Infinitely many solutions for an elliptic problem involving critical nonlinearity

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1. Background

Yamabe Conjecture

Manifolds without boundary

Let (M, g) be an N-dimensional,smooth, compact Riemannian manifold without boundary. For $N \geq 3$, the Yamabe conjecture states that there exist Riemannian metrics which are pointwise conformal to g and have a constant scalar curvature.

Let R_g denote the scalar curvature of g and Δ_g denote the Laplace-Beltrami operator of g. For $N \geq 3$, let $g' = u^{\frac{4}{N-2}} g$ for some positive function u, we have

$$R_{g'} = u^{-\frac{N+2}{N-2}}(R_g u - \frac{4(N-1)}{N-2}\Delta_g u),$$

The Yamabe conjecture is therefore equivalent to the solvability of

$$-\Delta_g u + \frac{N-2}{4(N-1)} R_g u = \bar{R} u^{\frac{N+2}{N-2}},\ u > 0,\ in\, M \tag{1.1}$$

for $\bar{R} = 1, 0$, or -1.

Manifold with boundary – an example

Consider a smooth metric g on $B = \{x \in \mathbb{R}^N : |x| < 1\}, N \geq 3$. Let Δ_g, R_g, ν_g, h_g denote, respectively the Laplace-Beltrami operator, the scalar curvature of (B, g),the outward normal to $\partial B = S^{N-1}$ with respect to g and the mean curvature of (S^{N-1}, g). Given two smooth functions R' and h', we will consider the existence of positive solutions $u \in H^1(B)$ of

$$\begin{cases} -4\frac{N-1}{N-2}\Delta_g u + R_g u = R' u^{\frac{N+2}{N-2}} \quad in\, B, \\ \\ \frac{2}{N-2}\partial_{\nu_g} u + h_g u = h' u^{\frac{N}{N-2}} \;\; on\, \partial B. \end{cases} \tag{1.2}$$

It is well known that such a solution is C^∞ provided g, R' and h' are. If $u > 0$ is a smooth solution of (1.2) then $g' = u^{\frac{4}{N-2}} g$ is a metric,conformally equivalent to g, such that R' and h' are, respectively, the scalar curvature of (B, g') and the mean curvature of (S^{N-1}, g').

Suppose $M = S^N, g = g_0$, if we allow $\bar{R}$ to be general positive function, then (1.3) becomes

$$-\Delta_{g_0} u + \frac{N(N-2)}{4} u = \frac{N-2}{4(N-1)} \bar{R} u^{\frac{N+2}{N-2}}, \ u > 0, \ in \, S^N \tag{1.3}$$

Making the stereographic projection $\pi = \pi_N$, and setting $K_0(x) = K(\pi^{-1}x))$, we get the following

$$\begin{cases} -\Delta u = K_0(x) u^{2^*-1}, & x \in \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $2^* = \frac{2N}{N-2}, N \geq 3$.

There have been many works on (1.4).

For example, Scheon, Scheon and Yau, Chang and Yang, Bahri and Coron,etc

The main difficult is that suppose there exists a sequence of approximating solutions, how can we get a subsequence which converges strongly in a suitable space?

Usually we use the variational method. Let H be a Hilbert space and $I \in C^1(H, \mathbb{R})$. We can use many ways (constrained minimization, mountain pass lemma, Galekin method, etc) to obtain Palais-Smale sequence $\{u_k\}$, that is $\{u_k\}$ satisfies

$$I(u_k) \to c, I'(u_k) \to 0$$

The Palais-Smale sequence is the corresponding sequence of approximating solutions.

If we can always choose a strongly convergent subsequence from any given Palais-Smale sequence, then I is said to satisfy Palais-Smale condition. Unfortunately, for most problem with critical nonlinearity, the corresponding functional does not satisfy Palais-Smale condition.

2. **Problem, Previous Results and Our Result**

Let $N \geq 3$, $2^* = \frac{2N}{N-2}$, and Ω be an open bounded domain in $\mathbb{R}^N$. We consider the following elliptic problem:

$$\begin{cases} -div(a(x)Du) = Q(x)|u|^{2^*-2}u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $a,\ Q \in C^4(\bar{\Omega})$, $a(x) \geq a_0 > 0$, $Q(x) \geq Q_0 > 0$, and $\lambda > 0$ is a positive constant.

The functional corresponding to (2.1) is

$$I(u) = \frac{1}{2}\int_\Omega \big(a(x)|Du|^2 - \lambda u^2\big)dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}dx, \quad u \in H_0^1(\Omega). \tag{2.2}$$

Since the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is not compact, the functional $I(u)$ does not satisfies the Palais-Smale condition((PS) condition for short). This loss of compactness creates a lot of difficulties when variational method is used to obtain the existence result for (2.1).

The case of constant coefficients

$$a(x) \equiv 1, \;\; Q(x) \equiv 1.$$

Let λ_1 denote the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Pohozaev (1965): (2.1) has no nontrivial solution if $N \geq 4$ and $\lambda < 0$.

Brezis-Nirenberg (1983):
If $N \geq 4$ and $\lambda \in (0, \lambda_1)$, then (2.1) has a positive solution.

Capozzi, Fortunato and **Palmieri** (1985), **Ambrosetti** and **Struwe**(1986)
Existence of nontrivial solutions were obtained when $N \geq 4$.

Cerami, Solimini and **Struwe**(1986)
The existence of solutions of changing sign was studied when $N \geq 7$.

Fortunato and **Jannelli**(1987)
For any real positive parameter λ and for all bounded domains Ω, which have suitable symmetry properties, (2.1) has infinitely many solutions when $N \geq 4$. When $N = 3$ it is shown that the number of solutions increases with λ.

Let S be the best Sobolev constant.

The fact that for $c < \frac{1}{N} S^{\frac{N}{2}}$, $I(u)$ satisfies $(PS)_c$ condition were established and used in all the mentioned papers.

Recently, **Devillanova** and **Solimini** (2003) proved that (2.1) has infinitely many solutions if $N \geq 7$ and $\lambda > 0$.

Framework of **Devillanova** *and* **Solimini**

First consider the existence of infinitely many solutions of the following perturbed problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2-\varepsilon}u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $\varepsilon > 0$ is a small constant.

By the result of **Ambrosetti - Rabinowitz**, for fixed ε, (2.3) has solutions $u_{\varepsilon,k}$, $k = 1, 2, \cdots$,

The strong convergence of $u_{\varepsilon,k}$ as $\varepsilon \to 0$ was proved by show that $|u_{\varepsilon,k}|$ has a uniform bound.

