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On Complex Finsler Manifolds

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Contents

1. Introduction

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- In [4] S.Kobayashi gave two good reasons for considering complex Finsler structures in a complex manifold. One is that every hyperbolic complex manifold M carries a natural complex Finsler metric in a broad sense. The second reason is as differential geometric tool in the study of complex vector bundles.
- There are many very famous classical metrics on the Teichmüller and the moduli spaces, among which there are three complex Finsler metrics: Teichmüller metric; Caratheodory metric; and Kobayashi metric.
- Recently, J.-G. Cao and Pit-Mann Wong ([1]) studied Finsler geometry of projective vector bundle and proposed the following question: Suppose that M is a Kähler manifold and E is a holomorphic vector bundle over M. Is E Kähler? They gave some partial results and showed some equivalent conditions for E to be Kähler.

- Let E be a holomorphic vector bundle of rank r over a complex manifold M of complex dimension n with the natural projection π. We denote a point of E by (z, v), where z represents a point of M and v is a vector in the fibre $E_z = \pi^{-1}(z)$ of E over $z \in M$. Let $o : M \to E$ be the zero section of E and set $E^o = E \setminus \{o\}$.

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- **Definition.** A complex Finsler metric on E is a real function $G : E \to \mathbf{R}$ which satisfies the following conditions:
(1) $G(z, v) \geq 0$, where the equality holds if and only if $v = 0$;
(2) $G \in C^\infty(E^o)$, that is, G is smooth in E^o;
(3) $G(z, \lambda v) = |\lambda|^2 G(z, v)$ for all $(z, v) \in E$, $\lambda \in \mathbf{C} \setminus \{0\}$.

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(3) $G(z, \lambda v) = |\lambda|^2 G(z, v)$ for all $(z, v) \in E$, $\lambda \in \mathbf{C} \setminus \{0\}$.
- **Theorem(Shen-Du,[6]). Let (M, G) be a complex Finsler manifold of dimension n and TM its holomorphic tangent bundle. Then the Hermitian metric**

$$h_{TM} = G_{i\bar{j}}(z, v) dz^i \otimes d\bar{z}^j + G_{i\bar{j}}(z, v) \delta v^i \otimes \delta \bar{v}^j$$

on TM is Kählerian if and only if (M, G) is a Kähler manifold with zero holomorphic sectional curvature, where $G_{i\bar{j}} = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^j}$, $1 \leq i, j, \cdots \leq n$.
- **Moreover, we shall consider complex Randers metrics.**

2. Hermitian Metrics

- Let $(z, v) = (z^1, \cdots, z^n, v^1, \cdots, v^r)$ be a local coordinate system for E. A complex Finsler metric G on E is said to be strongly pseudo-convex if the complex Hessian

$$(G_{i\bar{j}}) = \left(\frac{\partial^2 G}{\partial v^i \partial \bar{v}^j} \right)$$

of G is positively definite on E^o. In particular, if $G(z, v) = h_{i\bar{j}}(z) v^i \bar{v}^j$ is a Hermitian metric on E, then $G(z, v)$ defines a strongly pseudo-convex Finsler metric on E.

$$1 \leq \alpha, \beta, \cdots \leq n; \qquad 1 \leq i, j, k, \cdots \leq r.$$

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$$1\leq \alpha,\beta,\cdots\leq n; \qquad 1\leq i,j,k,\cdots\leq r.$$

- Introduce the following notations:

$$G_i=\frac{\partial G}{\partial v^i},\quad G_{\bar{j}}=\frac{\partial G}{\partial \bar{v}^j},\quad G_{i\bar{j}}=\frac{\partial^2 G}{\partial v^i\partial \bar{v}^j},$$

$$G_{,\alpha}=\frac{\partial G}{\partial z^\alpha},\quad G_{,\bar{\beta}}=\frac{\partial G}{\partial \bar{z}^\beta},\quad G_{i,\bar{\alpha}}=\frac{\partial^2 G}{\partial v^i\partial \bar{z}^\alpha},\quad G_{i\bar{j},\bar{\beta}}=\frac{\partial G_{i\bar{j}}}{\partial \bar{z}^\beta},\quad etc.,$$

