HODGE METRIC COMPLETION OF THE TEICHMÜLLER SPACE OF CALABI–YAU MANIFOLDS

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ABSTRACT. We prove that the Hodge metric completion of the Teichmüller space of polarized and marked Calabi–Yau manifolds is a complex affine manifold. We also show that the extended period map from the completion space is injective into the period domain, and that the completion space is a domain of holomorphy and admits a complete Kähler-Einstein metric.

CONTENTS

1. INTRODUCTION

A compact projective manifold M of complex dimension n with $n \geq 3$ is called Calabi–Yau in this paper, if it has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$. A polarized and marked Calabi–Yau manifold is a triple consisting of a Calabi–Yau manifold M, an ample line bundle L over M, and a basis of the integral middle homology group modulo torsion, $H_n(M, \mathbb{Z})/\mathrm{Tor}$.

We will denote by $\mathcal{T}$ the Teichmüller space for the deformation of the complex structure on the polarized and marked Calabi–Yau manifold M. Actually the Teichmüller space is precisely the universal cover of the smooth moduli space $\mathcal{Z}_m$ of polarized Calabi–Yau manifolds with level m structure with $m \geq 3$, which is constructed by Popp, Viehweg, and Szendröi, for example in Section 2 of [13]. The versal family $\mathcal{U} \to \mathcal{T}$ of the polarized and marked Calabi–Yau manifolds is the pull-back of the versal family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$, which is also introduced in [13]. Therefore $\mathcal{T}$ is a connected and simply connected smooth complex manifold with $\dim_{\mathbb{C}} \mathcal{T} = h^{n-1,1}(M) = N$, where $h^{n-1,1}(M) = \dim_{\mathbb{C}} H^{n-1,1}(M)$.

Let D be the period domain of polarized Hodge structures of the n-th primitive cohomology of M. The period map $\Phi : \mathcal{T} \to D$ is defined by assigning to each point in $\mathcal{T}$ the Hodge structure of the corresponding fiber. Let us denote the period map on the smooth moduli space by $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$, where Γ denotes the global monodromy group which acts properly and discontinuously on D. Then $\Phi : \mathcal{T} \to D$ is the lifting of $\Phi_{\mathcal{Z}_m} \circ \pi_m$, where $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ is the universal covering map. There is Hodge metric h on D, which is a complete homogeneous metric and is studied in [4]. By local Torelli theorem of Calabi–Yau manifolds, both $\Phi_{\mathcal{Z}_m}$ and Φ are locally injective. Thus the pull-backs of h on $\mathcal{Z}_m$ and $\mathcal{T}$ are both well-defined Kähler metrics, and they are still called Hodge metrics.

In this paper, one of our essential constructions is the global holomorphic affine structure on the Teichmüller space, which can be outlined as follows: by using the local Kuranishi deformation theory of Calabi–Yau manifolds, we construct Kuranishi coordinate cover on $\mathcal{T}$. We then verify that the transition maps between any two local Kuranishi coordinate charts are holomorphic affine maps. The computation of transition maps is based the local Torelli theorem for Calabi–Yau manifolds and the Griffiths transversality for variations of Hodge structures. We point out that the construction of the holomorphic affine structure on the Teichmüller space of Calabi–Yau type manifolds in [1] is similar to this construction. However, one may notice that such construction is algebraically using local Torelli Theorem for the Calabi–Yau manifolds and the Griffiths transversality, while the construction of holomorphic affine structure on $\mathcal{T}$ in [5] is analytically using the canonical $(n, 0)$-forms on the local Kuranishi family of Calabi–Yau manifolds.

As a consequence, we get that Φ is a map from $\mathcal{T}$ to $N_+ \cap D$, where N_+ is identified with its unipotent orbit in $\check{D}$ by fixing a base point $\Phi(p) \in D$ with $p \in \mathcal{T}$. Based on this result, we first define a holomorphic map Ψ from $\mathcal{T}$ to $\mathbb{C}^N$ by composing the period map with a projection $P : N_+ \to \mathbb{C}^N$ and then show that Ψ is a holomorphic affine local embedding with respect to the holomorphic affine structure on $\mathcal{T}$. One will see the property that Ψ is a local embedding is crucial in our further constructions.

Let $\mathcal{Z}^H_m$ be the Hodge metric completion of the smooth moduli space $\mathcal{Z}_m$ and let $\mathcal{T}^H_m$ be the universal cover of $\mathcal{Z}^H_m$ with the universal covering map $\pi^H_m : \mathcal{T}^H_m \to \mathcal{Z}^H_m$. It is easy to see that $\mathcal{Z}^H_m$ is a connected and complete smooth complex manifold, and thus $\mathcal{T}^H_m$ is a connected and simply connected complete smooth complex manifold. We also obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}^H_m & \xrightarrow{\Phi^H_m} & D \\ \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi^H_m} & & \downarrow{\scriptstyle \pi_D} \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}^H_m & \xrightarrow{\Phi^H_{\mathcal{Z}_m}} & D/\Gamma, \end{array} \tag{1}$$

where $\Phi^H_{\mathcal{Z}_m}$ is the continuous extension map of the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$, i is the inclusion map, i_m is a lifting of $i \circ \pi_m$, and Φ^H_m is a lifting of $\Phi^H_{\mathcal{Z}_m} \circ \pi^H_m$. We prove in Lemma A.1 that there is a suitable choice of i_m and Φ^H_m such that $\Phi = \Phi^H_m \circ i_m$. It is not hard to see that Φ^H_m is actually a holomorphic map from $\mathcal{T}^H_m$ to $N_+ \cap D$, where N_+ is also identified with its unipotent orbit in $\check{D}$ by fixing the base point $\Phi(p) \in D$ with $p \in \mathcal{T}$.

Proposition 1.1. *For any $m \geq 3$, the complete complex manifold $\mathcal{T}^H_m$ is a complex affine manifold, which can be embedded into $\mathbb{C}^N$. Moreover, the holomorphic map $\Phi^H_m : \mathcal{T}^H_m \to N_+ \cap D$ is a injection. As a consequence, the complex manifolds $\mathcal{T}^H_m$ and $\mathcal{T}^H_{m'}$ are biholomorphic to each other for any $m, m' \geq 3$.*

This proposition allows us to define the complete complex manifold $\mathcal{T}^H$ by $\mathcal{T}^H = \mathcal{T}^H_m$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \to D$ by $\Phi^H = \Phi^H_m$ for any $m \geq 3$. By these definitions, Proposition 1.1 implies that $\mathcal{T}^H$ is a complex affine manifold and that $\Phi^H : \mathcal{T}^H \to N_+ \cap D$ is a holomorphic injection. The main result of this paper is the following.

Theorem 1.2. *The complete complex affine manifold $\mathcal{T}^H$ is the completion space of $\mathcal{T}$ with respect to the Hodge metric, and it can be embedded into $\mathbb{C}^N$. Moreover, the extended period map $\Phi^H : \mathcal{T}^H \to N_+ \cap D$ is a holomorphic injection.*

Here we remark that one technical difficulty of our arguments is to directly prove that $\mathcal{T}^H$ is the smooth Hodge metric completion space of $\mathcal{T}$, which is to prove that $\mathcal{T}^H$ is smooth and that the map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ is an embedding.

To show the smoothness of $\mathcal{T}^H$, we need to show that the definition $\mathcal{T}^H = \mathcal{T}^H_m$ does not depend on the choice of the level structure m. Therefore, we first have to introduce the smooth complete manifold $\mathcal{T}^H_m$. Then by realizing the fact that the image of the period map in D is independent of the level structure and that the metric completion space is unique, this independence of level structure can be reduced to show that $\Phi^H_m : \mathcal{T}^H_m \to N_+ \cap D$ is injective. To achieve this, we first define the map $\Psi^H_m : \mathcal{T}^H_m \to \mathbb{C}^N$ by composing the map Φ^H_m with the projection P as above. Then by crucially using the property that Ψ is local embedding, we show that Ψ^H_m is a local embedding as well. Thus we conclude that there exists a holomorphic affine structure on $\mathcal{T}^H_m$ such that Ψ^H_m is a holomorphic affine map. Finally, the affineness of Ψ^H_m and completeness of $\mathcal{T}^H_m$ gives the injectivity of Ψ^H_m, which implies the injectivity of Φ^H_m.

To overcome the difficulty of showing $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ is embedding, we first note that $\mathcal{T}_0 = \mathcal{T}_m := i_m(\mathcal{T})$ is also well-defined. Then it is not hard to show that $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ is a covering map. Moreover, we prove that $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ is actually one-to-one by showing that the fundamental group of $\mathcal{T}_0$ is trivial. Here the markings of the Calabi–Yau manifolds and the simply connectedness of $\mathcal{T}$ come into play substantially. Therefore, we can conclude that $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ is an embedding. It would substantially simplify our arguments if one can directly prove that the Hodge metric completion space of $\mathcal{T}$ is smooth without using $\mathcal{T}_m$ and $\mathcal{T}^H_m$.

As a direct corollary of this theorem, we easily deduce that the period map $\Phi = \Phi^H \circ i_{\mathcal{T}} : \mathcal{T} \to D$ is also injective since it is a composition of two injective maps. This is the global Torelli theorem for the period map from the Teichmüller space to the period domain. In the case that the period domain D is Hermitian symmetric and that it has the same dimension as $\mathcal{T}$, the above theorem implies that the extended period map Φ^H is biholomorphic, in particular it is surjective. Moreover, we prove another important result as follows.

Theorem 1.3. *The completion space $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$; thus there exists a complete Kähler-Einstein metric on $\mathcal{T}^H$.*

To prove this theorem, we construct a plurisubharmonic exhaustion function on $\mathcal{T}^H$ by using Proposition 5.9, the completeness of $\mathcal{T}^H$, and the injectivity of Φ^H. This shows that $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$. The existence of the Kähler-Einstein metric follows directly from the well-known theorem of Cheng–Yau in [2].

This paper is organized as follows. In Section 2, we review the definition of the period domain of polarized Hodge structures and briefly describe the construction of the Teichmüller space of polarized and marked Calabi–Yau manifolds, the definition of the period map and the Hodge metrics on the moduli space and the Teichmüller space respectively. In Section 3, we construct a holomorphic affine structure on the Teichmüller space and show that the image of the period map is in $N_+ \cap D$. Then we define the map Ψ from $\mathcal{T}$ to $\mathbb{C}^N$ and show it is a holomorphic affine local embedding. In Section 4, we prove that there exists a global holomorphic affine structure on $\mathcal{T}^H_{_m}$ and that the map $\Phi^H_{_m} : \mathcal{T}^H_{_m} \to D$ is an injective map. In Section 5, we define the completion space $\mathcal{T}^H$ and the extended period map Φ^H. We then show our main result that $\mathcal{T}^H$ is the Hodge metric completion space of $\mathcal{T}$, which is also a complex affine manifold, and that Φ^H is a holomorphic injection, which extends the period map $\Phi : \mathcal{T} \to D$. Therefore, the global Torelli theorem for Calabi–Yau manifolds on the Teichmüller space follows directly. Finally, we prove $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$, and thus it admits a complete Kähler-Einstein metric. In the appendix, we include two topological lemmas: a lemma shows the existence of the choices of $\Phi^H_{_m}$ and i_m satisfying $\Phi = \Phi^H_{_m} \circ i_m$ and a lemma that relates the fundamental group of the moduli space $\mathcal{Z}_m$ to that of its completion space $\mathcal{Z}^H_{_m}$ with respect to the Hodge metric.

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2. Teichmüller space and the period map of Calabi–Yau manifolds

In Section 2.1, we recall the definition and some basic properties of the period domain. In Section 2.2, we discuss the construction of the Teichmüllller space of Calabi–Yau manifolds based on the works of Popp [10], Viehweg [14] and Szendröi [13] on the moduli spaces of Calabi–Yau manifolds. In Section 2.3, we define the period map from the Teichmüller space to the period domain. We remark that most of the results in this section are standard and can be found from the literature in the subjects.

2.1. **Period domain of polarized Hodge structures.**

We first review the construction of the period domain of polarized Hodge structures. We refer the reader to Section 3 in [11] for more details.

A pair (M, L) consisting of a Calabi–Yau manifold M of complex dimension n with $n \geq 3$ and an ample line bundle L over M is called a polarized Calabi–Yau manifold. By abuse of notation, the Chern class of L will also be denoted by L and thus $L \in H^2(M, \mathbb{Z})$. Let $\{\gamma_1, \cdots, \gamma_{h^n}\}$ be a basis of the integral homology group modulo torsion, $H_n(M, \mathbb{Z})/\mathrm{Tor}$ with $\dim H_n(M, \mathbb{Z})/\mathrm{Tor} = h^n$.

Definition 2.1. *Let the pair (M, L) be a polarized Calabi–Yau manifold, we call the triple $(M, L, \{\gamma_1, \cdots, \gamma_{h^n}\})$ a polarized and marked Calabi–Yau manifold.*

For a polarized and marked Calabi–Yau manifold M with background smooth manifold X, we identify the basis of $H_n(M, \mathbb{Z})/\mathrm{Tor}$ to a lattice Λ as in [13]. This gives us a canonical identification of the middle dimensional de Rahm cohomology of M to that of the background manifold X, that is,

$$H^n(M) \cong H^n(X)$$

where the coefficient ring can be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Since the polarization L is an integer class, it defines a map

$$L:\ H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q}), \qquad A \mapsto L \wedge A.$$

We denote by $H^n_{pr}(X) = \ker(L)$ the primitive cohomology groups where, the coefficient ring can also be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. We let $H^{k,n-k}_{pr}(M) = H^{k,n-k}(M) \cap H^n_{pr}(M, \mathbb{C})$ and denote its dimension by $h^{k,n-k}$. We have the Hodge decomposition

$$H^n_{pr}(M, \mathbb{C}) = H^{n,0}_{pr}(M) \oplus \cdots \oplus H^{0,n}_{pr}(M). \tag{2}$$

It is easy to see that for a polarized Calabi–Yau manifold, since $H^2(M, \mathcal{O}_M) = 0$, we have

$$H^{n,0}_{pr}(M) = H^{n,0}(M),\ H^{n-1,1}_{pr}(M) = H^{n-1,1}(M).$$

The Poincaré bilinear form Q on $H^n_{pr}(X, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d-closed n-forms u, v on X. The bilinear form Q is symmetric if n is even and is skew-symmetric if n is odd. Furthermore, Q is nondegenerate and can be extended to $H^n_{pr}(X, \mathbb{C})$ bilinearly. Moreover, it also satisfies the Hodge-Riemann relations

$$Q\left(H^{k,n-k}_{pr}(M), H^{l,n-l}_{pr}(M)\right) = 0 \ \text{ unless } \ k + l = n, \quad \text{and} \tag{3}$$

$$\left(\sqrt{-1}\right)^{2k-n} Q\left(v, \overline{v}\right) > 0 \ \text{ for } \ v \in H^{k,n-k}_{pr}(M) \setminus \{0\}. \tag{4}$$

Let $f^k = \sum_{i=k}^n h^{i,n-i}$, $f^0 = m$, and $F^k = F^k(M) = H^{n,0}_{pr}(M) \oplus \cdots \oplus H^{k,n-k}_{pr}(M)$, from which we have the decreasing filtration $H^n_{pr}(M, \mathbb{C}) = F^0 \supset \cdots \supset F^n$. We know that

$$\dim_{\mathbb{C}} F^k = f^k, \tag{5}$$

$$H^n_{pr}(X, \mathbb{C}) \ = F^k \oplus \overline{F^{n-k+1}}, \quad \text{and} \quad H^{k,n-k}_{pr}(M) = F^k \cap \overline{F^{n-k}}. \tag{6}$$

In terms of the Hodge filtration, the Hodge-Riemann relations (3) and (4) are

$$Q\left(F^k, F^{n-k+1}\right) = 0, \quad \text{and} \tag{7}$$

$$Q\left(Cv, \overline{v}\right) > 0 \quad \text{if} \quad v \neq 0, \tag{8}$$

where C is the Weil operator given by $Cv = \left(\sqrt{-1}\right)^{2k-n} v$ for $v \in H^{k,n-k}_{pr}(M)$. The period domain D for polarized Hodge structures with data (5) is the space of all such Hodge filtrations

$$D = \left\{F^n \subset \cdots \subset F^0 = H^n_{pr}(X, \mathbb{C}) \mid (5), (7) \text{ and } (8) \text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{F^n \subset \cdots \subset F^0 = H^n_{pr}(X, \mathbb{C}) \mid (5) \text{ and } (7) \text{ hold}\right\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. We note that the conditions (7) and (8) imply the identities in (6). From the definition of period domain we naturally get the Hodge bundles on $\check{D}$ by associating to each point in $\check{D}$ the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point. Without confusion we will also denote by F^k the bundle with F^k as the fiber for each $0 \leq k \leq n$.