The procedure of obtaining the uniform bound consists of two steps.

Step 1: find a safe region, where solutions of (2.3) are uniformly bounded.

The main ingredient used to achieve this goal is the mean value theorem for the Laplacian operator.

Step 2: establish a local Pohozaev identity in a small ball, whose boundary lies entirely in the safe region, to reach a contradiction.

The case of non-constant coefficients

that is, either $a(x)$, or $Q(x)$ is not identically constant.

For such a case, it is even more difficult to obtain a sign-changing solution for (2.1), because $I(u)$ does not satisfy $(PS)_c$ condition for any c larger than the smallest number $min\{\frac{(a(x)S)^{\frac{N}{2}}}{N(Q(x))^{\frac{N-2}{2}}} \mid x \in \Omega\}$, where the (PS) condition fails. The aim of this talk is to show that (2.1) has infinitely many solutions if $N \geq 7, a(x)$ and $Q(x)$ satisfy some degenerate conditions near their critical points.

Consider the following perturbed problem:

$$\begin{cases} -div(a(x)Du) = Q(x)|u|^{2^*-2-\varepsilon}u + \lambda u & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\varepsilon > 0$ is a small constant.

The functional corresponding to (2.4) becomes

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \big(a(x)|Du|^2 - \lambda u^2\big)dx - \frac{1}{2^*-\varepsilon}\int_\Omega Q(x)|u|^{2^*-\varepsilon}dx, \quad u \in H_0^1(\Omega). \tag{2.5}$$

Now $I_\varepsilon(u)$ is an even functional and satisfies the (PS) condition. So from **Ambrosetti-Rabinowitz**, (2.4) has infinitely many solutions. More precisely, there are positive numbers $c_{\varepsilon,l}$, $l = 1, 2, \cdots$, with $c_{\varepsilon,l} \to +\infty$ as $l \to +\infty$, and a solution $u_{\varepsilon,l}$ for (2.4), satisfying

$$I_\varepsilon(u_{\varepsilon,l}) = c_{\varepsilon,l}.$$

Moreover, $c_{\varepsilon,l} \to c_l < +\infty$ as $\varepsilon \to 0$.

Question:

For each fixed l, as $\varepsilon \to 0$, does $u_{\varepsilon,l}$ converges strongly in $H_0^1(\Omega)$ to u_l?

If the answer is yes, then u_l is a solution of (2.1) with $I(u_l) = c_l$.

If we can prove that under suitable assumptions on a and Q, $u_{\varepsilon,l}$ as $\varepsilon \to 0$, then This will imply that (2.1) has infinitely many solutions.

Define

$$\Sigma(x) = \frac{a^{N/2}(x)}{Q^{(N-2)/2}(x)}. \tag{2.6}$$

Let C_Σ be the set of all the critical points of $\Sigma(x)$. Let $\langle x, y\rangle$ denote the inner product of $x, y \in \mathbb{R}^N$.

(C_1) There is a constant $\delta > 0$, such that for any $x \in \bar{\Omega}$ with $d(x, C_\Sigma) \le \delta$,

$$\langle Da(x), DQ(x)\rangle \le 0.$$

(C_2) For each $x_0 \in C_\Sigma$, there is a number $m(x_0) > 2$ and $q(x_0) > 2$, such that for x near x_0,

$$|D^j a(x)| \le C|Da(x)|^{(m(x_0)-j)/(m(x_0)-1)},\ j = 2, 3;$$
$$|D^j Q(x)| \le C|DQ(x)|^{(q(x_0)-j)/(q(x_0)-1)},\ j = 2, 3.$$

(C_3) There is a small $\tau > 0$, such that for any $y \in \Omega$ with $d(y, \partial\Omega) < \tau$, $t \in (0, \tau]$,

$$\langle \tilde{y} - x, Da(x)\rangle \ge 0, \quad \forall\, x \in B_t(y),$$
$$\langle \tilde{y} - x, DQ(x)\rangle \le 0, \quad \forall\, x \in B_t(y),$$

where $\tilde{y} = y + 2tn$, and n is the outward unit normal of $\partial\Omega$ at $\bar{y}$, $|\bar{y} - y| = d(y, \partial\Omega)$.

Remarks on the conditions (C_2) and (C_3).

Remark 2.1. If $a(x) \equiv 1$, then Σ and $Q(x)$ has the same critical points. Condition (C_2) implies that any critical point of $Q(x)$ must be degenerate. On the other hand, (C_3) implies that $Q(x)$ must be non-increasing near the boundary at the direction of the outward normal of $\partial\Omega$. Note that if $Q \equiv 1$, (C_2) and (C_3) hold. So Theorem 2.6 is a generalization of the result of Devillanova and Solimini.

Remark 2.2. If $Q \equiv 1$, then Σ and $a(x)$ has the same critical points. Condition (C_2) implies that any critical point of $a(x)$ must be degenerate, while (C_3) implies that $a(x)$ must be non-decreasing near the boundary at the direction of the outward normal of $\partial\Omega$.

Remark 2.3. By (C_3), it is easy to check that any critical point of Σ must be a critical point for $a(x)$, and $Q(x)$.

Remark 2.4. If $\frac{\partial a(x)}{\partial n} > 0$ and $\frac{\partial Q(x)}{\partial n} < 0$ for any $x \in \partial\Omega$, where n is the outward unit normal of $\partial\Omega$ at x, then (C_3) holds.

Examples:

Suppose $\forall x_0 \in C_\Sigma$, there are $\theta > 2, \hat{\theta} > 2$ and constants $q_{x_0,1} > 0, q_{x_0,2}, a_{x_0,1} > 0, a_{x_0,2}$ such that for x near x_0,

$$Q(x) = q_{x_0,1} - q_{x_0,2}|x - x_0|^{\theta} + q(x),\ q(x) = o(|x - x_0|^{\theta}),$$

$$a(x) = a_{x_0,1} + a_{x_0,2}|x - x_0|^{\hat{\theta}} + \hat{a}(x),\ \hat{a}(x) = o(|x - x_0|^{\hat{\theta}})$$

with $q(x)$ and $\hat{a}(x)$ satisfy certain conditions of degenerate at x_0, in particular $q(x) \equiv 0,\ \hat{a}(x) \equiv 0$. Then C_1, C_2 are satisfied.

