ON f-BI-HARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

WEI-JUN LU

Abstract. Both bi-harmonic map and f-harmonic map have nice physical motivation and applications. In this paper, by combination of these two harmonic maps, we introduce and study f-bi-harmonic maps as the critical points of the f-bi-energy functional $\frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g$. This class of maps generalizes both concepts of harmonic maps and biharmonic maps. We first derive the f-biharmonic map equation and then use it to study f-bi-harmonicity of some special maps, including conformal maps between manifolds of same dimensions, some product maps between direct product manifold and singly warped product manifold, some projection maps from and some inclusion maps into a warped product manifold.

1. Introduction

First motivated by the physical interpretation of f-harmonic map (see [Ou1],[LW], [RV]), we borrowed from the method for studying bi-harmonic maps in [PK, BMO] to investigate the behaviors of f-harmonic maps from or into doubly warped product manifold (WPM). We derived some characteristic equations for f-harmonicity and also constructed some examples [Lu1]. Subsequently, we found that f-tension field don't involve the Riemannian curvature tensor $\bar{R}$ on WPM unlike [PK] and [BMO].

To make amends for this shortcoming, we wanted to formulate a new type of tension field which contains the Rimannian curvature component like bi-tension field. Naturally, we focused on constructing a field so-called bi-f-tension field or f-bi-tension field via combining f-tension field and bi-tension field. Thus we attempted to derive the Euler-Lagrangian equation by the first variation for corresponding energy functional $\frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g$ or $\frac{1}{2}\int_M |\tau_f(\phi)|^2 dv_g$ according to the canonical methods as same as bi-energy functional and f-energy functional.

At that time,since the deduction is very complicated, together with our poor processing techniques, we had attacked this problem vainly for two weeks. At the very moment, Ou sent us the scan PDF file of Ouakkas-Nasri-Djaa's article [OND]. Although the terminology about f-bi-tension field in [OND] is not the terminology we expected (from now on, we change it as bi-f-tension field), we could directly use the already bi-f-tension field deduced by them to discuss bi-f-harmonic maps whose domain or codomain is doubly WPM (see [Lu2]). With much more complicated and tedious computations, together without any non-trivial example for bi-f-harmonic map, the referee gave an unfavorable review of the paper [Lu2] and so far we dared not submit it to the Journal.

During on writing our Ph.D thesis ([Lu3]) recently, we rearranged the the paper [Lu2]. In order to obtain some simpler and interesting results, we made a modification by exchanging doubly WPM to singly WPM. Thus we arrived at our original goal. Based on

1991 *Mathematics Subject Classification.* 58E20; 53C12.

Key words and phrases. f-Harmonic; bi-f-tension field; f-bi-tension field; bi-$f$harmonic map; f-biharmonic map; conformal dilation; singly warped product manifold .

this work, we again developed another new tension field so-called f-bi-tension field (different from the terminology in [OND]) which comes from the first variation of energy functional

$$\frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g.$$

Subsequently, we employed the method of [BMO, Lu1] to discuss the behaviors of f-bi-harmonic maps from or into singly WPM.

Just noted that the progress in the topic on bi-f-harmonic map such as [CET, Ch1, Ch2], together with [BFO], we further consider f-bi-harmonic maps with conformal dilation.

In this paper, our main results are listed below:

(i) f-B i-tension field $\tau_{2,f}(\phi)$ by the first variation of f-bi-energy functional $\frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g$, attached to Propositions 4.2 and 4.3;

(ii) Results on f-bi-harmonic maps with conformal dilation, attached to Propositions 4.8 and 4.11;

(iii) Characteristic behaviors of bi-f-harmonic maps from or into singly WPM, attached to Theorem 5.1, Corollaries 5.4 and 5.5, Propositions 5.9 and 5.10;

(iv) Characteristic behaviors of f-bi-harmonic maps from or into singly WPM, attached to Corollaries 6.2 and 6.3, Proposition 6.6, Propositions 6.7 and 6.8, Propositions 6.10 and 6.11.

The organization of this paper is as follows. In the second section is preliminary, which reviews some basic definitions on biharmonic maps, f-harmonic maps, also gives the definition of doubly/singly WPMs and the more explicit expressions of the connection $\bar{\nabla}$ and curvature tensor $\bar{R}$ on singly WPM. Section 3 is devoted to briefly recall the first variation of bi-f-harmonic map, the formula of bi-f-tension field and bi-f-harmonic map, also includes the results for bi-f-harmonic maps with conformal dilation and some examples. In section 4, we deduce f-bi-tension field from the corresponding energy functional and discuss f-bi-harmonic maps with conformal dilation. In Section 5, we discuss the behaviors of bi-f-harmonic maps whose domain or codomain is singly WPM and also construct some non-trivial example. Section 6 is devoted to discuss the behaviors of f-bi-harmonic maps from or into singly WPM and also construct some non-trivial example. In the last section, we make some comparisons on f-bi-harmonic map and bi-f-harmonic map, also present a prising expectation.

2. Preliminaries

2.1. **Harmonic, bi-harmonic and f-harmonic maps.** Recall that the energy of a smooth map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is defined by integral $E(\phi) = \int_\Omega e(\phi) dv_g$, for every compact domain $\Omega \subset M$ where $e(\phi) = \frac{1}{2}|d\phi|^2$ is energy density and ϕ is called *harmonic* if it's a critical point of energy. From the first variation formula for the energy, the Euler-Lagrange equation is given by the vanishing of the *tension field* $\tau(\phi) = \mathrm{Tr_g}\nabla \mathrm{d}\phi$ (see [ES]). As the generalizations of harmonic maps, we now recall the concepts of bi-harmonic maps and f-harmonic maps.

Definition 2.1. (i) Bi-harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_\Omega |\tau(\phi)|^2 dv_g,$$

for any compact domain $\Omega \subset M$.

(ii) An f-harmonic map with a positive function $f \in C^\infty(M)$ is a critical point of $f-$energy

$$E_f(\phi) = \frac{1}{2}\int_{\Omega\Subset M} f|d(\phi)|^2 dv_g. \tag{2.1}$$

The Euler-Lagrange equations give the bitension field $\tau_2(\phi)$([Ji]) and the f-tension field equation $\tau_f(\phi)$ (see [Co], [OND],[Ou1]), respectively,

$$\begin{aligned} &\tau_2(\phi) = -\mathrm{Tr_g}(\nabla^\phi_\cdot\nabla^\phi_\cdot\tau(\phi) - \nabla^\phi_{{}^M\nabla_\cdot\cdot}\tau(\phi)) - \mathrm{Tr_g}(\mathrm{R^N}(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi)) = 0,\\ &\tau_f(\phi) = f\tau(\phi) + d\phi(grad\ f) = 0. \end{aligned} \tag{2.2}$$

2.2. **Connection and Riemannian curvature tensor on singly WPM.** First we refer to [Un] and give the definition of doubly/singly WPM.

Definition 2.2. Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n respectively and let $\lambda : M \to (0, +\infty)$ and $\mu : N \to (0, +\infty)$ be smooth functions. A *doubly warped product manifold (WPM)* $\bar{G} = M \times_{(\mu,\lambda)} N$ is the product manifold $M \times N$ endowed with the doubly warped product metric $\bar{g} = \mu^2 g \oplus \lambda^2 h$ defined by

$$\bar{g}(X, Y) = (\mu\circ\pi_1)^2 g(d\pi_1(X), d\pi_1(Y)) + (\lambda\circ\pi_2)^2 h(d\pi_2(X), d\pi_2(Y))$$

for all $X, Y \in T_{(x,y)}(M\times N)$, where $\pi_1 : M\times N \to M$ and $\pi_2 : M\times N \to N$ are the canonical projections. The functions λ and μ are called the *warping functions*.

If either $\mu = 1$ or $\lambda = 1$ but not both we obtain a *singly WPM*. If both $\mu = 1$ and $\lambda = 1$ then we have a *direct product manifold*. If neither μ nor λ is constant, then we have a *non-trivial doubly WPM*.

We have known that preciously the formulas about Riemann curvature and Ricci curvature are spilt into several parts according to the horizontal lift or vertical lift of the tangent vectors attached to the initial space M or target space N. For this we first introduce the unified connection and unified Riemannian curvature on a general warped product manifold $\bar{G}$ (cf. [BG], [BMO]) by introducing a new notation of lift vector.

Proposition 2.3. *Let $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \mathscr{X}(\bar{G})$, where $X_1, Y_1 \in \mathscr{X}(M)$ and $X_2, Y_2 \in \mathscr{X}(N)$. Denote ∇ by the Levi-Civita connection on the Riemannian product $M\times N$ with respect to the direct product metric $g = g\oplus h$ and by R its curvature tensor field. Then the Levi-Civita connection $\bar{\nabla}$ of $\bar{G}$ is given by*

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \tfrac{1}{2\lambda^2}X_1(\lambda^2)(0, Y_2)\\ &\quad+\tfrac{1}{2\lambda^2}Y_1(\lambda^2)(0, X_2) - \tfrac{1}{2}h(X_2, Y_2)(grad\ \lambda^2, 0)\\ &= ({}^M\nabla_{X_1}Y_1 - \tfrac{1}{2}h(X_2, Y_2)grad\ \lambda^2, 0)\\ &\quad+(0, {}^N\nabla_{X_2}Y_2 + \tfrac{1}{2\lambda^2}X_1(\lambda^2)Y_2 + \tfrac{1}{2\lambda^2}Y_1(\lambda^2)X_2), \end{aligned} \tag{2.3}$$

and the relation between the curvature tensor fields of $\bar{G}$ and $M \times N$ is

$$\begin{aligned} \bar{R}_{XY} - R_{XY} = \frac{1}{2\lambda^2}\Big\{&\Big({}^M\nabla_{Y_1}grad_g\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)grad_g\lambda^2,\ 0\Big)\wedge_{\bar{g}}(0, X_2)\\ &-\Big({}^M\nabla_{X_1}grad_g\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)grad_g\lambda^2,\ 0\Big)\wedge_{\bar{g}}(0, Y_2)\\ &-\frac{1}{2\lambda^2}|grad_g\lambda^2|^2(0, X_2)\wedge_{\bar{g}}(0, Y_2)\Big\} \end{aligned} \tag{2.4}$$

where the wedge product $(X\wedge_{\bar{g}} Y)Z = \bar{g}(Y, Z)X - \bar{g}(X, Z)Y$, for all $X, Y, Z \in \mathscr{X}(\bar{G})$.

The detail proof see [Lu3] From (2.4), we easily obtain

Proposition 2.4.

$$\begin{aligned}
&\bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1,Z_2)\\
&= ({}^{M}R_{X_1Y_1}Z_1,\ {}^{N}R_{X_2Y_2}Z_2)\\
&+ \frac{1}{2}h(X_2,Z_2)({}^{M}\nabla_{Y_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad }\lambda^2, 0)\\
&- \frac{1}{2}h(Y_2,Z_2)({}^{M}\nabla_{X_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad }\lambda^2, 0)\\
&+ \Big(0, \frac{1}{2\lambda^2}g({}^{M}\nabla_{X_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad }\lambda^2,\ Z_1)Y_2)\\
&- \Big(0, \frac{1}{2\lambda^2}g({}^{M}\nabla_{Y_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad }\lambda^2,\ Z_1)X_2)\\
&+ (0, \frac{1}{4\lambda^2}\mid \text{grad }\lambda^2 \mid^2 h(X_2,Z_2)Y_2\Big)\\
&- (0, \frac{1}{4\lambda^2}\mid \text{grad }\lambda^2 \mid^2 h(Y_2,Z_2)X_2\Big).
\end{aligned} \tag{2.5}$$

Corollary 2.5.

$$\begin{aligned}
&\bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1,Z_2)\\
&= ({}^{M}R_{X_1Y_1}Z_1,\ {}^{N}R_{X_2Y_2}Z_2)\\
&+ \lambda h(X_2,Z_2)({}^{M}\nabla_{Y_1}\text{grad }\lambda, 0) - \lambda h(Y_2,Z_2)({}^{M}\nabla_{X_1}\text{grad }\lambda, 0)\\
&+ \frac{1}{\lambda}\text{Hess}(\lambda)(X_1,Z_1)(0,Y_2) - \frac{1}{\lambda}\text{Hess}(\lambda)(Y_1,Z_1)(0,X_2)\\
&+ \mid \text{grad }\lambda \mid^2 h(X_2,Z_2)(0,Y_2) - \mid \text{grad }\lambda \mid^2 h(Y_2,Z_2)(0,X_2).
\end{aligned} \tag{2.6}$$

Proof. Note that

$$\begin{aligned}
&{}^{M}\nabla_{X_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad }\lambda^2\\
&= {}^{M}\nabla_{X_1}(2\lambda\text{grad }\lambda) - \frac{1}{2\lambda^2}2\lambda X_1(\lambda)2\lambda\text{grad }\lambda\\
&= 2\lambda\, {}^{M}\nabla_{X_1}\text{grad }\lambda
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\lambda^2}g\left({}^{M}\nabla_{X_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}X_1(\lambda^2)\text{grad }\lambda^2,\ Z_1\right)\\
&= \frac{1}{\lambda}g\left({}^{M}\nabla_{X_1}\text{grad }\lambda,\ Z_1\right)\\
&= \frac{1}{\lambda}\text{Hess}(\lambda)(X_1,Z_1).
\end{aligned}$$

Exchanging X_1 with Y_1, we obtain

$${}^{M}\nabla_{Y_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad }\lambda^2 = 2\lambda\, {}^{M}\nabla_{Y_1}\text{grad }\lambda,$$

$$\frac{1}{2\lambda^2}g\left({}^{M}\nabla_{Y_1}\text{grad }\lambda^2 - \frac{1}{2\lambda^2}Y_1(\lambda^2)\text{grad }\lambda^2,\ Z_1\right) = \frac{1}{\lambda}\text{Hess}(\lambda)(Y_1,Z_1).$$

Putting these facts together, (2.5) can reduce to

$$\begin{aligned}
&\bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1,Z_2)\\
&= ({}^{M}R_{X_1Y_1}Z_1,\ {}^{N}R_{X_2Y_2}Z_2)\\
&+ \lambda h(X_2,Z_2)({}^{M}\nabla_{Y_1}\mathrm{grad}\,\lambda,0) - \lambda h(Y_2,Z_2)({}^{M}\nabla_{X_1}\mathrm{grad}\,\lambda,0)\\
&+ \frac{1}{\lambda}\mathrm{Hess}(\lambda)(X_1,Z_1)(0,Y_2) - \frac{1}{\lambda}\mathrm{Hess}(\lambda)(Y_1,Z_1)(0,X_2)\\
&+ \mid \mathrm{grad}\,\lambda\mid^2 h(X_2,Z_2)(0,Y_2) - \mid \mathrm{grad}\,\lambda\mid^2 h(Y_2,Z_2)(0,X_2),
\end{aligned}$$

as claimed (2.6). □

3. The first variation of Bi-f-energy functional and properties of bi-f-harmonicity with conformal dilation

A more natural generalization of f-harmonic maps and bi-harmonic maps is given by integrating the square of the norm of the f-tension field, introduced recently in [OND]. The authors of [OND] give the first and second variations of bi-f-energy functional. But they cannot give any example about bi-f-harmonic maps. Here, we have exchanged the terminology "f-bi-harmonic" in [OND] for "bi-f-harmonic", which the reasons see Remark 3.2.