Remark 2.2. We remark the notation change for the primitive cohomology groups. As mentioned above that for a polarized Calabi–Yau manifold,

$$H^{n,0}_{pr}(M) = H^{n,0}(M),\ H^{n-1,1}_{pr}(M) = H^{n-1,1}(M).$$

For the reason that we mainly consider these two types of primitive cohomology group of a Calabi–Yau manifold, by abuse of notation, we will simply use $H^n(M,\mathbb{C})$ and $H^{k,n-k}(M)$ to denote the primitive cohomology groups $H^n_{pr}(M,\mathbb{C})$ and $H^{k,n-k}_{pr}(M)$ respectively. Moreover, we will use cohomology to mean primitive cohomology in the rest of the paper.

2.2. **Construction of the Teichmuller space.** We first recall the concept of Kuranishi family of compact complex manifolds. We refer to page 8-10 in [12], page 94 in [10] or page 19 in [14] for equivalent definitions and more details.

A family of compact complex manifolds $\pi : \mathcal{W} \to \mathcal{B}$ is *versal* at a point $p \in \mathcal{B}$ if it satisfies the following conditions:

(1) If given a complex analytic family $\iota : \mathcal{V} \to \mathcal{S}$ of compact complex manifolds with a point $s \in \mathcal{S}$ and a biholomorphic map $f_0 : V = \iota^{-1}(s) \to U = \pi^{-1}(p)$, then there exists a holomorphic map g from a neighbourhood $\mathcal{N} \subseteq \mathcal{S}$ of the point s to $\mathcal{T}$ and a holomorphic map $f : \iota^{-1}(\mathcal{N}) \to \mathcal{W}$ with $\iota^{-1}(\mathcal{N}) \subseteq \mathcal{V}$ such that they satisfy that $g(s) = p$ and $f|_{\iota^{-1}(s)} = f_0$, with the following commutative diagram

$$\begin{array}{ccc} \iota^{-1}(\mathcal{N}) & \xrightarrow{f} & \mathcal{W} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \pi} \\ \mathcal{N} & \xrightarrow{g} & \mathcal{B}. \end{array}$$

(2) For all g satisfying the above condition, the tangent map $(dg)_s$ is uniquely determined.

If a family $\pi : \mathcal{W} \to \mathcal{B}$ is versal at every point $p \in \mathcal{B}$, then it is a *versal family* on $\mathcal{B}$. If a complex analytic family satisfies the above condition (1), then the family is called *complete* at p. If a complex analytic family $\pi : \mathcal{W} \to \mathcal{B}$ of compact complex manifolds is complete at each point of $\mathcal{B}$ and versal at the point $p \in \mathcal{B}$, then the family $\pi : \mathcal{W} \to \mathcal{B}$ is called the *Kuranishi family* of the complex manifold $V = \pi^{-1}(p)$. The base space $\mathcal{B}$ is called the *Kuranishi space*. If the family is complete at each point in a neighbourhood of $p \in \mathcal{B}$ and versal at p, then the family is called a *local Kuranishi family* at $p \in \mathcal{B}$.

Let (M, L) be a polarized Calabi–Yau manifold. We call a basis of the quotient space $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m(H_n(M,\mathbb{Z})/\mathrm{Tor})$ a level m structure on the polarized Calabi–Yau manifold with $m \geq 3$. For deformation of polarized Calabi–Yau manifold with level m structure, we have the following theorem, which is a reformulation of Theorem 2.2 in [13]. We just take the statement we need in this paper. One can also look at [10] and [14] for more details about the construction of moduli spaces of Calabi–Yau manifolds.

Theorem 2.3. *Let $m \geq 3$ and M be a polarized Calabi–Yau manifold with level m structure. Then there exists a connected quasi-projective complex manifold $\mathcal{Z}_m$ with a versal family of Calabi–Yau manifolds,*

$$\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m, \tag{9}$$

containing M as a fiber, and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$.

We define the *Teichmüller space* $\mathcal{T}$ to be the universal cover of $\mathcal{Z}_m$ with the covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. We denote by the family $\varphi : \mathcal{U} \to \mathcal{T}$ the pull-back of the family (9) via the projection π_m.

Proposition 2.4. *The Teichmüller space $\mathcal{T}$ is a connected and simply connected smooth complex manifold and the family*

$$\varphi : \mathcal{U} \to \mathcal{T}, \tag{10}$$

which contains M as a fiber, is local Kuranishi at each point of $\mathcal{T}$.

Proof. For the first part, because $\mathcal{Z}_m$ is a connected and smooth complex manifold, its universal cover $\mathcal{T}$ is thus a connected and simply connected smooth complex manifold. For the second part, we know that the family (9) is a versal family at each point of $\mathcal{Z}_m$ and that π_m is locally biholomorphic, therefore the pull-back family via π_m is also versal at each point of $\mathcal{T}$. By the definition of local Kuranishi family, we get that $\mathcal{U} \to \mathcal{T}$ is local Kuranishi at each point of $\mathcal{T}$. □

Remark 2.5. We remark that the family $\varphi : \mathcal{U} \to \mathcal{T}$ being local Kuranishi at each point is essentially due to the local Torelli theorem for Calabi–Yau manifolds. In fact, we know that for the family $\mathcal{U} \to \mathcal{T}$, the Kodaira-Spencer map

$$\kappa : T_p^{1,0}\mathcal{T} \to H^{0,1}(M_p, T^{1,0}M_p),$$

is an isomorphism for each $p \in \mathcal{T}$. Then by theorems in page 9 of [12], we conclude that $\mathcal{U} \to \mathcal{T}$ is versal at each $p \in \mathcal{T}$. We refer the reader to Chapter 4 in [8] for more details about deformation of complex structures and the Kodaira-Spencer map. In particular, by Lemma 3.3 in the next section, it is easy to see that $\dim_{\mathbb{C}} \mathcal{T} = \dim_{\mathbb{C}} H^{n-1,1}(M_p) = N$.

We also remark that the Teichmüller space $\mathcal{T}$ does not depend on the choice of m. In fact, let m_1 and m_2 be two different integers, and $\mathcal{U}_1 \to \mathcal{T}_1$, $\mathcal{U}_2 \to \mathcal{T}_2$ be two versal families constructed via level m_1 and level m_2 structures respectively as above, and both of which contain M as a fiber. By using the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are simply connected and the definition of versal family, we have a biholomorphic map $f : \mathcal{T}_1 \to \mathcal{T}_2$, such that the versal family $\mathcal{U}_1 \to \mathcal{T}_1$ is the pull back of the versal family $\mathcal{U}_2 \to \mathcal{T}_2$ by f. Thus these two families are biholomorphic to each other.

In the rest of the paper, we will simply use "level m structure" to mean "level m structure with $m \geq 3$".

2.3. **The period map and the Hodge metric on the Teichmüller space.** For any point $p \in \mathcal{T}$, let M_p be the fiber of family $\varphi : \mathcal{U} \to \mathcal{T}$, which is a polarized and marked Calabi–Yau manifold. Since the Teichmüller space is simply connected and we have fixed the basis of the middle homology group modulo torsions, we identify the basis of $H_n(M, \mathbb{Z})/\mathrm{Tor}$ to a lattice Λ as in [13]. This gives us a canonical identification of the middle dimensional cohomology of M to that of the background manifold X, that is,

$H^n(M,\mathbb{C}) \simeq H^n(X,\mathbb{C})$. Therefore, we can use this to identify $H^n(M_p,\ \mathbb{C})$ for all fibers on $\mathcal{T}$. Thus we get a canonical trivial bundle $H^n(M_p,\mathbb{C}) \times \mathcal{T}$.

The period map from $\mathcal{T}$ to D is defined by assigning each point $p \in \mathcal{T}$ the Hodge structure on M_p, that is

$$\Phi : \mathcal{T} \to D, \qquad p \mapsto \Phi(p) = \{F^n(M_p) \subset \cdots \subset F^0(M_p)\}$$

We denote $F^k(M_p)$ by F_p^k for simplicity.

The period map has several good properties, and we refer the reader to Chapter 10 in [15] for details. Among them, one of the most important is the following Griffiths transversality: the period map Φ is a holomorphic map and its tangent map satisfies that

$$\Phi_*(v) \in \bigoplus_{k=1}^{n} \mathrm{Hom}\left(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k\right), \quad \text{for any} \quad p \in \mathcal{T} \ \text{ and } \ v \in T_p^{1,0}\mathcal{T}$$

with $F^{n+1} = 0$, or equivalently, $\Phi_*(v) \in \bigoplus_{k=0}^n \mathrm{Hom}(F_p^k, F_p^{k-1})$.

In [4], Griffiths and Schmid studied the so-called Hodge metric on the period domain D. We denote it by h. In particular, this Hodge metric is a complete homogeneous metric. Let us denote the period map on the moduli space by $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$, where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D. Since $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ is the universal covering map, the period map $\Phi : \mathcal{T} \to D$ is the lifting of $\Phi_{\mathcal{Z}_m} \circ \pi_m$. By local Torelli theorem for Calabi–Yau manifolds, we know that $\Phi_{\mathcal{Z}_m}, \Phi$ are both locally injective. Thus it follows from [4] that the pull-backs of h by $\Phi_{\mathcal{Z}_m}$ and Φ on $\mathcal{Z}_m$ and $\mathcal{T}$ respectively are both well-defined Kähler metrics. By abuse of notation, we still call these pull-back metrics the *Hodge metrics*, and they are both denoted by h.

3. Holomorphic affine structure on the Teichmüller space

In Section 3.1, we will give a brief review on definitions and some basic properties of affine manifolds and affine maps. In Section 3.2, we construct the Kuranishi coordinate cover on $\mathcal{T}$ and prove that this coordinate cover gives a global holomorphic affine structure on $\mathcal{T}$. In Section 3.3, we define the map Ψ from $\mathcal{T}$ to $\mathbb{C}^N$ by composing the period map with a projection map. We then show that Ψ is a holomorphic affine local embedding by using the affine structure and simply connectedness of $\mathcal{T}$ as well as the local property of the period map Φ.

3.1. Affine manifolds and affine maps.

We first review the definition of complex affine manifolds. One may refer to page 215 of [9] for more details.

Definition 3.1. *Let M be a complex manifold of complex dimension n. If there is a coordinate cover $\{(U_i,\ \varphi_i);\ i \in I\}$ of M such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i,\ \varphi_i);\ i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M.*

A differentiable map $f : M \to M'$ between two holomorphic affine manifolds is called holomorpic affine, if f is holomorphic affine when restricted to the complex affine coordinate cover. We also recall an extension theorem of holomorphic affine maps on affine manifolds, which is the holomorphic analogue of Theorem 6.1 in [7]. We refer the reader to page 252-255 of [7] for more details.

Theorem 3.2. *Let M be a connected and simply connected complex affine manifold. Let M' be a complete complex affine manifold. Then any holomorphic affine map f_U on a connected open subset U of M to M' can be uniquely extended to a holomorphic affine map f on M to M'.*

Proof. Because both M and M' are complex affine manifolds, they are naturally real affine manifolds. The map f_U is automatically a real affine map. Then Theorem 6.1 in [7] implies that there exists a global real affine map

$$f:\, M \to M'$$

which is an extension of f_U on M. We know that f is real analytic on M and holomorphic on U, so f is globally holomorphic on M. If there is another holomorphic affine extension $g:\, M \to M'$ of f_U, then g is holomorphic and $g = f$ on U. This implies $g = f$ on M. □

3.2. **Holomorphic affine structure on the Teichmüller space.** We know that there is the pull-back bundle of the Hodge bundle $F^n \to D$ on $\mathcal{T}$ via the period map Φ. Notice that as $H^{n,0}$ is isomorphic F^n as holomorphic bundles, we denote the restriction of the pull-back bundle on U_p by $H^{n,0} \to U_p$. Then there exists a local holomorphic section,

$$[\Omega_p]:\, U_p \to H^{n,0}, \quad \text{with} \quad [\Omega_p](q) \in H^{n,0}_q, \quad \text{for each} \quad q \in U_p.$$

Let us choose orthonormal bases $\{\eta_0\}$ and $\{\eta_1, \cdots, \eta_N\}$ of $H^{n,0}_p$ and $H^{n-1,1}_p$ respectively. Without loss of generality we assume $[\Omega_p](p) = \eta_0$. As vector spaces, $H^n(M_q, \mathbb{C})$ are the same for all $q \in U_p$. Therefore, using $P^{n-j,j}_p$ to denote projection

$$P^{n-j,j}_p :\, \bigoplus_{k=0}^{n} H^{n-k,k}_p \to H^{n-j,j}_p, \quad \text{for any} \quad 0 \le j \le n, \tag{11}$$

we then have the decomposition of the holomorphic section $[\Omega_p](q)$ for any $q \in U_p$, and

$$P^{n,0}_p([\Omega_p](q)) = a_0(q)\eta_0, \quad P^{n-1,1}_p([\Omega_p](q)) = a_1(q)\eta_1 + \cdots + a_N(q)\eta_N,$$

with holomorphic coefficient functions $\{a_i(q)\}_{i=0}^N$. We may assume $a_0(q) \neq 0$ for any $q \in U_p$ on a small enough neighborhood U_p since a_0 is a continuous function with $a_0(p) = 1$. This gives us a collection of local holomorphic functions on U_p as follows,

$$\tau_i :\, U_p \to \mathbb{C}, \quad \tau_i(q) = a_i(q)/a_0(q), \quad \text{for } 1 \le i \le N. \tag{12}$$

In particular, $\tau_i(p) = 0$ for $1 \le i \le N$. In order to show that $(\tau_1, \cdots, \tau_N)$ gives a local holomorphic coordinate system, we first show the following two lemmas.

Let us denote by (M_p, L) the corresponding polarized and marked Calabi–Yau manifold as the fiber over $p \in \mathcal{T}$. Yau's solution of the Calabi conjecture implies that there exists a unique Calabi–Yau metric h_p on M_p, and the imaginary part $\omega_p = \operatorname{Im} h_p \in L$ is the corresponding Kähler form. We denote by $A^{0,1}(M, T^{1,0}M)$ the space of $T^{1,0}M$-valued smooth $(0,1)$-forms, $H^{0,1}(M,\, T^{1,0}M)$ the space of harmonic $(0,1)$-forms with values in a holomorphic tangent bundle $T^{1,0}M$, and $A^{n-1,1}(M)$ the space of smooth $(n-1,1)$-forms on M. By using the Calabi–Yau metric we have the following lemma,

Lemma 3.3. *Let Ω_p be a nowhere vanishing holomorphic $(n,0)$-form on M_p such that*

$$\left(\frac{\sqrt{-1}}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega_p \wedge \overline{\Omega}_p = \omega_p^n. \tag{13}$$

Then the map $\iota: A^{0,1}(M, T^{1,0}M) \to A^{n-1,1}(M)$ *given by* $\iota(\varphi) = \varphi \lrcorner \Omega_p$ *is an isometry with respect to the natural Hermitian inner product on both spaces induced by* ω_p*. Furthermore,* ι *preserves the Hodge decomposition.*

Recall that we can always pick local coordinates $z_1, \cdots, z_n$ on M such that the $(n,0)$-form $\Omega_p = dz_1 \wedge \cdots \wedge dz_n$ locally and $\omega_p = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, then the condition (13) implies that $\det[g_{i\bar{j}}] = 1$. Then the lemma follows from direct computations.

