(α, β)-METRICS WITH ISOTROPIC S-CURVATURE

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1 Introduction

A (positive definite) **Finsler metric** on a manifold M is a C^∞ scalar function $F = F(x, y)$ on $TM\backslash\{0\}$:

$$F(x, y) > 0, \quad y \neq 0$$

$$F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0$$

$$(g_{ij}(x, y)) \;\; \text{positive definite}$$

where $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$.

Inner product: $g_y : T_xM \times T_xM \to R :$

$$g_y(u, v) = g_{ij}(x, y)u^i u^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x, v = v^j \frac{\partial}{\partial x^j}|_x$.

Geodesic Equation:

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where

$$G^i = \frac{1}{4} g^{il} \{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\}.$$

Riemann Curvature $\mathbf{R}_y : T_xM \to T_xM$,

$$\mathbf{R}_y(u) := R^i{}_k u^k \frac{\partial}{\partial x^i}|_x, \quad u = u^i \frac{\partial}{\partial x^i}|_x,$$

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Flag Curvature:

$$\mathbf{K}(P, y) := \frac{g_y(\mathbf{R}_y(u), u)}{g_y(y, y)g_y(u, u) - [g_y(u, y)]^2},$$

where $P := span\{y, u\} \subset T_xM$.

F is of scalar flag curvature: $\mathbf{K}(P, y) = \mathbf{K}(y)$
(independent of P)

F is of constant flag curvature: $\mathbf{K}(P, y) =$constant

To characterize Riemannian metrics among Finsler metrics, we introduce the quantity

$$\tau(x, y) := \ln \left[\frac{\sqrt{det(g_{ij}(x, y))}}{\sigma_F(x)} \right],$$

where

$$\sigma_F(x) := \frac{Vol(B^n)}{Vol\left\{(y^i) \in R^n | F(x, y) < 1\right\}}$$

characterizes the Busemann-Hausdorff volume form. τ is called the **distortion**.

F **is Riemannian if and only if** $\tau = constant$([Shen-2]).

$$\begin{array}{ccccc}
\mathbf{C} & \longrightarrow & \mathbf{I} : I_i = g^{jk} C_{ijk} = \tau_{y^i} & \longleftarrow & \tau \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{L} : L_{ijk} := C_{ijk|m} y^m & \longrightarrow & \mathbf{J} : J_i := g^{jk} L_{ijk} = I_{i|m} y^m & & \mathbf{S} := \tau_{|m} y^m
\end{array}$$

where $(g^{ij}(x,y)) := (\frac{1}{2}\left[F^2\right]_{y^i y^j}(x,y))^{-1}$.

L : *Landsberg curvature*
J : *mean Landsberg curvature*
S : S-*curvature* (**Z. Shen,** 1997 [Shen-1])

- We say S-curvature is **isotropic** if there exists a scalar function $c(x)$ on M such that

$$\mathbf{S}(x,y) = (n+1)c(x)F(x,y),$$

  If $c(x) = constant$, we say that F has constant S-curvature.

- S-curvature $\mathbf{S}(x,y)$ is the rate of change of τ along geodesics and measures the averages rate of change of (T_xM, F_x) in the direction $y \in T_xM$.

- Let $G^i(x, y)$ be the *geodesic coefficients* of F. By the definition of S-curvature, we have

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m}(x, y) - y^m \frac{\partial}{\partial x^m}(\ln \sigma_F(x)).$$

- ([Shen-2][Shen-3]) For any Berwald metric,the S-curvature vanishes, $\mathbf{S} = 0$.

2 Why do we study S-curvature?

S-curvature and flag curvature have many affinities
([Cheng-Mo-Shen][Mo])

$$\begin{aligned} J_{k;m}y^m + I_m R^m_k &= -\frac{1}{3}\{2R^m_{k.m} + R^m_{m.k}\} \\ \mathbf{S}_{.k;m}y^m - \mathbf{S}_{;k} &= -\frac{1}{3}\{2R^m_{k.m} + R^m_{m.k}\} : \end{aligned}$$

If F is of scalar curvature with flag curvature $\mathbf{K} = \mathbf{K}(x, y)$:

$$R^i_k = \mathbf{K}F^2 h^i_k,$$

where $h^i_k := g^{ij}h_{jk}$ and $h_{jk} := g_{jk} - F^{-2}g_{js}y^s g_{kt}y^t$, then we have

$$\mathbf{S}_{.k;l}y^l - \mathbf{S}_{;k} = -\frac{n+1}{3}\mathbf{K}_{.k}F^2.$$

S-curvature has important influence on the geometric structure of Finsler metrics

- **Finsler metrics of positive flag curvature**

  ♣ [Kim-Yim] Finsler manifold (M, F):

  (1) reversible $(F(-y) = F(y))$;

  (2) $\mathbf{S} = 0$,

  (3) flag curvature $\mathbf{K} = constant > 0$.
  $\Longrightarrow F$ is a Riemannian.

