Contact Discontinuities for Gas Motion

Feimin Huang

fhuang@amt.ac.cn

Academy of Mathematics & Systems Science, CAS,
Beijing 100080, P. R. China

♠ **Hyperbolic conservation laws**

$$u_t + f(u)_x = 0, \ u \in R^n. \qquad (1)$$

All eigenvalues of $\nabla f(u)$ are real.

Three elementary hyperbolic waves:

- **Shock wave**
- **Rarefaction wave**
- **Contact discontinuity**

The typical hyperbolic system is Euler equations.

$$v_t - u_x = 0, \tag{2}$$
$$u_t + p_x = 0, \tag{3}$$
$$(e + \frac{u^2}{2})_t + (pu)_x = 0, \tag{4}$$

v: **specific volume,**
u: **velocity,**
θ: **temperature,**
p: **pressure,**
e: **internal energy.**

Polytropic gas case:

$$p = R\theta/v, \tag{5}$$

$$e = \frac{R}{\gamma - 1}\theta + \text{const.}. \tag{6}$$

$R > 0$: the gas constant
$\gamma > 1$: the adiabatic exponent.

The Euler system includes all three elementary hyperbolic waves: Shock wave, rarefaction wave, contact discontinuity.

♠ **Viscous conservation laws**

$$u_t + f(u)_x = (B(u)u_x)_x, \; u \in R^n. \tag{7}$$

All eigenvalues of $\nabla f(u)$ are real and $B(u) \geq 0$.

A typical artificial viscosity system is

$$u_t + f(u)_x = u_{xx}, \; u \in R^n. \tag{8}$$

The typical physical model is the Compressible N-S equations

$$v_t - u_x = 0, \tag{9}$$

$$u_t + p(v, \theta)_x = \mu(\frac{u_x}{v})_x, \tag{10}$$

$$(e + \frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x, \tag{11}$$

v: specific volume,
u: velocity,
θ: temperature,
p: pressure,
e: internal energy,
$\nu > 0$: viscosity constant,
$\kappa > 0$: heat conductivity.

♠ Stability of the hyperbolic waves

The basic waves and their linear combinations, called Riemann solutions, govern both local and large time asymptotic behavior of general solutions to the inviscid hyperbolic system (1). Since the inviscid system (1) is an idealization when the dissipative effects are neglected, thus it is of great importance to study the large time asymptotic behavior of solutions to the corresponding viscous systems, such as (7), toward the viscous versions of these basic waves.

• **Shock and rarefaction wave:**

Since 1985 initiated by Goodman and Matsumura-Nishihara, deeper understanding has been achieved on the asymptotic stability toward nonlinear waves, viscous shock profiles and viscous rarefaction waves, which have been shown to be nonlinearly stable for quite general perturbation for general viscous hyperbolic system, and new phenomena have been discovered and new techniques, such as weighted characteristic energy methods and uniform approximate Green's functions, have been developed based on the intrinsic properties of the underlying nonlinear waves, see

Goodman, Kawashima, Liu, Matsumura, Nishihara, Szepessy, Xin, Zumbrun, $\cdots$,

♣ **Contact discontinuity:**

• **Artificial viscosity:**

[Xin, 1994] and [Liu,Xin, Asian J. Math., 1997].

In fact, the studies on the stability of contact discontinuity began with [Xin, 1994]. In [Liu,Xin, Asian J. Math., 1997], a general hyperbolic system of a uniform artificial viscosity is investigated, i.e.,

$$u_t + f(u)_x = u_{xx}, \tag{12}$$

with a structure condition. The compressible Euler equations with uniformly viscosity naturally satisfies the structure condition.

It is shown that although the contact discontinuity to the inviscid hyperbolic system is not an asymptotic attractor for the viscous system, yet a viscous wave, which approximates the contact discontinuity on any finite time interval, is asymptotically nonlinear stable for small generic perturbations and the detail asymptotic behavior can be determined a priorily by initial mass distribution. The pointwise asymptotic behavior toward viscous contact wave by approximate fundamental solutions was also obtained. This also leads to the nonlinear stability of the viscous contact wave in L^p-norms for all $p \geq 1$.

However, the theory in [Liu,Xin,1997] and [Xin,1994] does not apply to the compressible Navier-Stokes system since its viscosity matrix is only semi-positive definite.