3.1. Bi-f-tension field and the first variation.

The precise definitions of bi-f-tension field and bi-f-harmonic map are as follows [OND].

Definition 3.1. Bi-f-energy functional of smooth map $\phi : (M, g) \to (N, h)$ is defined by

$$E_{f,2}(\phi) = \frac{1}{2}\int_{\Omega} |\tau_f(\phi)|^2 dv_g \tag{3.1}$$

for every compact domain $\Omega \subset M$. A map ϕ is called bi-f-harmonic map if it the critical point of bi-f-energy functional.

Remark 3.2. Here we have changed the terminology "$f-$bi-energy functional" and "f-bi-harmonic map" in [OND] into "bi-$f-$energy functional" and "bi-f-harmonic map". Since we think it is suitable for the constructing models of bi-harmonic map and $f-$harmonic map. "$f-$Bi-energy functional" is weighted too heavily toward "$f-$energy functional E_f" with factor f in the integrand while bi-$f-$energy functional " toward "*bi*−energy functional" without factor f. Based on above views, f-bi-energy functional seems to be the form

$$\frac{1}{2}\int_{\Omega} f|\tau(\phi)|^2 dv_g \tag{3.2}$$

not the form (3.1).

Now we are to briefly derive the Euler-Lagrange equation which gives the bi-f-harmonic map equation by using the first variation (c.f.[OND])

Proposition 3.3. *Let $\phi : (M, g) \to (N, h)$ be a smooth map. Then bi-f-tension field of ϕ is*

$$\tau_{f,2}(\phi) = \Delta_f^2\tau_f(\phi) - f\mathrm{Tr_g R^N}(\tau_\mathrm{f}(\phi), \mathrm{d}\phi)\mathrm{d}\phi = 0 \tag{3.3}$$

where

$$\Delta_f^2\tau_f(\phi) = -\mathrm{Tr_g}(\nabla^{\phi}\mathrm{f}\nabla^{\phi}\tau_\mathrm{f}(\phi) - \mathrm{f}\nabla^{\phi}_{{}^{M}\nabla_{\cdot\cdot}}\tau_\mathrm{f}(\phi)).$$

For an orthonormal frame $\{e_i\}_{i=1}^m$, *we have*

$$\begin{aligned}\mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi \tau_\mathrm{f}(\phi) \quad &-f\nabla^\phi_{\nabla_{\cdot}\cdot}\tau_f(\phi)) = \sum_{i=1}^m \left(\nabla^\phi_{e_i} f\nabla^\phi_{e_i}\tau_f(\phi) - f\nabla^\phi_{{}^M\nabla_{e_i}e_i}\tau_f(\phi)\right)\\ &= \sum_{i=1}^m \left(f\nabla^\phi_{e_i}\nabla^\phi_{e_i}\tau_f(\phi) - f\nabla^\phi_{\nabla_{e_i}e_i}\tau_f(\phi) + \nabla^\phi_{grad\, f}\tau_f(\phi)\right).\end{aligned} \tag{3.4}$$

Here we only give the outline proof of Proposition 3.3, for detailed, refer to Proposition 6 in [OND]. We mainly stress on the key points in the proof.

Proof. Let $\{\phi_t\}_{t\in I}$ is a smooth variation of ϕ and $I = (-\varepsilon, \varepsilon)$ for some small sufficiently positive number ε. Denote $\Phi(t, p) = \phi_t(p)$ and the variation vector field $V \in \Gamma(\Phi^{-1}TN)$ associated to $\{\phi_t\}_{t\in I}$ by

$$V_p = \frac{d}{dt}\Big|_{t=0}\phi_t(p) = d\Phi_{(0,p)}(\frac{\partial}{\partial t}), \qquad \forall p \in M.$$

Taking a normal orthonormal frame $\{e_i\}_{1\le i\le m}$ at p, we have

$$\begin{aligned}\frac{d}{dt}E_{f,2}(\phi_t)\Big|_{t=0} &= \int_M \left\langle \nabla^\Phi_{\frac{\partial}{\partial t}}\tau_f(\phi_t)\Big|_{t=0}, \tau_f(\phi)\right\rangle dv_g\\ &= \int_M \left\langle \sum_i \left(\nabla^\Phi_{\frac{\partial}{\partial t}}\nabla^\Phi_{e_i} f d\Phi(e_i) - \nabla^\Phi_{\frac{\partial}{\partial t}} f d\Phi(\nabla_{e_i}e_i)\right)\Big|_{t=0}, \tau_f(\phi)\right\rangle dv_g\\ &= \int_M \left\langle \sum_i \nabla^\phi_{e_i} f\nabla^\phi_{e_i} V_p + f\sum_i R^N(V_p, d\phi(e_i))d\phi(e_i), \tau_f(\phi)\right\rangle dv_g\end{aligned} \tag{3.5}$$

Now we manage to isolate V_p from $< R^N(V_p, d\phi(e_i))d\phi(e_i), \tau_f(\phi) >$ and $< \mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi \mathrm{V_p}), \tau_\mathrm{f}(\phi) >$. On one hand, by the symmetric properties of Riemann-Christoffel tensor field, we have

$$< f\sum_i R^N(V, d\phi(e_i))d\phi(e_i), \tau_f(\phi) > = < f\mathrm{Tr_g R^N}(\tau_\mathrm{f}(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V} > .$$

On the other hand, noting that the following fact

$$\langle \mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi \mathrm{V}), \tau_\mathrm{f}(\phi)\rangle = \langle \mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi \tau_\mathrm{f}(\phi)), \mathrm{V}\rangle - \mathrm{d}^\phi < \mathrm{f}\nabla\tau_\mathrm{f}(\phi), \mathrm{V} > + \mathrm{d}^\phi < \tau_\mathrm{f}(\phi), \mathrm{f}\nabla \mathrm{V} > .$$

by using the Divergence Theorem, (3.5) yields

$$\begin{aligned}&\frac{d}{dt}E_{f,2}(\phi_t)\Big|_{t=0}\\ &= \int_M \langle \mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi\tau_\mathrm{f}(\phi)), \mathrm{V}\rangle \mathrm{dv_g} - \int_\mathrm{M} \mathrm{d}^\phi < \mathrm{f}\nabla\tau_\mathrm{f}(\phi), \mathrm{V} > \mathrm{dv_g}\\ &\quad + \int_M d^\phi < \tau_f(\phi), f\nabla V > dv_g + \int_M \left\langle f\mathrm{Tr_g R^N}(\tau_\mathrm{f}(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\right\rangle \mathrm{dv_g}\\ &= \int_M \langle \mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi\tau_\mathrm{f}(\phi)), \mathrm{V}\rangle \mathrm{dv_g} - \int_{\partial\mathrm{M}} < \mathrm{f}\nabla_\nu\tau_\mathrm{f}(\phi), \mathrm{V} > \mathrm{dv_{i^*g}}\\ &\quad + \int_{\partial M} < \tau_f(\phi), f\nabla_\nu V > dv_{i^*g} + \int_M \left\langle f\mathrm{Tr_g R^N}(\tau_\mathrm{f}(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\right\rangle \mathrm{dv_g}\\ &= \int_M \langle \mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi - \mathrm{f}\nabla_{{}^M\nabla_{\cdot}\cdot})\tau_\mathrm{f}(\phi)) + \mathrm{fTr_g R^N}(\tau_\mathrm{f}(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\rangle \mathrm{dv_g}\end{aligned} \tag{3.6}$$

where ν is the outward unit normal vector of ∂M in M and $i : \partial M \to M$ is the canonical inclusion. Hence, we obtain

$$\begin{aligned}\tau_{f,2}(\phi) &= -\mathrm{Tr_g}(\nabla^\phi \mathrm{f}\nabla^\phi - \mathrm{f}\nabla_{{}^M\nabla_{\cdot}\cdot})\tau_\mathrm{f}(\phi)) - \mathrm{fTr_g R^N}(\tau_\mathrm{f}(\phi)\\ &= \Delta_f^2\tau_f(\phi)) - f\mathrm{Tr_g R^N}(\tau_\mathrm{f}(\phi), \mathrm{d}\phi)\mathrm{d}\phi,\end{aligned} \tag{3.7}$$

as claimed. □

From (3.3 and (3.4), $\tau_{f,2}(\phi)$ can simplified as the expression

$$\tau_{f,2}(\phi) = -f\mathrm{Tr_g}(\nabla^{\phi})^2\tau_{\mathrm{f}}(\phi)) - \mathrm{fTr_g R^N}(\tau_{\mathrm{f}}(\phi) - \nabla^{\phi}_{\mathrm{grad\,f}}\tau_{\mathrm{f}}(\phi)). \tag{3.8}$$

Remark 3.4. From (3.8), we can easily see that $f-$harmonic map must be bi-f-harmonic map. Conversely, it is not true. From the right hand side in 3.8, we argue that there exist at least a non-zero $f-$tension field $\tau_f(\phi)$ such that $\tau_{f,2}(\phi) = 0$.

3.2. **Properties of bi-f-harmonic maps with dilation.** In order to increase some sense of bi-f-harmonic map, we character some properties on conformal map between equi-dimensional manifolds.

Proposition 3.5. *(*[OND]*) Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal maps with dilation λ, i.e., $\phi^*h = \lambda^2 g$. Then phi is a bi-f-harmonic map if and only if*

$$\begin{aligned}
0 = {}& (n-2)f^2 d\phi(\mathrm{grad_g}(\Delta\log\lambda)) - \mathrm{(n-2)^2f^2}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{d}\phi(\mathrm{grad_g}\log\lambda)\\
&+4(n-2)f\nabla_{\mathrm{grad_g f}}d\phi(\mathrm{grad_g}\log\lambda) + \mathrm{(n-2)fd}\phi(\mathrm{grad_g}\lambda)\Delta\mathrm{f}\\
&-fd\phi(\mathrm{grad_g}(\Delta\mathrm{f})) + 2\mathrm{(n-2)f^2}\left\langle\nabla\mathrm{d}\phi, \nabla\mathrm{d}\log\lambda\right\rangle\\
&-2\left\langle\nabla d\phi, \nabla df\right\rangle + (n-2)|\mathrm{grad_g f}|^2\mathrm{d}\phi(\mathrm{grad_g}\log\lambda)\\
&+2(n-2)f^2 d\phi({}^{M}\mathrm{Ric}(\mathrm{grad_g}\log\lambda))\\
&-\nabla_{\mathrm{grad_g f}}d\phi(\mathrm{grad_g f}) - \mathrm{fd}\phi({}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\lambda)),
\end{aligned} \tag{3.9}$$

where for a local orthonormal frame $\{e_i\}_{i=1,\ldots,n}$ on M, ${}^{M}\mathrm{Ric(X)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{R^M(X, e_i)e_i}$,

$$\begin{aligned}
\left\langle\nabla d\phi, \nabla d\log\lambda\right\rangle &= \sum_{i,j=1}^{n}\nabla d\phi(e_i, e_j)\nabla d\log\lambda(e_i, e_j)\\
&= \sum_{i,j=1}^{n}\nabla d\phi(e_i, e_j) g({}^{M}\nabla_{e_i}\mathrm{grad_g}\log\lambda, \mathrm{e_j})\\
&= \sum_{i=1}^{n}\nabla d\phi(e_i, {}^{M}\nabla_{e_i}\mathrm{grad_g}\log\lambda).
\end{aligned} \tag{3.10}$$

similar to $\langle\nabla d\phi, \nabla df\rangle$.