The norm of $H_0^1(\Omega)$

$$\|u\| = (\int_\Omega |Du|^2 dx)^{\frac{1}{2}};$$

The norm of $L^p(\Omega)(1 \leq p < \infty)$

$$\|u\|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}.$$

The main result is the following:

Theorem 2.5. *Suppose that $N \geq 7$ and (C_1)–(C_3) hold with $\inf_{x_0 \in S}(m(x_0), q(x_0)) > \frac{2(N-2)}{N-4}$, $\lambda > 0$. Then for any sequence u_n, which is a solution of* (2.4) *with $\varepsilon = \varepsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant independent of n, u_n converges strongly in $H_0^1(\Omega)$ as $n \to +\infty$.*

A direct consequence of Theorem 2.5 is the following multiplicity result.

Theorem 2.6. *Suppose that $N \geq 7$ and (C_1)–(C_3) hold with $\inf_{x_0 \in S}(m(x_0), q(x_0)) > \frac{2(N-2)}{N-4}$, $\lambda > 0$. Then* (2.1) *has infinitely many solutions.*

Difference with previous work

It appears the techniques used in DEVILLANOVA and SOLIMINI can not be applied to study (2.1).

1. it seems that result similar to the mean value theorem for the Laplacian operator is still unknown for elliptic operator in divergent form.

2. under our assumptions on a and Q, we can not obtain a contradiction by applying the local Pohozaev identity in a small ball whose boundary lies entirely in the region where the solutions are uniformly bounded.

To overcome these difficulties, in our paper, we will use the frozen coefficients technique, together with the mean value theorem for elliptic operator of constant coefficient, to obtain the desired local L^1 estimate for solutions of (2.4). And then we estimate the solutions of (2.4) in a carefully defined safe region for problem (2.4), where a local Pohozaev identity will be used to reach a contradiction. It is worthwhile to point out that in the safe region we choose for (2.4), the solutions are not uniformly bounded.

Framework of proof

Suppose u_n is a solution of the perturbed problem with $\varepsilon = \varepsilon_n$ and $\|u_n\| \le C$ for $C > 0$ independent of n. Suppose u_n blow-up at some point. Then we have x_n and $\sigma_n \to \infty$ such that u_n behavior like $\sigma_n^{\frac{N-2}{2}} U(\sigma_n(x - x_n))$ near x_n.

Let $\alpha > \frac{1}{2}$, $B_n = B_{t_n \sigma_n^{-\alpha}}(x_n) \bigcap \Omega$. We have a local Phozaev identity

$$\begin{aligned}
&\lambda \int_{B_n} u_n^2 + \Big(\frac{N}{2^* - \varepsilon_n} - \frac{N-2}{2}\Big) \int_{B_n} Q(x)|u_n|^{2^* - \varepsilon_n} \\
&- \frac{1}{2} \int_{B_n} \langle Da(x), x - x_0 \rangle |Du_n|^2 + \frac{1}{2^* - \varepsilon_n} \int_{B_n} \langle DQ(x), x - x_0 \rangle |u_n|^{2^* - \varepsilon_n} \\
= &\int_{\partial B_n} \Big(a(x) \langle Du_n, x - x_0 \rangle + \frac{N-2}{2} a(x) u_n \Big) D_i u_n n_i \\
&- \int_{\partial B_n} \Big(\frac{1}{2} a(x) |Du_n|^2 - \frac{1}{2} \lambda u_n^2 - \frac{1}{2^* - \varepsilon_n} Q(x) |u_n|^{2 - \varepsilon_n} \Big) \langle n, x - x_0 \rangle,
\end{aligned}$$

which deduces from $2^* - \varepsilon_n - \frac{N-2}{2} > 0$

$$\begin{aligned}
&\lambda \int_{B_n} |u_n|^2 \, dx \\
\le &\int_{\partial B_n} \Big(a(x) \langle Du_n, x - x_0 \rangle + \frac{N-2}{2} a(x) u_n \Big) D_i u_n n_i \\
&- \int_{\partial B_n} \Big(\frac{1}{2} a(x) |Du_n|^2 - \frac{1}{2} \lambda u_n^2 - \frac{1}{2^* - \varepsilon_n} Q(x) |u_n|^{2 - \varepsilon_n} \Big) \langle n, x - x_0 \rangle.
\end{aligned}$$

We try to obtain a contradiction by estimating each term in the above inequality.

3. **Some Integral Estimates**

For any $\sigma > 0$ and $y \in \mathbb{R}^N$, we define

$$\rho_{y,\sigma}(u) = \sigma^{\frac{N}{2^*}} u\big(\sigma(\cdot - y)\big), \quad u \in H_0^1(\Omega).$$

First, we need the following decomposition result for the solutions of (2.4).

Proposition 3.1. *Suppose $N \geq 3$. Let u_n be a solution of (2.4) with $\varepsilon = \varepsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant C. Then*

(i) *u_n can be decomposed as*

$$u_n = u_0 + \sum_{j=1}^{k} \rho_{x_{n,j},\sigma_{n,j}}(U_j) + \omega_n. \tag{3.1}$$

where $\omega_n \to 0$ in $H^1(\Omega)$, u_0 is a solution for (2.1). For $j = 1, \cdots, k$, $x_{n,j} \in \Omega$, $\sigma_{n,j} d(x_{n,j}, \partial\Omega) \to +\infty$, $x_{n,j} \to x_j \in \bar{\Omega}$, as $n \to \infty$, and U_j is a solution of

$$\begin{cases} -a(x_j)\Delta u = b_j Q(x_j)|u|^{2^*-2}u, & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{3.2}$$

for some $b_j \in (0, 1]$.

(ii) *For $i, j = 1, \cdots, k$, if $i \neq j$, then, as $n \to \infty$,*

$$\frac{\sigma_{n,j}}{\sigma_{n,i}} + \frac{\sigma_{n,i}}{\sigma_{n,j}} + \sigma_{n,j}\sigma_{n,i}|x_{n,i} - x_{n,j}|^2 \to \infty. \tag{3.3}$$

To prove the strong convergence of u_n in $H^1(\Omega)$, we just need to show that the bubbles $\rho_{x_{n,j},\sigma_{n,j}}(U_j)$ will not appear in the decomposition of u_n.

Among all the bubbles $\rho_{x_{n,j},\sigma_{n,j}}(U_j)$, we can choose a bubble, such that this bubble has the slowest concentration rate. That is, the corresponding $\sigma_{n,i}$ is the lowest order infinity among all the $\sigma_{n,j}$ appearing in the bubbles. For simplicity, we denote σ_n the slowest concentration rate and x_n the corresponding concentration point.