• Physical system:

[Huang-Matumura-Shi, Osaka J. Math.,2004] and [Huang-Zhao, Rend. Sem.Mat.Univ.Padova, 2003]: free boundary (N-S system)

[Huang-Matsumura-Xin, Arch.Rat.Mech.Anal., 2005]: Cauchy problem with zero mass condition (N-S system).

[Huang-Yang, J. Diff. Equa., 2006]: Cauchy problem with zero mass condition (Boltzmann equation).

Question: How about the stability of contact discontinuity for N-S system and Boltzmann equation with general perturbation? It is a long standing open problem.

Recently Zhouping Xin, Tong Yang and myself solved this open problem, i.e., we successfully obtained the nonlinear stability of contact wave of the compressible Navier-Stokes system and the Boltamann equation for the general initial perturbations. The decay rate is also obtained.

Today's talk is based on the work of [Huang-Xin-Yang].

♠ Main difficulties:

1. The contact discontinuities are associated with linear degenerate fields and are less stable compared with the nonlinear waves for the inviscid Euler system. Thus the stabilizing effects around a contact discontinuity for Navier-Stokes equations come mainly from the viscosity and heat conductivity. In another word, unlike the shock and rarefaction wave, the space derivative of the viscous contact wave changes sign, while the definite sign plays the leading role for the stability analysis of shock and rarefaction wave.

2. A general perturbation of a contact wave may introduce waves in the nonlinear sound wave families, and interactions of these waves with the linear contact wave are some of the major difficulties to overcome.

3. The viscosity matrix is only semi-positive definite.

Consider the compressible N-S equations

$$v_t - u_x = 0, \tag{13}$$

$$u_t + p(v,\theta)_x = \mu(\frac{u_x}{v})_x, \tag{14}$$

$$(e + \frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x. \tag{15}$$

Here we consider the polytropic gas case, i.e., $p = R\theta/v$, $e = \frac{R}{\gamma-1}\theta + \text{const.}$.

The initial data (v_0, u_0, θ_0) satisfies $p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}$ and

$$(v_0, u_0, \theta_0) \to (v_+, 0, \theta_+), \text{ as } x \to +\infty. \tag{16}$$

• **Contact discontinuity for Euler equations.**

Consider the Euler equations with the Riemann initial data

$$v_t - u_x = 0, \tag{17}$$
$$u_t + p(v,\theta)_x = 0, \tag{18}$$
$$(e + \frac{u^2}{2})_t + (pu)_x = 0, \tag{19}$$
$$(v,u,\theta)(x,0) = (v_-, 0, \theta_-),\ x < 0, \tag{20}$$
$$(v,u,\theta)(x,0) = (v_+, 0, \theta_+),\ x > 0. \tag{21}$$

If

$$p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+} \tag{22}$$

holds, then there is a contact discontinuity

$$(\bar{V}, \bar{U}, \bar{\Theta}) = (v_-, 0, \theta_-), \ x < 0, \tag{23}$$

$$(\bar{V}, \bar{U}, \bar{\Theta}) = (v_+, 0, \theta_+), \ x > 0. \tag{24}$$

- **Viscous contact wave:**

Let $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$ **be the viscous contact wave. we expect**

$$\bar{p}(\bar{v}, \bar{\theta}) = \frac{R\bar{\theta}}{\bar{v}} \approx p_+, \ \bar{u} << 1, \text{ as } t \to \infty. \tag{25}$$

Note that the third equation of N-S system can be reduced to

$$\frac{R}{\gamma - 1}\theta_t + p(v, \theta)u_x = (\kappa\frac{\theta_x}{v})_x + \mu\frac{u_x^2}{v}. \tag{26}$$

We expect that the leading part is

$$\frac{R}{\gamma - 1}\theta_t + p_+ u_x = (\kappa \frac{\theta_x}{v})_x. \tag{27}$$

By the conservation of mass

$$v_t - u_x = 0, \tag{28}$$

we obtain a nonlinear diffusion equation

$$\Theta_t = a(\frac{\Theta_x}{\Theta})_x, \ a = \frac{\kappa p_+(\gamma - 1)}{\gamma R^2} > 0. \tag{29}$$

It is easy to check that there exists a unique self-similar solution $\Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$ with the condition $\Theta(-\infty, t) = \theta_-, \Theta(+\infty, t) = \theta_+$. This profile $\Theta(\xi)$ is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$.