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$$1 \le \alpha, \beta, \cdots \le n; \qquad 1 \le i, j, k, \cdots \le r.$$

- Introduce the following notations:

$$G_i = \frac{\partial G}{\partial v^i}, \quad G_{\bar{j}} = \frac{\partial G}{\partial \bar{v}^j}, \quad G_{i\bar{j}} = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^j},$$

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- In particular, we can take $E = TM$, the holomorphic tangent bundle of M, so that $r = n$.

- Suppose that a strongly pseudoconvex complex Finsler metric $G(z,v)$ is given on TM. The pair (M,G) is called a complex Finsler manifold. Let $\tilde{M} = TM \setminus \{o\}$ denote TM without the zero section. $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial v^j}\}(1 \leq i,j \leq n)$ give a local frame field of the holomorphic tangent bundle $T\tilde{M}$ of $\tilde{M}$.

- Suppose that a strongly pseudoconvex complex Finsler metric $G(z,v)$ is given on TM. The pair (M,G) is called a complex Finsler manifold. Let $\tilde{M} = TM \setminus \{o\}$ denote TM without the zero section. $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial v^j}\}(1 \le i,j \le n)$ give a local frame field of the holomorphic tangent bundle $T\tilde{M}$ of $\tilde{M}$.
- Let $\tilde{\pi} : T\tilde{M} \to \tilde{M}$ denote the holomorphic tangent bundle of $\tilde{M}$. Then the differential $d\pi : T^{\mathbf{C}}\tilde{M} \to T^{\mathbf{C}}M$ of $\pi : \tilde{M} \to M$ defines the vertical bundle $\mathcal{V}$ over $\tilde{M}$ by

$$\mathcal{V} = \ker d\pi \cap T\tilde{M},$$

which yields a holomorphic vector bundle of rank n over $\tilde{M}$. A local frame field of $\mathcal{V}$ is given by $\{\frac{\partial}{\partial v^j}\}(1 \le j \le n)$, and a natural section $\iota : \tilde{M} \to \mathcal{V}$, called the radial vertical field, is well-defined for $(z,v) \in \tilde{M}$ by

$$\iota(v) = \iota(v^i(\frac{\partial}{\partial z^i})_z) = v^i(\frac{\partial}{\partial v^i})_v.$$

Associated with G, we now define a Hermite metric on the vertical bundle $\mathcal{V}$ by

$$< X, Y >_v = G_{i\bar{j}}(z,v) X^i \bar{Y}^j,$$

where $(z,v) \in \tilde{M}$ and $X, Y \in \mathcal{V}_v \cap \tilde{\pi}^{-1}(z,v)$.

- Let $D:\Gamma(\mathcal{V})\to\Gamma(T^{*\mathbf{C}}\tilde{M}\otimes\mathcal{V})$ be the Hermitian connection of the Hermitian vector bundle $(\mathcal{V},<,>)$, where $\Gamma(\cdot)$ denotes the space of smooth sections. Let ∇ denote the covariant differentiation defined by D, and define a bundle map $\bigwedge: T\tilde{M}\to\mathcal{V}$ by $\bigwedge(X)=\nabla_X\iota$. The horizontal bundle $\mathcal{H}$ over $\tilde{M}$ is then defined by $\mathcal{H}=\ker\bigwedge$, which is the subbundle of $T\tilde{M}$ consisting of vectors with respect to which ι is parallel. Then it is verified that

$$T\tilde{M}=\mathcal{V}\oplus\mathcal{H}$$

and a natural local frame field $\{\frac{\delta}{\delta z^i}\},(1\le i\le n)$ of $\mathcal{H}$ is given by

$$\frac{\delta}{\delta z^i}=\frac{\partial}{\partial z^i}-N_i^j\frac{\partial}{\partial v^j},\qquad N_j^i=G^{i\bar{l}}G_{\bar{l},\,j}=G^{i\bar{l}}\frac{\partial^2 G}{\partial\bar{v}^l\partial z^j}.$$