Let $w_n(x) = |u_n(x)|$ in Ω; $w_n(x) = 0$ in $\mathbb{R}^N \setminus \Omega$. We have

Lemma 3.2. *Let D be any bounded domain in $\mathbb{R}^N$. Then for any $\phi \in H^1(\mathbb{R}^N)$ with $\phi \geq 0$,*

$$\int_D a(x) Dw_n D\phi \, dx - \int_{\partial D} a(x) \langle Dw_n, n \rangle \phi \leq A \int_D \big(w_n^{2^*-1} + 1 \big) \phi dx, \tag{3.4}$$

where $A > 0$ is a large constant, n is the outward unit normal of ∂D. In particular,

$$\int_{\mathbb{R}^N} a(x) Dw_n D\phi \, dx \leq A \int_{\mathbb{R}^N} \big(w_n^{2^*-1} + 1 \big) \phi dx, \quad \forall \, \phi \in C_0^\infty(\mathbb{R}^N), \ \phi \geq 0. \tag{3.5}$$

To analyze the asymptotic behaviors of the solutions of (2.4), we need some integral estimates for u_n.

For any $p_2 < 2^* < p_1$, $\alpha > 0$ and $\sigma > 0$, we consider the following relation:

$$\begin{cases} \|u_1\|_{p_1} \le \alpha, \\ \|u_2\|_{p_2} \le \alpha\sigma^{\frac{N}{2^*}-\frac{N}{p_2}}. \end{cases} \tag{3.6}$$

Define

$$\|u\|_{p_1,p_2,\sigma} = \inf\{\alpha > 0 : \text{ there are } u_1 \text{ and } u_2 \text{, such that } (3.6) \text{ holds and } |u| \le u_1 + u_2\}.$$

The main result of this section is the following proposition.

Proposition 3.3. *Let w_n be a weak solution of* (3.5)*. For any $p_2 < 2^* < p_1 < +\infty$, there is a constant C, depending on p_1 and p_2, such that*

$$\|w_n\|_{p_1,p_2,\sigma_n} \le C.$$

Proof. By using the estimates in Appendix A, we can prove Proposition 3.3 in exactly the same way as in the paper of Devillanova-Solimini.

□

4. **Estimates on Safe Regions**

Let $B_t(y)$ be the open ball centered at y with radius $t > 0$ and Denote $B_t^*(y) = B_{\sqrt{a(y)}t}(y)$.

Let $\alpha \in \left(\frac{1}{2}, \frac{N-4}{N-2}\right)$ be a constant, which is to be determined later. Since the number of the bubbles of u_n is finite, we may always find a constant $\bar{C} > 0$, independent of n, such that the region

$$\mathcal{A}_n^1 = \left(B^*_{(\bar{C}+5)\sigma_n^{-\alpha}}(x_n) \setminus B^*_{\bar{C}\sigma_n^{-\alpha}}(x_n)\right) \cap \Omega,$$

does not contain any concentration point of u_n for any n. We call this region a safe region for u_n.

Let

$$\mathcal{A}_n^2 = \left(B^*_{(\bar{C}+4)\sigma_n^{-\alpha}}(x_n) \setminus B^*_{(\bar{C}+1)\sigma_n^{-\alpha}}(x_n)\right) \cap \Omega.$$

In this section, we will prove the following result.

Proposition 4.1. *Let w_n be a weak solution of* (3.5). *Then, there is a constant $C > 0$, independent of n, such that*

$$w_n(x) \le C\sigma_n^{(N-2)(\alpha-\frac{1}{2})}, \quad \forall\, x \in \mathcal{A}_n^2.$$

To prove Proposition 4.1, we need the following lemma.

Lemma 4.2. *Suppose that w_n satisfies* (3.5). *Then, there is a constant $C > 0$, independent of n, such that*

$$\frac{1}{r^{N-1}} \int_{\partial B_r^*(x_n)} w_n \, dS \le C\sigma_n^{(N-2)(\alpha-\frac{1}{2})},$$

for all $r \in [\bar{C}\sigma_n^{-\alpha}, (\bar{C}+5)\sigma_n^{-\alpha}]$.

Proof. From $\int_{B_1(x_n)} w_n dx \le C$, we can find a $r_n \in \left[\frac{1}{2}, 1\right]$, such that

$$\frac{1}{r_n^{N-1}} \int_{\partial B_{r_n}(x_n)} w_n dS \le C.$$

We make use of the following formula:
For any C^2 function v, we have

$$\int_{r_n}^{r} \frac{d}{dt}\Big(\frac{1}{t^{N-1}}\int_{B_t(x_n)} v dx\Big)\, dt = \int_{r_n}^{r} \frac{1}{\omega_N t^{N-1}}\int_{B_t(x_n)} \Delta v dx\, dt,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. As a result, there are constants $C_1 > 0$, such that

$$\int_{r_n}^{r} \frac{d}{dt}\Big(\frac{1}{t^{N-1}}\int_{B_t^*(x_n)} v dx\Big)\, dt = \int_{r_n}^{r} \frac{C_1}{\omega_N t^{N-1}}\int_{B_t^*(x_n)} a(x_n)\Delta v dx\, dt. \tag{4.1}$$

Since w_n does not belong to C^2, we need to modify it.
Anyway we can obtain

$$\begin{aligned}&\frac{1}{r^{N-1}}\int_{\partial B_r^*(x_n)} w_n\, dS\\ =&\frac{1}{r_n^{N-1}}\int_{\partial B_{r_n}^*(x_n)} w_n\, dS + \int_{r}^{r_n} \frac{C_1}{\omega_N t^{N-1}}\int_{\partial B_t^*(x_n)} (-a(x_n)D_i w_n n_i) dS\, dt,\end{aligned} \tag{4.2}$$

which, together with Lemma 3.2, gives

$$\begin{aligned}
&\frac{1}{r^{N-1}}\int_{\partial B_r^*(x_n)} w_n\,dS\\
=&\frac{1}{r_n^{N-1}}\int_{\partial B_{r_n}^*(x_n)} w_n\,dS+\int_r^{r_n}\frac{C_1}{\omega_N t^{N-1}}\int_{\partial B_t^*(x_n)}(-a(x)D_iw_nn_i)dS\,dt\\
&+\int_r^{r_n}\frac{C_1}{\omega_N t^{N-1}}\int_{\partial B_t^*(x_n)}(a(x)-a(x_n))D_iw_nn_idS\,dt\\
\le& C+C\int_r^{r_n}\frac{1}{t^{N-1}}\int_{B_t^*(x_n)}(w_n^{2^*-1}+1)dx\,dt\\
&+C\int_r^{r_n}\frac{1}{t^{N-2}}\int_{\partial B_t^*(x_n)}|Dw_n|dS\,dt\\
=&C+C\int_r^{r_n}\frac{1}{t^{N-1}}\int_{B_t^*(x_n)}(w_n^{2^*-1}+1)dx\,dt\\
&+C\int_r^{r_n}\frac{1}{t^{N-2}}\int_{\partial B_t^*(x_n)}|Du_n|dS\,dt.
\end{aligned} \tag{4.3}$$