- **Finsler metrics of negative flag curvature**

♣ [Akbar-Zadeh 1988] Finsler manifold (M, F):

(1) closed;

(2) flag curvature $\mathbf{K}$=constant;

(3) $\mathbf{K} < 0$.
$\Longrightarrow$ F is Riemannian.

♣ [Shen-6] Finsler manifold (M, F):

(1) closed;

(2) $\mathbf{S} = (n+1)cF, \quad c = constant$;

(3) flag curvature $\mathbf{K}(P, y) < 0$.
$\Longrightarrow$ F is Riemannian.

- If Randers metric $F = \alpha + \beta$ (α: Riemann metric; β: 1-form) is of constant flag curvature $\Longrightarrow$ F is of constant S-curvature ([Bao-Robles-Shen])

  More general, Many known Finsler metrics of constant/scalar flag curvature actually have constant/isotropic S-curvature ([Cheng-Mo-Shen][Shen-4]).

- **(Z. Shen, 1997)** the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing S-curvature

3 Why do we study (α, β)-metrics?

Given a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$ on a manifold M. Let

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a C^∞ positive function on $(-b_o, b_o)$. It is known that $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_o$ if and only if ϕ satisfies the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \leq b < b_o).$$

Such metric is called an (α, β)-metric.

Example 3.1 *Some important (α, β)-metrics:*

- **Randers metric:** $F = \alpha + \beta$; $\phi = 1 + s$.
  More general,
  $$F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta.$$
- **Matsumoto metric:** $F = \frac{\alpha^2}{\alpha - \beta}$; $\phi = \frac{1}{1-s}$.
- $F = \alpha + \epsilon\beta + k\beta^2/\alpha$; $\phi = 1 + \epsilon s + ks^2$, *where* ϵ *and* $k \neq 0$ *are constants.*
  In particular,
  $$F = \frac{(\alpha + \beta)^2}{\alpha}.$$

A. (α, β)**-metrics are " computable"**

B. **The study for** (α, β)**-metircs can help us to understand better and deeply geometric properties of Finsler metrics in general case**

C. (α, β)**-metrics have important applications in physics and biology(ecology)** ([Antonelli-Miron][Asanov2006])

D. **Some important progress of the study on (α, β)-metrics**

(a) **Randers metrics of constant/scalar flag curvature**

♣ **(Z. Shen, 2003)** classified locally projectively flat Randers metrics with constant Ricci curvature

Remark: The solutions to the Hilbert's Fourth Problem in the regular case are projectively flat Finsler metrics

♣ **(Cheng-Mo-Shen, 2003)** classified locally projectively flat Randers metrics with isotropic S-curvature
More general, we characterized projectively flat Finsler metrics with isotropic S-curvature ([Cheng-Shen2006(1)])

♣ **(Bao-Robles-Shen, 2004)** classified Randers metrics of constant flag curvature

♣ **(Cheng-Shen, 2005)** classified Randers metrics of scalar flag curvature with isotropic S-curvature (This class contains all Randers metrics of constant flag curvature)

(b) **Projectively flat (α, β)-metrics**

Berwald's metric (Berwald, 1929)

$$B = \frac{(\alpha+\beta)^2}{\alpha}, \quad y \in T_x\mathbf{B}^n,$$

where $\alpha = \lambda\bar{\alpha}, \beta = \lambda\bar{\beta}$ and

$$\bar{\alpha} = \frac{\sqrt{(1-|x|^2)|y|^2 + < x, y >^2}}{1-|x|^2}, \quad \bar{\beta} = \frac{< x, y >}{1-|x|^2}, \quad \lambda = \frac{1}{1-|x|^2}.$$

$\diamondsuit$ projectively flat
$\diamondsuit$ $\mathbf{K} = 0$

♣ **(Shen-Zhao, 2005)** determined completely the local structure of a projectively flat Finsler metric F in the form $F = \alpha + C_1\beta + \frac{1}{2p}\frac{\beta^2}{\alpha} - \frac{1}{48p^2}\frac{\beta^4}{\alpha^3}$ of constant flag curvature

♣ **(Z. Shen, 2006)** studied and characterized projectively flat (α, β)-metrics in dimension $n \geq 3$

♣ **(Shen-Yildirim, 2006)** determined completely the local structure of a projectively flat Finsler metric F in the form $F = (\alpha + \beta)^2/\alpha$ of constant flag curvature