Let us denote the following projection map for any point $p \in \mathcal{T}$,

$$P_p^j : \bigoplus_{k=1}^{n-1} \mathrm{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k) \to \mathrm{Hom}(F_p^j/F_p^{j+1}, F_p^{j-1}/F_p^j), \quad \text{for any} \quad 0 \le j \le n$$

where $F_p^{n+1} = 0$. We then have the following lemma.

Lemma 3.4. *For any* $p \in \mathcal{T}$ *and any generator* $[\Omega_p]$ *of* F_p^n*, the map*

$$P_p^n \circ \Phi_* : T_p^{1,0}\mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p) \to Hom(F_p^n, F_p^{n-1}/F_p^n) \cong H_p^{n-1,1}$$

is an isomorphism, where Φ_* *is the tangent map of* Φ*.*

Proof. The first isomorphism $T_p^{1,0}\mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p)$ follows from that Kodaira-Spencer map is an isomorphism. The second isomorphism

$$\mathrm{Hom}(F_p^n, F_p^{n-1}/F_p^n) \cong H_p^{n-1,1}$$

follows from the property that $\dim F_p^n = 1$ for Calabi–Yau manifolds, where this isomorphism is determined by the choice of the generator $[\Omega_p]$. It is clear now that the map

$$P_p^n \circ \Phi_* : H^{0,1}(M_p, T^{1,0}M_p) \to H_p^{n-1,1}$$

is given by contraction $P_p^n \circ \Phi_*(v) = [\kappa(v) \lrcorner \Omega_p]$. This contraction map is an isomorphism by Lemma 3.3. □

Lemma 3.5. *The holomorphic map* $(\tau_1, \cdots, \tau_N) : U_p \to \mathbb{C}^N$ *is a local embedding.*

Proof. The proof follows directly from Lemma 3.4, which states that the map

$$P_p^n \circ \Phi_* : T_p^{1,0}\mathcal{T} \to H_p^{n-1,1}$$

is an isomorphism. Letting $\{t_1, \cdots, t_N\}$ be an arbitrary local holomorphic coordinate system around p, then the Jacobian matrix $[\partial a_i/\partial t_j]_{i,j=0}^N$ is nonsingular at p. This means that $\{a_1, \cdots, a_N\}$ gives a local holomorphic coordinate system around p. Since $\tau_i = a_i/a_0$, we have

$$\frac{\partial \tau_i}{\partial t_j} = \left(\frac{\partial a_i}{\partial t_j} a_0 - \frac{\partial a_0}{\partial t_j} a_i \right) / a_0^2, \quad \text{for any} \quad 1 \le i \le N.$$

As $a_0(p) = 1$, and $a_i(p) = 0$ for $i > 0$, we have $\frac{\partial \tau_i}{\partial t_j}(p) = \frac{\partial a_i}{\partial t_j}(p)$. Thus the Jacobian matrix $[\partial \tau_i/\partial t_j]_{i,j=0}^N$ is nonsingular at p. This proves that the map $(\tau_1, \cdots, \tau_N) : U_p \to \mathbb{C}^N$ is a local embedding. Therefore $(\tau_1, \cdots, \tau_N)$ also gives a local holomorphic coordinate system around p. □

In general, for any point $q \in \mathcal{T}$, by choosing bases of $H_q^{n,0}$ and $H_q^{n-1,1}$, we can define a local holomorphic coordinate $(U_q, (\sigma_1, \cdots, \sigma_N))$ around q in the same way. Thus the collection $\{(U_q, (\sigma_1, \cdots, \sigma_N)), q \in \mathcal{T}\}$ forms a holomorphic coordinate system on $\mathcal{T}$, which we call the *Kuranishi coordinate cover* over $\mathcal{T}$. The local holomorphic coordinate system $(U_q, (\sigma_1, \cdots, \sigma_N))$ a *local Kuranishi coordinate chart* around q.

For any $p \in \mathcal{T}$, by using the coordinate chart $(U_p, (\tau_1, \cdots, \tau_N))$, we can express a holomorphic class in $H_q^{n,0}$ at each point $q \in U_p$ by the following Taylor expansion,

$$[\Omega_{p,\tau}^c](q) = [\Omega_p](q)/a_0(q) = \eta_0 + \tau_1(q)\eta_1 + \cdots + \tau_N(q)\eta_N + g(q), \tag{14}$$

with $g(q) \in \bigoplus_{k=2}^n H_p^{n-k,k}$. This gives us a local holomorphic section of F^n on U_p, and thus a holomorphic section of F^n on U_p,

$$[\Omega_{p,\tau}^c] : U_p \to F^n, \quad q \mapsto [\Omega_{p,\tau}^c](q).$$

We call $[\Omega_{p,\tau}^c]$ the *local canonical section* of F^n around the point p in the local Kuranishi coordinate chart $(U_p, (\tau_1, \cdots, \tau_N))$.

Let us introduce the notion of an adapted basis for the given Hodge decomposition. For any $p \in \mathcal{T}$ and $f^k = \dim F_p^k$ for any $0 \leq k \leq n$, we call a basis

$$\xi = \left\{\xi_0, \xi_1, \cdots, \xi_N, \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^2}, \cdots, \xi_{f^1-1}, \xi_{f^0-1}\right\}$$

of $H^n(M_p, \mathbb{C})$ an *adapted basis* for the given Hodge decomposition

$$H^n(M_p, \mathbb{C}) = H_p^{n,0} \oplus H_p^{n-1,1} \oplus \cdots \oplus H_p^{1,n-1} \oplus H_p^{0,n},$$

if it satisfies $H_p^{k,n-k} = \mathrm{Span}_{\mathbb{C}}\left\{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\right\}$ with $\dim H_p^{k,n-k} = f^k - f^{k+1}$.

Moreover, unless otherwise pointed out, the matrices in this paper are $m \times m$ matrices, where $m = f^0$. The blocks of the $m \times m$ matrix T is set in the following way: for each $0 \leq \alpha, \beta \leq n$, the (α, β)-th block $T^{\alpha,\beta}$ is

$$T^{\alpha,\beta} = \left[T_{ij}(\tau)\right]_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n}-1,\ f^{-\beta+n+1} \leq j \leq f^{-\beta+n}-1}, \tag{15}$$

where T_{ij} is the entries of the matrix T, and f^{n+1} is defined to be zero. In particular, $T = [T^{\alpha,\beta}]$ is called a *block lower triangular matrix* if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

Remark 3.6. Let Ω^k denote the sheaf of holomorphic k-forms on complex manifolds for each $0 \leq k \leq n$, especially Ω^0 is the sheaf of holomorphic functions. We use $H^{n-k}(M_q, \Omega^k)$ to denote the $(n-k)$-th sheaf cohomology group of Ω^k on the Calabi–Yau manifold M_q. For the local Kuranishi family $\varphi : \mathcal{U} \to \mathcal{T}$ introduced in Proposition 2.4, let $\Omega_{\mathcal{U}/\mathcal{T}}^k$ be the sheaf of relative holomorphic k-forms on $\mathcal{U}$. It is well-known that the push-forward sheaf $R^{n-k}\varphi_*\Omega_{\mathcal{U}/\mathcal{T}}^k$, which is a sheaf of free $\mathcal{O}_{\mathcal{T}}$- modules of finite rank with fiber at q, is isomorphic to $H^{n-k}(M_q, \Omega^k)$. By abuse of notation, the holomorphic vector bundle over $\mathcal{T}$ corresponding to $R^{n-k}\varphi_*\Omega_{\mathcal{U}/\mathcal{T}}^k$ is also denoted by $R^{n-k}\varphi_*\Omega_{\mathcal{U}/\mathcal{T}}^k$.

For any point $p \in \mathcal{T}$, each Dolbeault cohomology group $H_p^{k,n-k}$ is identified with the sheaf cohomology group $H^{n-k}(M_p, \Omega^k)$ for $0 \leq k \leq n$, which is defined by $H^{n-k}(M_p, \Omega^k) = R^{n-k}\varphi_*\Omega_{\mathcal{U}/\mathcal{T}}^k$. See Chapter 8 in [15]. Globally, on the Teichmüller space $\mathcal{T}$, we can identify each quotient F^k/F^{k+1} with $R^{n-k}\varphi_*\Omega_{\mathcal{U}/\mathcal{T}}^k$ as holomorphic vector bundles. Therefore we will be able to choose local holomorphic sections $\{\xi_0, \cdots, \xi_{m-1}\}$ of $R^{n-k}\varphi_*\Omega_{\mathcal{U}/\mathcal{T}}^k$ such that, when restricted to each point p, these holomorphic sections form an adapted basis of the Hodge decomposition of $\Phi(p)$.

Lemma 3.7. *For any $p, q \in \mathcal{T}$, suppose $(\eta_0, \cdots, \eta_{m-1})$ and $(\zeta_0, \cdots, \zeta_{m-1})$ are adapted bases for the Hodge decompositions of the Hodge structures $\Phi(p)$ and $\Phi(q)$ respectively. Then there exists a nonsingular block lower triangular matrix T such that*

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T.$$

Proof. This lemma is a direct corollary of the Griffiths transversality for the period map. Let $(U_p, (\tau_1, \cdots, \tau_N))$ be the local Kuranishi coordinate chart around the point p. We use the same notation for a point $\tau \in U_p$ and its coordinate in the coordinate chart.

We first consider the case when $q \in U_p$. As U_p is contractible, the holomorphic Hodge bundles $F^n \cong R^0\varphi_*\Omega^n_{\mathcal{U}/\mathcal{T}}, F^{n-1}/F^n \cong R^1\varphi_*\Omega^{n-1}_{\mathcal{U}/\mathcal{T}}, \cdots, F^0/F^1 \cong R^n\varphi_*\Omega^0_{\mathcal{U}/\mathcal{T}}$ are trivial bundles on U_p. Therefore for each $\tau \in U_p$, there exist holomorphic frames $\{\xi_0(\tau)\}$, $\{\xi_1(\tau), \cdots, \xi_N(\tau)\}$, $\cdots$, $\{\xi_{m-1}(\tau)\}$ of $R^0\varphi_*\Omega^n_{\mathcal{U}/\mathcal{T}}, R^1\varphi_*\Omega^{n-1}_{\mathcal{U}/\mathcal{T}}, \cdots, R^n\varphi_*\Omega^0_{\mathcal{U}/\mathcal{T}}$ respectively, that is,

$$\mathrm{Span}_{\mathbb{C}}\{\xi_j(\tau)\}_{f^{k+1}\le j\le f^k-1} = H^{n-k}(M_\tau, \Omega^k) \quad \text{for each} \quad 0 \le k \le n.$$

According to Remark 3.6, we have the linear space isomorphism

$$H^{n-k}(M_\tau, \Omega^k) \cong H^{k,n-k}_\tau \quad \text{for each} \quad 0 \le k \le n \quad \text{and each} \quad \tau \in U_p,$$

and the evaluation $(\xi_0(\tau), \cdots, \xi_{m-1}(\tau))$ of the family at τ forms an adapted basis for the Hodge decomposition $\oplus_{k=0}^n H^{k,n-k}_\tau$ of $\Phi(\tau)$. In particular, we can choose the family such that $\xi_j(p) = \eta_j$ at p for all $0 \le j \le m-1$.

Now for each $0 \le k \le n$, we look at the holomorphic families $\{\xi_j(\tau)\}_{f^{k+1}\le j\le f^k-1} \subseteq H^{n-k}(M_\tau, \Omega^k)$. Using the Griffiths transversality, we have

$$\tfrac{\partial}{\partial\tau}\xi_j(\tau) \in H^{n-k}(M_\tau, \Omega^k) \oplus H^{n-k+1}(M_\tau, \Omega^{k-1}).$$

More generally, by taking higher order derivatives, we get

$$\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(\tau) \in H^{n-k}(M_\tau, \Omega^k) \oplus \cdots \oplus H^{n-k+|I|}(M_\tau, \Omega^{k-|I|}),$$

with $I = (i_1, \cdots, i_N), |I| = i_1 + \cdots + i_N$. Therefore, we can write

$$\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(\tau) = \sum_{f^{k+1}\le i\le f^{k-|I|}-1} J^I_{ij}(\tau)\xi_i(\tau) \quad \text{for each } |I| \ge 0,$$

where $J^I_{ij}(\tau)$ are holomorphic functions of τ. Let us look at the Taylor expansion of $\xi_j(\tau)$ at p. For U_p small enough, we get for each $f^{k+1} \le j \le f^k - 1$ that

$$\xi_j(\tau) = \sum_{|I|\ge 0} \frac{1}{I!}\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(p)\tau^I = \sum_{|I|\ge 0}\left(\sum_{f^{k+1}\le i\le f^{k-|I|}-1} \frac{1}{I!}J^I_{ij}(p)\xi_i(p)\right)\tau^I,$$

where $\xi_i(\tau) \in H^{n-k}(M_p, \Omega^{n-k}) \oplus \cdots \oplus H^0(M_p, \Omega^n)$ and $I! = i_1! \cdots i_N!$. Thus, combining the coefficients for each $\xi_i(p)$, we can write

$$\xi_j(\tau) = \sum_{f^{k+1}\le i\le m-1} J_{ij}(\tau)\xi_i(p) = \sum_{f^{k+1}\le i\le m-1} J_{ij}(\tau)\eta_i, \quad \text{for each} \quad f^{k+1} \le j \le f^k - 1. \tag{16}$$

By considering all $0 \le k \le n$, (16) gives us an $m \times m$ matrix $J(\tau)$, such that

$$(\xi_0(\tau), \cdots, \xi_{m-1}(\tau)) = (\eta_0, \cdots, \eta_{m-1})J(\tau).$$

From (16), we also get that given any $0 \leq \alpha \leq n$ and $f^{n-\alpha+1} \leq j \leq f^{n-\alpha} - 1$, if $i \leq f^{n-\alpha+1} - 1$, then $J_{ij}(\tau) = 0$. Recalling the definition of matrix blocks given by (15), we may rephrase this as follows: if $f^{n-\alpha+1} < f^{n-\beta+1}$, then $J^{\alpha,\beta} = 0$. As f^k is decreasing with respect to k, we then have that for $\alpha < \beta$, $J^{\alpha,\beta} = 0$. Thus we conclude that $J(\tau)$ is a block lower triangular matrix. In particular, for the point $q \in U_p$, we have

$$(\xi_0(q), \cdots, \xi_{m-1}(q)) = (\eta_0, \cdots, \eta_{m-1}) J(q). \tag{17}$$

Moreover, since $(\xi_0(q), \cdots, \xi_{m-1}(q))$ and $(\eta_0, \cdots, \eta_{m-1})$ are both bases for $H^n(M, \mathbb{C})$, the matrix $J(q)$ is nonsingular. As both $(\xi_0(q), \cdots, \xi_{m-1}(q))$ and $(\zeta_0, \cdots, \zeta_{m-1})$ are adapted bases of the Hodge decomposition $\oplus_{k=0}^n H_q^{k,n-k}$ of $\Phi(q)$, they differ from each other by a nonsingular block diagonal matrix A, that is,

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\xi_0(q), \cdots, \xi_{m-1}(q)) A. \tag{18}$$