By the Sobolev embedding theorem,

$$\int_{\partial B_t^*(x_n)}|Du_n|dS\le C\int_{B_t^*(x_n)}|D^2u_n|+\frac{C}{t^2}\int_{B_t^*(x_n)}|u_n|. \tag{4.4}$$

Let $\bar{\theta} > 0$ be a fixed small constant and let $q = 1 + \bar{\theta} > 1$. By the L^q estimate for the elliptic equation, we have

$$\begin{aligned}
&\int_{B_t^*(x_n)} |D^2 u_n| \le t^{N(1-\frac{1}{q})}\Big(\int_{B_t^*(x_n)} |D^2 u_n|^q\Big)^{1/q} \\
\le& C t^{N(1-\frac{1}{q})}\Big(\int_{B_{2t}^*(x_n)} \big(|u_n|^q + (|u_n|^{2^*-1}+1)^q\big)\Big)^{1/q} \\
\le& C t^{N(1-\frac{1}{q})}\Big(\int_{B_{2t}^*(x_n)} \big(|u_n|^{(2^*-1)q}+1\big)\Big)^{1/q}.
\end{aligned} \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we are led to

$$\begin{aligned}
&\frac{1}{r^{N-1}}\int_{\partial B_r^*(x_n)} w_n\, dS \\
\le& C + C\int_r^{r_n} \frac{1}{t^{N-1}}\int_{B_t^*(x_n)} w_n^{2^*-1} dx\, dt \\
&+ C\int_r^{r_n} \frac{t^{N(1-1/q)}}{t^{N-2}}\Big(\int_{B_{2t}^*(x_n)} |u_n|^{(2^*-1)q} dx\Big)^{1/q} dt \\
&+ C\int_r^{r_n} \frac{1}{t^N}\int_{B_{2t}^*(x_n)} |u_n| dx\, dt.
\end{aligned} \tag{4.6}$$

We can use Proposition 3.3 to estimate each term in the right hand side of (4.6) and finish our proof of Lemma 4.2.

□

Proof of Proposition 4.1. It follows from Lemma 4.2 that for any $y \in \mathcal{A}_n^2$, we have

$$\frac{1}{\sigma_n^{\alpha N}}\int_{B^*_{\bar{C}\sigma_n^{-\alpha}(y)}} |w_n| \le C\sigma_n^{(N-2)(\alpha-\frac{1}{2})}.$$

Let
$$v_n(x) = w_n(\bar{C}\sigma_n^{-\alpha}x + y), \quad x \in \Omega_n,$$
where $\Omega_n = \{x : \bar{C}\sigma_n^{-\alpha}x + y \in \Omega\}$. Then v_n satisfies $\forall\ \eta \in H^1_{loc}(\mathbb{R}^N),\ \eta \geq 0$
$$\int_{\mathbb{R}^N} a(\bar{C}\sigma_n^{-\alpha}x + y)Dv_n D\eta \leq C\sigma_n^{-2\alpha}\int_{\mathbb{R}^N}\big(|v_n|^{2^*-2} + \lambda\big)v_n\eta.$$

Since $B^*_{\sigma_n^{-\alpha}}(y)$, $y \in \mathcal{A}_n^2$, does not contains any concentration point of u_n, we can deduce
$$\int_{B_1(0)} |\sigma_n^{-2\alpha}\big(|v_n|^{2^*-2}+\lambda\big)|^{\frac{N}{2}}dx \leq C\int_{B_{\bar{C}\sigma_n^{-\alpha}(y)}} |u_n|^{2^*}dx + C\sigma_n^{-\alpha N} \to 0$$
as $n \to +\infty$. Thus, by Moser iteration, we obtain
$$\|v_n\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C\Big(\int_{B_1(0)} |v_n|dx + 1\Big) = C\Big(\frac{1}{\sigma_n^{\alpha N}}\int_{B^*_{\bar{C}\sigma_n^{-\alpha N}(y)}} |w_n|dx + 1\Big)$$
$$\leq C\sigma_n^{(N-2)(\alpha-\frac{1}{2})}.$$

□

Let

$$\mathcal{A}_n^3 = \Big(B^*_{(\bar{C}+3)\sigma_n^{-\alpha}}(x_n) \setminus B^*_{(\bar{C}+2)\sigma_n^{-\alpha}}(x_n) \Big) \cap \Omega.$$

Proposition 4.3. *We have*

$$\int_{\mathcal{A}_n^3} |Du_n|^2 dx \leq C \int_{\mathcal{A}_n^2} \big(|u_n|^{2^*} + 1) dx + C\sigma_n^{2\alpha} \int_{\mathcal{A}_n^2} |u_n|^2 dx. \quad (4.7)$$

In particular,

$$\int_{\mathcal{A}_n^3} |Du_n|^2 dx \leq C\sigma_n^{-(N-2)(1-\alpha)}. \quad (4.8)$$

Corollary 4.4. *There exists* $t_n \in [\bar{C}+2, \bar{C}+3]$*, such that*

$$\int_{\partial B^*_{t_n\sigma_n^{-\alpha}}(x_n)} |Du_n|^2 dS \leq C\sigma_n^{-(N-2)(1-\alpha)+\alpha}. \quad (4.9)$$

5. **Local Pohozaev Identities and Location of Concentration Point**

In this section, we will first find two identities by applying the general Pucci and Serrin identity. We then use one of these two identities to locate the concentration points. The other one will be used to prove the main result in the next section.