After $\Theta(x,t) = \Theta(\xi)$ is obtained, we define

$$\bar{v} = \frac{R}{p_+}\Theta(x,t), \tag{30}$$

$$\bar{u} = \frac{Ra}{p_+\Theta}\Theta_x, \tag{31}$$

$$\bar{\theta} = \Theta - \frac{\gamma - 1}{2R}\bar{u}^2. \tag{32}$$

It is easy to see $(\bar{v}, \bar{u}, \bar{\theta})$ satisfies

$$\|(\bar{v} - \bar{V}, \bar{u} - \bar{U}, \bar{\theta} - \bar{\Theta})\|_{L^p} = [O(\kappa)(1+t)]^{\frac{1}{2p}}, p \geq 1 \tag{33}$$

which means the nonlinear wave $(\bar{v}, \bar{u}, \bar{\theta})$ approximates the contact discontinuity $(\bar{V}, \bar{U}, \bar{\Theta})$ in L^p norm, $p \geq 1$ on any finite time interval as κ tends to zero.

The viscous contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ satisfies

$$\bar{v}_t - \bar{u}_x = 0, \tag{34}$$

$$\bar{u}_t + \bar{p}_x = \mu(\frac{\bar{u}_x}{\bar{v}})_x + R_{1x}, \tag{35}$$

$$(\bar{e} + \frac{\bar{u}^2}{2})_t + (\bar{p}\bar{u})_x = \kappa(\frac{\bar{\theta}_x}{\bar{v}})_x + (\frac{\mu}{\bar{v}}\bar{u}\bar{u}_x)_x + R_{2x}, \tag{36}$$

where

$$R_1 = O(\delta)(1+t)^{-1} e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, \tag{37}$$

$$R_2 = O(\delta)(1+t)^{-\frac{3}{2}} e^{-\frac{\theta_\pm x^2}{4a(1+t)}}. \tag{38}$$

Here $\delta = |\theta_+ - \theta_-|$ is the strength of the contact discontinuity.

Let

$$m(x,t) = (v, u, \theta + \frac{\gamma - 1}{2R} u^2)^t, \tag{39}$$

$$\bar{m}(x,t) = (\bar{v}, \bar{u}, \bar{\theta} + \frac{\gamma - 1}{2R} \bar{u}^2)^t, \tag{40}$$

Since the integral $\int_{-\infty}^{\infty} m(x,0) - \bar{m}(x,0) dx$ is not zero, we have to introduce additional diffusion wave to carry the excessive mass in the nonlinear sound families. Let $A(v,u,\theta)$ be the Jacobi matrix of the flux $(-u, p, \frac{\gamma-1}{R} pu)^t$. The first eigenvalue of $A(v_-, 0, \theta_-)$ is $\lambda_1^- = -\sqrt{\frac{\gamma p_-}{v_-}}$ and the corresponding right eigenvector is

$$r_1^- = (-1, \lambda_1^-, \frac{\gamma - 1}{R} p_-)^t. \tag{41}$$

Similarly, the third eigenvalue and right eigenvector of

$A(v_+, 0, \theta_+)$ are $\lambda_3^+ = \sqrt{\frac{\gamma p_+}{v_+}}$ and

$$r_3^+ = (-1, \lambda_3^+, \frac{\gamma - 1}{R} p_+)^t. \quad (42)$$

The vectors $r_1^-, r_2 = (v_+ - v_-, 0, \theta_+ - \theta_-)^t$ and r_3^+ are linearly independent in R^3. Assume that

$$\int m(x, 0) - \bar{m}(x, 0) dx = \bar{\theta}_1 r_1^- + \bar{\theta}_2 r_2 + \bar{\theta}_3 r_3^+ \quad (43)$$

with some constants $\bar{\theta}_i, i = 1, 2, 3$. We define the ansatz $\tilde{m}(x, t)$ as follows:

$$\tilde{m}(x, t) = \bar{m}(x + \bar{\theta}_2, t) + \bar{\theta}_1 \theta_1 r_1^- + \bar{\theta}_3 \theta_3 r_3^+, \quad (44)$$

where

$$\theta_1(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}}, \tag{45}$$

$$\theta_3(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{46}$$

satisfying

$$\theta_{1t} + \lambda_1^- \theta_{1x} = \theta_{1xx}, \tag{47}$$

$$\theta_{3t} + \lambda_3^+ \theta_{3x} = \theta_{3xx}, \tag{48}$$

and $\int_{-\infty}^{\infty} \theta_i(x,t)dx = 1$ for $i = 1, 3$ and all $t \geq 0$. We have