Combining (17) and (18), we get $(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1}) J(q) A$. Let $T(q) = J(q)A$. Then $T(q)$ is a nonsingular block lower triangular matrix such that

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1}) T(q).$$

For the general case, since $\mathcal{T}$ is a connected complex manifold, it is path-connected. Therefore, for any $q \in \mathcal{T}$, there exists a path $\gamma : [0,1] \to \mathcal{T}$ connecting p and q, that is, $\gamma(0) = p$ and $\gamma(1) = q$. We choose a partition $0 = s_0 < s_1 < \cdots < s_l = 1$ of $[0,1]$ so that $\gamma(s_{k+1}) \in U_{\gamma(s_k)}$ with $(U_{\gamma(s_{k+1})}, (\tau_1^{(k+1)}, \cdots, \tau_N^{(k+1)}))$ a local Kuranishi coordinate chart for each k. Therefore for any adapted bases $(\xi_0^{(k)}, \cdots, \xi_{m-1}^{(k)})$ and $(\xi_0^{(k+1)}, \cdots, \xi_{m-1}^{(k+1)})$ and the Hodge decomposition of the Hodge decompositions of $\Phi(\gamma(s_k))$ and $\Phi(\gamma(s_{k+1}))$, respectively, we can apply the argument above repeatedly to all $0 \leq k \leq l-1$ to conclude that there exists some nonsingular block lower triangular matrix T_k such that

$$(\xi_0^{(k+1)}, \cdots, \xi_{m-1}^{(k+1)}) = (\xi_0^{(k)}, \cdots, \xi_{m-1}^{(k)}) T_k, \quad \text{for} \quad 0 \leq k \leq l-1.$$

In particular, we may assume $(\xi_0^{(0)}, \cdots, \xi_{m-1}^{(n-1)}) = (\eta_0, \cdots, \eta_{m-1})$ and $(\xi_0^{(l)}, \cdots, \xi_{m-1}^{(l)}) = (\zeta_0, \cdots, \zeta_{m-1})$. Thus we have

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\xi_0^{(l)}, \cdots, \xi_{m-1}^{(l)}) = (\xi_0^{(0)}, \cdots, \xi_{m-1}^{(0)}) \prod_{k=0}^{l-1} T_k = (\eta_0, \cdots, \eta_{m-1}) \prod_{k=0}^{l-1} T_k.$$

Let $T(q) = \prod_{k=0}^{l-1} T_k$, which is a product of nonsingular block lower triangular matrices. Therefore $T(q)$ is a nonsingular block lower triangular matrix such that $(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1}) T(q)$. □

Theorem 3.8. *The Kuranishi coordinate cover gives a global holomorphic affine structure on $\mathcal{T}$.*

Proof. Let $p, q \in \mathcal{T}$. Let $(U_p, (\tau_1, \cdots, \tau_N))$ and $(U_q, (\sigma_1, \cdots, \sigma_N))$ be the local Kuranishi coordinate charts around p and q, respectively with $U_p \cap U_q \neq \emptyset$. To prove the theorem, we need to show that given $r \in U_p \cap U_q$, the transition map between $(\tau_1(r), \cdots, \tau_N(r))$ and $(\sigma_1(r), \cdots, \sigma_N(r))$ is holomorphic affine.

Suppose $(U_p, (\tau_1, \cdots, \tau_N))$ and $(U_q, (\sigma_1, \cdots, \sigma_N))$ are defined by choosing the bases $(\eta_0, \cdots, \eta_N)$ of $H_p^n \oplus H_p^{n-1,1}$ and $(\zeta_0, \cdots, \zeta_N)$ of $H_q^n \oplus H_q^{n-1,1}$ respectively. We then complete them to adapted bases $(\eta_0, \cdots, \eta_{m-1})$ and $(\zeta_0, \cdots, \zeta_{m-1})$ for the Hodge decompositions of the Hodge structures $\Phi(p)$ and $\Phi(q)$ respectively. Let $[\Omega^c_{p,\tau}](r) \in H^{n,0}_r$ be the local canonical $(n,0)$-class defined by (14) in the local coordinate chart $(U_p, (\tau_1, \cdots, \tau_N))$. Then

$$[\Omega^c_{p,\tau}](r) = \eta_0 + \sum_{i=1}^N \tau_i(r)\eta_i + \sum_{l\geq N+1} g_l(\tau(r))\eta_l.$$

Similarly, letting $[\Omega^c_{q,\sigma}](r) \in H^{n,0}_r$ be the local canonical $(n,0)$ class in the local coordinate chart $(U_q, (\sigma_1, \cdots, \sigma_N))$, we have

$$[\Omega^c_{q,\sigma}](r) = \zeta_0 + \sum_{i=1}^N \sigma_i(r)\zeta_i + \sum_{l\geq N+1} f_l(\sigma(r))\zeta_l.$$

Since $[\Omega^c_{p,\tau}](r), [\Omega^c_{q,\sigma}](r) \in H^{n,0}_r$, with $\dim H^{n,0}_r = 1$, there exists $\lambda \in \mathbb{C}$ such that

$$\eta_0 + \sum_{i=1}^N \tau_i(r)\eta_i + \sum_{l\geq N+1} g_l(\tau(r))\eta_l = \lambda\left(\zeta_0 + \sum_{i=1}^N \sigma_i(r)\zeta_i + \sum_{l\geq N+1} f_l(\sigma(r))\zeta_l\right). \tag{19}$$

On the other hand, by Lemma 3.7, there exists a nonsingular block lower triangular matrix T such that

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T. \tag{20}$$

We can write (20) explicitly as follows,

$$\zeta_j = \sum_{i\geq f^{k+1}} T_{ij}\eta_i \quad \text{for each} \quad f^{k+1} \leq j \leq f^k - 1 \quad \text{and for each} \quad 0 \leq k \leq n.$$

Substituting ζ_j in (19) by using (20), then we get

$$\begin{aligned} &\eta_0 + \sum_{i=1}^N \tau_i(r)\eta_i + \sum_{l\geq N+1} g_l(\tau(r))\eta_l \\ &= \lambda\left(\sum_{j\geq 0} T_{j0}\eta_j + \sum_{i=1}^N \left(\sigma_i(r)\sum_{j\geq 1} T_{ji}\eta_j\right) + \sum_{l\geq N+1}\left(f_l(\sigma(r))\sum_{j\geq N+1} T_{jl}\eta_j\right)\right). \end{aligned}$$

By considering the coefficient of η_0, we get $\lambda = T_{00}^{-1}$. Projecting both sides of the above equation to $H^{n-1,1}_p$, we get

$$\sum_{i=1}^N \tau_i(r)\eta_i = \lambda\left(\sum_{i\geq 1}^N \left(T_{i0} + \sum_{j=1}^N \sigma_j(r)T_{ij}\right)\eta_i\right).$$

By considering the coefficient of each η_i, and substituting $\lambda = T_{00}^{-1}$, we get

$$T_{00}\tau_i(r) = T_{i0} + \sum_{j=1}^N \sigma_j(r)T_{ij}, \quad \text{for any} \quad 1 \leq i \leq N. \tag{21}$$

Since for each $0 \leq i, j \leq N$, T_{ij} is a constant depending only on p and q, the equality (21) implies that the transition map between $(\tau_1(r), \cdots, \tau_N(r))$ and $(\sigma_1(r), \cdots, \sigma_N(r))$ is a holomorphic affine map. □

Remark 3.9. We can not expect the existence of such holomorphic affine structure on the moduli space $\mathcal{Z}_m$. In fact, we crucially used Lemma 3.7 in the computation of transition maps in the proof of Theorem 3.8. Let $(\zeta_0, \cdots \zeta_{m-1})$ and $(\eta_0, \cdots, \eta_{m-1})$ be adapted bases for Hodge decomposition of $\Phi(p)$ and $\Phi(p')$, respectively, which are both bases of the same trivialized space $H^n(M, \mathbb{C})$. Then we can relate them by $(\zeta_0, \cdots \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T$ using a nonsingular block lower triangular matrix T as was proved in Lemma 3.7. However, for any $q, q' \in \mathcal{Z}_m$, the corresponding Hodge decompositions of $\Phi_{\mathcal{Z}_m}(q)$ and $\Phi_{\mathcal{Z}_m}(q')$ are both in D/Γ. In this case, we don't have the concepts adapted bases for the Hodge decomposition of $H^n(M, \mathbb{C})$, as elements in D/Γ can no longer be identified with Hodge decompositions of $H^n(M, \mathbb{C})$. Thus, we have neither the concept of adapted bases nor the trivialization of space $H^n(M, \mathbb{C})$. Therefore, our construction of the holomorphic affine structure on $\mathcal{T}$ dose not apply to $\mathcal{Z}_m$.

3.3. **The local embedding Ψ on the Teichmüller space.** In this section, we first give a general review about Hodge structures and period domain from Lie group point of view. One may refer to [11] for more details. In particular, we describe in detail the identification of the unipotent group N_+ with its unipotent orbit in $\check{D}$.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over $\mathbb{Q}$. Let us simply denote $H_{\mathbb{C}} = H^n(M, \mathbb{C})$ and $H_{\mathbb{R}} = H^n(M, \mathbb{R})$. The group of the $\mathbb{C}$-rational points is

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

which acts on $\check{D}$ transitively. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which acts transitively on D as well.

Consider the period map $\Phi : \mathcal{T} \to D$. Fix a point $p \in \mathcal{T}$, with the image $o := \Phi(p) = \{F_p^n \subset \cdots \subset F_p^0\} \in D$. The points $p \in \mathcal{T}$ and $o \in D$ may be referred as the base points or the reference points. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF_p^k = F_p^k$ for each k. Thus it gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B, \quad \text{with} \quad B = \{g \in G_{\mathbb{C}}|\ gF_p^k = F_p^k, \text{ for any } k\}.$$

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D}, \text{ with } V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subseteq G_{\mathbb{C}}/B$. The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\mathfrak{g} = \{X \in \mathrm{End}(H_{\mathbb{C}})|\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\}.$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g}|\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$, $\mathfrak{g}_0$ becomes Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_p^{k,n-k}\}_{k=0}^n$ of $H^n(M, \mathbb{C})$ induces a Hodge structure of weight zero on $\mathrm{End}(H^n(M, \mathbb{C}))$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k}, \text{ with } \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g}|XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}\}.$$

Since the Lie algebra $\mathfrak{b}$ of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^n \subset \cdots \subset F_p^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}.$$

The Lie algebra $\mathfrak{v}$ of V is $\mathfrak{v} = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \oplus_{k \geq 1} \mathfrak{g}^{-k,k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We take the unipotent group $N_+ = \exp(\mathfrak{n}_+)$.

As $\mathrm{Ad}(g)(\mathfrak{g}^{k,-k})$ is in $\bigoplus_{i \geq k} \mathfrak{g}^{i,-i}$ for each $g \in B$, the sub-Lie algebra $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \subseteq \mathfrak{g}/\mathfrak{b}$ defines an $\mathrm{Ad}(B)$-invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$-invariant holomorphic subbundle of the holomorphic tangent bundle at the base point. It will be denoted by $T^{1,0}_{o,h}\check{D}$, and will be referred to as the holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle at the base point o, restricted to D, determines a subbundle $T^{1,0}_{o,h}D$ of the holomorphic tangent bundle $T^{1,0}_o D$ of D at the base point. The $G_{\mathbb{C}}$-invariace of $T^{1,0}_{o,h}\check{D}$ implies the $G_{\mathbb{R}}$-invariance of $T^{1,0}_{o,h}D$. As another interpretation of this holomorphic horizontal bundle at the base point, one has

$$T^{1,0}_{o,h}\check{D} \simeq T^{1,0}_o \check{D} \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k).$$

It is easy to see that the Griffiths transversality of the period map $\Phi : \mathcal{T} \to D$ is equivalent to $\Phi_*(T^{1,0}_p \mathcal{T}) \subseteq T^{1,0}_{o,h} D$ for any $p \in \mathcal{T}$. Since D is an open set in $\check{D}$, we have the following relation:

$$T^{1,0}_{o,h}D = T^{1,0}_{o,h}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \tag{22}$$

We remark that the Lie algebras $\mathfrak{n}_+$, $\mathfrak{g}^{-1,1}$ are originally defined to be subsets in $\mathfrak{g}$, and that the Lie group N_+ is defined to be a subgroup of $G_{\mathbb{C}}$. However, by fixing a base point, we can identify them with their orbits in the corresponding quotient Lie algebras or Lie groups. For example, $\mathfrak{n}_+ \cong \mathfrak{g}/\mathfrak{b}$, $\mathfrak{g}^{-1,1} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$, and $N_+ \cong N_+B/B \subseteq \check{D}$.

Remark 3.10. With a fixed base point, we can identify N_+ with its unipotent orbit in $\check{D}$ by dentifying an element $c \in N_+$ with $[c] = cB$ in $\check{D}$; that is, $N_+ = N_+(\text{ base point }) \cong N_+B/B \subseteq \check{D}$. In particular, when the base point o is in D, we have $N_+ \cap D \subseteq D$. In the rest of the paper, the base point is always chosen to be in D; thus we can view $N_+ \cap D = N_+(\text{ base point }) \cap D$ as a subset of D.

Recall that we can identify a point $\Phi(p) = \{F_p^n \subset F_p^{n-1} \subset \cdots \subset F_p^0\} \in D$ with its Hodge decomposition $\bigoplus_{k=0}^n H_p^{k,n-k}$, and thus with any fixed adapted basis of the corresponding Hodge decomposition. With a fixed adapted basis for the base point, we will matrix representations of elements in the above Lie groups and Lie algebras. Moreover, we have the following remark regarding the forms of matrix representation of elements in some of the Lie groups or Lie algebras mentioned above.

Remark 3.11. If we fix an adapted basis for the Hodge decomposition of the base point, and we identify the following quotient Lie groups or quotient Lie algebras with their orbits, then we have the following descriptions of matrix representation of elements: elements in

V are nonsingular block diagonal matrix and elements in N_+ will be nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix.

Let us fix a base point $\Phi(p) \in D$ with $p \in \mathcal{T}$. Then N_+ can be identified with its unipotent orbit in $\check{D}$ as in Remark 3.10. We also fix an adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition at the base point $\Phi(p) \in D$. Then according to Remark 3.11, we can identify elements in D with its matrix representation under the fixed adapted basis of the base point. In particular, the base point $\Phi(p)$ is identified with the identity matrix and elements in $N_+ \cap D$ can be represented by nonsingular block lower triangular matrices whose diagonal blocks are all identity matrix. With these assumptions, we have the following lemma.

Lemma 3.12. *The image of the period map $\Phi : \mathcal{T} \to D$ is in $N_+ \cap D$.*

Proof. For any $q \in \mathcal{T}$, according to Lemma 3.7, there exists a nonsingular block lower triangular matrix T such that $(\eta_0, \cdots \eta_{m-1})T$ gives an adapted basis for the Hodge decomposition of $\Phi(q)$. Thus the matrix T represents the point $\Phi(q) \in D$. Let A be the nonsingular block diagonal matrix which consists of the diagonal blocks of T. Then TA^{-1} is a nonsingular block lower triangular matrix with identies on the diagonal blocks. Thus the matrix TA^{-1} represents an element in N_+. As A^{-1} is a matrix in V and $D \cong G_{\mathbb{R}}/V$, $(\eta_0, \cdots, \eta_{m-1})TA^{-1}$ still represents the point $\Phi(q) \in D$. Therefore, the matrix TA^{-1} represents the point $\Phi(q)$ as well as a point in N_+. Thus we conclude that $\Phi(q) \in N_+$. ☐

Let p be the fixed base point in $\mathcal{T}$, then the isomorphism $\mathrm{Hom}(H_p^{n,0}, H_p^{n-1,1}) \cong H_p^{n-1,1} \cong \mathbb{C}^N$ follows from $\dim H_p^{n,0} = 1$. We now define the following projection map

$$P : N_+ \cap D \to \mathrm{Hom}(H_p^{n,0}, H_p^{n-1,1}) \cong H_p^{n-1,1} \cong \mathbb{C}^N, \tag{23}$$

$$F \mapsto (\eta_1, \cdots, \eta_N)F^{(1,0)} = F_{10}\eta_1 + \cdots F_{N0}\eta_N, \tag{24}$$

where $F^{(1,0)}$ is the $(1,0)$-block of the unipotent matrix F, according to our convention about block matrices in (15). Based on Lemma 3.12, we can define the map:

$$\Psi := P \circ \Phi : \mathcal{T} \to \mathbb{C}^N.$$

Therefore the map Ψ can also be describe as $\Psi(q) = P_p^{n-1,1}(P_q^{n,0}((\eta_0, \cdots, \eta_{m-1})TA^{-1}))$, for any $q \in \mathcal{T}$, where $\Phi(q) = (\eta_0, \cdots, \eta_{m-1})TA^{-1}$ according to Lemma 3.12 and $P_p^{n-1,1}$ and $P_q^{n,0}$ are the projections defined in (11).