Thus we get a local frame field $\{\frac{\delta}{\delta z^i},\frac{\partial}{\partial v^i}\},(1\le i\le n)$ of $T\tilde{M}$. Let $\{dz^i,\delta v^i\}$ denote the dual frame field of $\{\frac{\delta}{\delta z^i},\frac{\partial}{\partial v^i}\}$, where

$$\delta v^i=dv^i+N_j^i dz^j.$$

- Associated with the decomposition $T\tilde{M} = \mathcal{V} \oplus \mathcal{H}$, we have the horizontal map $\Theta : \mathcal{V} \to \mathcal{H}$ given locally by $\Theta(\frac{\partial}{\partial v^i}) = \frac{\delta}{\delta z^i}$ for $1 \leq i \leq n$, and a natural section

$$\chi = \Theta \circ \iota : \tilde{M} \to \mathcal{H},$$

called the radial horizontal field, such that

$$\chi(v^i \frac{\partial}{\partial z^i}) = v^i \frac{\delta}{\delta z^i}.$$

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$$\chi(v^i \frac{\partial}{\partial z^i}) = v^i \frac{\delta}{\delta z^i}.$$

- Using the horizontal map $\Theta : \mathcal{V} \to \mathcal{H}$, we can transfer the Hermitian metric $<,>$ on $\mathcal{V}$ to $\mathcal{H}$ by setting

$$< X, Y >_v = < \Theta^{-1}(X), \Theta^{-1}(Y) >_v,$$

where $(z, v) \in \tilde{M}$, $X, Y \in \mathcal{H}_v \cap \tilde{\pi}^{-1}(z, v)$. Then a Hermitian metric h_{TM} on $\tilde{M}$ canonically associated with G is defined by requiring $\mathcal{H}$ to be orthogonal to $\mathcal{V}$, so that $\Theta : \mathcal{V} \to \mathcal{H}$ and $\chi : \tilde{M} \to \mathcal{H}$ are isometric embeddings. h_{TM} is given in local coordinates by

$$h_{TM} = G_{i\bar{j}}(z, v) dz^i \otimes d\bar{z}^j + G_{i\bar{j}}(z, v) \delta v^i \otimes \delta \bar{v}^j. \qquad (2.1)$$

3. Holomorphic Curvature

- Then the connection form $\omega = (\omega^i_j)$ of the Hermitian connection D of the Hermitian vector bundle $(\tilde{M}, h_{TM})$ is given by

$$\omega^i_j = G^{\bar{k}i}\partial G_{j\bar{k}} = \Gamma^i_{jk}dz^k + \gamma^i_{jk}dv^k, \tag{3.1}$$

where

$$\Gamma^i_{jk} = G^{\bar{l}i}\frac{\partial G_{j\bar{l}}}{\partial z^k}, \qquad \gamma^i_{jk} = G^{\bar{l}i}\frac{\partial G_{j\bar{l}}}{\partial v^k}.$$

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- Then the connection form $\omega = (\omega_j^i)$ of the Hermitian connection D of the Hermitian vector bundle $(\tilde{M}, h_{TM})$ is given by

$$\omega_j^i = G^{\bar{k}i}\partial G_{j\bar{k}} = \Gamma_{jk}^i dz^k + \gamma_{jk}^i dv^k, \tag{3.1}$$

where

$$\Gamma_{jk}^i = G^{\bar{l}i}\frac{\partial G_{j\bar{l}}}{\partial z^k}, \qquad \gamma_{jk}^i = G^{\bar{l}i}\frac{\partial G_{j\bar{l}}}{\partial v^k}.$$