$$\int_{-\infty}^{\infty} m(x,0) - \tilde{m}(x,0)dx = 0 \tag{49}$$

Without loss generality, we assume that $\bar{\theta}_2 = 0$. By a straightforward computation, the ansatz $\tilde{m}$ satisfies

$$\tilde{v}_t - \tilde{u}_x = \tilde{R}_{1x}, \tag{50}$$

$$\tilde{u}_t + \tilde{p}_x = \mu(\frac{\tilde{u}_x}{\tilde{v}})_x + \tilde{R}_{2x}, \tag{51}$$

$$(\tilde{e} + \frac{\tilde{u}^2}{2})_t + (\tilde{p}\tilde{u})_x = \kappa(\frac{\tilde{\theta}_x}{\tilde{v}})_x + (\frac{\mu}{\tilde{v}}\tilde{u}\tilde{u}_x)_x + \tilde{R}_{3x}, \tag{52}$$

where $\tilde{m} = (\tilde{v}, \tilde{u}, \tilde{\theta})$ and for $i = 1, 2, 3,$

$$|\tilde{R}_i| = O(\delta + \bar{\theta}_1^2 + \bar{\theta}_3^2)\frac{1}{1+t}(e^{-\frac{cx^2}{1+t}} \tag{53}$$

$$+e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}} + e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}}). \tag{54}$$

Remark: The decay rate of error terms $\tilde{R}_i$ (53) is not enough for the usual energy estimate. People also meet the same problem in the study of viscous shock wave. So Szepessy and Xin in 1993 introduced coupled diffusion waves to improve the decay rate of error terms like $\tilde{R}_i$ for general artificial viscous system. However our results below will show that the decay rate of error terms $\tilde{R}_i$ in (53) is suitable for our new method for the stability of viscous contact wave. That is the energy method can be applied to capture the coupling of the viscous contact wave with the diffusion waves created by the perturbation in the other two characteristic families so that a priori estimate can be closed with a convergence rate on the solution to the wave profile time asymptotically.

Let

$$\phi(x,t) = v - \tilde{v}, \psi(x,t) = u - \tilde{u}, \zeta(x,t) = \theta - \tilde{\theta}. \tag{55}$$

Set

$$\Phi(x,t) = \int_{-\infty}^{x} \phi dy, \Psi(x,t) = \int_{-\infty}^{x} \psi dy, \tag{56}$$

$$\bar{W}(x,t) = \int_{-\infty}^{x} (e + \frac{|u|^2}{2} - \tilde{e} - \frac{|\tilde{u}|^2}{2}) dy. \tag{57}$$

The quantities $(\Phi, \Psi, \bar{W})$ are well defined in some Sobolev space because $(\Phi, \Psi, \bar{W})(\pm\infty, 0) = 0$.

Theorem 1. **Let $(\tilde{v}, \tilde{u}, \tilde{\theta})(x, t)$ be the ansatz above and $\delta = |\theta_+ - \theta_-|$. Then there exist some positive constants δ_0 and ϵ, such that if $\delta \le \delta_0$ and the initial data (v_0, u_0, θ_0) satisfies**

$$\|(\Phi, \Psi, \bar{W})\|_{L^2} + \|m - \bar{m}\|_{H^1} \le \epsilon, \tag{58}$$

then the compressible Navier-Stokes system admits a unique global solution (v, u, θ) satisfying

$$(\Phi, \Psi, \bar{W}) \in C(0, +\infty; H^2), \tag{59}$$

$$\phi \in L^2(0, +\infty; H^1), \tag{60}$$

$$(\psi, \zeta) \in L^2(0, +\infty; H^2). \tag{61}$$

Furthermore, the perturbation $(\Phi, \Psi, \bar{W})$ and (ϕ, ψ, ζ)

have the following decay rate,

$$\|(v-\tilde{v}, u-\tilde{u}, \theta-\tilde{\theta}\|_{L^\infty} \leq C(\epsilon+\delta_0^{\frac{1}{2}})(1+t)^{-\frac{1}{4}}. \tag{62}$$

Remark 1: Theorem 1 shows not only that the viscous contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ is nonlinear stable in super-norm with generic initial perturbations, but also a uniform rate of convergence (62), is obtained. This is somewhat surprising given that the convergence rate to either the viscous shock wave or viscous rarefaction wave has not been achieved yet for the compressible Navier-Stokes system. Moreover, the rate of decay in (62) may not be optimal. Motivated by the pointwise behavior toward viscous contact waves for solutions to the Euler system with uniform viscosity [Liu,Xin,1997], one would conjecture that its de-

cay rate in (62) should be improved to $(1+t)^{-\frac{1}{2}}$.