Proof. By the assumption of ϕ and (2.2), the $f-$tension field of ϕ is given by

$$\tau_f(\phi) = (2-n)fd\phi(\mathrm{grad_g}\log\lambda) + \mathrm{d}\phi(\mathrm{grad_g f}). \tag{3.11}$$

Using (3.3), a long deduction (for detail, see [OND], pp.22-24) gives

$$\begin{aligned}
\tau_{f,2}(\phi) = {}& (n-2)f^2 d\phi(\mathrm{grad_g}(\Delta\log\lambda)) - \mathrm{(n-2)^2f^2}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{d}\phi(\mathrm{grad_g}\log\lambda)\\
&+4(n-2)f\nabla_{\mathrm{grad_g f}}d\phi(\mathrm{grad_g}\log\lambda) + \mathrm{(n-2)fd}\phi(\mathrm{grad_g}\lambda)\Delta\mathrm{f}\\
&-fd\phi(\mathrm{grad_g}(\Delta\mathrm{f})) + 2\mathrm{(n-2)f^2}\left\langle\nabla\mathrm{d}\phi, \nabla\mathrm{d}\log\lambda\right\rangle\\
&-2\left\langle\nabla d\phi, \nabla df\right\rangle + (n-2)|\mathrm{grad_g f}|^2_{\mathrm{g}}\mathrm{d}\phi(\mathrm{grad_g}\log\lambda)\\
&+2(n-2)f^2 d\phi({}^{M}\mathrm{Ric}(\mathrm{grad_g}\log\lambda))\\
&-\nabla_{\mathrm{grad_g f}}d\phi(\mathrm{grad_g f}) - \mathrm{fd}\phi({}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\lambda)).
\end{aligned} \tag{3.12}$$

Thus the necessary and sufficient condition for bi-f-harmonic map of ϕ is clear. □

In particular, if we consider identity map $\phi = Id_M$, then from $\lambda = 1$ we have

Corollary 3.6. *([OND]) Identity map $Id_M : (M^n, g) \to (M^n, g)$ is a bi-f-harmonic map if and only if*

$$\begin{aligned} &2d\phi({}^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda)) + \mathrm{grad_g}(\Delta\log\lambda) \\ &+\tfrac{3}{2}\mathrm{grad_g}(|\mathrm{grad_g f}|^2_{\mathrm{g}}) + 2|\mathrm{grad_g f}|^2_{\mathrm{g}}\mathrm{grad_g}\log\mathrm{f} = 0. \end{aligned}$$

Observe that if $f = \lambda$, then (3.11) is of the form

$$\tau_f(\phi) = (3-n)f d\phi(\mathrm{grad_g}\log\lambda).$$

Hence, when $n \geq 4$, we obtain

Corollary 3.7. *([OND]) Let $\phi : (M^n, g) \to (N^n, g)$ $(n \geq 4)$ be a conformal map with dilation $\lambda = f$. Then ϕ is a bi-λ-harmonic map if and only if*

$$\begin{aligned} &d\phi({}^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda)) + \mathrm{grad_g}(\log\lambda) - (\Delta\log\lambda)\mathrm{grad_g}\log\lambda \\ &+\tfrac{9-n}{2}\mathrm{grad_g}(|\mathrm{grad_g f}|^2_{\mathrm{g}}) + (7-\mathrm{n})|\mathrm{grad_g f}|^2_{\mathrm{g}}\mathrm{grad_g}\log\mathrm{f} = 0. \end{aligned} \tag{3.13}$$

3.3. **Examples.** Following the method of Examples A and B in [CET, P107], and noting that the function $f : M \times N \to (0, +\infty)$ and $f_\phi : M \to (0, +\infty)$ which differs from our discussing function $f : M \to N$ in this paper, we give two relatively simple examples.

Example 3.8. Let $\phi : \mathbb{R} \to (N^2, h)$ be a conformal map with constant dilation λ. If $f_\phi = f_N \circ \phi = \gamma : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, where, at $(x, y) \in \mathbb{R}^2$, $f_N : N \to (0, +\infty)$ defined by $f_N(y) = f(x, y)$ for all $y \in N$, then ϕ is bi-f-harmonic if and only if

$$\begin{cases} \frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial x^2} + \frac{\partial\gamma}{\partial y}\frac{\partial^2\gamma}{\partial x\partial y} = 0, \\ \frac{\partial\gamma}{\partial y}\frac{\partial^2\gamma}{\partial y^2} + \frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial x\partial y} = 0. \end{cases} \tag{3.14}$$

Example 3.9. Let $\phi : (M^2, g) \to (N^2, h)$ be a conformal map with constant dilation λ. If $f_\phi = f_N \circ \phi = \log\lambda$ be a smooth function, then ϕ is bi-f-harmonic if and only if λ satisfies

$$\mathrm{grad_g}(\left|\mathrm{grad_g}\log\lambda\right|^2) = 0. \tag{3.15}$$

4. Definition and conformal properties of f-Bi-harmonic maps

4.1. **The first variation of f-bi-energy functional.** Another more natural generalization of combining f-harmonic maps and bi-harmonic maps is given by integrating the square of the norm of the tension field times f. The precise definition is follows.

Definition 4.1. f-Bi-energy functional of smooth map $\phi : (M, g) \to (N, h)$ is defined by

$$E_{2,f}(\phi) = \frac{1}{2}\int_\Omega f|\tau(\phi)|^2 dV \tag{4.1}$$

for every compact domain $\Omega \subset M$. A map ϕ is called f-bi-harmonic map if it the critical point of f-bi-energy functional.

Now we are to derive the Euler-Lagrange equation gives the f-bi-harmonic map equation by using the first variation.

Proposition 4.2. *Let $\phi : (M, g) \to (N, h)$ be a smooth map. Then f-bi-tension field of ϕ is*

$$\begin{aligned} \tau_{2,f}(\phi) &= -\mathrm{Tr_g}(\nabla^\phi)^2\mathrm{f}\tau(\phi) - \mathrm{fTr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi \\ &= -J^\phi(f\tau(\phi)), \end{aligned} \tag{4.2}$$

where J^ϕ is the Jacobi operator along the map ϕ.

Proof. Let $\Phi : I \times M \to M$ be a smooth map satisfying

$$\Phi(t, p) = \phi_t(p), \qquad \Phi(0, p) = \phi(p), \qquad \forall t \in I, p \in M,$$

where $\{\phi_t\}_{t\in I}$ is a smooth variation of ϕ and $I = (-\varepsilon, \varepsilon)$ for some small sufficiently positive number ε.

The variation vector field $V \in \Gamma(\Phi^{-1}TN)$ associated to $\{\phi_t\}_{t\in I}$ is given by

$$V_p = \frac{d}{dt}\Big|_{t=0}\phi_t(p) = d\Phi_{(0,p)}(\frac{\partial}{\partial t}), \qquad \forall p \in M.$$

We have

$$\begin{aligned} \frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0} &= \frac{1}{2}\int_M \frac{\partial}{\partial t} f\left\langle \tau(\phi_t), \tau(\phi_t)\right\rangle\Big|_{t=0} dv_g \\ &= \int_M f\left\langle \nabla^\Phi_{\frac{\partial}{\partial t}}\tau(\phi_t), \tau(\phi_t)\right\rangle\Big|_{t=0} dv_g \\ &= \int_M f\left\langle \nabla^\Phi_{\frac{\partial}{\partial t}}\tau(\phi_t)\Big|_{t=0}, \tau(\phi)\right\rangle dv_g. \end{aligned} \tag{4.3}$$

Now let $\{e_i\}_{1\le i\le m}$ be a normal orthonormal frame at $p \in M$. Then calculating at p gives

$$\begin{aligned} \nabla^\Phi_{\frac{\partial}{\partial t}}\tau(\phi_t) &= \nabla^\Phi_{\frac{\partial}{\partial t}}\mathrm{Tr_g}\nabla \mathrm{d}\Phi \\ &= \nabla^\Phi_{\frac{\partial}{\partial t}}\sum_i \nabla d\Phi(e_i, e_i) \\ &= \nabla^\Phi_{\frac{\partial}{\partial t}}\sum_i (\nabla^\Phi_{e_i} d\Phi)(e_i) \\ &= \sum_i \left(\nabla^\Phi_{\frac{\partial}{\partial t}}\nabla^\Phi_{e_i} d\Phi(e_i) - \nabla^\Phi_{\frac{\partial}{\partial t}} d\Phi(\nabla_{e_i}e_i)\right). \end{aligned} \tag{4.4}$$

For a given $X \in \Gamma(TM)$, note that $[\frac{\partial}{\partial t}, X] = 0$, we get

$$\begin{aligned} \nabla^\Phi_{\frac{\partial}{\partial t}} d\Phi(X) &= \nabla^\Phi_X d\Phi(\frac{\partial}{\partial t}) + d\Phi\left([\frac{\partial}{\partial t}, X]\right) \\ &= \nabla^\Phi_X d\Phi(\frac{\partial}{\partial t}). \end{aligned}$$

Thus at p, (4.4) becomes

$$\begin{aligned} \nabla^\Phi_{\frac{\partial}{\partial t}}\tau(\phi_t) &= \sum_i \left(\nabla^\Phi_{\frac{\partial}{\partial t}}\nabla^\Phi_{e_i} d\Phi(e_i) - \nabla^\Phi_{\nabla_{e_i}e_i} d\Phi(\frac{\partial}{\partial t})\right) \\ &= \sum_i \nabla^\Phi_{\frac{\partial}{\partial t}}\nabla^\Phi_{e_i} d\Phi(e_i) \\ &= \sum_i \left(\nabla^\Phi_{e_i}\nabla^\Phi_{\frac{\partial}{\partial t}} d\Phi(e_i) + \nabla^\Phi_{[\frac{\partial}{\partial t}, e_i]} d\Phi(e_i) + R^\Phi(\frac{\partial}{\partial t}, e_i) d\Phi(e_i)\right) \\ &= \sum_i \left(\nabla^\Phi_{e_i}\nabla^\Phi_{\frac{\partial}{\partial t}} d\Phi(e_i) + R^\Phi(\frac{\partial}{\partial t}, e_i) d\Phi(e_i)\right) \\ &= \sum_i \left(\nabla^\Phi_{e_i}\nabla^\Phi_{e_i} d\Phi(\frac{\partial}{\partial t}) + R^N(d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i)))f d\Phi(e_i)\right) \\ &= \sum_i \nabla^\Phi_{e_i}\nabla^\Phi_{e_i} d\Phi(\frac{\partial}{\partial t}) + \sum_i R^N(d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i)) d\Phi(e_i). \end{aligned} \tag{4.5}$$

Noticing the symmetric properties of Riemann-Christoffel tensor field, from (4.5), we have

$$\begin{aligned}
&\frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0}\\
&= \int_M f\left(\Big\langle \sum_i \nabla^\phi_{e_i}\nabla^\phi_{e_i}V, \tau(\phi)\Big\rangle + \Big\langle \sum_i R^N(V, d\phi(e_i))d\phi(e_i), \tau(\phi)\Big\rangle\right)dv_g\\
&= \int_M f\left(\langle \mathrm{Tr_g}(\nabla^\phi\nabla^\phi \mathrm{V}), \tau(\phi)\rangle - \Big\langle \sum_{\mathrm{i}} \mathrm{R^N}(\mathrm{d}\phi(\mathrm{e_i}), \tau(\phi))\mathrm{d}\phi(\mathrm{e_i}), \mathrm{V}\Big\rangle\right)dv_g\\
&= \int_M \Big(\langle \mathrm{Tr_g}(\nabla^\phi\nabla^\phi \mathrm{V}), \mathrm{f}\tau(\phi)\rangle + \Big\langle \mathrm{fTr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\Big\rangle\Big)dv_g,
\end{aligned} \tag{4.6}$$

where

$$\mathrm{Tr_g}(\nabla^\phi\nabla^\phi \mathrm{V}) = \sum_{\mathrm{i}} (\nabla^\phi_{\mathrm{e_i}}\nabla^\phi_{\mathrm{e_i}}\mathrm{V} - \nabla^\phi_{{}^{\mathrm{M}}\nabla_{\mathrm{e_i}}\mathrm{e_i}}\mathrm{V}) = \mathrm{Tr_g}(\nabla^\phi)^2\mathrm{V}.$$

Denote by ν the outward unit normal vector of ∂M in M and by $i : \partial M \hookrightarrow M$ the canonical inclusion. Note that

$$\langle \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{V}), \mathrm{f}\tau(\phi)\rangle = \langle \mathrm{V}, \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{f}\tau(\phi))\rangle + \mathrm{d}^\phi\langle\nabla^\phi\mathrm{V}, \mathrm{f}\tau(\phi)\rangle - \mathrm{d}^\phi\langle \mathrm{V}, \nabla^\phi\mathrm{f}\tau(\phi)\rangle,$$

where

$$\begin{aligned}
d^\phi\langle\nabla^\phi V, f\tau(\phi)\rangle &= \langle \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{V}), \mathrm{f}\tau(\phi)\rangle + \sum_{\mathrm{i}}\langle\nabla^\phi_{\mathrm{e_i}}\mathrm{V}, \nabla^\phi_{\mathrm{e_i}}\mathrm{f}\tau(\phi)\rangle,\\
d^\phi\langle V, \nabla^\phi f\tau(\phi)\rangle &= \Big\langle V, \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{f}\tau(\phi))\Big\rangle + \sum_{\mathrm{i}}\langle\nabla^\phi_{\mathrm{e_i}}\mathrm{V}, \nabla^\phi_{\mathrm{e_i}}\mathrm{f}\tau(\phi)\rangle.
\end{aligned}$$