- The curvature form $\Omega = (\Omega_j^i)$ of its curvature $R = D \circ D$ is given by $\Omega = (\Omega_j^i) = (\bar{\partial}\omega_j^i)$, which can be written as

$$\Omega_j^i = \kappa_{jk\bar{l}}^i dz^k \wedge d\bar{z}^l + \mu_{jk\bar{l}}^i dz^k \wedge d\bar{v}^l + \sigma_{jk\bar{l}}^i dv^k \wedge d\bar{z}^l + \tau_{jk\bar{l}}^i dv^k \wedge d\bar{v}^l,$$

where

$$\kappa_{jk\bar{l}}^i = -\frac{\partial \Gamma_{jk}^i}{\partial \bar{z}^l}, \qquad \mu_{jk\bar{l}}^i = -\frac{\partial \Gamma_{jk}^i}{\partial \bar{v}^l},$$

$$\sigma_{jk\bar{l}}^i = -\frac{\partial \gamma_{jk}^i}{\partial \bar{z}^l}, \qquad \tau_{jk\bar{l}}^i = -\frac{\partial \gamma_{jk}^i}{\partial \bar{v}^l}.$$

- Setting $\kappa_{i\bar{j}k\bar{l}} = G_{h\bar{j}}\kappa^h_{ik\bar{l}}$ and $\tau_{i\bar{j}k\bar{l}}v^i = G_{h\bar{j}}\tau^h_{ik\bar{l}}$, we have

$$\begin{aligned}\kappa_{i\bar{j}k\bar{l}}v^i\bar{v}^j &= (-G_{i\bar{j},\,k\bar{l}} + G^{h\bar{p}}G_{h\bar{j},\,\bar{l}}G_{\bar{p}i,\,k})v^i\bar{v}^j \\ &= -G_{,\,k\bar{l}} + G^{h\bar{p}}G_{h,\,\bar{l}}G_{\bar{p},\,k}, \qquad (3.2)\end{aligned}$$

$$\tau_{i\bar{j}k\bar{l}}v^i = \mu_{i\bar{j}k\bar{l}}v^i = \sigma_{i\bar{j}k\bar{l}}\bar{v}^j = 0.$$

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$$\begin{aligned} \kappa_{i\bar{j}k\bar{l}}v^i\bar{v}^j &= (-G_{i\bar{j},\,k\bar{l}} + G^{h\bar{p}}G_{h\bar{j},\,\bar{l}}G_{\bar{p}i,\,k})v^i\bar{v}^j \\ &= -G_{,\,k\bar{l}} + G^{h\bar{p}}G_{h,\,\bar{l}}G_{\bar{p},\,k}, \end{aligned} \tag{3.2}$$

$$\tau_{i\bar{j}k\bar{l}}v^i = \mu_{i\bar{j}k\bar{l}}v^i = \sigma_{i\bar{j}k\bar{l}}\bar{v}^j = 0.$$

- Corresponding to the decomposition $T\tilde{M} = \mathcal{V} \oplus \mathcal{H}$, The differential operator d on functions is decomposed as $d = d_{\mathcal{H}} + d_{\mathcal{V}}$. We also decompose $d_{\mathcal{H}}$ and $d_{\mathcal{V}}$ into (1,0)-part and (0,1)-part as

$$d_{\mathcal{H}} = \partial_{\mathcal{H}} + \bar{\partial}_{\mathcal{H}}, \quad and \quad d_{\mathcal{V}} = \partial_{\mathcal{V}} + \bar{\partial}_{\mathcal{V}}, \tag{3.3}$$

respectively, where we put $\partial_{\mathcal{H}} f = \frac{\delta f}{\delta z^i}dz^i$, $\partial_{\mathcal{V}} f = \frac{\partial f}{\partial v^i}\delta v^i$ for a C^∞ function $f(z, v)$ on TM.

