Generalized Ricci flow: Local existence and uniqueness

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1. Introduction

- **In 1982, R. Hamilton introduced the Ricci flow**

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

to construct canonical metrics for some manifolds.

- **Hamilton, Yau, Perelman and other mathematicians developed many tools and techniques to study the Ricci flow.**

- **Recently, based on Perelman's breakthrough, Cao and Zhu offered a complete proof of Poincare's conjecture and Thurston's geometrization conjecture.**

In Perelman's work, a key step is to introduce a functional

$$W(g,f)=\int_{M^3} d^3x\sqrt{g}e^{-f}(R+|\nabla f|^2).$$

The corresponding gradient flow:

$$\dot{g}_{ij}=-2(R_{ij}+\nabla_i\nabla_j f),$$

$$\dot{f}=-(R+\triangle f).$$

In this way, we express the Ricci flow as a gradient flow.

Dynamics of a gradient flow is much easier to handle.

- The functional generating the flow is monotone along the orbit of the flow automatically.

- If the flow exists for all time, then it shall flow to a critical point which lead to the existence of a canonical metric.

- If the flow does not exist for all time, the generating functional helps very much in the analysis of singularities.

For a three-manifold M^3, J.Gegenberg et al. present a action:

$$S = \int_M d^3x\sqrt{g}e^{-f}(\chi + R + |\nabla f|^2) - \frac{\epsilon_H}{2}e^{-f}H \wedge *H$$
$$- \epsilon_F e^{-f}F \wedge *F + \frac{e}{2}A \wedge F.$$

$$F = dA, H = dB.$$

- J. Gegenberg and G. Kunstatter, Using 3D stringy gravity to understand the Thurston conjecture, 2003.
- J. Gegenberg, S. Vaidya and J. F. V$\acute{a}$zquez-Poritz, Thurston geometries from eleven dimensions, 2002.

The corresponding gradient flow:

$$
\begin{cases}
\dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} + 2\nabla_i\nabla_j\phi - \dfrac{\epsilon_H}{4}H_{ikl}H_j{}^{kl} - \epsilon_F F_i{}^k F_{jk}], \\
\dfrac{\partial B_{ij}}{\partial t} = \epsilon_H e^{2\phi}\nabla_k(e^{-2\phi}H^k{}_{ij}), \\
\dfrac{\partial A_i}{\partial t} = -\epsilon_F e^{2\phi}\nabla_k(e^{-2\phi}F_i{}^k) + \dfrac{e}{2}e^{2\phi}\eta_i{}^{kl}F_{kl}, \\
\dfrac{\partial \phi}{\partial t} = -\chi + R(g) + 4\triangle\phi - 4\mid\nabla\phi\mid^2 - \dfrac{\epsilon_H}{12}H^2 - \dfrac{\epsilon_F}{2}F^2.
\end{cases}
\tag{1}
$$

J.Gegenberg et al. found that Thurston's eight geometries appear as critical points of the above functional.

Under the condition $e = 0$, they show that there are no other critical points.

Basically critical points of the above functional are eight geometries of Thurston.

We consider a flow for a similar functional for a four-dimension manifold:

$$S = \int_M d^4x\sqrt{g}e^{-f}(\chi + R + |\nabla f|^2) - \frac{\epsilon_H}{2}e^{-f}H \wedge *H$$
$$- \epsilon_F e^{-f}F \wedge *F + \frac{e}{2}F \wedge F.$$

The generalization to four-manifolds is probably more interesting. It may offer a systematic way to study four-manifolds.

The corresponding gradient flow:

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} + 2\nabla_i\nabla_j\phi - \dfrac{\epsilon_H}{4}H_{ikl}H_j{}^{kl} - \epsilon_F F_i{}^k F_{jk}], \\ \dfrac{\partial B_{ij}}{\partial t} = \epsilon_H e^{2\phi}\nabla_k(e^{-2\phi}H^k{}_{ij}), \\ \dfrac{\partial A_i}{\partial t} = -\epsilon_F e^{2\phi}\nabla_k(e^{-2\phi}F_i{}^k) + e\, e^{2\phi}\eta_i{}^{kjl}\nabla_k(F_{jl}), \\ \dfrac{\partial \phi}{\partial t} = -\chi + R(g) + 4\triangle\phi - 4\mid\nabla\phi\mid^2 - \dfrac{\epsilon_H}{12}H^2 - \dfrac{\epsilon_F}{2}F^2. \end{cases} \quad (2)$$

♣ We prove that the system of PDEs (2) are strictly and uniformly parabolic in some sence.