Lemma 5.1. *Suppose that u_ε is a solution of* (2.4)*. Then for any bounded domain B contained in Ω,*

$$\lambda\int_B u_\varepsilon^2+\Big(\frac{N}{2^*-\varepsilon}-\frac{N-2}{2}\Big)\int_B Q(x)|u_\varepsilon|^{2^*-\varepsilon}$$
$$-\frac{1}{2}\int_B\langle Da(x),x-x_0\rangle|Du_\varepsilon|^2+\frac{1}{2^*-\varepsilon}\int_B\langle DQ(x),x-x_0\rangle|u_\varepsilon|^{2^*-\varepsilon}$$
$$=\int_{\partial B}\Big(a(x)\langle Du_\varepsilon,x-x_0\rangle+\frac{N-2}{2}a(x)u_\varepsilon\Big)D_iu_\varepsilon n_i$$
$$-\int_{\partial B}\Big(\frac{1}{2}a(x)|Du_\varepsilon|^2-\frac{1}{2}\lambda u_\varepsilon^2-\frac{1}{2^*-\varepsilon}Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)\langle n,x-x_0\rangle$$

where n is the outward unit normal vector of ∂B.

Lemma 5.2. *Suppose that u_ε is a solution of* (2.4)*. Then for any bounded domain B contained in Ω, $k=1,\cdots,N$,*

$$\frac{1}{2}\int_B D_ka(x)|Du_\varepsilon|^2-\frac{1}{2^*-\varepsilon}\int_B D_kQ(x)|u_\varepsilon|^{2^*-\varepsilon}$$
$$=\int_{\partial B}\Big(\frac{1}{2}a(x)|Du_\varepsilon|^2-\frac{1}{2}\lambda u_\varepsilon^2-\frac{1}{2^*-\varepsilon}Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)n_k$$
$$-\int_{\partial B}\Big(a(x)D_ku_\varepsilon+\frac{N-2}{2}u_\varepsilon a(x)-\frac{N-2}{2}a(x)\Big)D_iu_\varepsilon n_i.$$

where n is the outward unit normal vector of ∂B.

In the rest of this section and whole the next section, we will always assume that u_{ε_n} is a solution of (2.4) with $\varepsilon = \varepsilon_n$ and x_n is the concentration point of u_{ε_n} chosen as in section 2. To simplify notation we will write ε_n and u_{ε_n} as ε and u_ε respectively. Now, we locate the limit position of the concentration points.

Lemma 5.3. *Suppose that $x_n \to x_0$. Then x_0 is a critical point of $\Sigma(x)$.*

Proof. Taking $B = B^*_{t_n\sigma_n^{-\alpha}}(x_n)$ in Lemma 5.1, then from Proposition 4.1 and Corollary 4.4 we obtain

$$\begin{aligned}
&\frac{1}{2}\int_B D_k a(x)|Du_\varepsilon|^2 - \frac{1}{2^*-\varepsilon}\int_B D_k Q(x)|u_\varepsilon|^{2^*-\varepsilon}\\
=&O\Big(\int_{\partial B}\big(|Du_\varepsilon|^2 + |u_\varepsilon|^{2-\varepsilon} + 1\big)\Big)\\
=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha} + \sigma_n^{-N+(N+1)\alpha+o(1)} + \sigma_n^{-(N-1)\alpha}\Big)\\
=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha}\Big).
\end{aligned} \tag{5.1}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{2}\int_B D_k a(x)|Du_\varepsilon|^2 - \frac{1}{2^*-\varepsilon}\int_B D_k Q(x)|u_\varepsilon|^{2^*-\varepsilon}\\
=&\frac{1}{2}\int_B D_k a(x_n)|Du_\varepsilon|^2 - \frac{1}{2^*-\varepsilon}\int_B D_k Q(x_n)|u_\varepsilon|^{2^*-\varepsilon}\\
&+O\Big(|D^2a(x_n)|\sigma_n^{-\alpha} + |D^3a(x_n)|\sigma_n^{-2\alpha} + \sigma_n^{-3\alpha}\Big)\\
&+O\Big(|D^2Q(x_n)|\sigma_n^{-\alpha} + |D^3Q(x_n)|\sigma_n^{-2\alpha} + \sigma_n^{-3\alpha}\Big).
\end{aligned} \tag{5.2}$$

Moreover,

$$
\begin{aligned}
&\int_B D_k a(x_n)|Du_\varepsilon|^2 = \frac{D_k a(x_n)}{a(x_n)} \int_B a(x_n)|Du_\varepsilon|^2 \\
=&\frac{D_k a(x_n)}{a(x)} \int_B a(x_n)|Du_\varepsilon|^2 + o(1) \\
=&\frac{D_k a(x_n)}{a(x)} \Big(\int_B \big(Q(x)|u_\varepsilon|^{2^*-\varepsilon} + \lambda u_\varepsilon\big) + \int_{\partial B} a(x) D_i u_\varepsilon n_i \Big) + o(1) \\
&= \frac{Q(x_n) D_k a(x_n)}{a(x)} \int_B |u_\varepsilon|^{2^*-\varepsilon} + o(1).
\end{aligned} \tag{5.3}
$$

Combining (5.1), (5.2) and (5.3), we are led to

$$
\Big(\frac{1}{2} \frac{Q(x_n) D_k a(x_n)}{a(x)} - \frac{1}{2^* - \varepsilon} D_k Q(x_n) \Big) \int_B |u_\varepsilon|^{2^*-\varepsilon} = o(1). \tag{5.4}
$$

Letting $n \to +\infty$, we obtain

$$
Q(x_0) D_k a(x_0) - \frac{N-2}{N} a(x_0) D_k Q(x_0) = 0.
$$

So, the result follows. □

Let $\gamma = \min\big(\frac{m(x_0)-2}{m(x_0)-1}, \frac{q(x_0)-2}{q(x_0)-1}\big)$.

Proposition 5.4. *Suppose that $x_n \to x_0 \in \Omega$. Suppose that (C_1) and (C_2) hold. Then*

$$
|Da(x_n)| + |DQ(x_n)| = O\Big(\sigma_n^{-(N-2)+(N-1)\alpha} + \sigma_n^{-\frac{\alpha}{1-\gamma}} + \sigma_n^{-3\alpha} \Big).
$$

Proof. Combining (5.1) and (5.2), we obtain

$$\begin{aligned}&\frac{1}{2}D_k a(x_n)\int_B |Du_\varepsilon|^2-\frac{1}{2^*-\varepsilon}D_kQ(x_n)\int_B|u_\varepsilon|^{2^*-\varepsilon}\\=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha}\Big)+O\Big(|D^2a(x_n)|\sigma_n^{-\alpha}+|D^3a(x_n)|\sigma_n^{-2\alpha}+\sigma_n^{-3\alpha}\Big)\\&+O\Big(|D^2Q(x_n)|\sigma_n^{-\alpha}+|D^3Q(x_n)|\sigma_n^{-2\alpha}+\sigma_n^{-3\alpha}\Big).\end{aligned}\tag{5.5}$$