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- **Definition.** Let (M,G) be a complex Finsler manifold. The fundamental form associated with G is

$$\Phi = \sqrt{-1}G_{i\bar{j}}dz^i \wedge d\bar{z}^j$$

which is a real (1,1)-form on $\tilde{M}$. (M,G) is called a Finsler-Kähler manifold if $d_{\mathcal{H}}\Phi = 0$.
It is equivalent to

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It is equivalent to

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- **Definition.** Let (M, G) be a complex Finsler manifold. The holomorphic curvature $K(z, v)$ of G along v is given by

$$K(z, v) = \frac{2\psi_{i\bar{j}} v^i \bar{v}^j}{G^2(z, v)}, \tag{3.4}$$

where

$$\psi_{i\bar{j}} = G_{k\bar{l}} N^k_i N^{\bar{l}}_{\bar{j}} - G_{, i\bar{j}}.$$

- **Lemma.** For the non-linear connection N^i_j we have the following formulas:

$$
\begin{aligned}
&(1) \qquad \frac{\partial N^p_i}{\partial \bar{v}^j} G_p = 0, \\
&(2) \qquad \frac{\partial N^p_i}{\partial \bar{z}^j} G_p = -\psi_{i\bar{j}}, \\
&(3) \qquad \frac{\partial N^p_i}{\partial \bar{v}^q} G_{p\bar{j}} = \frac{\partial N^p_i}{\partial \bar{v}^j} G_{p\bar{q}}, \\
&(4) \qquad -N^p_k \frac{\delta G_{p\bar{j}}}{\delta z^l} + \frac{\delta G_{\bar{j},\,k}}{\delta z^l} = -N^p_l \frac{\delta G_{p\bar{j}}}{\delta z^k} + \frac{\delta G_{\bar{j},\,l}}{\delta z^k}.
\end{aligned}
$$

- **Lemma.** For the non-linear connection N_j^i we have the following formulas:

$$(1)\qquad \frac{\partial N_i^p}{\partial \bar{v}^j} G_p = 0,$$

$$(2)\qquad \frac{\partial N_i^p}{\partial \bar{z}^j} G_p = -\psi_{i\bar{j}},$$

$$(3)\qquad \frac{\partial N_i^p}{\partial \bar{v}^q} G_{p\bar{j}} = \frac{\partial N_i^p}{\partial \bar{v}^j} G_{p\bar{q}},$$

$$(4)\qquad -N_k^p \frac{\delta G_{p\bar{j}}}{\delta z^l} + \frac{\delta G_{\bar{j},\,k}}{\delta z^l} = -N_l^p \frac{\delta G_{p\bar{j}}}{\delta z^k} + \frac{\delta G_{\bar{j},\,l}}{\delta z^k}.$$

- **Proof.** A straightforward calculation can prove the lemma.

4. Proof of Main Theorem

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- Let ω be the fundamental 2-form of the Hermitian metric h_{TM}, that is, ω is defined by

$$\omega(X,Y)=h_{TM}(X,JY),$$

for $X,Y\in T^c(TM)$. In a local coordinate system for TM, ω can be expressed as

$$\omega=-\sqrt{-1}G_{i\bar{j}}(z,v)dz^i\wedge d\bar{z}^j-\sqrt{-1}G_{i\bar{j}}(z,v)\delta v^i\wedge\delta\bar{v}^j.$$

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for $X, Y \in T^c(TM)$. In a local coordinate system for TM, ω can be expressed as

$$\omega = -\sqrt{-1}G_{i\bar{j}}(z,v)dz^i \wedge d\bar{z}^j - \sqrt{-1}G_{i\bar{j}}(z,v)\delta v^i \wedge \delta\bar{v}^j.$$

- Taking exterior differentiation of ω, we have

$$\begin{aligned} d\omega &= (-\sqrt{-1})\{dG_{i\bar{j}} \wedge dz^i \wedge d\bar{z}^j + dG_{i\bar{j}} \wedge \delta v^i \wedge \delta\bar{v}^j \\ &\quad + G_{i\bar{j}} d(\delta v^i) \wedge \delta\bar{v}^j - G_{i\bar{j}}\delta v^i \wedge d(\delta\bar{v}^j)\} \\ &:= (-\sqrt{-1})(I + II + III - \overline{III}) \end{aligned}$$