By using the Divergence Theorem, (4.6) gives

$$\begin{aligned}
&\frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0}\\
&= \int_M \langle \mathrm{Tr_g}(\nabla^\phi\mathrm{f}\nabla^\phi\tau(\phi)), \mathrm{V}\rangle + \int_{\mathrm{M}} \mathrm{d}^\phi\langle\nabla^\phi\mathrm{V}, \mathrm{f}\tau(\phi)\rangle\mathrm{dv_g} - \int_{\mathrm{M}} \mathrm{d}^\phi\langle \mathrm{V}, \nabla^\phi\mathrm{f}\tau(\phi)\rangle\mathrm{dv_g}\\
&\quad + \int_M \Big\langle f\mathrm{Tr_g R^N}(\tau_{\mathrm{f}}(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\Big\rangle\mathrm{dv_g}\\
&= \int_M \langle \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{f}\tau(\phi)) + \mathrm{fTr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\rangle\mathrm{dv_g}\\
&\quad + \int_{\partial M}\langle\nabla^\phi_\nu V, f\tau(\phi)\rangle dv_{i^*g} - \int_{\partial M}\langle V, \nabla^\phi_\nu f\tau(\phi)\rangle dv_{i^*g}\\
&= \int_M \langle \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{f}\tau(\phi)) + \mathrm{fTr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\rangle\mathrm{dv_g}.
\end{aligned} \tag{4.7}$$

Thus, (4.7) reduces to

$$\begin{aligned}
\frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0} = -\int_M \Big\langle &- \mathrm{Tr_g}(\nabla^\phi\nabla^\phi\mathrm{f}\tau(\phi) - \nabla^\phi_{{}^{\mathrm{M}}\nabla_{\cdot}\cdot})\mathrm{f}\tau(\phi)\\
&- f\mathrm{Tr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi, \mathrm{V}\Big\rangle\mathrm{dv_g},
\end{aligned} \tag{4.8}$$

from which, the Euler-Lagrange operator associated with ϕ is given by

$$\begin{aligned}
\tau_{2,f}(\phi) &= -\mathrm{Tr_g}(\nabla^\phi\nabla^\phi - \nabla^\phi_{{}^{\mathrm{M}}\nabla_{\cdot}\cdot})\mathrm{f}\tau(\phi) - \mathrm{fTr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi\\
&= -\mathrm{Tr_g}(\nabla^\phi)^2\mathrm{f}\tau(\phi) - \mathrm{fTr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi.
\end{aligned}$$

□

Next, we give the relation between f-bi-tension field $\tau_{2,f}(\phi)$ and bi-tension field $\tau_2(\phi)$.

Proposition 4.3. *Let $\phi : (M, g) \to (N, h)$ be a smooth map. Then the relation between f-bi-tension field $\tau_{2,f}(\phi)$ and bi-tension field $\tau_2(\phi)$ is*

$$\tau_{2,f}(\phi) = f\tau_2(\phi) - \Delta(f)\tau(\phi) - 2\nabla^{\phi}_{\mathrm{grad_g f}}\tau(\phi). \tag{4.9}$$

where $\tau_2(\phi) = -\mathrm{Tr_g}(\nabla^{\phi}_{\cdot})^2\tau(\phi) - \mathrm{Tr_g R^N}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi$.

Proof. Under a local orthonormal basis $\{e_i\}_{i=1,\dots,m}$ over M, we have

$$\begin{aligned}
&\mathrm{Tr_g}(\nabla^{\phi})^2\mathrm{f}\tau(\phi)\\
&= \mathrm{Tr_g}(\nabla^{\phi}\nabla^{\phi}\mathrm{f}\tau(\phi) - \nabla^{\phi}_{{}^{\mathrm{M}}\nabla_{\cdot}\cdot}\mathrm{f}\tau(\phi))\\
&= \sum_i (\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}f\tau(\phi) - \nabla^{\phi}_{{}^{M}\nabla_{e_i}e_i}f\tau(\phi))\\
&= f\sum_i (\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\tau(\phi) - \nabla^{\phi}_{{}^{M}\nabla_{e_i}e_i}\tau(\phi))\\
&\quad + \sum_i (e_i(e_i f) - {}^{M}\nabla_{e_i}e_i(f))\tau(\phi) + 2\sum_i e_i(f)\nabla^{\phi}_{e_i}\tau(\phi)\\
&= f\mathrm{Tr_g}(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{{}^{\mathrm{M}}\nabla_{\cdot}\cdot})\tau(\phi) + \Delta_{\mathrm{M}}(\mathrm{f})\tau(\phi) + 2\nabla^{\phi}_{\mathrm{grad_g f}}\tau(\phi).
\end{aligned} \tag{4.10}$$

Combining (2.2) and (4.2), we obtain

$$\tau_{2,f}(\phi) = f\tau_2(\phi) - \Delta(f)\tau(\phi) - 2\nabla^{\phi}_{\mathrm{grad_g f}}\tau(\phi),$$

as claimed. □

Remark 4.4. In fact, the proof of Proposition (4.3) can also follow from Equation (7) in [Ou2].

Now we give an equivalent definition of f-bi-harmonic maps.

Definition 4.5. A map $\phi : M \to N$ is called f-bi-harmonic map if its f-bi-tension field $\tau_{2,f}$ vanishes on M, that is, ϕ satisfies the Euler-Lagrange equation

$$\tau_{2,f}(\phi) = 0. \tag{4.11}$$

Remark 4.6. From (4.9), we can easily see that the notion of f-bi-harmonic maps generalizes the notion of harmonic maps and that of bi-harmonic maps because a harmonic map is always an f-bi-harmonic map for any function $f > 0$. Also, an f-bi-harmonic map with constant f is nothing but a bi-harmonic map.

Remark 4.7. Proposition 4.3 provides a simple path to find a non-trivial f-bi-harmonic map. As long as let $\tau_2(\phi) = 0$ and test whether there exist a suitable map ϕ and a non-constant positive function f such that they are a solution to

$$\Delta_M(f)\tau(\phi) + 2\nabla^{\phi}_{\mathrm{grad_g f}}\tau(\phi) = 0.$$

4.2. **f-Bi-harmonic maps with conformal dilation.** In order to increase some sense of f-bi-harmonic map, we character some properties on conformal map between equi-dimensional manifolds by borrowing from the idea in [BFO, OND, CET].

Proposition 4.8. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal maps with dilation λ, i.e., $\phi^* h = \lambda^2 g$. Then under an assumption that $n \geq 3$, phi is a f-bi-harmonic map if and only if*

$$\begin{aligned}&(2-n)f\nabla_{\mathrm{grad_g}\log\lambda} d\phi(\mathrm{grad_g}\log\lambda) + \mathrm{fd}\phi(\mathrm{grad_g}(\Delta\log\lambda))\\ &+2d\phi({}^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda) + 2\left\langle\nabla\mathrm{d}\phi, \nabla\mathrm{d}\log\lambda\right\rangle\\ &+\Delta(f)d\phi(\mathrm{grad_g}\log\lambda) + 2\nabla^{\phi}_{\mathrm{grad_g f}}\mathrm{d}\phi(\mathrm{grad_g}\log\lambda) = 0,\end{aligned} \tag{4.12}$$

where $\left\langle\nabla d\phi, \nabla d\log\lambda\right\rangle$ is defined by (3.10).

Proof. Note that the fundamental equation for the tension field of horizontally conformal submersion to the mean curvature μ^{ν} of its fibres and the horizontal gradient of its dilation λ is given by

$$\tau(\phi) = (2-n)d\phi(\mathrm{grad_g}\log\lambda) - (\mathrm{m}-\mathrm{n})\mathrm{d}\phi(\mu^{\nu}), \tag{4.13}$$

(see Proposition 4.5.3 in [BW]). By the assumption of equiv-dimension, i.e., $m = n$, the tension field of ϕ is given by

$$\tau(\phi) = (2-n)d\phi(\mathrm{grad_g}\log\lambda). \tag{4.14}$$

Using (4.9) and (2.2), we have

$$\begin{aligned}&\tau_{2,f}(\phi) = (n-2)f\mathrm{Tr_g}(\nabla^{\phi})^2\mathrm{d}\phi(\mathrm{grad_g}\log\lambda)\\ &\quad+(n-2)f\mathrm{Tr_g}\mathrm{R^N}(\mathrm{d}\phi(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi)\mathrm{d}\phi\\ &+(n-2)\Delta(f)d\phi(\mathrm{grad_g}\log\lambda) + 2(\mathrm{n}-2)\nabla^{\phi}_{\mathrm{grad_g f}}\mathrm{d}\phi(\mathrm{grad_g}\log\lambda).\end{aligned} \tag{4.15}$$

By Lemma B in [CET] or Corollary 2.2 in [BFO], we have

$$\begin{aligned}&\mathrm{Tr_g}(\nabla^{\phi})^2\mathrm{d}\phi(\mathrm{grad_g}\log\lambda) + \mathrm{Tr_g}\mathrm{R^N}(\mathrm{d}\phi(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi)\mathrm{d}\phi\\ &= (2-n)\nabla_{\mathrm{grad_g}\log\lambda} d\phi(\mathrm{grad_g}\log\lambda) + \mathrm{d}\phi(\mathrm{grad_g}(\Delta\log\lambda))\\ &\quad+2d\phi({}^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda) + 2\left\langle\nabla\mathrm{d}\phi, \nabla\mathrm{d}\log\lambda\right\rangle.\end{aligned} \tag{4.16}$$

Substituting (4.16) into (4.15), we obtain

$$\begin{aligned}&\tau_{2,f}(\phi) = -(n-2)^2 f\nabla_{\mathrm{grad_g}\log\lambda} d\phi(\mathrm{grad_g}\log\lambda)\\ &\quad+(n-2)fd\phi(\mathrm{grad_g}(\Delta\log\lambda)) + 2(\mathrm{n}-2)\mathrm{d}\phi({}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda)\\ &+2(n-2)\left\langle\nabla d\phi, \nabla d\log\lambda\right\rangle + (n-2)\Delta(f)d\phi(\mathrm{grad_g}\log\lambda)\\ &+2(n-2)\nabla^{\phi}_{\mathrm{grad_g f}} d\phi(\mathrm{grad_g}\log\lambda),\end{aligned} \tag{4.17}$$

from which, ϕ is a f-bi-harmonic map $(n > 2)$ if and only if

$$\begin{aligned}&(2-n)f\nabla_{\mathrm{grad_g}\log\lambda} d\phi(\mathrm{grad_g}\log\lambda) + \mathrm{fd}\phi(\mathrm{grad_g}(\Delta\log\lambda))\\ &\quad+2d\phi({}^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda) + 2\left\langle\nabla\mathrm{d}\phi, \nabla\mathrm{d}\log\lambda\right\rangle\\ &\quad+\Delta(f)d\phi(\mathrm{grad_g}\log\lambda) + 2\nabla^{\phi}_{\mathrm{grad_g f}}\mathrm{d}\phi(\mathrm{grad_g}\log\lambda) = 0,\end{aligned}$$

as claimed.

□

In particular, if we consider identity map $\phi = Id_M$, then from $\lambda = 1$ we have

Corollary 4.9. *Identity map $Id_M : (M^n, g) \to (M^n, g)$ is a f-bi-harmonic map*

Remark 4.10. By Remark 4.6, Corollary 4.9 indeed is a trivial conclusion. More generally, any isometry is harmonic and hence an f-bi-harmonic map for any f.

Observe that (4.14) contains such two terms $\nabla^{\phi}_{\mathrm{grad_g f}} d\phi(\mathrm{grad_g}\log\lambda)$ and $f\nabla_{\mathrm{grad_g}\log\lambda} d\phi(\mathrm{grad_g}\log\lambda)$, when $f = \lambda$, we have

Proposition 4.11. *Let $\phi : (M^n, g) \to (N^n, g)$ $(n \geq 3)$ be a conformal map with dilation $\lambda = f$. Then ϕ is a f-bi-harmonic map if and only if*

$$\begin{aligned}&\Big((5-n)\lambda|\mathrm{grad_g}\log\lambda|^2 + (\lambda-2)(\Delta\log\lambda)\Big)\mathrm{grad_g}\log\lambda\\ &+\Big((4-n)\lambda+2\Big)^M\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda + \mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)\\ &+\lambda\mathrm{grad_g}(\Delta\log\lambda) + 2^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda) = 0,\end{aligned}\tag{4.18}$$

Proof. From (4.14), it is clear that if $n = 2$, then a conformal map ϕ is harmonic so λ-bi-harmonic.