♣ Based on this, we show that the generalized Ricci flow defined on a n-dimensional compact Riemannian manifold admits a unique short-time smooth solution.

♣ Moreover, we also derive the evolution equations for the curvatures, which play an important role in our future study.

2. Main results

Christoffel symbols

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\},$$

The Riemannian curvature tensors

$$R_{ijl}^{k} = \frac{\partial \Gamma_{jl}^{k}}{\partial x^i} - \frac{\partial \Gamma_{il}^{k}}{\partial x^j} + \Gamma_{ip}^{k}\Gamma_{jl}^{p} - \Gamma_{jp}^{k}\Gamma_{il}^{p} \quad R_{ijkl} = g_{kp} R_{ijl}^{p}$$

The Ricci tensor

$$R_{ik} = g^{jl} R_{ijkl}$$

The scalar curvature

$$R = g^{ij} R_{ij}$$

For each field we shall consider the gauge equivalent classes of fields.

- Two metrics g_1, g_2 are in the same equivalent class if and only if they are differ by a diffeomorphism, i.e., there exists a diffeomorphism $f : M \longrightarrow M$ such that $g_2 = f^* g_1$.
- Two gauge fields A_1 and A_2 are equivalent if and only if there exists a function α on M such that $A_2 = A_1 + d\alpha$.
- Two B-fields B_1 and B_2 are equivalent if and only if there exists an one-form β on M such that $B_2 = B_1 + d\beta$.

Theorem 1:(Local existences and uniqueness) Let $(M, g_{ij}(x))$ be a three-dimensional compact Riemannian manifold. Then there exists a constant $T > 0$ such that the system (1) of the evolution equations has a unique smooth solution on $M \times [0, T)$ for every initial fields.

Theorem 2:(Local existences and uniqueness) Let $(M, g_{ij}(x))$ be a four-dimensional compact Riemannian manifold. Then there exists a constant $T > 0$ such that the system (2) of the evolution equations has a unique smooth solution on $M \times [0, T)$ for every initial fields.

3. Method of proof

Lemma 1. For each gauge equivalent class of a gauge field A, there exists an $A^{'}$ such that $d(*A^{'}) = 0$.

The lemma can be proved by the Hodge decomposition.

Proof. For each one-form A, by the Hodge decomposition, there exists an one-form A_0, a function α and a two-form β such that

$$A = A_0 + d\alpha + d^*\beta,$$

$$dA_0 = 0, d^*A_0 = 0.$$

Let $A' = A - d\alpha$. A' is in the same gauge equivalent class of A. Since $d(*A_0) = 0$, $d(*d^*\beta) = 0$, then we have $d(*A') = 0$. $\square$

Lemma 2. The differential operator of the right hand of (2) with respect to the gauge equivalent class of a gauge field A is uniformly elliptic.

Proof. Let $A = A_i dx^i$ be a gauge field. From Lemma 3.1 we can choose an $A^{'}$ in the gauge equivalent class of A such that $d(*A^{'}) = 0$. Then $dd^* A^{'} = 0$. We still denote A' as A .

$$
\begin{aligned}
\frac{\partial A_i}{\partial t} &= -\epsilon_F e^{2\phi}\nabla_k(e^{-2\phi}F_i{}^k) + e\, e^{2\phi}\eta_i{}^{kjl}\nabla_k(F_{jl}) \\
&= -\epsilon_F \nabla_k(g^{kl}F_{il}) - \epsilon_F e^{2\phi}F_i{}^k\nabla_k(e^{-2\phi}) \\
&\quad + e\, e^{2\phi}\eta_i{}^{kjl}\nabla_k\left(\frac{\partial A_l}{\partial x^j} - \frac{\partial A_j}{\partial x^l}\right) \\
&= \epsilon_F (d^*F)_i + 2\epsilon_F F_i{}^k\frac{\partial \phi}{\partial x^k} + 2e\, e^{2\phi}\eta_i{}^{kjl}\frac{\partial^2 A_l}{\partial x^k \partial x^j} \\
&= \epsilon_F (d^*dA)_i + \epsilon_F (dd^*A)_i + 2\epsilon_F F_i{}^k\frac{\partial \phi}{\partial x^k} \\
&= \epsilon_F \triangle A_i + 2\epsilon_F F_i{}^k\frac{\partial \phi}{\partial x^k} \quad .
\end{aligned}
$$