♠ **Energy method.**

A priori assumptions:

$$N(T) = \sup_{0\le t\le T} \{\|(\Phi, \Psi, \bar{W})\|^2_{L^\infty} + \|(\phi, \psi, \zeta)\|^2_{H^1}\} \le \varepsilon_0^2, \tag{63}$$

where ε_0 is positive small constant depending on the initial data and the strength of the contact wave.

Key observations: By some complicated estimates, finally we obtain

$$E_{1t} + K_1 \le C\delta(1+t)^{-1}E_1 + C\delta(1+t)^{-\frac{1}{2}}, \tag{64}$$

were

$$E_1 \approx \|(\Phi, \Psi, \bar{W})\|^2_{H^2}, \tag{65}$$

$$K_1 \approx \|\Phi_x\|_{H^1}^2 + \|(\Psi_x, \bar{W}_x)\|_{H^2}^2. \qquad (66)$$

$C\delta(1+t)^{-1}E_1$ and $C\delta(1+t)^{-\frac{1}{2}}$ come from the terms $\int \bar{u}_x(\Psi^2+\bar{W}^2)dx$ and $\tilde{R}_i$, $i=1,2,3$ respectively. Thus we have

$$E_1 + \int_0^t K_1 dt \leq C(\delta+\epsilon^2)(1+t)^{\frac{1}{2}} \qquad (67)$$

or

$$\|(\Phi, \Psi, \bar{W})\|_{L^2}^2 \leq C(\delta+\epsilon^2)(1+t)^{\frac{1}{2}} \qquad (68)$$

which seems not good. However, when considering the derivative estimate, we have

$$E_{2t} + K_2 \leq C\delta(1+t)^{-1}K_1 + C\delta(1+t)^{-\frac{3}{2}}, \qquad (69)$$

$$E_2 \approx \|(\Phi_x, \Psi_x, \bar{W}_x)\|_{H^1}^2, \qquad (70)$$

$$K_2 \approx \|\Phi_{xx}\|_{L^2}^2 + \|(\Psi_{xx}, \bar{W}_{xx})\|_{H^1}^2. \tag{71}$$

We obtain

$$E_2 \leq C(\delta + \epsilon^2)(1+t)^{-\frac{1}{2}}, \tag{72}$$

$$\|(\Phi_x, \Psi_x, \bar{W}_x)\|_{H^1}^2 \leq C(\delta + \epsilon^2)(1+t)^{-\frac{1}{2}}. \tag{73}$$

Due to Sobolev inequality, (68) and (73) imply that

$$\|(\Phi, \Psi, \bar{W})\|_{L^\infty} \leq C(\delta^{\frac{1}{2}} + \epsilon) \tag{74}$$

which verifies the a priori assumption (63). From (73), we obtain the decay rate (62).

Remark 2: Because this approach allows that the L^2 norm of $(\Phi, \Psi, \bar{W})$ grows with the rate $(1+t)^{\frac{1}{2}}$, it is not necessary to require the accurate ansatz.

♣ **Boltzmann equation:**

$$f_t + \xi_1 f_x = Q(f, f), \tag{75}$$

$f(x, t, \xi)$: **distributional density of particles.**
ξ: **particle velocity.**
Q: **collision operator.**

$$Q(f, g) = \frac{1}{2} \int_{\mathrm{R}^3} \int_{\mathrm{S}^2_+} \Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') \tag{76}$$

$$-f(\xi)g(\xi_*) - f(\xi_*)g(\xi) \Big) B(|\xi - \xi_*|, \theta)\ d\xi_* d\Omega, \tag{77}$$

with θ **being the angle between the relative velocity and the unit vector** Ω. **Here** $\mathrm{S}^2_+ = \{\Omega \in \mathrm{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0\}$. **The conservation of momentum and energy**

gives the following relation between velocities before and after collision:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\, \Omega, \tag{78}$$

$$\xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\, \Omega. \tag{79}$$

Hard sphere:

$$B(|\xi - \xi_*|, \theta) = |(\xi - \xi_*, \Omega)|, \tag{80}$$

$$B(|\xi - \xi_*|, \theta) = |\xi - \xi_*|^{\frac{n-5}{n-1}} b(\theta), \tag{81}$$

$$b(\theta) \in L^1([0, \pi]),\ n > 5, \tag{82}$$

Macro-Micro decomposition (Liu-Yang-Yu, 2004), (Liu-Yu, 2004):

$$f(x, t, \xi) = M(x, t, \xi) + G(x, t, \xi), \tag{83}$$

where the local Maxwellian M and G represent the fluid and non-fluid components in the solution respectively.