- Calculating all of the terms one by one, we obtain

$$\begin{aligned} I &= \frac{\delta G_{i\bar{j}}}{\delta z^k} dz^k \wedge dz^i \wedge d\bar{z}^j + \frac{\delta G_{i\bar{j}}}{\delta \bar{z}^k} d\bar{z}^k \wedge dz^i \wedge d\bar{z}^j \\ &\quad + \frac{\partial G_{k\bar{l}}}{\partial v^j} dz^k \wedge d\bar{z}^l \wedge \delta v^j + \frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j} dz^k \wedge d\bar{z}^l \wedge \delta \bar{v}^j . \end{aligned}$$

$$II = \frac{\delta G_{i\bar{j}}}{\delta z^k} dz^k \wedge \delta v^i \wedge \delta \bar{v}^j + \frac{\delta G_{i\bar{j}}}{\delta \bar{z}^k} d\bar{z}^k \wedge \delta v^i \wedge \delta \bar{v}^j .$$

$$\begin{aligned} III &= N^p_k \frac{\delta G_{p\bar{j}}}{\delta \bar{z}^l} dz^k \wedge d\bar{z}^l \wedge \delta \bar{v}^j + N^p_k \frac{\partial G_{p\bar{j}}}{\partial v^i} dz^k \wedge \delta v^i \wedge \delta \bar{v}^j \\ &\quad - \frac{\delta G_{\bar{j},\,k}}{\delta \bar{z}^l} dz^k \wedge d\bar{z}^l \wedge \delta \bar{v}^j - \frac{\partial G_{\bar{j},\,k}}{\partial v^i} dz^k \wedge \delta v^i \wedge \delta \bar{v}^j . \end{aligned}$$

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$$\begin{aligned} III &= N_k^p \frac{\delta G_{p\bar{j}}}{\delta \bar{z}^l} dz^k \wedge d\bar{z}^l \wedge \delta \bar{v}^j + N_k^p \frac{\partial G_{p\bar{j}}}{\partial v^i} dz^k \wedge \delta v^i \wedge \delta \bar{v}^j \\ &\quad - \frac{\delta G_{\bar{j},\,k}}{\delta \bar{z}^l} dz^k \wedge d\bar{z}^l \wedge \delta \bar{v}^j - \frac{\partial G_{\bar{j},\,k}}{\partial v^i} dz^k \wedge \delta v^i \wedge \delta \bar{v}^j. \end{aligned}$$

- Therefore, the coefficient of $(-\sqrt{-1}) dz^k \wedge \delta v^i \wedge \delta \bar{v}^j$ in $d\omega$ is

$$\frac{\delta G_{i\bar{j}}}{\delta z^k} + N_k^p \frac{\partial G_{p\bar{j}}}{\partial v^i} - \frac{\partial G_{\bar{j},\,k}}{\partial v^i} = 0.$$

- Correspondingly, the coefficient of $(-\sqrt{-1}) d\bar{z}^k \wedge \delta v^i \wedge \delta \bar{v}^j$ is also 0.

- We know that $\frac{\partial}{\partial \bar{z}^l}(\frac{\partial N_k^p}{\partial \bar{v}^j} G_p) = 0$, from which it follows that the coefficient of $(-\sqrt{-1}) dz^k \wedge d\bar{z}^l \wedge \delta \bar{v}^j$ is

$$\frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j} + N_k^p \frac{\delta G_{p\bar{j}}}{\delta \bar{z}^l} - \frac{\delta G_{\bar{j},\,k}}{\delta \bar{z}^l} = \frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j} + \frac{\partial}{\partial \bar{v}^j} \psi_{k\bar{l}}.$$

- Hence, the coefficient of $(-\sqrt{-1}) dz^k \wedge d\bar{z}^l \wedge \delta v^j$ is

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- Hence, the coefficient of $(-\sqrt{-1}) dz^k \wedge d\bar{z}^l \wedge \delta v^j$ is

$$\frac{\partial G_{k\bar{l}}}{\partial v^j} + \frac{\partial \psi_{k\bar{l}}}{\partial v^j}.$$