Now, we consider $n \geq 3$ and calculate the λ-bi-tension field. By (4.17), we have

$$\begin{aligned}\tau_{2,\lambda}(\phi) &= (n-2)(4-n)\lambda\nabla_{\mathrm{grad_g}\log\lambda}d\phi(\mathrm{grad_g}\log\lambda)\\ &+(n-2)\lambda d\phi(\mathrm{grad_g}(\Delta\log\lambda)) + 2(\mathrm{n}-2)\mathrm{d}\phi(^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda)\\ &+2(n-2)\langle\nabla d\phi, \nabla d\log\lambda\rangle + (n-2)\Delta(\lambda)d\phi(\mathrm{grad_g}\log\lambda).\end{aligned}\tag{4.19}$$

In order to use the relation $\phi^*h = \lambda^2 g$, we consider the equivalence of $\tau_{2,\lambda} = 0$ to

$$h(\tau_{2,\lambda}(\phi), d\phi) = 0.$$

From (4.19), we conclude that ϕ is λ-bi-harmonic if and only if for any $X \in \Gamma(TM)$, we have

$$\begin{aligned}&(4-n)\lambda h(\nabla_{\mathrm{grad_g}\log\lambda}d\phi(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi(\mathrm{X}))\\ &+\lambda h\Big(d\phi(\mathrm{grad_g}(\Delta\log\lambda)), \mathrm{d}\phi(\mathrm{X})\Big) + 2\mathrm{h}(\mathrm{d}\phi(^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi(\mathrm{X}))\\ &+2h(\langle\nabla d\phi, \nabla d\log\lambda\rangle, d\phi(X)) + h(\Delta(\lambda)d\phi(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi(\mathrm{X})) = 0.\end{aligned}\tag{4.20}$$

We will study term by term of (4.20).

On one hand, note that Equation (i) of Lemma 4.5.1 in [BW] is stated by

$$\begin{aligned}\nabla d\phi(X,Y) &= X(\log\lambda)d\phi(Y) + Y(\log\lambda)d\phi(X) - g(X,Y)d\phi(\mathrm{grad_g}\log\lambda)\\ &= d\phi(X(\log\lambda)Y + Y(\log\lambda)X - g(X,Y)\mathrm{grad_g}\log\lambda), \quad \forall \mathrm{X}, \mathrm{Y} \in \Gamma(\mathrm{TM}),\end{aligned}\tag{4.21}$$

we have

$$\begin{aligned}&\nabla_{\mathrm{grad_g}\log\lambda}d\phi(\mathrm{grad_g}\log\lambda)\\ &= \nabla d\phi(\mathrm{grad_g}\log\lambda, \mathrm{grad_g}\log\lambda) + \mathrm{d}\phi(^{\mathrm{M}}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda)\\ &= d\phi(|\mathrm{grad_g}\log\lambda|^2\mathrm{grad_g}\log\lambda) + \mathrm{d}\phi(^{\mathrm{M}}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda)\end{aligned}\tag{4.22}$$

and

$$\begin{aligned}\langle\nabla d\phi, \nabla d\log\lambda\rangle &= \sum_{i=1}^{n}\nabla d\phi(e_i, \nabla_{e_i}\mathrm{grad_g}\log\lambda)\\ &= \sum_{i=1}^{n}(e_i(\log\lambda)d\phi(\nabla_{e_i}\mathrm{grad_g}\log\lambda)\\ &\quad+(\nabla_{e_i}\mathrm{grad_g}\log\lambda)(\log\lambda)\mathrm{d}\phi(\mathrm{e_i})\\ &\quad-g(e_i, \nabla_{e_i}\mathrm{grad_g}\log\lambda)\mathrm{d}\phi(\mathrm{grad_g}\log\lambda))\\ &= d\phi(^M\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda))\\ &\quad+d\phi(\tfrac{1}{2}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2) - \Delta(\log\lambda)\mathrm{d}\phi(\mathrm{grad_g}\log\lambda)\\ &= d\phi\Big(^M\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda)\\ &\quad+\tfrac{1}{2}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2) - \Delta(\log\lambda)\mathrm{grad_g}\log\lambda\Big).\end{aligned}\tag{4.23}$$

From (4.22) and (4.23),respectively, we get

$$
\begin{aligned}
&h(\nabla_{\mathrm{grad_g}\log\lambda} d\phi(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi(\mathrm{X}))\\
&= \lambda^2 g\big(|\mathrm{grad_g}\log\lambda|^2\mathrm{grad_g}\log\lambda + {}^{\mathrm{M}}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda, \mathrm{X}\big),\\
&h(\langle\nabla d\phi, \nabla d\log\lambda\rangle, d\phi(X))\\
&\quad = \lambda^2 g\Big({}^{M}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda)\\
&\quad + \frac{1}{2}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2) - \Delta(\log\lambda)\mathrm{grad_g}\log\lambda, \mathrm{X}\Big).
\end{aligned}
\tag{4.24}
$$

On the other hand, we have

$$
\begin{aligned}
&h\Big(d\phi(\mathrm{grad_g}(\Delta\log\lambda)), \mathrm{d}\phi(\mathrm{X})\Big) = \lambda^2 \mathrm{g}(\mathrm{grad_g}(\Delta\log\lambda), \mathrm{X}),\\
&h(d\phi({}^{M}\mathrm{Ric}(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi(\mathrm{X}) = \lambda^2\mathrm{g}({}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda), \mathrm{X})\\
&h((\Delta\lambda)d\phi(\mathrm{grad_g}\log\lambda), \mathrm{d}\phi(\mathrm{X})) = \lambda^2\mathrm{g}((\Delta\lambda)\mathrm{grad_g}\log\lambda, \mathrm{X}).
\end{aligned}
\tag{4.25}
$$

Substituting (4.24) and (4.25) into (4.20), we obtain

$$
\begin{aligned}
&(4-n)\lambda|\mathrm{grad_g}\log\lambda|^2\mathrm{grad_g}\log\lambda + (4-\mathrm{n})\lambda^{\mathrm{M}}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda\\
&\quad + \lambda\mathrm{grad_g}(\Delta\log\lambda) + 2{}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda)\\
&\quad + 2{}^{M}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda) + \mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)\\
&\quad - 2\Delta(\log\lambda)\mathrm{grad_g}\log\lambda + \Big(\lambda|\mathrm{grad_g}\log\lambda|^2 + \lambda(\Delta\log\lambda)\Big)\mathrm{grad_g}\log\lambda = 0,
\end{aligned}
\tag{4.26}
$$

which implies

$$
\begin{aligned}
&\Big((5-n)\lambda|\mathrm{grad_g}\log\lambda|^2 + (\lambda-2)(\Delta\log\lambda)\Big)\mathrm{grad_g}\log\lambda\\
&+ \Big((4-n)\lambda+2)\Big){}^{M}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda + \mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)\\
&\quad + \lambda\mathrm{grad_g}(\Delta\log\lambda) + 2{}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda) = 0,
\end{aligned}
$$

as claimed. □

Furthermore, applying the following identity

$$
{}^{M}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda = \frac{1}{2}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)
\tag{4.27}
$$

since

$$
\begin{aligned}
{}^{M}\nabla_{\mathrm{grad_g}\log\lambda}\mathrm{grad_g}\log\lambda &= \sum_{i,j=1}^{m} g({}^{M}\nabla_{e_j(\log\lambda)e_j}\mathrm{grad_g}\log\lambda, \mathrm{e_i})\mathrm{e_i}\\
&= \sum_{i,j=1}^{m} e_j(\log\lambda)\mathrm{Hess}(\log\lambda)(\mathrm{e_j},\mathrm{e_i})\mathrm{e_i}\\
&= \sum_{i=1}^{m}\mathrm{Hess}(\log\lambda)\left(\mathrm{e_i}, \sum_{\mathrm{i}}^{\mathrm{m}}\mathrm{e_j}(\log\lambda)\mathrm{e_j}\right)\mathrm{e_i}\\
&= \sum_{i=1}^{m}\mathrm{Hess}(\log\lambda)(\mathrm{e_i},\mathrm{grad_g}\log\lambda)\mathrm{e_i}\\
&= \sum_{i=1}^{m} g(\nabla_{e_i}\mathrm{grad_g}\log\lambda, \mathrm{grad_g}\log\lambda)\mathrm{e_i}\\
&= \frac{1}{2}\sum_{i=1}^{m} e_i\Big(g(\mathrm{grad_g}\log\lambda,\mathrm{grad_g}\log\lambda)\Big)\mathrm{e_i}\\
&= \frac{1}{2}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2),
\end{aligned} \tag{4.28}
$$

we have the following consequence.

Corollary 4.12. *Let $\phi : (M^n, g) \to (N^n, g) \quad (n \geq 3)$ be a conformal map with dilation $\lambda = f$. Then ϕ is a f-bi-harmonic map if and only if*

$$
\begin{aligned}
&\Big((5-n)\lambda|\mathrm{grad_g}\log\lambda|^2 + (\lambda-2)(\Delta\log\lambda)\Big)\mathrm{grad_g}\log\lambda\\
&+\Big(\frac{4-n}{2}\lambda+2\Big)\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)\\
&\quad+\lambda\mathrm{grad_g}(\Delta\log\lambda) + 2{}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g}\log\lambda) = 0.
\end{aligned} \tag{4.29}
$$

5. The behavior of bi-f-harmonic maps from or into singly WPM

In this section, we will use several special maps from or into WPM to study bi-f-harmonicity like the method in [PK, BMO, Lu1].

However, if we consider the doubly WPM to discuss the bi-f-harmonic maps by following the method of [Lu1] , then we will encounter a dreadful trouble [Lu2]. For example, we consider the inclusion map

$$i_{y_0} : (M, g) \to M \times_{(\mu,\lambda)} N, \quad i_{y_0}(x) = (x, y_0),$$

since by Equations (7) and (8) in [Lu1], we know that

$$\tau(i_{y_0}) = -\frac{m}{2}(0_1, \mathrm{grad_h}\mu^2)\Big|_{\mathrm{i_{y_0}}}, \tag{5.1}$$

$$\tau_f(i_{y_0}) = -\frac{m}{2}f(0_1,\ \mathrm{grad}_h\mu^2) + (\mathrm{grad}_g f, 0_2). \tag{5.2}$$

Since $\tau_f(i_{y_0})$ contains $f = f(x)$, in order to calculus $\tau_{f,2}(i_{y_0})$, we must compute the term "$\overline{\nabla}_{(e_j,0)}\overline{\nabla}_{(e_j,0)}f(0_1,\ \mathrm{grad}_h\mu^2)$" (see (3.8)). By using (2.3), we first obtain

$$\overline{\nabla}_{(e_j,0)}f(0_1,\ \mathrm{grad}_h\mu^2) = -e_j(f)(0_1, \mathrm{grad_h}\mu^2) + \frac{1}{2\mu^2}|\mathrm{grad_h}\mu^2|^2_{\mathrm{h}}\ \mathrm{f}(\mathrm{e_j}, 0_2).$$

If again, we will fall into exponential growth terms. A way to avoid this trouble is that we should restrict the term with f to occur and let $\mu = 1$. This implies that we'd better consider singly WPM but not doubly WPM.

Under the assumption that there exist some non-trivial bi-f-harmonic maps, we derive some behavior characteristics on bi-f-harmonic maps.

5.1. **f-Bi-harmonicity of the inclusion maps.** We present some non-existence results for bi-f-harmonicity of inclusion maps i_{y_0} of M and i_{x_0} of N under the singly warped product case. Firstly, we consider the inclusion map $i_{y_0} : (M, g) \to M \times_\lambda N, \quad i_{y_0}(x) = (x, y_0)$ for any $y_0 \in N$.

Theorem 5.1. *The inclusion map*

$$i_{y_0} : (M, g) \to M \times_\lambda N$$

is a non-trivial bi-f-harmonic map if and only if λ and f simultaneously satisfy

$$2f(\mathrm{Tr_g}\ {}^{\mathrm{M}}\nabla^2 \mathrm{grad_g f} + {}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g f}) + \mathrm{grad_g}(|\mathrm{grad_g f}|^2) = 0, \tag{5.3}$$

where $f : M \to \mathbb{R}$ is a smooth positive and non-constant function.

Proof. Let $\{e_j\}_{j=1}^m$ be an orthonormal frame on M. Then from 3.8, bi-f-harmonic map of i_{y_0} is

$$\begin{aligned} \tau_{f,2}(i_{y_0}) &= -f[Tr_g(\nabla^{i_{y_0}})^2 \tau_f(i_{y_0}) + Tr_g \bar{R}(di_{y_0}, \tau_f(i_{y_0}))di_{y_0}] - \nabla^{i_{y_0}}_{grad_g f} \tau_f(i_{y_0}) \\ &= -f \sum_{j=1}^m \{[\nabla^{i_{y_0}}_{e_j} \nabla^{i_{y_0}}_{e_j} - \nabla^{i_{y_0}}_{{}^M\nabla_{e_j} e_j}] \tau_f(i_{y_0}) \\ &\qquad + \bar{R}(\tau_f(i_{y_0}), (e_j, 0_2))(e_j, 0_2)\} - \overline{\nabla}_{(\mathrm{grad_g} f, 0)} \tau_f(i_{y_0}). \end{aligned} \tag{5.4}$$