♣ **(Li-Shen, 2006)** classified projectively flat (α, β)-metrics with constant flag curvature in dimension $n \geq 3$: one of the following holds

- ∘ α is projectively flat and β is parallel with respect to α
- ∘ $\phi = \sqrt{1 + ks^2} + \epsilon s, \quad k, \epsilon(\neq 0)$: constants; $\mathbf{K} < 0$
- ∘ $\phi = (\sqrt{1 + ks^2} + \epsilon s)^2/\sqrt{1 + ks^2}, \quad k, \epsilon(\neq 0)$: constants; $\mathbf{K}$=0

(c) (α, β)**-metrics of Landsberg type**

F is called a **Berwald metric** if its geodesic coefficients

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$$

are quadratic in $y \in T_xM$, or equivalently

$$[G^i]_{y^j y^k y^l} = 0.$$

Riemannian metrics are Berwald metrics.

F is called a **Landsberg metric** if its Landsberg curvature

$$L_{ijk} = 0.$$

Fact: Every Berwald metric is a Landsberg metric

A Long Existing Open Problem: Is there any Landsberg metric which is not Berwald metric?

• **(Z. Shen, 2006)**

characterized the Landsberg curvature of (α, β)-metrics

a regular (α, β)-metric is Landsbergian if and only if it is Berwaldian

• **(Li-Shen, 2006)** characterized weakly Landsberg (i.e. $\mathbf{J} = 0$) (α, β)-metrics and shown that there exist weakly Landsberg metrics which are not Landsberg metrics in dimension greater than two

4 (α, β)-metrics with isotropic S-curvature

Open Problems:

(1) Determine non-Randers (α, β)-metrics with vanishing S-curvature and constant flag curvature.

(2) Determine (α, β)-metrics of scalar flag curvature and isotropic S-curvature.

Let

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}.$$

We have the following formula for the spray coefficients G^i of F:

$$G^i = \bar{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + \alpha Q s^i{}_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i,$$

where $\bar{G}^i$ denote the spray coefficients of α and

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Theta = \frac{Q - sQ'}{2\Delta}, \qquad \Psi = \frac{Q'}{2\Delta},$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$ and $b = \|\beta_x\|_\alpha$.

Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where "|" denotes the covariant derivative with respect to α.

Let

$$r_{ij} := \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \quad s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right), \quad s^i{}_j := a^{ih}s_{hj},$$

$$s_j := b^i s_{ij}, \quad r_j := b^i r_{ij}, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Recall:

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m}(x,y) - y^m \frac{\partial}{\partial x^m}(\ln \sigma_F(x)).$$

Proposition 4.1 ([Cheng-Shen2006(2)]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M. Let $dV = dV_{BH}$ or dV_{HT}. Let*

$$f(b) := \begin{cases} \dfrac{\int_0^\pi \sin^{n-2}(t)dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b\cos(t))^n}dt} & \text{if } dV = dV_{BH} \\ \dfrac{\int_0^\pi \sin^{n-2}(t)T(b\cos t)dt}{\int_0^\pi \sin^{n-2}(t)dt} & \text{if } dV = dV_{TH}, \end{cases}$$

where $T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']$. Then the volume form dV is given by

$$dV = f(b)dV_\alpha,$$

where $dV_\alpha = \sqrt{\det(a_{ij})}dx$ denotes the Riemannian volume form of α.

A useful technique in the proof: take a local coordinate system at x such that

$$\alpha = \sqrt{\sum (y^i)^2}, \qquad \beta = by^1,$$

where $b = \|\beta_x\|_\alpha$. Then the volume form $dV_\alpha = \sigma_\alpha dx$ at x is given by

$$\sigma_\alpha = \sqrt{\det(a_{ij})} = 1.$$

In order to evaluate the integrals

$$\mathrm{Vol}\{(y^i) \in R^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\} = \int_{F(x,y)<1} dy = \int_{\alpha\phi(\beta/\alpha)<1} dy$$

and

$$\int_{F(x,y)<1} \det(g_{ij}) dy = \int_{\alpha\phi(\beta/\alpha)<1} \det(g_{ij}) dy,$$

we take the following coordinate transformation, $\psi : (s, u^a) \to (y^i)$:

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \qquad y^a = u^a,$$

where $\bar{\alpha} = \sqrt{\sum_{a=2}^n (u^a)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

Thus

$$F = \alpha\phi(\beta/\alpha) = \frac{b\phi(s)}{\sqrt{b^2 - s^2}} \bar{\alpha}$$

and the Jacobian of the transformation $\psi : (s, u^a) \to (y^i)$ is given by

$$\frac{b^2}{(b^2 - s^2)^{3/2}} \bar{\alpha}.$$

Q.E.D.