- If h_{TM} is a Kähler metric, then we have $d\omega = 0$, so that

$$\frac{\delta G_{i\bar{j}}}{\delta z^k} dz^k \wedge dz^i \wedge d\bar{z}^j + \frac{\delta G_{i\bar{j}}}{\delta \bar{z}^k} d\bar{z}^k \wedge dz^i \wedge d\bar{z}^j = 0, \qquad (4.1)$$

$$\frac{\partial G_{k\bar{l}}}{\partial v^j} + \frac{\partial \psi_{k\bar{l}}}{\partial v^j} = 0, \quad \frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j} + \frac{\partial \psi_{k\bar{l}}}{\partial \bar{v}^j} = 0, \qquad (4.2)$$

which implies that M is a Finsler-Kähler manifold.

- By the definition of $\psi_{k\bar{l}}$, one can check easily that $\psi_{k\bar{l}}$ have the same homogeneity as G, that is ,

$$\psi_{k\bar{l}}(z, \lambda v) = \lambda\bar{\lambda}\psi_{k\bar{l}}(z, v), \quad \forall\lambda \in \mathbf{C}^*.$$

Therefore, we have

$$\frac{\partial\psi_{k\bar{l}}}{\partial v^j}v^j = \psi_{k\bar{l}}, \tag{4.3}$$

from which we obtain

$$\frac{\partial G_{k\bar{l}}}{\partial v^j}v^j = 0, \quad \frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j}\bar{v}^j = 0. \tag{4.4}$$

Thus, we obtain

$$\psi_{k\bar{l}} = 0. \tag{4.5}$$

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Therefore, we have

$$\frac{\partial \psi_{k\bar{l}}}{\partial v^j} v^j = \psi_{k\bar{l}}, \tag{4.3}$$

from which we obtain

$$\frac{\partial G_{k\bar{l}}}{\partial v^j} v^j = 0, \quad \frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j} \bar{v}^j = 0. \tag{4.4}$$

Thus, we obtain

$$\psi_{k\bar{l}} = 0. \tag{4.5}$$

- Substituting (5.5) into (5.2) yields that

$$\frac{\partial G_{k\bar{l}}}{\partial v^j} = 0, \quad \frac{\partial G_{k\bar{l}}}{\partial \bar{v}^j} = 0,$$

which implies that (M, G) is a Hermitian manifold. Then, by (5.1), we see that (M, G) is a Kähler manifold. From (3.4) and (5.5) we know the holomorphic curvature K of M is zero, and the proof of the main Theorem is completed.

5. Complex Randers metrics

- Let M be a complex manifold of complex dimension n, and $\alpha = \sqrt{a_{i\bar{j}}(z)v^i\bar{v}^j}$ be a Hermitian metric on M. Suppose that $\beta = b_i(z)v^i$ is a holomorphic 1-form on M.

  Set

$$F = \alpha + \epsilon\sqrt{\beta\bar{\beta}} = \alpha + \epsilon|\beta|, \tag{5.1}$$

  where

$$\epsilon = \begin{cases} 1, & \text{for} \quad \beta \neq 0, \\ 0, & \text{for} \quad \beta = 0. \end{cases}$$

- **Definition.** The metrics

$$G := F^2 = \alpha^2 + 2\epsilon\alpha|\beta| + \epsilon|\beta|^2 \tag{5.2}$$

  are called complex Randers metrics.