It follows from (C_1) that

$$\begin{aligned}&|Da(x_n)|+|DQ(x_n)|\\=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha}\Big)+O\Big(|D^2a(x_n)|\sigma_n^{-\alpha}+|D^3a(x_n)|\sigma_n^{-2\alpha}+\sigma_n^{-3\alpha}\Big)\\&+O\Big(|D^2Q(x_n)|\sigma_n^{-\alpha}+|D^3Q(x_n)|\sigma_n^{-2\alpha}+\sigma_n^{-3\alpha}\Big).\end{aligned}\tag{5.6}$$

Let $\gamma_1=\frac{m(x_0)-2}{m(x_0)-1}$, and $\gamma_2=\frac{q(x_0)-2}{q(x_0)-1}$. By (C_2), (5.6) becomes

$$\begin{aligned}&|Da(x_n)|+|DQ(x_n)|\\=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha}\Big)+O\Big(|Da(x_n)|^{\gamma_1}\sigma_n^{-\alpha}+|Da(x_n)|^{2\gamma_1-1}\sigma_n^{-2\alpha}+\sigma_n^{-3\alpha}\Big)\\&+O\Big(|DQ(x_n)|^{\gamma_2}\sigma_n^{-\alpha}+|DQ(x_n)|^{2\gamma_2-1}\sigma_n^{-2\alpha}+\sigma_n^{-3\alpha}\Big),\end{aligned}$$

from which, we obtain

$$\begin{aligned}&|Da(x_n)|+|DQ(x_n)|\\=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha}+\sigma_n^{-\alpha/(1-\gamma_1)}+\sigma_n^{-\alpha/(1-\gamma_2)}+\sigma_n^{-3\alpha}\Big)\\=&O\Big(\sigma_n^{-(N-2)+(N-1)\alpha}+\sigma_n^{-\alpha/(1-\gamma)}+\sigma_n^{-3\alpha}\Big).\end{aligned}$$

□

6. **Proof of the Main Result**

In this section, we will prove Theorem 2.5.

Proof of Theorem 2.5. We have two different cases: (i) $d(x_n, \partial\Omega) \leq \tau$; (ii) $d(x_n, \partial\Omega) \geq \tau$, where $\tau > 0$ is the constant in (C_3).

Let $B_n = B_{t_n\sigma_n^{-\alpha}}(x_n) \bigcap \Omega$. By Lemma 5.1

$$\begin{aligned}
&\lambda \int_{B_n} u_\varepsilon^2 + \big(\frac{N}{2^* - \varepsilon} - \frac{N-2}{2}\big) \int_{B_n} Q(x)|u_\varepsilon|^{2^*-\varepsilon} \\
&- \frac{1}{2}\int_{B_n} \langle Da(x), x - x_0\rangle |Du_\varepsilon|^2 + \frac{1}{2^* - \varepsilon}\int_{B_n} \langle DQ(x), x - x_0\rangle |u_\varepsilon|^{2^*-\varepsilon} \\
= &\int_{\partial B_n} \Big(a(x)\langle Du_\varepsilon, x - x_0\rangle + \frac{N-2}{2} a(x) u_\varepsilon\Big) D_i u_\varepsilon n_i \\
&- \int_{\partial B_n} \Big(\frac{1}{2} a(x)|Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)\langle n, x - x_0\rangle,
\end{aligned} \tag{6.1}$$

where ν is the outward normal to ∂B_n. The point x_0 in (6.1) is chosen as follows. In case (i), we take $x_0 = x_n + 2t_n\sigma_n^{-\alpha}\nu$, where ν is the outward unit normal of $\partial\Omega$ at $\bar{x}_n$ $|\bar{x}_n - x_n| = d(x_n, \partial\Omega)$. In case (ii), we take a point $x_0 = x_n$.

We first consider case (i).

Since $2^* - \varepsilon < 2^*$, the second term in the left hand side of (6.1) is non-negative. By the choice of x_0 and (C_3), the last two terms in the left hand side of (6.1) are non-negative as well. We thus obtain from (6.1) that

$$\lambda\int_{B_n}|u_n|^2\,dx$$
$$\leq\int_{\partial B_n}\Big(a(x)\langle Du_\varepsilon,x-x_0\rangle+\frac{N-2}{2}a(x)u_\varepsilon\Big)D_iu_\varepsilon n_i$$
$$-\int_{\partial B_n}\Big(\frac{1}{2}a(x)|Du_\varepsilon|^2-\frac{1}{2}\lambda u_\varepsilon^2-\frac{1}{2^*-\varepsilon}Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)\langle n,x-x_0\rangle. \tag{6.2}$$

Now we decompose ∂B_n into

$$\partial B_n=\partial_iB_n\cup\partial_eB_n,$$

where $\partial_iB_n=\partial B_n\cap\Omega$ and $\partial_eB_n=\partial B_n\cap\partial\Omega$.
Noting $u_n=0$ on $\partial\Omega$, we find

$$\int_{\partial_eB_n}\Big(a(x)\langle Du_\varepsilon,x-x_0\rangle+\frac{N-2}{2}a(x)u_\varepsilon\Big)D_iu_\varepsilon n_i$$
$$-\int_{\partial_eB_n}\Big(\frac{1}{2}a(x)|Du_\varepsilon|^2-\frac{1}{2}\lambda u_\varepsilon^2-\frac{1}{2^*-\varepsilon}Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)\langle n,x-x_0\rangle$$
$$=\frac{1}{2}\int_{\partial_eB_n}a(x)|Du_n|^2\langle n,x-x_0\rangle\leq 0. \tag{6.3}$$

So, we can rewrite (6.2) as

$$
\begin{aligned}
&\lambda \int_{B_n} |u_n|^2\,dx \\
\le &\int_{\partial_i B_n} \Big(a(x)\langle Du_\varepsilon, x-x_0\rangle + \frac{N-2}{2}a(x)u_\varepsilon\Big) D_i u_\varepsilon n_i \\
&- \int_{\partial_i B_n} \Big(\frac{1}{2}a(x)|Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^*-\varepsilon}Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)\langle n, x-x_0\rangle.
\end{aligned} \tag{6.4}
$$

Now we consider case (ii).