We recall that in Section 3.2, fixing the basis $(\eta_0, \cdots, \eta_N)$ of $H_p^{n,0} \oplus H_p^{n-1,1}$ at the reference point p, we defined the Kuranishi coordinate map in (12) as follows:

$$(\tau_1, \cdots, \tau_N) : U_p \to \mathbb{C}^N, \quad q \mapsto (\tau_1(q), \cdots, \tau_N(q)).$$

Moreover, we have the Taylor expansion of the local canonical section of Hodge bundle F^n over U_p as given in (14),

$$[\Omega^c_{p,\tau}](q) = [\Omega_p](q)/a_0(q) = \eta_0 + (\eta_1, \cdots, \eta_N)(\tau_1(q), \cdots, \tau_N(q))^T + g(q),$$

where η_0 has constant coefficient 1 and $g(q) \in \bigoplus_{k=2}^n H_p^{n-k,k}$.

On the other hand, let us denote by $[\widetilde{\Omega}^c_{p,\tau}](q) \in H_q^{n,0}$ the first element of the adapted basis $(\eta_0, \cdots, \eta_{m-1})TA^{-1}$. Let us set $(TA^{-1})^{(1,0)} = ((TA^{-1})_{10}, (TA^{-1})_{20}, \cdots, (TA^{-1})_{N0})^T$, which is the $(1,0)$-block of the matrix TA^{-1}. Since all the diagonal blocks of TA^{-1} are identity submatrix, we have

$$[\widetilde{\Omega}^c_{p,\tau}](q) = \eta_0 + (\eta_1, \cdots, \eta_N)(TA^{-1})^{(1,0)} + f(q), \quad \text{with} \quad f(q) \in \bigoplus_{k=2}^{n} H_p^{n-k,k}. \tag{25}$$

However, the fact $\dim H_q^{n,0} = 1$ implies that there exists $\lambda \in \mathbb{C}$ such that $[\Omega^c_{p,\tau}](q) = \lambda[\widetilde{\Omega}^c_{p,\tau}](q)$. Then by comparing the coefficient of η_0 in the expression of $[\Omega^c_{p,\tau}](q)$ and $[\widetilde{\Omega}^c_{p,\tau}](q)$, we get $\lambda = 1$. Thus we have

$$[\widetilde{\Omega}^c_{p,\tau}](q) = [\Omega^c_{p,\tau}](q) = \eta_0 + (\eta_1, \cdots, \eta_N)(\tau_1(q), \cdots, \tau_N(q))^T + g(q), \tag{26}$$

with $g(q) \in \oplus_{k=2}^n H_p^{n-k,k}$. Now by comparing the coefficients of $(\eta_1, \cdots, \eta_N)$ in (25) and (26), we get that

$$(TA^{-1})^{(1,0)} = (\tau_1(q), \cdots, \tau_N(q))^T. \tag{27}$$

Let us denote the restriction map $\psi := \Psi|_{U_p} : U_p \to \mathbb{C}^N \cong \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \cong H_p^{n-1,1}$. Then $\psi(q) = (\eta_1, \cdots, \eta_N)(TA^{-1})^{(1,0)} = \tau_1(q)\eta_1 + \cdots + \tau_N(q)\eta_N$. Therefore, with respect to the affine structure on U_p given by the Kuranishi coordinate cover, the restriction map ψ is a holomorphic affine map. In particular, the tangent map of ψ at p

$$(\psi_*)_p = P_p^n \circ \Phi_* : T_p^{1,0}U_p \cong H^{0,1}(M_p, T^{1,0}M_p) \to \mathrm{Hom}(F_p^n, F_p^{n-1}/F_p^n) \cong H_p^{n-1,1}.$$

is an isomorphism, since $P_p^n \circ \Phi_*$ is an isomorphism according to Lemma 3.3. In particular, $(\psi_*)_p$ is nondegenerate. We now apply Theorem 3.2 to prove the following proposition.

Proposition 3.13. *The holomorphic map $\Psi : \mathcal{T} \to \mathbb{C}^N$ is an affine map with respect to the affine structure on $\mathcal{T}$ given by the Kuranishi coordinate cover. Moreover, the map Ψ is a local embedding.*

Proof. As was shown above, we have that $\psi : U_p \to \mathbb{C}^N$ is a holomorphic affine map with respect to the holomorphic affine structure on U_p given by the Kuranishi coordinate cover. Since $\mathcal{T}$ is a simply connected complex affine manifold and that $\mathbb{C}^N$ is a complete manifold, there exists a holomorphic affine extension map $\Psi' : \mathcal{T} \to \mathbb{C}^N$ with respect to the affine structure on $\mathcal{T}$ given by the Kuranishi coordinate cover such that $\Psi'|_{U_p} = \psi : U_p \to \mathbb{C}^N$. Since both Ψ and Ψ' are holomorphic maps from $\mathcal{T}$ to $\mathbb{C}^N$, and they agree on the open set U_p, they must be the same map on the whole set of $\mathcal{T}$, that is, $\Psi = \Psi' : \mathcal{T} \to \mathbb{C}^N$. Thus we conclude that Ψ is a holomorphic affine map.

We recall that the tangent map of the restriction $\psi : U_p \to \mathbb{C}^N$ is nondegenerate at $p \in \mathcal{T}$. Then the tangent map of Ψ is also nondegerate at p. Therefore, since $\Psi : \mathcal{T} \to \mathbb{C}^N$ is a holomorphic affine map, the tangent map of Ψ is actually nondegenerate at any point of $\mathcal{T}$. This shows that Ψ is a local embedding. □

4. Hodge metric completion of the Teichmüller space with level structure

In Section 4.1, we introduce the Hodge metric completion of the Teichmüller space with level m structure $\mathcal{T}^H_{_m}$, which is the universal cover of $\mathcal{Z}^H_{_m}$, where $\mathcal{Z}^H_{_m}$ is the completion space of the smooth moduli space $\mathcal{Z}_m$ with respect to the Hodge metric. We denote the lifting maps $i_m : \mathcal{T} \to \mathcal{T}^H_{_m}$ and $\Phi^H_{_m} : \mathcal{T}^H_{_m} \to D$ and take $\mathcal{T}_m := i_m(\mathcal{T})$ and $\Phi_m := \Phi^H_{_m}|_{\mathcal{T}_m}$.

We prove that Φ_m^H is a holomorphic map from $\mathcal{T}_m^H$ to $N_+ \cap D$. In Section 4.2, we first define the map Ψ_m^H from $\mathcal{T}_m^H$ to $\mathbb{C}^N$ and its restriction Ψ_m on the submanifold $\mathcal{T}_m$. We then use the local embeddedness of Ψ to show that Ψ_m is also a local embedding and conclude that there is a holomorphic affine structure on $\mathcal{T}_m$ with Ψ_m natually being affine on $\mathcal{T}_m$. Then the affineness of Ψ_m shows that the the extension map Ψ_m^H is also a local embedding. We then analogously conclude the affineness of $\mathcal{T}_m^H$ and Ψ_m^H. In Section 4.3, we prove that Ψ_m^H is an injection and it can be embedded into $\mathbb{C}^N$ by using the Hodge metric completeness and the global holomorphic affine structure on $\mathcal{T}_m^H$ as well as the affineness of Ψ_m^H. As a corollary, we show that the holomorphic map Φ_m^H is an injection.

4.1. **Definitions and basic properties.** Recall in Section 2.2, $\mathcal{Z}_m$ is the smooth moduli space of polarized Calabi–Yau manifolds with level m structure, which is introduced in [13]. We then defined the Teichmüller space $\mathcal{T}$ to be the universal cover of $\mathcal{Z}_m$.

By the work of Viehweg in [14], we know that $\mathcal{Z}_m$ is quasi-projective and that we can find a smooth projective compactification $\overline{\mathcal{Z}}_m$ such that $\mathcal{Z}_m$ is open in $\overline{\mathcal{Z}}_m$ and the complement $\overline{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ is a divisor of normal crossing. Therefore, $\mathcal{Z}_m$ is dense and open in $\overline{\mathcal{Z}}_m$ with the complex codimension of the complement $\overline{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ at least one. Moreover, as $\overline{\mathcal{Z}}_m$ a compact space, it is a complete space.

Recall at the end of Section 2.3, we pointed out that there is induced Hodge metric on $\mathcal{Z}_m$. Let us now take $\mathcal{Z}_m^H$ to be the completion of $\mathcal{Z}_m$ with respect to the Hodge metric. Then $\mathcal{Z}_m^H$ is the smallest complete space with respect to the Hodge metric that contains $\mathcal{Z}_m$. Although the compact space $\overline{\mathcal{Z}}_m$ may not be unique, the Hodge metric completion space $\mathcal{Z}_m^H$ is unique up to isometry. In particular, $\mathcal{Z}_m^H \subseteq \overline{\mathcal{Z}}_m$ and thus the complex codimension of the complement $\mathcal{Z}_m^H \backslash \mathcal{Z}_m$ is at least one. Given a fixed base point p, then any point in $\mathcal{Z}_m^H$ that is of Hodge finite distance from p has a neighborhood $U \subseteq \overline{\mathcal{Z}}_m$, which is also at finite Hodge distance from the reference point p. As $\overline{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ is at least of complex codimension one in $\overline{\mathcal{Z}}_m$ and that any two points in U have finite Hodge distance, we have $U \subseteq \mathcal{Z}_m^H$. This implies that $\mathcal{Z}_m^H$ is an open submanifold of $\overline{\mathcal{Z}}_m$. In particular, any two points in $\mathcal{Z}_m^H$ are of Hodge finite distance. We summarize the above observations in the following lemma.

Lemma 4.1. *The Hodge metric completion $\mathcal{Z}_m^H$ is a dense and open smooth submanifold in $\overline{\mathcal{Z}}_m$ and the complex codimenison of $\mathcal{Z}_m^H \backslash \mathcal{Z}_m$ is at least one.*

Remark 4.2. We recall some basic properties about metric completion space we are using in this paper. We know that the metric completion space of a connected space is still connected. Therefore, $\mathcal{Z}_m^H$ is connected since $\mathcal{Z}_m$ is connected. In particular, the universal cover $\mathcal{T}_m^H$ of the completion space $\mathcal{Z}_m^H$, $\mathcal{T}_m^H$ is also connected.

Suppose (X, d) is a metric space with the metric d. Then the metric completion space of (X, d) is unique in the following sense: if $\overline{X}_1$ and $\overline{X}_2$ are complete metric spaces that both contain X as a dense set, then there exists an isometry $f : \overline{X}_1 \to \overline{X}_2$ such that $f|_X$ is the identity map on X. Moreover, as mentioned above that the metric completion space $\overline{X}$ of X is the smallest complete metric space containing X in the sense that any other complete space that contains X as a subspace must also contains $\overline{X}$ as a subspace.

Moreover, suppose $\overline{X}$ is the metric completion space of the metric space (X, d). If there is a continuous map $f : X \to Y$ which is a local isometry with Y a complete space, then there exists a continuous extension $\overline{f} : \overline{X} \to Y$ such that $\overline{f}|_X = f$.

In the rest of the paper, unless otherwise pointed out, when we mention a complete space, the completeness is always with respect to the Hodge metric.

Let $\mathcal{T}^H_{_m}$ be the universal cover of $\mathcal{Z}^H_{_m}$ with the universal covering map $\pi^H_{_m} : \mathcal{T}^H_{_m} \to \mathcal{Z}^H_m$. Thus $\mathcal{T}^H_{_m}$ is a connected and simply connected complete smooth complex manifold with respect to the Hodge metric. We will call $\mathcal{T}^H_{_m}$ the *Hodge metric completion space with level m structure* of $\mathcal{T}$. Since $\mathcal{Z}^H_m$ is the Hodge metric completion of $\mathcal{Z}_m$, there exists the continuous extension map $\Phi^H_{_{\mathcal{Z}_m}} : \mathcal{Z}^H_m \to D/\Gamma$. Moreover, recall that the Teichmüller space $\mathcal{T}$ is the universal cover of the moduli space $\mathcal{Z}_m$ with the universal covering denoted by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}^H_{_m} & \xrightarrow{\Phi^H_{_m}} & D \\ \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi^H_{_m}} & & \downarrow{\scriptstyle \pi_D} \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}^H_{_m} & \xrightarrow{\Phi^H_{_{\mathcal{Z}_m}}} & D/\Gamma, \end{array} \tag{28}$$

where i is the inclusion map, $i_{_m}$ is a lifting map of $i \circ \pi_m$, π_D is the covering map and $\Phi^H_{_m}$ is a lifting map of $\Phi^H_{_{\mathcal{Z}_m}} \circ \pi^H_{_m}$. In particular, $\Phi^H_{_m}$ is a continuous map from $\mathcal{T}^H_{_m}$ to D. We notice that the lifting maps $i_{_\mathcal{T}}$ and $\Phi^H_{_m}$ are not unique, but Lemma A.1 implies that there exists a suitable choice of i_m and $\Phi^H_{_m}$ such that $\Phi = \Phi^H_{_m} \circ i_m$. We will fix the choice of i_m and Φ^H_m such that $\Phi = \Phi^H_{_m} \circ i_m$ in the rest of the paper. Let us denote $\mathcal{T}_m := i_m(\mathcal{T})$ and the restriction map $\Phi_m = \Phi^H_{_m}|_{\mathcal{T}_m}$. Then we also have $\Phi = \Phi_m \circ i_m$.

Proposition 4.3. *The image $\mathcal{T}_m$ equals to the preimage $(\pi^H_{_m})^{-1}(\mathcal{Z}_m)$.*

Proof. Because of the commutativity of diagram (28), we have that $\pi^H_{_m}(i_m(\mathcal{T})) = i(\pi_m(\mathcal{T})) = \mathcal{Z}_m$. Therefore, $\mathcal{T}_m = i_m(\mathcal{T}) \subseteq (\pi^H_{_m})^{-1}(\mathcal{Z}_m)$. For the other direction, we need to show that for any point $q \in (\pi^H_{_m})^{-1}(\mathcal{Z}_m) \subseteq \mathcal{T}^H_{_m}$, then $q \in i_m(\mathcal{T}) = \mathcal{T}_m$.

Let $p = \pi^H_{_m}(q) \in i(\mathcal{Z}_m)$, Let $x_1 \in \pi_m^{-1}(i^{-1}(p)) \subseteq \mathcal{T}$ be an arbitrary point, then $\pi^H_{_m}(i_m(x_1)) = i(\pi_m(x_1)) = p$ and $i_m(x_1) \in (\pi^H_{_m})^{-1}(p) \subseteq \mathcal{T}^H_{_m}$.