Since (5.1) and (5.2) with $\mu = 1$ give

$$\tau(i_{y_0}) = 0, \quad \tau_f(i_{y_0}) = (\mathrm{grad}_g f, 0_2),$$

using (2.3) and (2.5), we have

$$\begin{aligned} &\nabla^{i_{y_0}}_{e_j} \tau_f(i_{y_0}) = \overline{\nabla}_{(e_j, 0_2)}(\mathrm{grad_g f}, 0_2) = ({}^{\mathrm{M}}\nabla_{\mathrm{e_j}} \mathrm{grad_g f}, 0_2), \\ &\nabla^{i_{y_0}}_{e_j} \nabla^{i_{y_0}}_{e_j} \tau_f(i_{y_0}) = \overline{\nabla}_{(e_j, 0)}({}^M\nabla_{e_j} \mathrm{grad_g f}, 0_2) = ({}^{\mathrm{M}}\nabla_{\mathrm{e_j}}\, {}^{\mathrm{M}}\nabla_{\mathrm{e_j}} \mathrm{grad_g f}, 0_2), \\ &\overline{\nabla}_{(\mathrm{grad_g f}, 0_2)} \tau_f(i_{y_0}) = ({}^M\nabla_{\mathrm{grad_g f}} \mathrm{grad_g f}, 0_2) = (\frac{1}{2} \mathrm{grad_g}(|\mathrm{grad_g f}|^2), 0_2), \\ &\nabla^{i_{y_0}}_{{}^M\nabla_{e_j} e_j}(\tau_f(i_{y_0})) = \overline{\nabla}_{({}^M\nabla_{e_j} e_j, 0_2)}(\mathrm{grad_g f}, 0_2) = ({}^{\mathrm{M}}\nabla_{{}^{\mathrm{M}}\nabla_{\mathrm{e_j}} \mathrm{e_j}} \mathrm{grad_g f}, 0_2), \\ &\sum_{j=1}^m \overline{R}(\tau_f(i_{y_0}), (e_j, 0_2))(e_j, 0_2) = (\sum_{j=1}^m R^M(\mathrm{grad_g f}, \mathrm{e_j})\mathrm{e_j}, 0_2) = ({}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g f}), 0_2). \end{aligned}$$

Thus we obtain

$$\tau_{f,2}(i_{y_0}) = -f(\mathrm{Tr_g}\ {}^{\mathrm{M}}\nabla^2 \mathrm{grad_g f} + {}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g f}) - \tfrac{1}{2} \mathrm{grad_g}(|\mathrm{grad_g f}|^2), 0_2), \tag{5.5}$$

from which, we conclude that i_{y_0} is a non-trivial bi-f-harmonic map ($\tau_{f,2}(i_{y_0}) = 0$) if and only if

$$2f(\mathrm{Tr_g}\ {}^{\mathrm{M}}\nabla^2 \mathrm{grad_g f} + {}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g f}) + \mathrm{grad_g}(|\mathrm{grad_g f}|^2) = 0. \tag{5.6}$$

□

Remark 5.2. If PDE (5.3) has a solution besides $f = const$, then we really find a non-trivial bi-f-harmonic map which is usual harmonic ($\tau(i_{y_0}) = 0$) but not f-harmonic ($\tau_{f,2}(i_{y_0}) \neq 0$). This is a very interesting phenomenon that bi-f-harmonic map is only an extension to f-harmonic map but not harmonic map.

For inclusion map $i_{x_0} : (N, h) \to M \times_\lambda N, i_{x_0}(y) = (x_0, y)$ for any $x_0 \in M$. Note that i_{x_0} is no longer harmonic like i_{y_0}, we first give the bi-f-tension field of i_{x_0}.

Theorem 5.3. *Let $f : (N, h) \to (0, +\infty)$ be a smooth function. The bi-f-tension field of the inclusion map $i_{x_0} : (N, h) \to M \times_\lambda N$ is given by*

(5.7)
$$\begin{aligned}\tau_{f,2}(i_{y_0}) &= \Big(\big(\tfrac{n+2}{2} f(y)\Delta_N f(y) + \tfrac{n+1}{2}|\mathrm{grad_h f(y)}|^2\big)\mathrm{grad_g}\lambda^2 - \tfrac{n^2}{8}\mathrm{f^2(y)grad_g}(|\mathrm{grad_g}\lambda^2|^2, 0_2\Big)\\ &\quad +(0_1, \tfrac{3n+1}{4\lambda^2} f(y)|\mathrm{grad_g}\lambda^2|^2\mathrm{grad_h f(y)} - \mathrm{f(y)}{}^{\mathrm{N}}\mathrm{Ric(grad_g f(y))}\Big|_{\mathrm{i_{x_0}}}.\end{aligned}$$

Proof. Let $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be an orthonormal frame on N. Then from (3.8), bi-f-harmonic map of i_{x_0} is

$$\begin{aligned}\tau_{f,2}(i_{x_0}) &= -f\sum_{\alpha=1}^n\Big((\nabla^{i_{x_0}}_{\bar{e}_\alpha}\nabla^{i_{x_0}}_{\bar{e}_\alpha} - \nabla^{i_{x_0}}_{\nabla_{\bar{e}_\alpha}\bar{e}_\alpha})\tau_f(i_{x_0})\\ &\qquad +\bar{R}(\tau_f(i_{x_0}), (0_1, \bar{e}_\alpha))(0_1, \bar{e}_\alpha)\Big) - \overline{\nabla}_{(\mathrm{grad_g f},0)}\tau_f(i_{x_0}).\end{aligned} \tag{5.8}$$

Since

$$\begin{aligned}\tau(i_{x_0}) = \mathrm{Tr_h}\nabla\mathrm{dx_{x_0}} &= \sum_{\alpha=1}^{\mathrm{n}}\{(\bar{\nabla}_{(0_1,\bar{\mathrm{e}}_\alpha)}(0_1, \bar{\mathrm{e}}_\alpha) - (0_1, {}^{\mathrm{N}}\nabla_{\bar{\mathrm{e}}_\alpha}\bar{\mathrm{e}}_\alpha))\\ &= (-\frac{n}{2}\mathrm{grad_g}\lambda^2, 0_2) \circ \mathrm{i_{x_0}}, \qquad \text{(by (2.3))}\end{aligned} \tag{5.9}$$

(2.2) gives

$$\tau_f(i_{x_0}) = -\frac{n}{2}f(y)(\mathrm{grad_g}\lambda^2, 0_2) \circ \mathrm{i_{x_0}} + (0_1, \mathrm{grad_h f(y)}) \circ \mathrm{i_{x_0}}. \tag{5.10}$$

Thus by (2.3) we have

$$\begin{aligned}
\nabla^{i_{x_0}}_{\bar{e}_\alpha}\tau_f(i_{x_0}) &= -\frac{n}{2}\overline{\nabla}_{(\bar{e}_\alpha,0_2)}f(y)(\mathrm{grad_g}\lambda^2, 0_2) \circ \mathrm{i_{x_0}} + \overline{\nabla}_{(\bar{\mathrm{e}}_\alpha,0_2)}(0_1, \mathrm{grad_h f(y)}) \circ \mathrm{i_{x_0}}\\
&= -\frac{n+1}{2}\bar{e}_\alpha(f(y))(\mathrm{grad_g}\lambda^2, 0_2) - \frac{\mathrm{n}}{4\lambda^2}\mathrm{f(y)}|\mathrm{grad_g}\lambda^2|^2(0_1, \bar{\mathrm{e}}_\alpha)\\
&\quad + (0_1, {}^N\nabla_{\bar{e}_\alpha}grad_h f)\Big|_{i_{x_0}},\\
\nabla^{i_{x_0}}_{{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha}\tau_f(i_{x_0}) &= -\frac{n+1}{2}{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha(f(y))(\mathrm{grad_g}\lambda^2, 0_2)\Big|_{\mathrm{i_{x_0}}}\\
&\quad - \frac{n}{4\lambda^2}f(y)|\mathrm{grad_g}\lambda^2|^2(0_1, {}^{\mathrm{N}}\nabla_{\bar{\mathrm{e}}_\alpha}\bar{\mathrm{e}}_\alpha) + (0_1, {}^{\mathrm{N}}\nabla_{{}^{\mathrm{N}}\nabla_{\bar{\mathrm{e}}_\alpha}\bar{\mathrm{e}}_\alpha}\mathrm{grad_h f})\Big|_{\mathrm{i_{x_0}}},\\
\nabla^{i_{x_0}}_{\mathrm{grad_h f(y)}}\tau_f(i_{x_0}) &= -\frac{n+1}{2}\mathrm{grad_h f(y)(f(y))(grad_g}\lambda^2, 0_2)\\
&\quad - \frac{n}{4\lambda^2}f(y)|\mathrm{grad_g}\lambda^2|^2(0_1, \mathrm{grad_h f(y)}) + (0_1, {}^{\mathrm{N}}\nabla_{\mathrm{grad_h f(y)}}\mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}}\\
&= -\frac{n+1}{2}|grad_h f(y)|^2(\mathrm{grad_g}\lambda^2, 0_2) + (0_1, \frac{1}{2}\mathrm{grad_h}(|\mathrm{grad_h f(y)}|^2)\\
&\quad - \frac{n}{4\lambda^2}f(y)|\mathrm{grad_g}\lambda^2|^2(0_1, \bar{\mathrm{e}}_\alpha(\mathrm{grad_h f(y)}\bar{\mathrm{e}}_\alpha)\Big|_{\mathrm{i_{x_0}}},
\end{aligned} \tag{5.11}$$

$$\nabla^{i_{x_0}}_{\bar e_\alpha}\nabla^{i_{x_0}}_{\bar e_\alpha}\tau_f(i_{x_0}) = \Big(-\frac{n+1}{2}\bar e_\alpha(\bar e_\alpha(f(y))) - \frac{1}{2}{}^N\mathrm{Hess(f(y))(\bar e_\alpha, \bar e_\alpha)}\Big)(\mathrm{grad_g}\lambda^2, 0_2)$$
$$+ \frac{n}{8\lambda^2}f(y)|\mathrm{grad_g}\lambda^2|^2(\mathrm{grad_g}\lambda^2, 0_2) - \frac{2n+1}{4\lambda^2}|\mathrm{grad_g}\lambda^2|^2(0_1, \mathrm{\bar e_\alpha(f(y))\bar e_\alpha})$$
$$- \frac{n}{4\lambda^2}f(y)|\mathrm{grad_g}\lambda^2|^2(0_1, {}^{\mathrm{N}}\nabla_{\mathrm{\bar e_\alpha}}\mathrm{\bar e_\alpha}) + (0_1, {}^{\mathrm{N}}\nabla_{\mathrm{\bar e_\alpha}}\mathrm{grad_h f})\Big|_{\mathrm{i_{x_0}}},$$

$$\mathrm{Tr_h}(\nabla^{\mathrm{i_{x_0}}})^2\tau_{\mathrm{f}}(\mathrm{i_{x_0}}) = -\frac{n+2}{2}(\Delta_N f(y)(\mathrm{grad_g}\lambda^2, 0_2) + \frac{\mathrm{n}^2}{8\lambda^2}\mathrm{f(y)}|\mathrm{grad_g}\lambda^2|^2(\mathrm{grad_g}\lambda^2, 0_2)$$
$$- frac2n + 14\lambda^2|\mathrm{grad_g}\lambda^2|^2(0_1, \mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}}. \tag{5.12}$$

On the other hand, by (2.5) we have

$$\sum_{\alpha=1}^{n}\bar R(\tau_f(i_{x_0}), (0_1, \bar e_\alpha))(0_1, \bar e_\alpha)$$
$$= \tfrac{n^2}{4}f(y)\Big({}^M\nabla_{\mathrm{grad}_g\lambda^2}\mathrm{grad}_g\lambda^2 - \tfrac{1}{2\lambda^2}\mathrm{grad}_g\lambda^2(\lambda^2)\mathrm{grad}\ \lambda^2, 0_2\Big)$$
$$+\Big(0_1, \sum_{\alpha=1}^{n}R^N(\mathrm{grad_g f(y), \bar e_\alpha)\bar e_\alpha}\Big)\Big|_{i_{x_0}}$$
$$= \tfrac{n^2}{8}f(y)(\mathrm{grad}_g(|\mathrm{grad_g}\lambda^2|^2), 0_2) - \tfrac{n^2}{8\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(\mathrm{grad}_g\lambda^2, 0_2)$$
$$+(0_1, {}^N Ric(\mathrm{grad}_g f(y)))\Big|_{i_{x_0}}. \tag{5.13}$$

Substituting (5.11), (5.12) and (5.13) into (5.8), we obtain

$$\tau_{f,2}(i_{y_0}) = \Big((\tfrac{n+2}{2}f(y)\Delta_N f(y) + \tfrac{n+1}{2}|\mathrm{grad_h f(y)}|^2)\mathrm{grad_g}\lambda^2 - \tfrac{\mathrm{n}^2}{8}\mathrm{f^2(y)grad_g}(|\mathrm{grad_g}\lambda^2|^2, 0_2\Big)$$
$$+(0_1, \tfrac{3n+1}{4\lambda^2}f(y)|\mathrm{grad_g}\lambda^2|^2\mathrm{grad_h f(y) - f(y)}^{\mathrm{N}}\mathrm{Ric(grad_g f(y))}\Big|_{\mathrm{i_{x_0}}},$$

as claimed. □

Corollary 5.4. *Let $f : (N, h) \to (0, +\infty)$ be a smooth function. The inclusion map i_{x_0} : $(N, h) \to M \times_\lambda N$ is bi-f-harmonic map if and only if λ and f satisfy*

$$\begin{cases} \Big(4(n+2)f(y)\Delta_N f(y) + 4(n+1)|\mathrm{grad_h f(y)}|^2)\mathrm{grad_g}\lambda^2 - \mathrm{n^2f^2(y)grad_g}(|\mathrm{grad_g}\lambda^2|^2\Big|_{\mathrm{i_{x_0}}} = 0, \\ (3n+1)f(y)|\mathrm{grad_g}\lambda^2|^2\mathrm{grad_h f(y)} - 4\mathrm{f(y)}\lambda^{2\mathrm{N}}\mathrm{Ric(grad_g f(y))}\Big|_{\mathrm{i_{x_0}}} = 0. \end{cases} \tag{5.14}$$

Corollary 5.5. *If x_0 is a critical point of $\mathrm{grad_g}\lambda^2$ but not a critical point of λ^2 and, ${}^N Ric(\mathrm{grad}_g f(y)) = 0$ but $\mathrm{grad}_g f(y) \neq 0$, then the inclusion map $i_{x_0} : (N, h) \to M \times_\lambda N$ is non-trivial bi-f-harmonic map.*

5.2. **Bi-f-harmonicity of the projection maps.** In this subsection, we attempt two methods so-called projection maps related to singly WPM to discuss bi-f-harmonic maps. We first give two lemmas.