By Proposition 5.4 and (C_2), we obtain

$$
\begin{aligned}
&\Big|\int_{B_n} \langle Da(x), x-x_0\rangle |Du_\varepsilon|^2\Big| + \Big|\int_{B_n} \langle DQ(x), x-x_0\rangle |u_\varepsilon|^{2^*-\varepsilon}\Big| \\
=&O\Big(|Da(x_n)|\sigma_n^{-\alpha} + |D^2a(x_n)|\sigma_n^{-2\alpha} + |D^3a(x_n)|\sigma_n^{-3\alpha} + \sigma_n^{-4\alpha}\Big) \\
&+ O\Big(|DQ(x_n)|\sigma_n^{-\alpha} + |D^2Q(x_n)|\sigma_n^{-2\alpha} + |D^3Q(x_n)|\sigma_n^{-3\alpha} + \sigma_n^{-4\alpha}\Big) \\
=&O\Big(\sigma_n^{-\frac{2-\gamma}{1-\gamma}\alpha} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha}\Big).
\end{aligned} \tag{6.5}
$$

As a result, we can rewrite (6.2) as

$$
\begin{aligned}
&\lambda \int_{B_n} |u_n|^2\,dx \\
&\leq \int_{\partial_i B_n} \Big(a(x)\langle Du_\varepsilon, x - x_0\rangle + \frac{N-2}{2} a(x) u_\varepsilon \Big) D_i u_\varepsilon n_i \\
&\quad - \int_{\partial_i B_n} \Big(\frac{1}{2} a(x)|Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x)|u_\varepsilon|^{2-\varepsilon} \Big) \langle n, x - x_0\rangle \\
&\quad + O\Big(\sigma_n^{-\frac{2-\gamma}{1-\gamma}\alpha} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha} \Big).
\end{aligned}
\tag{6.6}
$$

So, we have proved that in both cases, we always have

$$
\begin{aligned}
&\lambda \int_{B_n} |u_n|^2\,dx \\
&\leq \int_{\partial_i B_n} \Big(a(x)\langle Du_\varepsilon, x - x_0\rangle + \frac{N-2}{2} a(x) u_\varepsilon \Big) D_i u_\varepsilon n_i \\
&\quad - \int_{\partial_i B_n} \Big(\frac{1}{2} a(x)|Du_\varepsilon|^2 - \frac{1}{2}\lambda u_\varepsilon^2 - \frac{1}{2^* - \varepsilon} Q(x)|u_\varepsilon|^{2-\varepsilon} \Big) \langle n, x - x_0\rangle \\
&\quad + O\Big(\sigma_n^{-\frac{2-\gamma}{1-\gamma}\alpha} + \sigma_n^{-(N-2)(1-\alpha)} + \sigma_n^{-4\alpha} \Big).
\end{aligned}
\tag{6.7}
$$

Using Corollary 4.4, noting that $|x-x_0|\le C\sigma_n^{-\alpha}$ for $x\in\partial_i B_n$, we see

$$\begin{aligned}
&\int_{\partial_i B_n}\Big(a(x)\langle Du_\varepsilon, x-x_0\rangle+\frac{N-2}{2}a(x)u_\varepsilon\Big)D_iu_\varepsilon n_i\\
&\quad-\int_{\partial_i B_n}\Big(\frac{1}{2}a(x)|Du_\varepsilon|^2-\frac{1}{2}\lambda u_\varepsilon^2-\frac{1}{2^*-\varepsilon}Q(x)|u_\varepsilon|^{2-\varepsilon}\Big)\langle n, x-x_0\rangle\\
\le& C\sigma_n^{-\alpha}\int_{\partial_i B_n}\big(|u_n|^{2^*-\varepsilon}+u_n^2+|Du_n|^2\big)\,d\sigma+C\int_{\partial_i B_n}|Du_n||u_n|d\sigma,\\
\le& O\Big(\sigma_n^{-N(1-\alpha)+o(1)}+\sigma_n^{-(N-2)+(N-4)\alpha}+\sigma_n^{-(N-2)(1-\alpha)}\Big)\\
=&O\Big(\sigma_n^{-(N-2)(1-\alpha)}\Big),
\end{aligned}\tag{6.8}$$

which, together with (6.7), implies

$$\int_{B_n}|u_n|^2\,dx\le +O\Big(\sigma_n^{-\frac{2-\gamma}{1-\gamma}\alpha}+\sigma_n^{-(N-2)(1-\alpha)}+\sigma_n^{-4\alpha}\Big).\tag{6.9}$$

Since

$$\int_{B_n}|u_n|^2\,dx\ge c'\sigma_n^{-2},$$

for some $c'>0$, we obtain from (6.9),

$$\sigma_n^{-2}\le C\Big(\sigma_n^{-\frac{2-\gamma}{1-\gamma}\alpha}+\sigma_n^{-(N-2)(1-\alpha)}+\sigma_n^{-4\alpha}\Big).\tag{6.10}$$

Choose $\alpha=\frac{N-4}{N-2}-\theta$, where $\theta>0$ is a small constant. Then $\alpha>\frac{1}{2}$, since $N\ge 7$. So, we have

$$4\alpha>2,\quad (N-2)(1-\alpha)>2.$$

Moreover, from

$$\gamma \geq \min_{x_0 \in S} \Big(\min \big(\frac{m(x_0)-2}{m(x_0)-1}, \frac{q(x_0)-2}{q(x_0)-1} \big) \Big),$$

and

$$\min_{x_0 \in S} \big(\min(m(x_0), q(x_0)) \big) > \frac{2(N-2)}{N-4},$$

we deduce

$$\gamma > \frac{4}{N}.$$

So,

$$\frac{2-\gamma}{1-\gamma}\alpha > 2.$$

Thus, (6.10) is a contradiction. So, we have proved Theorem 2.5.

□

Proof of Theorem 2.6. As pointed out in the introduction, by the result of Ambrosetti and Rabinowitz [1], for any given positive integer k and $\varepsilon > 0$ small, (2.4) has a solution $u_{k,\varepsilon}$ such that $I_\varepsilon(u_{k,\varepsilon}) = c_{k,\varepsilon}$ and $c_{k,\varepsilon} \to +\infty$ as $k \to +\infty$ independent of ε small. From Theorem 2.5 we can choose subsequence $\{u_{k,\varepsilon_n}\}$ such that $u_{k,\varepsilon_n} \to u_k$ strongly in $H_0^1(\Omega)$ for some u_k and $c_{k,\varepsilon_n} \to c_k$. u_k is a solution of (2.1) and $I_0(u_k) = c_k$. Since c_k goes to ∞ we get infinitely many solutions.

□

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