$$M \equiv \frac{\rho}{\sqrt{(2\pi R\theta^3}} \exp\left(-\frac{|\xi - u|^2}{2R\theta}\right). \tag{84}$$

The macroscopic space is spanned by the following five pairwise orthogonal functions

$$\chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}} M, \tag{85}$$

$$\chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}} M \ \text{ for } \ i = 1, 2, 3, \tag{86}$$

$$\chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}} (\frac{|\xi - u|^2}{R\theta} - 3) M, \tag{87}$$

$$< \chi_i, \chi_j >= \delta_{ij}, \ i, j = 0, 1, 2, 3, 4. \tag{88}$$

Macroscopic projection P_0:

$$P_0 h \equiv \sum_{j=0}^{4} < h, \chi_j > \chi_j. \tag{89}$$

where the inner product is defined by $< f, g >= \int \frac{fg}{M} d\xi$.
Microscopic projection P_1:

$$P_1 h \equiv h - P_0 h. \tag{90}$$

Under this decomposition, the Boltzmann equation can be rewritten as

$$\rho_t + (\rho u_1)_x = 0, \tag{91}$$

$$(\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4}{3}(\mu(\theta) u_{1x})_x - \int \xi_1^2 \Theta_x d\xi, \tag{92}$$

$$(\rho u_i)_t + (\rho u_1 u_i)_x = (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Theta_x d\xi, \; i = 2, 3, \tag{93}$$

$$(\rho(e + \frac{|u|^2}{2}))_t + (\rho u_1(e + \frac{|u|^2}{2}) + p u_1)_x = (\lambda(\theta)\theta_x)_x \tag{94}$$

$$+ \frac{4}{3}(\mu(\theta) u_1 u_{1x})_x + \sum_{i=2}^{3} (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_x d\xi, \tag{95}$$

together with the equation for the non-fluid component G:

$$G_t + P_1(\xi_1 M_x) + P_1(\xi_1 G_x) = L_M G + Q(G, G). \tag{96}$$

Thus

$$G = L_M^{-1}(P_1(\xi_1 M_x)) + \Theta$$

with

$$\Theta = L_M^{-1}(G_t + P_1(\xi_1 G_x) - Q(G, G)). \qquad (97)$$

Here L_M is the linearized operator of the collision operator with respect to the local Maxwellian M:

$$L_M h = Q(M, h) + Q(h, M).$$

Lagrangian **coordinates:**

$$x \Rightarrow \int_0^x \rho(y, t)dy, \qquad t \Rightarrow t. \qquad (98)$$

Boltzmann equation

$$v_t - u_{1x} = 0, \qquad (99)$$

$$u_{1t} + p_x = \frac{4}{3}(\frac{\mu(\theta)}{v}u_{1x})_x - \int \xi_1^2 \Theta_{1x} d\xi, \quad (100)$$

$$u_{it} = (\frac{\mu(\theta)}{v}u_{ix})_x - \int \xi_1 \xi_i \Theta_{1x} d\xi, \; i = 2, 3 \quad (101)$$

$$(e + \frac{|u|^2}{2})_t + (pu_1)_x = (\frac{\lambda(\theta)}{v}\theta_x)_x + \frac{4}{3}(\frac{\mu(\theta)}{v}u_1 u_{1x})_x \quad (102)$$

$$+ \sum_{i=2}^{3}(\frac{\mu(\theta)}{v}u_i u_{ix})_x - \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_{1x} d\xi, \quad (103)$$

$$G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 M_x) + \frac{1}{v}P_1(\xi_1 G_x) \quad (104)$$

$$= L_M G + Q(G, G), \quad (105)$$

with

$$G = L_M^{-1}(\frac{1}{v}P_1(\xi_1 M_x)) + \Theta_1,$$

$$\Theta_1 = L_M^{-1}(G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 G_x) - Q(G, G)).$$