Lemma 5.6. *For given projection map*

$$\bar\pi_1 : M \times_\lambda N \to M, \quad \bar\pi_1(x, y) = x,$$

let $f : M \times_\lambda N \to (0, +\infty)$ be smooth function. Then bi-f-bitension field of $\bar\pi_1$ is

$$\tau_{f,2}(\overline{\pi}_1) = -f\mathrm{Tr_g}({}^{\mathrm{M}}\nabla^2)\mathrm{f} \cdot \mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}) - \mathrm{f}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g f}\ \log(\mathrm{f}\lambda^{\mathrm{n}}))$$
$$-fn\,{}^M\nabla_{\mathrm{grad_g}\log\lambda}f \cdot \mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}) - {}^{\mathrm{M}}\nabla_{\mathrm{grad_g f}}\mathrm{f} \cdot \mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}). \tag{5.15}$$

Proof. Let $\{e_j\}_{j=1}^m$ and $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be local orthonormal frame fields on (M,g) and (N,h), respectively. Then $\{(e_j,0_2),(0_1,\frac{1}{\lambda}\bar{e}_\alpha)\}_{j=1,\dots,m,\alpha=1,\dots,n}$ is a local orthonormal frame on $M^m\times_\lambda N^n$. By a similar calculation as (5.9) and (5.10), we have

$$\begin{aligned}\tau(\bar{\pi}_1) &= Tr_{\bar{g}}\nabla d\bar{\pi}_1 = \sum_{j=1}^m \big({}^M\nabla_{d\bar{\pi}_1(e_j,0_2)}d\bar{\pi}_1(e_j,0_2) - d\bar{\pi}_1(\bar{\nabla}_{(e_j,0_2)}(e_j,0_2))\big)\\ &\quad+\frac{1}{\lambda^2}\sum_{\alpha=1}^n \big({}^M\nabla_{d\bar{\pi}_1(0_1,\bar{e}_\alpha)}d\bar{\pi}_1(0_1,\bar{e}_\alpha) - d\bar{\pi}_1(\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha))\big)\\ &= \frac{n}{2\lambda^2}\mathrm{grad_g}\lambda^2 \mid \bar{\pi}_1\\ &= n\mathrm{grad_g}\log\lambda \mid \bar{\pi}_1,\end{aligned}\tag{5.16}$$

$$\begin{aligned}\tau_f(\bar{\pi}_1) &= fn\mathrm{grad_g}\log\lambda\circ\bar{\pi}_1 + \mathrm{d}\bar{\pi}_1(\mathrm{grad_g f}, \tfrac{1}{\lambda^2}\mathrm{grad_h f})\\ &= f\mathrm{grad_g}\log(\lambda^{\mathrm{n}}\mathrm{f}) \mid \bar{\pi}_1.\end{aligned}\tag{5.17}$$

Thus we have

$$\nabla^{\bar{\pi}_1}_{(e_j,0_2)}\tau_f(\bar{\pi}_1) = {}^M\nabla_{d\bar{\pi}_1(e_j,0)}\tau_f(\bar{\pi}_1) = {}^M\nabla_{e_j}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}),$$
$$\nabla^{\bar{\pi}_1}_{\bar{\nabla}_{e_j,0_2)}(e_j,0_2)}\tau_f(\bar{\pi}_1) = \nabla^{\bar{\pi}_1}_{({}^M\nabla_{e_j},0_2)}\tau_f(\bar{\pi}_1) = {}^M\nabla_{{}^M\nabla_{e_j}e_j}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}),$$
$$\begin{aligned}\nabla^{\bar{\pi}_1}_{(e_j,0_2)}\nabla^{\bar{\pi}_1}_{e_j,0_2)}\tau_f(\bar{\pi}_1) &= \nabla^{\bar{\pi}_1}_{(e_j,0_2)}\,{}^M\nabla_{e_j}f\cdot\mathrm{grad}_{\mu^2\mathrm{g}}\log(\mathrm{f}\lambda^{\mathrm{n}})\\ &= {}^M\nabla_{e_j}{}^M\nabla_{e_j}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}),\end{aligned}$$
$$\begin{aligned}\nabla^{\bar{\pi}_1}_{\mathrm{grad}_{\bar{g}}\mathrm{f}}\tau_f(\bar{\pi}_1) &= \nabla^{\bar{\pi}_1}_{(\mathrm{grad_g f},0_2)+(0_1,\frac{1}{\lambda^2}\mathrm{grad_h f})}\tau_f(\bar{\pi}_1)\\ &= {}^M\nabla_{\mathrm{grad_g f}}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}),\end{aligned}$$
$$\nabla^{\bar{\pi}_1}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)}\tau_f(\bar{\pi}_1) = {}^M\nabla_{0_1}\tau_f(\bar{\pi}_1) = 0,$$
$$\begin{aligned}\nabla^{\bar{\pi}_1}_{\bar{\nabla}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)}(0_1,\frac{1}{\lambda}\bar{e}_\alpha)}\tau_f(\bar{\pi}_1) &= \frac{1}{\lambda^2}\nabla^{\bar{\pi}_1}_{(0_1,{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)-\frac{1}{2}(\mathrm{grad_g}\lambda^2,0_2)}\tau_f(\bar{\pi}_1)\\ &= -{}^M\nabla_{\mathrm{grad_g}\log\lambda}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}),\end{aligned}$$
$$\nabla^{\bar{\pi}_1}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)}\nabla^{\bar{\pi}_1}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)}\tau_f(\bar{\pi}_1) = 0,$$

which imply that

$$\begin{aligned}Tr_{\bar{g}}(\nabla^{\bar{\pi}_1})^2\tau_f(\bar{\pi}_1) &= Tr_g({}^M\nabla^2)f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}})\\ &\quad+n\,{}^M\nabla_{\mathrm{grad_g}\log\lambda}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}).\end{aligned}\tag{5.18}$$

On the other hand, since

$$\begin{aligned}Tr_{\bar{g}}R^M(d\bar{\pi}_1,\tau_f(\bar{\pi}_1))d\bar{\pi}_1 &= \sum_{j=1}^m R^M(d\pi_1(e_j,0_2),\tau_f(\bar{\pi}_1))d\pi_1(e_j,0_2)\\ &\quad+\sum_{\alpha=1}^n R^M(d\pi_1(0_1,\tfrac{1}{\lambda}\bar{e}_\alpha),\tau_f(\bar{\pi}_1))d\pi_1(0_1,\tfrac{1}{\lambda}\bar{e}_\alpha)\\ &= \sum_{j=1}^m R^M(e_j,\mathrm{grad_g f}\ \log(\mathrm{f}\lambda^{\mathrm{n}}))\mathrm{e_j}\\ &= -{}^M\mathrm{Ric}(\mathrm{grad_g f}\ \log(\mathrm{f}\lambda^{\mathrm{n}})),\end{aligned}\tag{5.19}$$

we get

$$\begin{aligned}\tau_{f,2}(\bar{\pi}_1) &= -f(\mathrm{Tr}_{\bar{g}}(\nabla^{\bar{\pi}_1})^2\tau_{\mathrm{f}}(\bar{\pi}_1) + \mathrm{fTr}_{\bar{g}}\mathrm{R^M}(\mathrm{d}\bar{\pi}_1,\tau_{\mathrm{f}}(\bar{\pi}_1))\mathrm{d}\bar{\pi}_1) - \nabla^{\bar{\pi}_1}_{\mathrm{grad}_{\bar{g}}\mathrm{f}}\tau_{\mathrm{f}}(\bar{\pi}_1)\\ &= -f\mathrm{Tr_g}({}^{\mathrm{M}}\nabla^2)\mathrm{f}\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}) - \mathrm{f}^{\mathrm{M}}\mathrm{Ric}(\mathrm{grad_g f}\ \log(\mathrm{f}\lambda^{\mathrm{n}}))\\ &\quad-fn\,{}^M\nabla_{\mathrm{grad_g}\log\lambda}f\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}) - {}^{\mathrm{M}}\nabla_{\mathrm{grad_g f}}\mathrm{f}\cdot\mathrm{grad_g}\log(\mathrm{f}\lambda^{\mathrm{n}}).\end{aligned}$$

Thus we complete the proof. □

For another projection map $\bar{\pi}_2 : M \times_\lambda N \to N$, note that at this time there are some differences between $\bar{\pi}_1$ and $\bar{\pi}_2$, that is $\tau(\bar{\pi}_2)$ and $\tau_f(\bar{\pi}_2)$ respectively satisfy

$$\tau(\bar{\pi}_2) = 0,$$
$$\tau_f(\bar{\pi}_2) = \frac{1}{\lambda^2} \mathrm{grad_h f} \mid \bar{\pi}_2,$$

we have

Lemma 5.7. *Given a projection map $\bar{\pi}_2 : (M \times_\lambda N \to (N, h)$, $\bar{\pi}_2(x, y) = y$, let $f : M \times_\lambda N \to (0, +\infty)$ be smooth function. Then bi-f-tension field of $\bar{\pi}_2$ is*

$$\begin{aligned} \tau_{f,2}(\overline{\pi}_2) \quad &= -\tfrac{f}{\lambda^2} Tr_h\, {}^N\nabla^2 \mathrm{grad_h f} - \tfrac{\mathrm{f}}{\lambda^2} {}^{\mathrm{N}}\mathrm{Ric}(\mathrm{grad_h f}) \\ &\quad -\tfrac{1}{2\lambda^2} \mathrm{grad_h f}(|\mathrm{grad_h f}|^2). \end{aligned} \tag{5.20}$$

From Lemmas 5.6 and 5.7, we easily conclude that

Corollary 5.8. *(i) If λ and f are non-constant function and* $\mathrm{grad_g} \log(\mathrm{f}\lambda^{\mathrm{n}}) \circ \bar{\pi}_1 = 0$*, then $\bar{\pi}_1$ is a non-trivial bi-f-harmonic map.*
(ii) Suppose λ and f are non-constant function. If $\mathrm{grad_h}$ *is non-zero constant and* ${}^N\mathrm{Ric}(\mathrm{grad_h f}) = 0$*, then $\bar{\pi}_2$ is a non-trivial bi-f-harmonic map.*

5.3. **Bi-f-harmonicity of the product maps with harmonic factor.** Now, we turn to consider a type of product map such as

$$\overline{\Phi} = \varphi_M \times \varphi_N : M \times_\lambda N \to (M \times N, g \oplus h)$$

defined by

$$\varphi_M \times \varphi_N(x, y) = (\varphi_M(x), \varphi_N(y)),$$

where $\varphi_M : M \to M$ and $\varphi_N : N \to N$ are smooth maps. In order to get some interesting results, we usually make some restrictions for φ_M and φ_N. Since in advance we observe that $\tau(\overline{\Phi})$ contains $\tau(\varphi_M)$ and $\tau(\varphi_N)$ (see (5.23), typically, φ_M and φ_N should be chosen as harmonic maps so that $\tau(\overline{\Phi}_M)$ has a simpler form. Thus we have

Proposition 5.9. *Suppose that $\varphi_M : (M, g) \to M$, $\varphi_N : N \to N$ are two harmonic maps. Let the product map $\overline{\Phi} = \varphi_M \times \varphi_N : M \times_\lambda N \to (M \times N, g \oplus h)$ be defined by $\overline{\Phi}(x, y) = (\varphi_M(x), \varphi_N(y))$ and $f : M \times_\lambda N \to \mathbb{R}$ smooth positive function . Then the bi-f-tension field of $\overline{\Phi}$ is*

$$\tau_f(\overline{\Phi}) = \big(d\varphi_M(\tau_{f,2}(\overline{\pi}_1)), d\varphi_N(\tau_{f,2}(\overline{\pi}_2))\big) \tag{5.21}$$

under some conventions below:

$$\begin{aligned} &d\varphi_L({}^L\nabla_\cdot) := {}^L\nabla_{d\varphi_L(\cdot)}, \quad L = M, N, \\ &d\varphi_L({}^L\nabla^2_\cdot) := {}^L\nabla_{d\varphi_L(\cdot)} {}^L\nabla_{d\varphi_L(\cdot)}, \\ &d\varphi_L({}^LR(\tau_f(\overline{\pi}_i), \cdot)\cdot = {}^LR(\tau_f(\overline{\pi}_i), d\varphi_L(\cdot))d\varphi_L(\cdot), \quad i = 1, 2. \end{aligned} \tag{5.22}$$