As $\mathcal{T}^H_{_m}$ is a connected smooth complex manifold, $\mathcal{T}^H_{_m}$ is path connected. Therefore, for $i_m(x_1), q \in \mathcal{T}^H_{_m}$, there exists a curve $\gamma : [0, 1] \to \mathcal{T}^H_{_m}$ with $\gamma(0) = i_m(x_1)$ and $\gamma(1) = q$. Then the composition $\pi^H_{_m} \circ \gamma$ gives a loop on $\mathcal{Z}^H_{_m}$ with $\pi^H_{_m} \circ \gamma(0) = \pi^H_{_m} \circ \gamma(1) = p$. Lemma A.2 implies that there is a loop Γ on $\mathcal{Z}_m$ with $\Gamma(0) = \Gamma(1) = i^{-1}(p)$ such that

$$[i \circ \Gamma] = [\pi^H_{_m} \circ \gamma] \in \pi_1(\mathcal{Z}^H_{_m}),$$

where $\pi_1(\mathcal{Z}^H_{_m})$ denotes the fundamental group of $\mathcal{Z}^H_{_m}$. Because $\mathcal{T}$ is universal cover of $\mathcal{Z}_m$, there is a unique lifting map $\widetilde{\Gamma} : [0, 1] \to \mathcal{T}$ with $\widetilde{\Gamma}(0) = x_1$ and $\pi_m \circ \widetilde{\Gamma} = \Gamma$. Again since $\pi^H_{_m} \circ i_m = i \circ \pi_m$, we have

$$\pi^H_{_m} \circ i_m \circ \widetilde{\Gamma} = i \circ \pi_m \circ \widetilde{\Gamma} = i \circ \Gamma : [0, 1] \to \mathcal{Z}_m.$$

Therefore $[\pi_m^H \circ i_m \circ \widetilde{\Gamma}] = [i \circ \Gamma] \in \pi_1(\mathcal{Z}_m)$, and the two curves $i_m \circ \widetilde{\Gamma}$ and γ have the same starting points $i_m \circ \widetilde{\Gamma}(0) = \gamma(0) = i_m(x_1)$. Then the homotopy lifting property of the covering map π_m^H implies that $i_m \circ \widetilde{\Gamma}(1) = \gamma(1) = q$. Therefore, $q \in i_m(\mathcal{T})$, as needed. □

Since $\mathcal{Z}_m$ is an open submanifold of $\mathcal{Z}_m^H$ and π_m^H is a holomorphic covering map, the preimage $\mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m)$ is a connected open submanifold of $\mathcal{T}_m^H$. Furthermore, because the complex codimension of $\mathcal{Z}_m^H \backslash \mathcal{Z}_m$ is as least one in $\mathcal{Z}_m^H$, the complex codimension of $\mathcal{T}_m^H \backslash \mathcal{T}_m$ is also as least one in $\mathcal{T}_m^H$.

We recall in Remark 3.10 that we fix a base point $\Phi(p) \in D$ with $p \in \mathcal{T}$ and identify the affine group N_+ with its unipotent orbit in $\check{D}$. In the following proposition, let us still view N_+ in this way by fixing the base point. First, it is not hard to see that the restriction map Φ_m is holomorphic. Indeed, we know that $i_m : \mathcal{T} \to \mathcal{T}_m$ is the lifting of $i \circ \pi_m$ and $\pi_m^H|_{\mathcal{T}_m} : \mathcal{T}_m \to \mathcal{Z}_m$ is a holomorphic covering map, thus i_m is also holomorphic. Since $\Phi = \Phi_m \circ i_m$ with both Φ, i_m holomorphic and i_m locally invertible, we can conclude that $\Phi_m : \mathcal{T}_m \to D$ is a holomorphic map. Moreover, we have $\Phi_m(\mathcal{T}_m) = \Phi_m(i_m(\mathcal{T})) = \Phi(\mathcal{T}) \subseteq N_+ \cap D$ as $\Phi = i_m \circ \Phi_m$. Therefore, Φ_m is a holomorphic map from $\mathcal{T}_m$ to $N_+ \cap D$. Then by applying the Riemann extension theorem to $\Phi_m : \mathcal{T}_m \to N_+$, we conclude the following result.

Proposition 4.4. *The map Φ_m^H is a holomorphic map from $\mathcal{T}_m^H$ to $N_+ \cap D$.*

Proof. According to the above discussion, we know that the complex codimension of the complement $\mathcal{T}_m^H \backslash \mathcal{T}_m$ is at least one, and $\Phi_m : \mathcal{T}_m \to N_+ \cap D$ is a locally bounded holomorphic map. Therefore, simply applying the Riemman extension theorem to the holomorphic map $\Phi_m : \mathcal{T}_m \to N_+ \cap D$, we conclude that there exists a holomorphic map $\Phi'_m : \mathcal{T}_m^H \to N_+ \cap D$ such that $\Phi'_m|_{\mathcal{T}_m} = \Phi_m$. We know that both Φ_m^H and Φ'_m are continuous maps defined on $\mathcal{T}_m^H$ that agree on the dense subset $\mathcal{T}_m$. Therefore, they must agree on the whole $\mathcal{T}_m^H$, that is, $\Phi_m^H = \Phi'_m$ on $\mathcal{T}_m^H$. Therefore, Φ_m^H is a holomorphic map from $\mathcal{T}_m^H$ to $N_+ \cap D$. □

4.2. **Holomorphic affine structure on the Hodge metric completion space.** In this section, we fix the base point $\Phi(p) \in D$ with $p \in \mathcal{T}$ and an adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition of $\Phi(p)$. Based on Proposition 4.4, we can analogously define the holomorphic map

$$\Psi_m^H = P \circ \Phi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N, \tag{29}$$

where P is the projection map given by (23) in Section 3.3 with the same the fixed base point $\Phi(p) \in D$ and the fixed adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition of $\Phi(p)$. Moreover, we also have $\Psi = P \circ \Phi = P \circ \Phi_m^H \circ i_m = \Psi_m^H \circ i_m$. Let us denote the restriction map $\Psi_m = \Psi_m^H|_{\mathcal{T}_m} : \mathcal{T}_m \to \mathbb{C}^N$ in the following context. Then Ψ_m^H is the continuous extension of Ψ_m and $\Psi = \Psi_m \circ i_m$. By the definition of Ψ_m^H, we can easily conclude the following lemma.

Lemma 4.5. *If the holomorphic map Ψ_m^H is injective, then $\Phi_m^H : \mathcal{T}_m^H \to N_+ \cap D$ is also injective.*

In the following lemma, we will crucially use the fact that the holomorphic map $\Psi : \mathcal{T} \to \mathbb{C}^N \cong H_p^{n-1,1}$ is a local embedding, which is based on the holomorphic affine structure on $\mathcal{T}$.

Lemma 4.6. *For any $m \geq 3$, the holomorphic map $\Psi_m : \mathcal{T} \to \mathbb{C}^N$ is a local embedding. Therefore, there exists a holomorphic affine structure on $\mathcal{T}_m$ and that the holomorphic map Ψ_m is an affine map with respect to this holomorphic affine structure on $\mathcal{T}_m$.*

Proof. Since $i \circ \pi_m = \pi_m^H \circ i_m$ with $i : \mathcal{Z}_m \to \mathcal{Z}_m^H$ the natural inclusion map and π_m, π_m^H both universal covering maps, i_m is a lifting of the inclusion map. Thus i_m is locally biholomorphic. On the other hand, we showed in Proposition 3.13 that Ψ is a local embedding. We may choose an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{T}_m$ such that for each $U_\alpha \subseteq \mathcal{T}_m$, i_m is biholomorphic on U_α and thus the inverse $(i_m)^{-1}$ is also an embedding on U_α. Obviously we may also assume that Ψ is an embedding on $(i_m)^{-1}(U_\alpha)$. In particular, the relation $\Psi = \Psi_m \circ i_m$ implies that $\Psi_m|_{U_\alpha} = \Psi \circ (i_m)^{-1}|_{U_\alpha}$ is also an embedding on U_α. In this way, we showed Ψ_m is a local embedding on $\mathcal{T}_m$.

Let $z = (z_1, \cdots, z_N)$ be the standard linear coordinate functions on $\mathbb{C}^N$. Since Ψ_m is an embedding on each U_α and $\dim \mathcal{T}_m = N$, the composition map $(z_1 \circ \Psi_m, \cdots, z_N \circ \Psi_m)$ gives a bijection from each U_α onto an open subset of $\mathbb{C}^N$. Thus this gives a local coordinate chart on each U_α. Therefore, we obtain a coordinate cover on $\mathcal{T}_m$ with trivial transition maps and this coordinate cover determines the desired holomorphic affine structure on $\mathcal{T}_m$. In particular, with respect to this holomorphic affine structure on $\mathcal{T}_m$, the holomorphic map $\Psi_m : \mathcal{T}_m \to \mathbb{C}^N$ is naturally an affine map. □

Lemma 4.7. *The holomorphic map $\Psi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N \cong H_p^{n-1,1}$ is a local embedding.*

Proof. The proof uses mainly the affineness of $\Psi_m : \mathcal{T}_m \to \mathbb{C}^N \cong H_p^{n-1,1}$, where $p \in \mathcal{T}$ is the base point in the definition of the projection map P. By Proposition 4.3, we know that $\mathcal{T}_m$ is dense and open in $\mathcal{T}_m^H$. Thus for any point $q \in \mathcal{T}_m^H$, there exists $\{q_k\}_{k=1}^\infty \subseteq \mathcal{T}_m$ such that $\lim_{k\to\infty} q_k = q$. Because $\Psi_m^H(q) \in H_p^{n-1,1}$, we can take a neighborhood $W \subseteq H_p^{n-1,1}$ of $\Psi_m^H(q)$ with $W \subseteq \Psi_m^H(\mathcal{T}_m^H)$.

Consider the projection map $P : N_+ \to \mathbb{C}^N$ with $P(F) = F^{(1,0)}$ the $(1,0)$ block of the matrix F, and the decomposition of the holomorphic tangent bundle

$$T^{1,0}N_+ = \bigoplus_{0\leq l\leq k\leq n} \mathrm{Hom}(F^k/F^{k+1}, F^l/F^{l+1}).$$

In particular, the subtangent bundle $\mathrm{Hom}(F^n, F^{n-1}/F^n)$ over N_+ is isomorphic to the pull-back bundle $P^*(T^{1,0}\mathbb{C}^N)$ of the holomorphic tangent bundle of $\mathbb{C}^N$ through the projection P. On the other hand, the holomorphic tangent bundle of $\mathcal{T}_m$ is also isomorphic to the holomorphic bundle $\mathrm{Hom}(F^n, F^{n-1}/F^n)$, where F^n and F^{n-1} are pull-back bundles on $\mathcal{T}_m$ via Φ_m from $N_+ \cap D$. Since the holomorphic map $\Psi_m = P \circ \Phi_m$ is a composition of P and Φ_m, the pull-back bundle of $T^{1,0}W$ through Ψ_m is also isomorphic to the tangent bundle of $\mathcal{T}_m$.

Now with the fixed adapted basis $\{\eta_1, \cdots, \eta_N\}$, one has a standard coordinate $(z_1, \cdots, z_N)$ on $\mathbb{C}^N \cong H_p^{n-1,1}$ such that each point in $\mathbb{C}^N \cong H_p^{n-1,1}$ is of the form $z_1\eta_1 + \cdots + z_N\eta_N$. Let us choose one special trivialization of

$$T^{1,0}W \cong \mathrm{Hom}(F_p^n, F_p^{n-1}/F_p^n) \times W$$

by the standard global holomorphic frame $(\Lambda_1, \cdots, \Lambda_N) = (\partial/\partial z_1, \cdots, \partial/\partial z_N)$ on $T^{1,0}W$. Under this trivialization, we can identify $T_o^{1,0}W$ with $\mathrm{Hom}(F_p^n, F_p^{n-1}/F_p^n)$ for any $o \in W$. Then $(\Lambda_1, \cdots, \Lambda_N)$ are parallel sections with respect to the trivial affine connection on

$T^{1,0}W$. Let $U_q \subseteq (\Psi_m^H)^{-1}(W)$ be a neighborhood of q and let $U = U_q \cap \mathcal{T}_m$. Then the pull back sections $(\Psi_m^H)^*(\Lambda_1, \cdots, \Lambda_N) : U_q \to T^{1,0}U_q$ are tangent vectors of U_q, we denote them by $(\mu_1^H, \cdots, \mu_N^H)$ for convenience.

According to the proof of Lemma 4.6, we know that the restriction map Ψ_m is a local embedding. Therefore the tangent map $(\Psi_m)_* : T_{q'}^{1,0}U \to T_o^{1,0}W$ is an isomorphism, for any $q' \in U$ and $o = \Psi_m(q')$. Moreover, since Ψ_m is a holomorphic affine map, the holomorphic sections $(\mu_1, \cdots, \mu_N) := (\mu_1^H, \cdots, \mu_N^H)|_U$ form a holomorphic parallel frame for $T^{1,0}U$. Under the parallel frames $(\mu_1, \cdots, \mu_N)$ and $(\Lambda_1, \cdots, \Lambda_N)$, there exists a nonsingular matrix function $A(q') = (a_{ij}(q'))_{1\le i\le N, 1\le j\le N}$, such that the tangent map $(\Psi_m)_*$ is given by

$$(\Psi_m)_*(\mu_1, \cdots, \mu_N)(q') = (\Lambda_1(o), \cdots, \Lambda_N(o))A(q'), \quad \text{with } q' \in U \text{ and } o = \Psi_m(q') \in D.$$

Moreover, since $(\Lambda_1, \cdots, \Lambda_N)$ and $(\mu_1, \cdots, \mu_N)$ are parallel frames for $T^{1,0}W$ and $T^{1,0}U$ respectively and Ψ_m is a holomorphic affine map, the matrix $A(q') = A$ is actually a constant nonsingular matrix for all $q' \in U$. In particular, for each $q_k \in U$, we have $((\Psi_m)_*\mu_1, \cdots, (\Psi_m)_*\mu_N)(q_k) = (\Lambda_1(o_k), \cdots, \Lambda_N(o_k))A$, where $o_k = \Psi_m(q_k)$. Because the tangent map $(\Psi_m^H)_* : T^{1,0}U_q \to T^{1,0}W$ is a continuous map, we have that

$$\begin{aligned}(\Psi_m^H)_*(\mu_1^H(q), \cdots, \mu_N^H(q)) &= \lim_{k\to\infty}(\Psi_m)_*(\mu_1(q_k), \cdots, \mu_N(q_k)) = \lim_{k\to\infty}(\Lambda_1(o_k), \cdots, \Lambda_N(o_k))A \\ &= (\Lambda_1(\overline{o}), \cdots, \Lambda_N(\overline{o}))A, \quad \text{where} \quad o_k = \Psi_m(q_k) \text{ and } \overline{o} = \Psi_m^H(q).\end{aligned}$$

As $(\Lambda_1(\overline{o}), \cdots, \Lambda_N(\overline{o}))$ forms a basis for $T_{\overline{o}}^{1,0}W = \mathrm{Hom}(F_p^n, F_p^{n-1}/F_p^n)$ and A is nonsingular, we can conclude that $(\Psi_m^H)_*$ is an isomorphism from $T_q^{1,0}U_q$ to $T_{\overline{o}}^{1,0}W$. This shows that $\Psi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N \cong H_p^{n-1,1}$ is a local embedding. □

Theorem 4.8. *There exists a holomorphic affine structure on $\mathcal{T}_m^H$. Moreover, the holomorphic map $\Psi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N$ is a holomorphic affine map with respect to this holomorphic affine structure on $\mathcal{T}_m^H$.*

Proof. Since $\Psi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N$ is a local embedding and $\dim \mathcal{T}_m^H = N$, thus the same arguments as the proof of Lemma 4.6 can be applied to conclude that there exists an induced holomorphic affine structure on $\mathcal{T}_m^H$ from the affine structure on $\mathbb{C}^N$ via the local embedding Ψ_m^H. In particular, with respect to this affine structure on $\mathcal{T}_m^H$, the holomorphic map Ψ_m^H is also an affine map. □

It is important to note that the flat connections which correspond to the global holomorphic affine structures on $\mathcal{T}$, on $\mathcal{T}_m$ or on $\mathcal{T}_m^H$ are in general not compatible with respect to the corresponding Hodge metric on them.