Viscous contact wave:

Analogous to N-S system, we construct the ansatz for the wave profile of the Boltzmann equation as follows. First, let $\Theta(\frac{x}{\sqrt{1+t}})$ be the unique self-similarity solution of the following nonlinear diffusion equation

$$\Theta_t = (a(\Theta)\Theta_x)_x, \Theta(-\infty, t) = \theta_-, \Theta(+\infty, t) = \theta_+, \tag{106}$$

where the function $a(s) = \frac{9p_+\lambda(s)}{10s} > 0$. We then define

$$\bar{v} = \frac{2}{3p_+}\Theta, \bar{u}_1 = \frac{2a(\Theta)}{3p_+}\Theta_x, \bar{u}_i = 0,\ i = 2, 3, \tag{107}$$

$$\bar{\theta} = \Theta - \frac{1}{2}|\bar{u}|^2. \tag{108}$$

Let $m = (v, u_1, \theta + \frac{1}{2}|u|^2)$ and $\bar{m} = (\bar{v}, \bar{u}_1, \bar{\theta} + \frac{1}{2}|\bar{u}|^2)$. Since $\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx$ is usually not zero, we have to introduce two diffusion waves in the sound wave families like N-S system. Let A_- and A_+ be the Jacobi matrices of the flux $(-u_1, p, pu_1)^t$ at $(v_-, 0, \theta_-)$ and $(v_+, 0, \theta_+)$ respectively. It is easy to check that $\lambda_1^- = -\sqrt{\frac{5p_-}{3v_-}}$ is the first eigenvalue of A_- with $r_1^- = (-1, \lambda_1^-, p_-)^t$ being the corresponding eigenvector. And $\lambda_3^+ = \sqrt{\frac{5p_+}{3v_+}}$ and $r_3^+ = (-1, \lambda_3^+, p_+)^t$ are those of the third family of A_+. Since $r_1^-, (v_+ - v_-, 0, \theta_+ - \theta_-)^t$ and r_3^+ are linearly independent in R^3, we have

$$\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx = \bar{\theta}_1 r_1^- \qquad (109)$$

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

$$+\bar{\theta}_2(v_+ - v_-, 0, \theta_+ - \theta_-)^t + \bar{\theta}_3 r_3^+ \tag{110}$$

with unique constants $\bar{\theta}_i$, $i = 1, 2, 3$. The ansatz $\tilde{m}(x,t)$ for m is defined as

$$\tilde{m}(x,t) = \bar{m}(x + \bar{\theta}_2, t) + \bar{\theta}_1 \theta_1 r_1^- + \bar{\theta}_3 \theta_3 r_3^+, \tag{111}$$

where

$$\theta_1(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}}, \tag{112}$$

$$\theta_3(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{113}$$

Thus we have $\int_{-\infty}^{\infty}(m(x,0) - \tilde{m}(x,0))dx = 0$. Notice that $\int_{-\infty}^{\infty} u_i(x,0)dx$ may not be zero either for $i = 2, 3$.

For this, we define

$$\bar{u}_i(x,t) = \bar{\theta}_{i+2}\frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{x^2}{4(1+t)}},\ i = 2,3, \quad (114)$$

where $\bar{\theta}_{i+2} = \int_{-\infty}^{\infty} u_i(x,0)dx$. It is obvious that

$$\int_{-\infty}^{\infty}(u_i(x,0) - \tilde{u}_i(x,0))dx = 0, i = 2,3. \quad (115)$$

Finally, our ansatz is defined as

$$\tilde{v}(x,t) = \bar{v}(x+\bar{\theta}_2,t) - \bar{\theta}_1\theta_1 - \bar{\theta}_3\theta_3, \quad (116)$$

$$\tilde{u}_1(x,t) = \bar{u}_1(x+\bar{\theta}_2,t) + \lambda_1^-\bar{\theta}_1\theta_1 + \lambda_3^+\bar{\theta}_3\theta_3, \quad (117)$$

$$\tilde{u}_i = \frac{\bar{\theta}_{i+2}}{\sqrt{4\pi(1+t)}}e^{-\frac{x^2}{4(1+t)}},\ i = 2,3, \quad (118)$$

$$\tilde{\theta}(x,t) = \bar{\theta}(x+\bar{\theta}_2,t) + \frac{1}{2}|\bar{u}|^2(x+\bar{\theta}_2,t) \quad (119)$$

$$+ p_+(\bar{\theta}_1\theta_1 + \bar{\theta}_3\theta_3) - \frac{1}{2}|\tilde{u}|^2. \quad (120)$$