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- **Definition.** The metrics
$$G := F^2 = \alpha^2 + 2\epsilon\alpha|\beta| + \epsilon|\beta|^2 \tag{5.2}$$
  are called complex Randers metrics.
- It is easy to see that
$$G_{i\bar{j}} = \frac{F}{\alpha}h_{i\bar{j}} + \frac{\epsilon F}{2|\beta|}b_i b_{\bar{j}} + \frac{1}{2G}G_i G_{\bar{j}}, \tag{5.3}$$
  where

$$h_{i\bar{j}} := a_{i\bar{j}} - \frac{1}{2\alpha^2}\ell_i \ell_{\bar{j}},$$

$$\ell_i := \dot{\partial}_i \alpha^2 = 2\alpha \dot{\partial}_i \alpha = a_{i\bar{j}} \bar{v}^j, \quad \ell_{\bar{j}} := \dot{\partial}_{\bar{j}} \alpha^2.$$

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- $$\begin{aligned} G^{\bar{j}i} &= \frac{\alpha}{F}a^{\bar{j}i} + \frac{|\beta|(\alpha||\beta||_\alpha^2 + |\beta|)}{G\gamma}v^i\bar{v}^j - \frac{\alpha^3}{F\gamma}b^i\bar{b}^j \\ &\quad -\frac{\alpha}{F\gamma}(\bar{\beta}v^i\bar{b}^j + \beta b^i\bar{v}^j), \end{aligned} \tag{5.4}$$

$$\det(G_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^n \frac{\epsilon\gamma}{2\alpha|\beta|}\det(a_{i\bar{j}}), \tag{5.5}$$

where

$$\gamma := G + \alpha^2(||\beta||_\alpha^2 - 1).$$

$$h_{i\bar{j}} := a_{i\bar{j}} - \frac{1}{2\alpha^2}\ell_i\ell_{\bar{j}},$$

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- $$G^{\bar{j}i} = \frac{\alpha}{F}a^{\bar{j}i} + \frac{|\beta|(\alpha||\beta||_\alpha^2 + |\beta|)}{G\gamma}v^i\bar{v}^j - \frac{\alpha^3}{F\gamma}b^i\bar{b}^j - \frac{\alpha}{F\gamma}(\bar{\beta}v^i\bar{b}^j + \beta b^i\bar{v}^j), \tag{5.4}$$

$$\det(G_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^n \frac{\epsilon\gamma}{2\alpha|\beta|}\det(a_{i\bar{j}}), \tag{5.5}$$

where

$$\gamma := G + \alpha^2(||\beta||_\alpha^2 - 1).$$

- Hence, G is strongly pseudo-convex if

$$G > \alpha^2(1 - ||\beta(v)||_\alpha^2).$$

- For the complex Randers metric, the coefficients of the Chern-Finsler connection are

$$N_j^i = {}^{\alpha}N_j^i + \frac{\eta^i}{\gamma}\left(\ell_{\bar{k}}\frac{\partial b^{\bar{k}}}{\partial z^j} - \frac{\beta^2}{2|\beta|}\frac{\partial b_{\bar{k}}}{\partial z^j}\bar{v}^k\right) + \frac{\beta}{2|\beta|}g^{\bar{k}i}\frac{\partial b_{\bar{k}}}{\partial z^j}, \quad (5.6)$$

where

$$\eta^i := \bar{\beta}v^i + \alpha^2 b^i, \qquad {}^{\alpha}N_j^i := a^{\bar{k}i}\frac{\partial a_{l\bar{k}}}{\partial z^j}v^l,$$

$$g^{\bar{j}i} := 2\alpha a^{\bar{j}i} + \frac{2(\alpha||\beta||_\alpha^2 + 2|\beta|)}{\gamma}v^i\bar{v}^j - \frac{2\alpha^3}{\gamma}b^i\bar{b}^j - \frac{2\alpha}{\gamma}(\bar{\beta}v^i\bar{b}^j + \beta b^i\bar{v}^j).$$

- From this we can calculus the holomorphic curvature of the complex Randers metric. It is a technical computation which involves a lot of long-winded calculus. So, we will not dwell too much on it.

6. Problems

- **Question 1.** Construct complex Randers metrics with constant holomorphic curvature.

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- **Question 1.** Construct complex Randers metrics with constant holomorphic curvature.
- **Question 2.** Consider the relation between the Kobayashi metric and the complex Randers metric with constant negative holomorphic curvature.
- **Question 3.** Consider complex projectively flat Randers metrics with constant holomorphic curvature.

7. Main References

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The End

Thank You!