The right hand side is clearly elliptic at point x. If we apply a diffeomorphism to the metric it won't change the positivity property of the second order operator of the right hand side. □

Lemma 3. For each gauge equivalent class of a B-field B, i.e., a two-form B on M, there exists a $B^{'}$ such that $d(*B^{'}) = 0$.

Proof. Again we use the Hodge decomposition. For a two-form B, there exist a one-form α, a two-form B_0 and a three-form β such that

$$B = B_0 + d\alpha + d^*\beta,$$

$$dB_0 = 0, d^*B_0 = 0.$$

Let $B^{'} = B - d\alpha$. $B^{'}$ is in the same gauge equivalent class of B, then we have $d(*B^{'}) = 0$. $\square$

Lemma 4. The differential operator of the right hand side of (2) with respect to the gauge equivalent class of a B-field B is uniformly elliptic.

Proof. Let us consider the equation for B-field. Without loss of generality, we assume $d(*B) = 0$. Then $dd^*B = 0$. We have

$$
\begin{aligned}
\frac{\partial B_{ij}}{\partial t} &= \epsilon_H e^{2\phi} \nabla_k (e^{-2\phi} H^k_{\ ij}) \\
&= \epsilon_H \nabla_k (H_{lij} g^{kl}) + \epsilon_H e^{2\phi} H^k_{\ ij} \nabla_k (e^{-2\phi}) \\
&= -\epsilon_H (d^* dB)_{ij} - \epsilon_H (dd^* B)_{ij} - 2\epsilon_H H^k_{\ ij} \frac{\partial \phi}{\partial x^k} \\
&= -\epsilon_H \triangle B_{ij} - 2\epsilon_H H^k_{\ ij} \frac{\partial \phi}{\partial x^k}
\end{aligned}
$$

The right hand side is clearly elliptic at the point x. If we apply a diffeomorphism to the metric it does not change the positivity property of the second order operator of the right hand side. □

Suppose $(\hat{g}_{ij}(x,t), \hat{\phi}(x,t))$ is a solution of the equations (2), and $\varphi_t : M \to M$ is a family of diffeomorphisms of M. Let

$$g_{ij}(x,t) = \varphi_t^* \hat{g}_{ij}(x,t), \quad \phi(x,t) = \varphi_t^* \hat{\phi}(x,t),$$

where φ_t^* is the pull-back operator of φ_t. We now want to find the evolution equations for the metric $g_{ij}(x,t)$ and $\phi(x,t)$.

Denote

$$y(x,t) = \varphi_t(x) = \{y^1(x,t), y^2(x,t), \cdots, y^n(x,t)\}$$

in local coordinates.

If we define $y(x,t)=\varphi_t(x)$ by the equations

$$\begin{cases} \dfrac{\partial y^\alpha}{\partial t} = \dfrac{\partial y^\alpha}{\partial x^k}(2\nabla_j \phi g^{kj} + g^{jl}(\Gamma^k_{jl} - \tilde{\Gamma}^k_{jl})), \\ y^\alpha(x,0) = x^\alpha \end{cases}$$

and $V_i = g_{ik} g^{jl}(\Gamma^k_{jl} - \tilde{\Gamma}^k_{jl})$, then we have

$$\begin{cases} \dfrac{\partial g_{ij}(x,t)}{\partial t} = g^{kl}\dfrac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \dfrac{\epsilon_H}{2} H_{ikl} H_j{}^{kl} + 2\epsilon_F F_i{}^k F_{jk}, \\ \dfrac{\partial \phi}{\partial t} = -\dfrac{1}{2} g^{ij} g^{kl} \dfrac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + 2 g^{kl} \dfrac{\partial^2 \phi}{\partial x^k \partial x^l} \\ \qquad -2g^{kl}\dfrac{\partial \phi}{\partial x^k}\dfrac{\partial \phi}{\partial x^l} - g^{jl}\tilde{\Gamma}^k_{jl}\dfrac{\partial \phi}{\partial x^k} - \chi - \dfrac{\epsilon_H}{12} H^2 - \dfrac{\epsilon_F}{2} F^2. \end{cases} \tag{6}$$