4.3. **Injectivity of the period map on the Hodge metric completion space.**

Theorem 4.9. *For any $m \ge 3$, the holomorphic map $\Psi_m^H : \mathcal{T}_m^H \to \mathbb{C}^N$ is an injection. In particular, the completion space $\mathcal{T}_m^H$ can be embedded into $\mathbb{C}^N$ via Ψ_m^H.*

To prove this theorem, we will first prove the following lemma, where we mainly use the completeness and the holomorphic affine structure on $\mathcal{T}_m^H$ as well as the affineness of Ψ_m^H.

Lemma 4.10. *For any two points in $\mathcal{T}_m^H$, there is a straight line in $\mathcal{T}_m^H$ connecting them.*

We remark that as $\mathcal{T}^H_m$ is a complex affine manifold, we have the notion of straight lines in it with respect to the affine structure.

Proof. Let p be an arbitrary point in $\mathcal{T}^H_m$, and let $S \subseteq \mathcal{T}^H_m$ be the collection of points that can be connected to p by straight lines in $\mathcal{T}^H_m$. We need to show that $S = \mathcal{T}^H_m$.

We first show that S is a closed set. Let $\{q_i\}_{i=1}^{\infty} \subseteq S$ be a Cauchy sequence with respect to the Hodge metric. Then for each i we have the straight line l_i connecting p and q_i such that $l_i(0) = p$, $l_i(T_i) = q_i$ for some $T_i \geq 0$ and $v_i := \frac{\partial}{\partial t} l_i(0)$ a unit vector. We can view these straight lines $l_i : [0, T_i] \to \mathcal{T}^H_m$ as the solutions of the affine geodesic equations $l_i''(t) = 0$ with initial conditions $v_i := \frac{\partial}{\partial t} l_i(0)$ and $l_i(0) = p$. It is well-known that solutions of these geodesic equations analytically depend on their initial data.

Let d_i denote the distance between q_i and p, for each i. As $\{q_i\}_{i=1}^{\infty}$ is a Cauchy sequence, $\lim_{i\to\infty} d_i = d_\infty$ for some d_∞. Since l_i is a straight line and v_i is a unit vector, we know that $d_i = T_i$ for each i. Indeed, since $l_i'(t) = l_i'(0) = v_i$, we have

$$d_i = \int_0^{T_i} ||l_i'(t)||\, dt = \int_0^{T_i} ||v_i||\, dt = T_i,$$

where the norm is the Hodge norm. Thus, as the set $\{d_i\}_{i=1}^{\infty}$ is bounded, the set $\{T_i\}_{i=1}^{\infty}$ is also bounded. Passing to a subsequence, we may therefore assume that $\{T_i\}$ and $\{v_i\}$ converge, with $\lim_{i\to\infty} T_i = T_\infty$ and $\lim_{i\to\infty} v_i = v_\infty$, respectively. Let $l_\infty(t)$ be the local solution of the affine geodesic equation with initial conditions $\frac{\partial}{\partial t} l_\infty(0) = v_\infty$ and $l_\infty(0) = p$. We claim that the solution $l_\infty(t)$ exists for $t \in [0, T_\infty]$. Consider the set

$$E_\infty := \{a : \, l_\infty(t) \quad \text{exists for} \quad t \in [0, a)\}.$$

If E_∞ is unbounded above, then the claim clearly holds. Otherwise, we let $a_\infty = \sup E_\infty$, and our goal is to show $a_\infty > T_\infty$. Suppose towards a contradiction that $a_\infty \leq T_\infty$. We then define the sequence $\{a_k\}_{k=1}^{\infty}$ so that $a_k/T_k = a_\infty/T_\infty$. We have $a_k \leq T_k$ and $\lim_{k\to\infty} a_k = a_\infty$. Using the continuous dependence of solutions of the geodesic equation on initial data, we conclude that the sequence $\{l_k(a_k)\}_{k=1}^{\infty}$ is a Cauchy sequence. As $\mathcal{T}^H_m$ is a complete space, the sequence $\{l_k(a_k)\}_{k=1}^{\infty}$ converges to some $q' \in \mathcal{T}^H_m$. Let us define $l_\infty(a_\infty) := q'$. Then the solution $l_\infty(t)$ exists for $t \in [0, a_\infty]$. On the other hand, as $\mathcal{T}^H_m$ is a smooth manifold, we have that q' is an inner point of $\mathcal{T}^H_m$. Thus the affine geodesic equation has a local solution at q' which extends the geodesic l_∞. That is, there exists $\epsilon > 0$ such that the solution $l_\infty(t)$ exists for $t \in [0, a_\infty + \epsilon)$. This contradicts the fact that a_∞ is an upper bound of E_∞. We have therefore proven that $l_\infty(t)$ exists for $t \in [0, T_\infty]$.

Using the continuous dependence of solutions of the affine geodesic equations on the initial data again, we get

$$l_\infty(T_\infty) = \lim_{k\to\infty} l_k(T_k) = \lim_{k\to\infty} q_k = q_\infty.$$

This means the limit point $q_\infty \in S$, and hence S is a closed set.

Let us now show that S is an open set. Let $q \in S$. Then there exists a straight line l connecting p and q. For each point $x \in l$ there exists an open neighborhood $U_x \subseteq \mathcal{T}^H_m$ with diameter $2r_x$. The collection of $\{U_x\}_{x\in l}$ forms an open cover of l. But l is a compact set, so there is a finite subcover $\{U_{x_i}\}_{i=1}^{K}$ of l. Then the straight line l is covered by a cylinder C_r in $\mathcal{T}^H_m$ of radius $r = \min\{r_{x_i} : \; 1 \leq i \leq K\}$. As C_r is a convex set, each point in C_r can be connected to p by a straight line. Therefore we have found an open neighborhood C_r of $q \in S$ such that $C_r \subseteq S$, which implies that S is an open set.

As S is a non-empty, open and closed subset in the connected space $\mathcal{T}^H_{_m}$, we conclude that $S = \mathcal{T}^H_{_m}$, as we desired. □

Proof of Theorem 4.9. Let $p, q \in \mathcal{T}^H_{_m}$ be two different points. Then Lemma 4.10 implies that there is a straight line $l \subseteq \mathcal{T}^H_{_m}$ connecting p and q. Since $\Psi^H_{_m} : \mathcal{T}^H_{_m} \to \mathbb{C}^N$ is affine, the restriction $\Psi^H_{_m}|_l$ is a linear map. Suppose towards a contradiction that $\Psi^H_{_m}(p) = \Psi^H_{_m}(q) \in \mathbb{C}^N$. Then the restriction of $\Psi^H_{_m}$ to the straight line l is a constant map as $\Psi^H_{_m}|_l$ is linear. By Lemma 4.7, we know that $\Psi^H_{_m} : \mathcal{T}^H_{_m} \to \mathbb{C}^N$ is locally injective. Therefore, we may take U_p to be a neighborhood of p in $\mathcal{T}^H_{_m}$ such that $\Psi^H_{_m} : U_p \to \mathbb{C}^N$ is injective. However, the intersection of U_p and l contains infinitely many points, but the restriction of $\Psi^H_{_m}$ to $U_p \cap l$ is a constant map. This contradicts the fact that when we restrict $\Psi^H_{_m}$ to $U_p \cap l$, $\Psi^H_{_m}$ is an injective map. Thus $\Psi^H_{_m}(p) \neq \Psi^H_{_m}(q)$ if $p \neq q \in \mathcal{T}^H_{_m}$. □

By Lemma 4.5, Theorem 4.9 implies the following corollary.

Corollary 4.11. *The holomorphic map $\Phi^H_{_m} : \mathcal{T}^H_{_m} \to N_+ \cap D$ is also an injection.*

5. Domain of holomorphy

In this section, we define the completion space $\mathcal{T}^H$ by $\mathcal{T}^H = \mathcal{T}^H_{_m}$, and the extended period map Φ^H by $\Phi^H = \Phi^H_{_m}$ for any $m \geq 3$ after proving that $\mathcal{T}^H_{_m}$ doesn't depend on the choice of the level structure. Therefore $\mathcal{T}^H$ is a complex affine manifold and that Φ^H is a holomorphic injection. We then prove the main result, which is Theorem 5.3: $\mathcal{T}^H$ is the completion space of $\mathcal{T}$ with respect to the Hodge metric and it is a domain of holomorphy in $\mathbb{C}^N$. As a direct corollary, we get the global Torelli theorem of the period map from the Teichmüller space to the period domain.

For any two integers $m, m' \geq 3$, let $\mathcal{Z}_m$ and $\mathcal{Z}_{m'}$ be the smooth quasi-projective manifolds as in Theorem 2.3 and $\mathcal{Z}^H_{_m}$ and $\mathcal{Z}^H_{m'}$ their completions with respect to the Hodge metric. Let $\mathcal{T}^H_{_m}$ and $\mathcal{T}^H_{m'}$ be the universal cover spaces of $\mathcal{Z}^H_{_m}$ and $\mathcal{Z}^H_{m'}$ respectively, then we have the following.

Proposition 5.1. *The complete complex manifolds $\mathcal{T}^H_m$ and $\mathcal{T}^H_{m'}$ are biholomorphic to each other.*

Proof. By defintion, $\mathcal{T}_m = i_m(\mathcal{T})$, $\mathcal{T}_{m'} = i_{m'}(\mathcal{T})$ and $\Phi_m = \Phi^H_{_m}|_{\mathcal{T}_m}$, $\Phi_{m'} = \Phi^H_{m'}|_{\mathcal{T}_{m'}}$. Because Φ^H_m and $\Phi^H_{m'}$ are embeddings, $\mathcal{T}_m \cong \Phi^H_{_m}(\mathcal{T}_m)$ and $\mathcal{T}_{m'} \cong \Phi^H_{_m}(\mathcal{T}_m)$. Since the composition maps $\Phi^H_{_m} \circ i_m = \Phi$ and $\Phi^H_{m'} \circ i_{m'} = \Phi$, we get $\Phi^H_{_m}(i_m(\mathcal{T})) = \Phi(\mathcal{T}) = \Phi^H_{m'}(i_{m'}(\mathcal{T}))$. Since Φ and $\mathcal{T}$ are both independent of the choice of the level structures, so is the image $\Phi(\mathcal{T})$. Then $\mathcal{T}_m \cong \Phi(\mathcal{T}) \cong \mathcal{T}_{m'}$ biholomorphically, and they don't depend on the choice of the level structures. Moreover, Proposition 4.3 implies that $\mathcal{T}^H_m$ and $\mathcal{T}^H_{m'}$ are Hodge metric completion spaces of $\mathcal{T}_m$ and $\mathcal{T}_{m'}$, respectively. Thus the uniqueness of the metric completion spaces implies that $\mathcal{T}^H_m$ is biholomorphic to $\mathcal{T}^H_{m'}$. □

Proposition 5.1 shows that $\mathcal{T}^H_{_m}$ is independent of the choice of the level m structure, and it allows us to give the following definitions.

Definition 5.2. *We define the complete complex manifold $\mathcal{T}^H = \mathcal{T}^H_m$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \to D$ by $\Phi^H = \Phi^H_m$ for any $m \geq 3$. In particular, with these new notations, we have the commutative diagram*

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}^H & \xrightarrow{\Phi^H} & D \\ \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi^H_m} & & \downarrow{\scriptstyle \pi_D} \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}^H_m & \xrightarrow{\Phi^H_{\mathcal{Z}_m}} & D/\Gamma. \end{array}$$

Theorem 5.3. *The complex manifold $\mathcal{T}^H$, which is a complex affine manifold and can be embedded into $\mathbb{C}^N$, is the completion space of $\mathcal{T}$ with respect to the Hodge metric. Moreover, the extended period map $\Phi^H : \mathcal{T}^H \to N_+ \cap D$ is a holomorphic injection.*

Proof. By the definition of $\mathcal{T}^H$ and Theorem 4.8, it is easy to see that $\mathcal{T}^H_m$ is a complex affine manifold, which can be embedded into $\mathbb{C}^N$. It is also not hard to see that the injectivity of Φ^H follows from Corollary 4.11 by the definition of Φ^H. Now we only need to show that $\mathcal{T}^H$ is the Hodge metric completion space of $\mathcal{T}$. This suffices to prove the following lemma. □

Lemma 5.4. *The map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ is an embedding.*

Proof. On one hand, define $\mathcal{T}_0$ to be $\mathcal{T}_0 = \mathcal{T}_m$ for any $m \geq 3$, as $\mathcal{T}_m$ doesn't depend on the choice of level m structure according to the proof of Proposition 5.1. Since $\mathcal{T}_0 = (\pi^H_m)^{-1}(\mathcal{Z}_m)$, $\pi^H_m : \mathcal{T}_0 \to \mathcal{Z}_m$ is a covering map. Thus the fundamental group of $\mathcal{T}_0$ is a subgroup of the fundamental group of $\mathcal{Z}_m$, that is, $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_m)$, for any $m \geq 3$. Moreover, the universal property of the universal covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ with $\pi_m = \pi^H_m|_{\mathcal{T}_0} \circ i_{\mathcal{T}}$ implies that $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ is also a covering map.

On the other hand, let $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $m_k < m_{k+1}$ and $m_k | m_{k+1}$ for each $k \geq 1$. Then there is a natural covering map from $\mathcal{Z}_{m_{k+1}}$ to $\mathcal{Z}_{m_k}$ for each k. In fact, because each point in $\mathcal{Z}_{m_{k+1}}$ is a polarized Calabi–Yau manifold with a basis $\gamma_{m_{k+1}}$ for the space $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m_{k+1}(H_n(M,\mathbb{Z})/\mathrm{Tor})$ and $m_k | m_{k+1}$, then the basis $\gamma_{m_{k+1}}$ induces a basis for the space $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m_k(H_n(M,\mathbb{Z})/\mathrm{Tor})$. Therefore we get a well-defined map $\mathcal{Z}_{m_{k+1}} \to \mathcal{Z}_{m_k}$ by assigning to a polarized Calabi–Yau manifold with the basis $\gamma_{m_{k+1}}$ the same polarized Calabi–Yau manifold with the basis $(\gamma_{m_{k+1}} (\mathrm{mod}\ m_k)) \in (H_n(M,\mathbb{Z})/\mathrm{Tor})/m_k(H_n(M,\mathbb{Z})/\mathrm{Tor})$. Moreover, using the versal properties of both the families $\mathcal{X}_{m_{k+1}} \to \mathcal{Z}_{m_{k+1}}$ and $\mathcal{X}_{m_k} \to \mathcal{Z}_{m_k}$, we can conclude that the map $\mathcal{Z}_{m_{k+1}} \to \mathcal{Z}_{m_k}$ is locally biholomorphic. Therefore, $\mathcal{Z}_{m_{k+1}} \to \mathcal{Z}_{m_k}$ is actually a covering map. Thus the fundamental group $\pi_1(\mathcal{Z}_{m_{k+1}})$ is a subgroup of $\pi_1(\mathcal{Z}_{m_k})$ for each k. Hence, the inverse system of fundamental groups

$$\pi_1(\mathcal{Z}_{m_1}) \longleftarrow \pi_1(\mathcal{Z}_{m_2}) \longleftarrow \cdots\cdots \longleftarrow \pi_1(\mathcal{Z}_{m_k}) \longleftarrow \cdots$$

has an inverse limit, which is the fundamental group of $\mathcal{T}$. Because $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_{m_k})$ for any k, we have the inclusion $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{T})$. But $\pi_1(\mathcal{T})$ is a trivial group since $\mathcal{T}$ is simply connected, thus $\pi_1(\mathcal{T}_0)$ is also a trivial group. Therefore the covering map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ is a one-to-one covering. This shows that $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ is an embedding. □

Remark 5.5. There is another approach to Lemma 5.4, which is a proof by contradiction. Suppose towards a contradiction that there were two points $p \neq q \in \mathcal{T}$ such that $i_{\mathcal{T}}(p) = i_{\mathcal{T}}(q) \in \mathcal{T}^H$.

