is a C^2 solution $\vec{r}$ to (DP), its Gauss map is injective and hence,

$$\int K \leq 4\pi$$

Theorem 3 (Li,Han,Hong)

(1) Any smooth $(\bar{D}, g)$ in Case b satisfying

$$\int K < 4\pi$$

always admits a solution in $\in C^\infty(D)$. If

(2)Any smooth $(\bar{D}, g)$ satisfying

$$\int K = 4\pi, K > 0 \text{ in } D,$$

$$K = 0 \neq dK, \text{ on } \partial D$$

admits a solution in $C^\infty(\bar{D})$ if and only if its geodesic curvature k_g is the curvature of a planar convex curve and moreover, solution is unique.

Remark Aleksandrov condition

Torus like surface:

Given a smooth closed surface M in R^3.

$$M^{\pm} = \{p \in M | K(p) > 0 (< 0)\} \text{ and}$$

$$M^0 = \{p \in M | K(p) = 0\}$$

Suppose that

$$\int_{M^+} K = 4\pi, K > 0 \text{ in } D,$$

$$K = 0 \neq dK, \text{ on } \partial D$$

We call such surface Torus like surface first studied by

Theorem(Aleksandrof) Torus like surface is global rigidity and moreover, M^0 must be composed of some planar convex curves,.,each component of $M^0 \subset \Pi$ for a plane Π which is tangent plane along this component