Let

$$u_1 = g_{11}, u_2 = g_{12}, u_3 = g_{13}, u_4 = g_{14}, u_5 = g_{22}, u_6 = g_{23},$$
$$u_7 = g_{24}, u_8 = g_{33}, u_9 = g_{34}, u_{10} = g_{44}, u_{11} = \phi \ .$$

The above equations can be rewritten as the following form

$$\frac{\partial u_i}{\partial t} = \sum_{j\,k\,l} a_{ikjl} \frac{\partial^2 u_j}{\partial x^k \partial x^l} + (lower\ order\ terms).$$

So for arbitrary $\xi \in \mathbb{R}^{4\times 11}$, it is easily verified that the eigenvalues of the above quadratic forms $\sum_{ijkl} a_{ikjl} \xi^i_k \xi^j_l$ read

$$\frac{3}{2} \pm \frac{\sqrt{2}}{2},\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1.$$

It means that the uniformly parabolic condition of the system (6) holds.

Lemma The differential operator of the right hand side of (6) with respect to the metric g and the dilaton ϕ is uniformly elliptic.

Summarizing the above discussions and by virtue of the standard theory of partial differential equations, we can obtain the existence and uniqueness of solution .

4. Evolution of curvatures

Theorem 1 **Under the gradient flow (2), the curvature tensor satisfies the evolution equation**

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ijkl} = {} & \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
& - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \\
& - 2g^{pq}(R_{pjkl}\nabla_q\nabla_i\phi + R_{ipkl}\nabla_q\nabla_j\phi + R_{ijpl}\nabla_q\nabla_k\phi + R_{ijkp}\nabla_q\nabla_l\phi) \\
& - 2g^{pq}\nabla_q\phi\nabla_p R_{ijkl} \\
& + \frac{\epsilon_H}{4}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) \\
& - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\
& + \frac{\epsilon_H}{4}g^{mn}(H_{kpq}H_m{}^{pq}R_{ijnl} + H_{mpq}H_l{}^{pq}R_{ijkn}) \\
& + \epsilon_F[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp})] \\
& + \epsilon_F g^{mn}(F_k{}^pF_{mp}R_{ijnl} + F_m{}^pF_{lp}R_{ijkn}),
\end{aligned}
$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and $\triangle$ is the Laplacian with respect to the evolving metric.

Theorem 2 The Ricci curvature satisfies the following evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ik} = {} & \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk} \\
& - 2g^{pq}(R_{pk}\nabla_q\nabla_i\phi + R_{ip}\nabla_q\nabla_k\phi) - 2g^{pq}\nabla_q\phi\nabla_p R_{ik} \\
& + \frac{\epsilon_H}{4}g^{jl}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) \\
& - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\
& + \frac{\epsilon_H}{4}g^{mn}(H_{kpq}H_m^{pq}R_{in} - g^{jl}H_{mpq}H_l{}^{pq}R_{ijkn}) \\
& + \epsilon_F g^{jl}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) \\
& - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp})] \\
& + \epsilon_F g^{mn}(F_k{}^pF_{mp}R_{in} - g^{jl}F_m^{\ p}F_{lp}R_{ijkn}).
\end{aligned}$$

Theorem 3 The scalar curvature satisfies the following evolution equation

$$
\begin{aligned}
\frac{\partial}{\partial t}R = \triangle R + 2|Ric|^2 - 2g^{pq}\nabla_q\phi\nabla_p R \\
+ \frac{\epsilon_H}{2}g^{jl}g^{ik}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq})] \\
+ 2\epsilon_F g^{jl}g^{ik}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp})] \\
- g^{ip}R_{ik}\left(\frac{\epsilon_H}{2}H_{pmn}H^{kmn} + 2\epsilon_F F_{pm}F^{km}\right).
\end{aligned}
$$

Thank you !