On one hand, since each point in $\mathcal{T}$ represents a polarized and marked Calabi–Yau manifold, p and q are actually triples (M_p, L_p, γ_p) and (M_q, L_q, γ_q) respectively, where γ_p and γ_q are two bases of $H_n(M,\mathbb{Z})/\mathrm{Tor}$. On the one hand, each point in $\mathcal{Z}_m$ represents a triple (M, L, γ_m) with γ_m a basis of $(H_n(M,\mathbb{Z})/\mathrm{Tor})/m(H_n(M,\mathbb{Z})\mathrm{Tor})$ for any $m \geq 3$. By the assumption that $i_{\mathcal{T}}(p) = i_{\mathcal{T}}(q)$ and the relation that $i \circ \pi_m = \pi_m^H \circ i_{\mathcal{T}}$, we have $i \circ \pi_m(p) = i \circ \pi_m(q) \in \mathcal{Z}_m$ for any $m \geq 3$. In particular, for any $m \geq 3$, the image of (M_p, L_p, γ_p) and (M_q, L_q, γ_q) under π_m are the same in $\mathcal{Z}_m$. This implies that there exists a biholomorphic map $f_{pq} : M_p \to M_q$ such that $f_{pq}^*(L_q) = L_p$ and $f_{pq}^*(\gamma_q) = \gamma_p \cdot A$, where A is an integer matrix satisfying

$$A = (A_{ij}) \equiv \mathrm{Id} \pmod{m} \qquad \text{for any} \quad m \geq 3. \tag{30}$$

Let $|A_{ij}|$ be the absolute value of the ij-th entry of the matrix (A_{ij}). Since (30) holds for any $m \geq 3$, we can choose an integer m_0 greater than any $|A_{ij}|$ such that

$$A = (A_{ij}) \equiv \mathrm{Id} \pmod{m_0}.$$

Since each $A_{ij} < m_0$ and $A = (A_{ij}) \equiv \mathrm{Id} \pmod{m_0}$, we have $A = \mathrm{Id}$. Therefore, we found a biholomorphic map $f_{pq} : M_p \to M_q$ such that $f_{pq}^*(L_q) = L_p$ and $f_{pq}^*(\gamma_q) = \gamma_p$. This implies that the two triples (M_p, L_p, γ_p) and (M_q, L_q, γ_q) are equivalent to each other. Therefore, p and q in $\mathcal{T}$ are actually the same point. This contradicts with our assumption that $p \neq q$.

Since $\Phi = \Phi^H \circ i_{\mathcal{T}}$ with both Φ^H and $i_{\mathcal{T}}$ embeddings, we get the global Torelli theorem for the period map from the Teichmüller space to the period domain as follows.

Corollary 5.6 (Global Torelli theorem)**.** *The period map $\Phi : \mathcal{T} \to D$ is injective.*

As another important result of this paper, we prove the following property of $\mathcal{T}^H$.

Theorem 5.7. *The completion space $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$; thus there exists a complete Kähler-Einstein metric on $\mathcal{T}^H$.*

We recall that a $\mathcal{C}^2$ function $\rho : \Omega \to \mathbb{R}$ on a domain $\Omega \subseteq \mathbb{C}^n$ is *plurisubharmornic* if and only if its Levi form is positive definite at each point in Ω. Given a domain $\Omega \subseteq \mathbb{C}^n$, a function $f : \Omega \to \mathbb{R}$ is called an *exhaustion function* if for any $c \in \mathbb{R}$, the set $\{z \in \Omega \,|\, f(z) < c\}$ is relatively compact in Ω. The following well-known theorem provides a definition for domains of holomorphy. For example, one may refer to [6] for details.

Proposition 5.8. *An open set $\Omega \in \mathbb{C}^n$ is a domain of holomorphy if and only if there exists a continuous plurisubharmonic function $f : \Omega \to \mathbb{R}$ such that f is also an exhaustion function.*

The following theorem from [4] gives us the basic ingredients to construct a plurisubharmoic exhaustion function on $\mathcal{T}^H$.

Proposition 5.9. *On every manifold D, which is dual to a Kähler C-space, there exists an exhaustion function $f : D \to \mathbb{R}$, whose Levi form, restricted to $T_h^{1,0}(D)$, is positive definite at every point of D.*

We remark that in this proposition, in order to show f is an exhaustion function on D, Griffiths and Schmid showed that the set $f^{-1}(-\infty, c]$ is compact in D for any $c \in \mathbb{R}$.

Lemma 5.10. *The extended period map $\Phi^H : \mathcal{T}^H \to D$ still satisfies the Griffiths transversality.*

Proof. Let us consider $T^{1,0}\mathcal{T}^H$ and $T^{1,0}D$ as two differential manifolds, and the tangent map

$$(\Phi^H)_* : T^{1,0}\mathcal{T}^H \to T^{1,0}D$$

as a continuous map. We only need to show that the image of $(\Phi^H)_*$ is contained in the horizontal tangent bundle $T^{1,0}_h D$.

The horizontal subbundle $T^{1,0}_h D$ is a close set in $T^{1,0}D$, so the preimage of $T^{1,0}_h D$ under $(\Phi^H)_*$ is a close set in $T^{1,0}\mathcal{T}^H$. On the other hand, because the period map Φ satisfies the Griffiths transversality, the image of Φ_* is in the horizontal subbundle $T^{1,0}_h D$. This means that the preimage of $T^{1,0}_h D$ under $(\Phi^H)_*$ contains both $T^{1,0}\mathcal{T}$ and the closure of $T^{1,0}\mathcal{T}$, which is $T^{1,0}\mathcal{T}^H$. This finishes the proof. □

Proof of Theorem 5.7. By Theorem 5.3, we can view the completion space $\mathcal{T}^H$ as a domain in $\mathbb{C}^N$. Hence, in order to show that $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$, it is enough to construct a plurisubharmonic exhaustion function on $\mathcal{T}^H$.

Let f be the exhaustion function on D constructed in Proposition 5.9, whose Levi form, when restricted to the horizontal tangent bundle $T^{1,0}_h D$ of D, is positive definite at each point of D. By the Griffiths transversality of Φ^H, the composition function $f \circ \Phi^H : \mathcal{T}^H \to \mathbb{R}$ is a plurisubharmonic function on $\mathcal{T}^H$. We claim that the composition function $f \circ \Phi^H$ is the demanded plurisubharmonic exhaustion function on $\mathcal{T}^H$. Thus it suffices to show that the function $f \circ \Phi^H$ is an exhaustion function on $\mathcal{T}^H$, and this is enough to show that to that for any constant $c \in \mathbb{R}$, $(f \circ \Phi^H)^{-1}(-\infty, c] = (\Phi^H)^{-1}\left(f^{-1}(-\infty, c]\right)$ is a compact set in $\mathcal{T}^H$.

Indeed, it has already been shown in [4] that the set $f^{-1}(-\infty, c]$ is a compact set in D. Now for any sequence $\{p_k\}_{k=1}^{\infty} \subseteq (f \circ \Phi^H)^{-1}(-\infty, c]$, we have $\{\Phi^H(p_k)\}_{k=1}^{\infty} \subseteq f^{-1}(-\infty, c]$. Since $f^{-1}(-\infty, c]$ is compact in D, the sequence $\{\Phi^H(p_k)\}_{k=1}^{\infty}$ has a convergent subsequence. We denote this convergent subsequence by $\{\Phi^H(p_{k_n})\}_{n=1}^{\infty} \subseteq \{\Phi^H(p_k)\}_{k=1}^{\infty}$ with $k_n < k_{n+1}$, and denote $\lim_{k\to\infty} \Phi^H(p_k) = o_\infty \in D$. On the other hand, since the map Φ^H is injective and the Hodge metric on $\mathcal{T}^H$ is induced from the Hodge metric on D, the extended period map Φ^H is actually a global isometry onto its image. Therefore the sequence $\{p_{k_n}\}_{n=1}^{\infty}$ is also a Cauchy sequence that converges to $(\Phi^H)^{-1}(o_\infty)$ with respect to the Hodge metric in $(f \circ \Phi^H)^{-1}(-\infty, c] \subseteq \mathcal{T}^H$. In this way, we showed that any sequence in $(f \circ \Phi^H)^{-1}(-\infty, c]$ has a convergent subsequence. Therefore $(f \circ \Phi^H)^{-1}(-\infty, c]$ is compact in $\mathcal{T}^H$, as was needed to show.

Because $\mathcal{T}^H$ is a domain of holomorphy in $\mathbb{C}^N$, the existence of a complete Kähler-Einstein metric on $\mathcal{T}^H$ follows directly from [2], which asserts the existence of such metric on any pseudo-convex domain. □

Appendix A. Two topological lemmas

In this appendix we first prove the existence of the choice of i_m and Φ^H_m in diagram (28) such that $\Phi = \Phi^H_m \circ i_m$. Then we show a lemma that relates the fundamental group of the moduli space of Calabi–Yau manifolds to that of completion space with respect to the Hodge metric on $\mathcal{Z}_m$. The arguments only use elementary topology and the results may be well-known. We include their proofs here for the sake of completeness.

Lemma A.1. *There exists a suitable choice of* i_m *and* Φ^H_m *such that* $\Phi^H_m \circ i_m = \Phi$.

Proof. Recall the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}^H_m & \xrightarrow{\Phi^H_m} & D \\ \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi^H_m} & & \downarrow{\scriptstyle \pi_D} \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}^H_m & \xrightarrow{\Phi^H_{\mathcal{Z}_m}} & D/\Gamma. \end{array}$$

Fix a reference point $p \in \mathcal{T}$. The relations $i \circ \pi_m = \pi^H_m \circ i_m$ and $\Phi^H_{\mathcal{Z}_m} \circ \pi^H_m = \pi_D \circ \Phi^H_m$ imply that $\pi_D \circ \Phi^H_m \circ i_m = \Phi^H_{\mathcal{Z}_m} \circ i \circ \pi_m = \Phi_{\mathcal{Z}_m} \circ \pi_m$. Therefore $\Phi^H_m \circ i_m$ is a lifting map of $\Phi_{\mathcal{Z}_m}$. On the other hand $\Phi : \mathcal{T} \to D$ is also a lifting of $\Phi_{\mathcal{Z}_m}$. In order to make $\Phi^H_m \circ i_m = \Phi$, one only needs to choose the suitable i_m and Φ^H_m such that these two maps agree on the reference point, i.e. $\Phi^H_m \circ i_m(p) = \Phi(p)$.

For an arbitrary choice of i_m, we have $i_m(p) \in \mathcal{T}^H_m$ and $\pi^H_m(i_m(p)) = i(\pi_m(p))$. Considering the point $i_m(p)$ as a reference point in $\mathcal{T}^H_m$, we can choose $\Phi^H_m(i_m(p))$ to be any point from $\pi_D^{-1}(\Phi^H_{\mathcal{Z}_m}(i(\pi_m(p)))) = \pi_D^{-1}(\Phi_{\mathcal{Z}_m}(\pi_m(p)))$. Moreover the relation $\pi_D(\Phi(p)) = \Phi_{\mathcal{Z}_m}(\pi_m(p))$ implies that $\Phi(p) \in \pi_D^{-1}(\Phi_{\mathcal{Z}_m}(\pi_m(p)))$. Therefore we can choose Φ^H_m such that $\Phi^H_m(i_m(p)) = \Phi(p)$. With this choice, we have $\Phi^H_m \circ i_m = \Phi$. □

Lemma A.2. *Let* $\pi_1(\mathcal{Z}_m)$ *and* $\pi_1(\mathcal{Z}^H_m)$ *be the fundamental groups of* $\mathcal{Z}_m$ *and* $\mathcal{Z}^H_m$ *respectively, and suppose the group morphism*

$$i_* : \pi_1(\mathcal{Z}_m) \to \pi_1(\mathcal{Z}^H_m)$$

is induced by the inclusion $i : \mathcal{Z}_m \to \mathcal{Z}^H_m$. *Then* i_* *is surjective.*

Proof. First notice that $\mathcal{Z}_m$ and $\mathcal{Z}^H_m$ are both smooth manifolds, and $\mathcal{Z}_m \subseteq \mathcal{Z}^H_m$ is open. Thus for each point $q \in \mathcal{Z}^H_m \setminus \mathcal{Z}_m$ there is a disc $D_q \subseteq \mathcal{Z}^H_m$ with $q \in D_q$. Then the union of these discs

$$\bigcup_{q \in \mathcal{Z}^H_m \setminus \mathcal{Z}_m} D_q$$

forms a manifold with open cover $\{D_q : q \in \cup_q D_q\}$. Because both $\mathcal{Z}_m$ and $\mathcal{Z}^H_m$ are second countable spaces, there is a countable subcover $\{D_i\}_{i=1}^\infty$ such that $\mathcal{Z}^H_m = \mathcal{Z}_m \cup \bigcup_{i=1}^{\infty} D_i$, where the D_i are open discs in $\mathcal{Z}^H_m$ for each i. Therefore, we have $\pi_1(D_i) = 0$ for all $i \geq 1$. Letting $\mathcal{Z}_{m,k} = \mathcal{Z}_m \cup \bigcup_{i=1}^{k} D_i$, we get

$$\pi_1(\mathcal{Z}_{m,k}) * \pi_1(D_{k+1}) = \pi_1(\mathcal{Z}_{m,k}) = \pi_1(\mathcal{Z}_{m,k-1} \cup D_k), \qquad \text{for any } k.$$

We know that $\operatorname{codim}_{\mathbb{C}}(\mathcal{Z}^H_m \backslash \mathcal{Z}_m) \geq 1$. Therefore since $D_{k+1} \backslash \mathcal{Z}_{m,k} \subseteq D_{k+1} \backslash \mathcal{Z}_m$, we have $\operatorname{codim}_{\mathbb{C}}[D_{k+1} \backslash (D_{k+1} \backslash \mathcal{Z}_{m,k})] \geq 1$ for any k. As a consequence we can conclude that $D_{k+1} \cap \mathcal{Z}_{m,k}$ is path-connected. Hence we can apply the Van Kampen Theorem on $X_k = D_{k+1} \cup \mathcal{Z}_{m,k}$ to conclude that for every k, the following group homomorphism is surjective:

$$\pi_1(\mathcal{Z}_{m,k}) = \pi_1(\mathcal{Z}_{m,k}) * \pi_1(D_{k+1}) \twoheadrightarrow \pi_1(\mathcal{Z}_{m,k} \cup D_{k+1}) = \pi_1(\mathcal{Z}_{m,k+1}).$$

Thus we get the directed system:

$$\pi_1(\mathcal{Z}_m) \twoheadrightarrow \pi_1(\mathcal{Z}_{m,1}) \twoheadrightarrow \cdots\cdots \twoheadrightarrow \pi_1(\mathcal{Z}_{m,k}) \twoheadrightarrow \cdots\cdots$$

By taking the direct limit of this directed system, we get the surjectivity of the group homomorphism $\pi_1(\mathcal{Z}_m) \to \pi_1(\mathcal{Z}_m^H)$. □

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