Here $(\tilde{v}, \tilde{u}, \tilde{\theta})$ satisfies

$$\tilde{m}(x,t) = (\tilde{v}, \tilde{u}_1, \tilde{\theta} + \frac{1}{2}|\tilde{u}|^2)^t(x,t). \quad (121)$$

Without loss of generality, we also assume that $\bar{\theta}_2 = 0$. It is straightforward to show that

$$\tilde{v}_t - \tilde{u}_x = \tilde{R}_{1x}, \quad (122)$$

$$\tilde{u}_{1t} + \tilde{p}_x = \frac{4}{3}(\mu(\tilde{\theta})\frac{\tilde{u}_{1x}}{\tilde{v}})_x + \tilde{R}_{2x}, \quad (123)$$

$$\tilde{u}_{it} = (\frac{\mu(\tilde{\theta})\tilde{u}_{ix}}{\tilde{v}})_x + (\tilde{R}_{i+1})_x, i = 2, 3, \quad (124)$$

$$(\tilde{e}+\frac{|\tilde{u}|^2}{2})_t+(\tilde{p}\tilde{u}_1)_x=(\lambda(\tilde{\theta})\frac{\tilde{\theta}_x}{\tilde{v}})_x+\frac{4}{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_1\tilde{u}_{1x})_x \tag{125}$$

$$+(\sum_{i=2}^{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_i\tilde{u}_{ix})_x+\tilde{R}_{5x}, \tag{126}$$

$$\tilde{R}_i=O(\delta+\sum_{i=1}^{5}|\bar{\theta}_i|)\frac{1}{1+t}(e^{-\frac{cx^2}{1+t}} \tag{127}$$

$$+e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}}+e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}}) \tag{128}$$

holds with some positive constant $c>0$. Denote the perturbation around the ansatz $(\tilde{v},\tilde{u},\tilde{\theta})$ by

$$\phi(x,t)=v-\tilde{v},\quad \psi(x,t)=u-\tilde{u},\quad \zeta(x,t)=\theta-\tilde{\theta}.$$

Then set

$$\Phi(x,t) = \int_{-\infty}^{x} \phi(y,t)dy, \Psi(x,t) = \int_{-\infty}^{x} \psi(y,t)dy, \tag{129}$$

$$\bar{W}(x,t) = \int_{-\infty}^{x} (e + \frac{|u|^2}{2} - \tilde{e} - \frac{|\tilde{u}|^2}{2})(y,t)dy, \tag{130}$$

so that the quantities Φ, Ψ ad $\bar{W}$ can be well defined in some Sobolev space.

Theorem 2. Let $(\tilde{v}, \tilde{u}, \tilde{\theta})(x,t)$ be the ansatz defined above with $\delta = |\theta_+ - \theta_-|$. Then there exist small positive constants δ_0, ϵ and global Maxwellian $M_* = M_{[\rho_*, u_*, \theta_*]}$, such that if $\delta \le \delta_0$ and the initial data satisfies

$$\{\|(\Phi, \Psi, \bar{W})\|_{H_x^2} + \sum_{|\alpha|=2} \|\partial^\alpha f\|_{L_x^2(L_\xi^2(\frac{1}{\sqrt{M_*}}))} \tag{131}$$

$$+ \sum_{0\le|\alpha|\le 1} \|\partial^\alpha G\|_{L_x^2(L_\xi^2(\frac{1}{\sqrt{M_*}}))}\}|_{t=0} \le \epsilon, \tag{132}$$

then the Boltzmann equation admits a unique global solution $f(x, t, \xi)$ satisfying

$$\|f(x, t, \xi) - M_{[\bar{v},\bar{u},\bar{\theta}]}\|_{L_x^\infty(L_\xi^2(\frac{1}{\sqrt{M_*}}))} \le C(\epsilon + \delta_0^{\frac{1}{2}})(1 + t)^{-\frac{1}{4}}. \tag{133}$$

Here $f(\xi) \in L_\xi^2(\frac{1}{\sqrt{M_*}})$ means that $\frac{f(\xi)}{\sqrt{M_*}} \in L_\xi^2(\mathrm{R}^3)$.

Thank you!