An important formula on S-curvature of (α, β)-metrics:

$$\mathbf{S} = \{2\Psi - \frac{f'(b)}{bf(b)}\}(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0),$$

where

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''.$$

Theorem 4.2 ([Cheng-Shen2003]) *Randers metric $F = \alpha + \beta$ is of isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F$, if and only if*

$$r_{ij} + b_i s_j + b_j s_i = 2c(a_{ij} - b_i b_j).$$

Theorem 4.3 ([Cheng-Shen2006(2)]) *Let*

$$F = k_1\sqrt{\alpha^2 + k_2\beta^2} + k_3\beta$$

be a Finsler metric of Randers type where $k_1 > 0$ and $k_3 \neq 0$. F is of isotropic S-curvature, $F = (n+1)cF$ if and only if β satisfies

$$r_{ij} + \tau(s_i b_j + s_j b_i) = \frac{2c(1 + k_2 b^2)k_1^2}{k_3}(a_{ij} - \tau b_i b_j),$$

where

$$\tau := \frac{k_3^2}{k_1^2} - k_2.$$

Lemma 4.4 ([Cheng-Shen2006(2)]) *Let β be a 1-form on a Riemannian manifold (M, α). Then $b(x) := \|\beta_x\|_\alpha = constnt$ if and only if β satisfies the following equation:*

$$r_j + s_j = 0.$$

In this case, the S-curvature is given by

$$\mathbf{S} = -\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0).$$

Proof: This follows immediately from $(b^2)_{|j} = 2(r_j + s_j)$. Q.E.D.

Theorem 4.5 ([Cheng-Shen2006(2)]) *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α, β)-metric on a manifold.* **Suppose** *that*

$$\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$$

for any constants k_1, k_2 and k_3.
Then *F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) β satisfies

$$r_j + s_j = 0 \tag{1}$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0. \tag{2}$$

In this case, $\mathbf{S} = 0$.

(ii) β *satisfies*

$$r_{ij} = \epsilon\{b^2 a_{ij} - b_i b_j\}, \qquad s_j = 0, \tag{3}$$

where $\epsilon = \epsilon(x)$ *is a scalar function, and* $\phi = \phi(s)$ *satisfies*

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{4}$$

where k *is a constant. In this case,* $\mathbf{S} = (n+1)cF$ *with* $c = k\epsilon$*.*

(iii) β *satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{5}$$

In this case, $\mathbf{S} = 0$*, regardless of the choice of a particular* ϕ*.*

Remark. It is easy to see that

$$r_{ij} = 0, \quad s_j = 0 \quad (5)$$

$$\Downarrow$$

$$r_{ij} = \epsilon\{b^2 a_{ij} - b_i b_j\}, \quad s_j = 0 \quad (3)$$

$$\Downarrow$$

$$r_j + s_j = 0 \quad (1)$$

$$(\Longleftrightarrow b := \|\beta_x\|_\alpha = constant)$$

If an (α, β)-metric of non-Randers type is of isotropic S-curvature, then $b := \|\beta_x\|_\alpha$= constant

Example 4.1 Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric defined on an open subset in R^3. At a point $\mathbf{x} = (x, y, z) \in R^3$ and in the direction $\mathbf{y} = (u, v, w) \in T_{\mathbf{x}}R^3$, $\alpha = \alpha(\mathbf{x}, \mathbf{y})$ and $\beta = \beta(\mathbf{x}, \mathbf{y})$ are given by

$$\begin{aligned} \alpha &:= \sqrt{u^2 + e^{2x}(v^2 + w^2)}, \\ \beta &:= u. \end{aligned}$$

Then β satisfies (3) with $\epsilon = 1$, $b = 1$. Thus if $\phi = \phi(s)$ satisfies (4) for some constant k, then $F = \alpha\phi(\beta/\alpha)$ is of constant S-curvature $\mathbf{S} = (n+1)cF$.

Example 4.2 *For $F = \alpha + \epsilon\beta + k(\beta^2/\alpha)$, $\epsilon, k(\neq 0)$ are constants, the following are equivalent:*

(i) *F is of isotropic* S*-curvature, $\mathbf{S} = (n+1)cF$;*

(ii) *β is a Killing 1-form with b=constant with respect to α, i.e.*

$$r_{ij} = 0, \quad s_j = 0;$$

(iii) $\mathbf{S} = 0$*;*

(iv) *F is of isotropic mean Berwald curvature, $\mathbf{E} = \frac{n+1}{2} cF^{-1}h$, where $\mathbf{E}$ denotes the mean Berwald curvature of F;*

(v) *F is a weakly Berwald metic, $\mathbf{E} = 0$.*

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Thank you very much for your attention!