Proof. Since φ_M and φ_N are harmonic, we have $\tau(\varphi_M) = \tau(\varphi_N) = 0$. As the trick of [Lu1], we have

$$\begin{aligned}\tau(\overline{\Phi}) &= \sum_{j=1}^{m} (\nabla_{d\overline{\Phi}(e_j,0_2)} d\overline{\Phi}(e_j,0_2) - d\overline{\Phi}(\,\bar{\nabla}_{(e_j,0_2)}(e_j,0_2)\,))\\ &+ \sum_{\alpha=1}^{n} \tfrac{1}{\lambda^2}\Big(\nabla_{d\varphi_M\times d\varphi_N(0_1,\bar{e}_\alpha)} d\varphi_M\times d\varphi_N(0_1,\bar{e}_\alpha)\\ &\quad - d\varphi_M\times d\varphi_N(\,\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha)\,)\Big)\\ &= (\tau(\varphi_M),0_2) + n(d\varphi_M(\mathrm{grad}_{\mu^2 g}\log\lambda),0_2\,)\\ &\quad + \tfrac{1}{\lambda^2}(0_1,\tau(\varphi_N)))\\ &= (n\, d\varphi_M(\mathrm{grad_g}\log\lambda),0_2\,).\end{aligned} \tag{5.23}$$

So

$$\begin{aligned}\tau_f(\overline{\Phi}) &= fn(d\varphi_M(\mathrm{grad_g}\log\lambda),0_2\,) + \mathrm{d}\varphi_\mathrm{M}\times\mathrm{d}\varphi_\mathrm{N}(\mathrm{grad_g f}, \tfrac{1}{\lambda^2}\mathrm{grad_h f})\\ &= (\,fnd\varphi_M(\mathrm{grad_g}\log\lambda) + \mathrm{d}\varphi_\mathrm{M}(\mathrm{grad_g f}), \tfrac{1}{\lambda^2}\mathrm{d}\varphi_\mathrm{N}(\mathrm{grad_h f})\,)\\ &= (f\, d\varphi_M(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f})), \tfrac{1}{\lambda^2}\mathrm{d}\varphi_\mathrm{N}(\mathrm{grad_h f}))\end{aligned} \tag{5.24}$$

Next we process $\tau_{f,2}(\overline{\Phi})$. To this end, we need to tackle two intricate terms by two steps:
Step 1 Consider $Tr_{\bar{g}}(\nabla^{\bar{\Phi}})^2\tau_f(\bar{\Phi})$. Since

$$\nabla^{\bar{\Phi}}_{(e_j,0_2)}\tau_f(\bar{\Phi}) = (\,{}^M\nabla_{d\varphi_M(e_j)} f\, d\varphi_M(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f})),\ 0_2),$$

$$\nabla^{\bar{\Phi}}_{(e_j,0_2)}\nabla^{\bar{\Phi}}_{(e_j,0_2)}\tau_f(\bar{\Phi}) = (\,{}^M\nabla_{d\varphi_M(e_j)}\,{}^M\nabla_{d\varphi_M(e_j)} f\, d\varphi_M(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f})),\ 0_2),$$

$$\begin{aligned}\nabla^{\bar{\Phi}}_{\bar{\nabla}_{(e_j,0_2)}(e_j,0_2)}\tau_f(\bar{\Phi}) &= \nabla^{\bar{\Phi}}_{({}^M\nabla_{e_j},0_2)}\tau_f(\bar{\Phi})\\ &= (\,{}^M\nabla_{d\varphi_M({}^M\nabla_{e_j}e_j)} f d\varphi_M(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f})),0_2)\end{aligned}$$

$$\nabla^{\bar{\Phi}}_{(0_1,\bar{e}_\alpha)}\tau_f(\bar{\Phi}) = \nabla_{(0_1,d\varphi_N(\bar{e}_\alpha)}\tau_f(\bar{\Phi}) = (\,0_1,\ {}^N\nabla_{d\varphi_N(\bar{e}_\alpha)}\tfrac{1}{\lambda^2} d\varphi_N(\mathrm{grad_h f}\,),$$

$$\nabla^{\bar{\Phi}}_{(0_1,\bar{e}_\alpha)}\nabla^{\bar{\Phi}}_{(0_1,\bar{e}_\alpha)}\tau_f(\bar{\Phi}) = (\,0_1,\ {}^N\nabla_{d\varphi_N(\bar{e}_\alpha)}\,{}^N\nabla_{d\varphi_N(\bar{e}_\alpha)}\tfrac{1}{\lambda^2} d\varphi_N(\,\mathrm{grad_h f})\,)$$

$$\begin{aligned}\nabla^{\bar{\Phi}}_{\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha)}\tau_f(\bar{\Phi}) &= \nabla^{\bar{\Phi}}_{(0_1,\,{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)-\frac{1}{2}(\mathrm{grad_g}\lambda^2,0_2)}\tau_f(\bar{\Phi})\\ &= (0_1,\ {}^N\nabla_{d\varphi_N({}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)}\tfrac{1}{\lambda^2} d\varphi_N(\,\mathrm{grad_h f}\,)\,)\\ &\quad -(\lambda\,{}^M\nabla_{d\varphi_M(\mathrm{grad_g}\lambda)} f\, d\varphi_M(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f})),\ 0_2\,),\end{aligned}$$

we have

$$\begin{aligned}\mathrm{Tr}_{\bar{g}}(\nabla^{\bar{\Phi}})^2\tau_\mathrm{f}(\bar{\Phi}) &= (\mathrm{Tr_g}(\,{}^\mathrm{M}\nabla_{\mathrm{d}\varphi_\mathrm{M}})^2\ \mathrm{fd}\varphi_\mathrm{M}(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f}))\\ &\qquad + n\,{}^M\nabla_{d\varphi_M(\mathrm{grad_g}\log\lambda)} f d\varphi_M(\mathrm{grad_g}\log(\lambda^\mathrm{n}\mathrm{f})),\ 0_2)\\ &\quad +(0_1,\ \tfrac{1}{\lambda^4}\mathrm{Tr_h}(\,{}^\mathrm{N}\nabla_{\mathrm{d}\varphi_\mathrm{N}}))^2\mathrm{d}\varphi_\mathrm{N}(\,\mathrm{grad_h f})).\end{aligned} \tag{5.25}$$

Step 2 Consider $\mathrm{Tr}_{\bar{g}}\mathrm{R}(\mathrm{d}\overline{\Phi},\tau_\mathrm{f}(\bar{\Phi})\mathrm{d}\overline{\Phi}$. Since

$$\begin{aligned}&\sum_{j=1}^{m} R(d\varphi_M(e_j),0_2),\tau_f(\bar{\Phi}))(d\varphi_M(e_j),0_2)\\ &= (\,\sum_{j=1}^{m}{}^M R(d\varphi_M(e_j),d\varphi_M(f\ \mathrm{grad_g}\log(\mathrm{f}\lambda^\mathrm{n})))\mathrm{d}\varphi_\mathrm{M}(\mathrm{e_j}),0_2)\\ &= (\mathrm{Tr_g}{}^\mathrm{M}\mathrm{R}(\mathrm{d}\varphi_\mathrm{M},\mathrm{d}\varphi_\mathrm{M}(\mathrm{f\ grad_g}\log(\mathrm{f}\lambda^\mathrm{n}))\,)\mathrm{d}\varphi_\mathrm{M},\ 0_2),\end{aligned}$$

$$\begin{aligned}&\sum_{\alpha=1}^{n}\tfrac{1}{\lambda^2}R((0_1,d\phi_N(\bar{e}_\alpha)\,),\tau_f(\bar{\Phi}))(0_1,d\phi_N(\bar{e}_\alpha)\,)\\ &= (\,0_1,\ \tfrac{1}{\lambda^4}\mathrm{Tr_h}\mathrm{R}^\mathrm{N}(\mathrm{d}\phi_\mathrm{N},\mathrm{grad_h f})\mathrm{d}\phi_\mathrm{N}\,),\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{Tr}_{\bar g}\mathrm{R}(\mathrm{d}\overline{\Phi},\tau_{\mathrm f}(\bar\Phi)\mathrm{d}\overline{\Phi} &= -(\mathrm{Tr}_{\mathrm g}{}^{\mathrm M}\mathrm{R}(\mathrm{d}\varphi_{\mathrm M}(\mathrm{f}\,\mathrm{grad}_{\mathrm g}\log(\mathrm f\lambda^{\mathrm n}),\mathrm{d}\varphi_{\mathrm M}))\mathrm{d}\varphi_{\mathrm M},\ 0_2)\\ &\quad -(\,0_1,\ \tfrac{1}{\lambda^2}\mathrm{Tr}_{\mathrm h}\mathrm{R}^{\mathrm N}(\mathrm{d}\phi_{\mathrm N}(\mathrm{grad}_{\mathrm h}\mathrm f),\mathrm{d}\phi_{\mathrm N})\mathrm{d}\phi_{\mathrm N}\,),\end{aligned} \tag{5.26}$$

Finally, note that

$$\begin{aligned}\nabla^{\bar\Phi}_{\mathrm{grad}_{\bar g}\mathrm f}\tau_f(\bar\Phi) &= (\,{}^M\nabla_{d\varphi_M(\mathrm{grad}_{\mathrm g}\mathrm f)}f\ d\varphi_M(\mathrm{grad}_{\mathrm g}\log(\mathrm f\lambda^{\mathrm n})),0_2)\\ &\quad +(0_1,\ \tfrac{1}{\lambda^4}\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_{\mathrm h}\mathrm f)}d\varphi_N(\,\mathrm{grad}_{\mathrm h}\mathrm f)).\end{aligned} \tag{5.27}$$

Putting (5.25), (5.25) and (5.25) together, we have

$$\tau_{f,2}(\overline{\Phi}) = (A,B), \tag{5.28}$$

where A and B denote by

$$\begin{aligned}A &= -f\ \mathrm{Tr}_{\mathrm g}(\,{}^{\mathrm M}\nabla_{\mathrm d\varphi_{\mathrm M}})^2\ \mathrm f\ \mathrm d\varphi_{\mathrm M}(\mathrm{grad}_{\mathrm g}\log(\lambda^{\mathrm n}\mathrm f))\\ &\quad -f\ \mathrm{Tr}_{\mathrm g}{}^{\mathrm M}\mathrm R(\mathrm d\varphi_{\mathrm M}(\mathrm f\ \mathrm{grad}_{\mathrm g}\log(\mathrm f\lambda^{\mathrm n})),\mathrm d\varphi_{\mathrm M}\,)\mathrm d\varphi_{\mathrm M}\\ &\quad -\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_{\mathrm g}\mathrm f)}f\ d\varphi_M(\mathrm{grad}_{\mathrm g}\log(\mathrm f\lambda^{\mathrm n})\\ &\quad -nf\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_{\mathrm g}\log\lambda)}f\ d\varphi_M(\mathrm{grad}_{\mathrm g}\log(\mathrm f\lambda^{\mathrm n})\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}B = -\tfrac{f}{\lambda^2}\mathrm{Tr}_{\mathrm h}(\,{}^{\mathrm N}\nabla_{\mathrm d\varphi_{\mathrm N})})^2\mathrm d\varphi_{\mathrm N}(\,\mathrm{grad}_{\mathrm h}\mathrm f) - \tfrac{\mathrm f}{\lambda^2}\mathrm{Tr}_{\mathrm h}\mathrm R^{\mathrm N}(\mathrm d\phi_{\mathrm N}(\mathrm{grad}_{\mathrm h}\mathrm f),\mathrm d\phi_{\mathrm N},)\mathrm d\phi_{\mathrm N}\\ -\tfrac{1}{\lambda^2}\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_{\mathrm h}\mathrm f)}d\varphi_N(\,\mathrm{grad}_{\mathrm h}\mathrm f).\end{aligned} \tag{5.30}$$

Connecting Lemmas 5.6 and 5.7 above, under the notation conventions 5.22, (5.28) can be singly written as

$$\tau_{f,2}(\overline{\Phi}) = (d\varphi_M(\tau_{f,2}(\overline{\pi}_1)), d\varphi_N(\tau_{f,2}(\overline{\pi}_2))),$$

as claimed. □

When $\varphi_M = Id_M$ or $\varphi_N = Id_N$, we easily obtain the following propositions.

Proposition 5.10. *(i) $\overline{\Phi}$ with $\varphi_M = Id_M$ is a bi-f-harmonic map if and only if the projection map $\overline{\pi}_1$ is bi-f-harmonic and $d\varphi_N(\tau_{f,2}(\overline{\pi}_2)) = 0$.*
(ii) $\overline{\Phi}$ with $\varphi_N = Id_N$ is a bi-f-harmonic map if and only if the projection map $\overline{\pi}_2$ is also and $d\varphi_N(\tau_{f,2}(\overline{\pi}_1)) = 0$.
(iii) $\overline{\Phi}$ with $\varphi_M = Id_M$ and $\varphi_N = Id_N$ is a bi-f-harmonic map if and only if both $\overline{\pi}_1$ and $\overline{\pi}_2$ are also.

Remark 5.11. In above propositions, neither $d\varphi_N(\tau_{f,2}(\overline{\pi}_2)) = 0$ nor $d\varphi_M(\tau_{f,2}(\overline{\pi}_1)) = 0$ implies $\tau_{f,2}(\overline{\pi}_2) \in \mathrm{Ker}(\overline{\phi}_2)$ or $\tau_{f,2}(\overline{\pi}_1)) \in \mathrm{Ker}(\overline{\phi}_1)$. Because $d\varphi_N(\tau_{f,2}(\overline{\pi}_2))$ and $d\varphi_M(\tau_{f,2}(\overline{\pi}_1))$ don't have the usual sense for differential map but only a kind of special notation, see already stipulations (5.22).

If we interchange the roles between the domain and codomain of $\overline{\Phi}$, we will obtain another type of product map such as

$$\widehat{\Psi} = \varphi_M \widehat{\times} \varphi_N : (M\times N, g\oplus h) \to M\times_\lambda N$$

defined by $\widehat{\Psi}(x,y) = (\varphi_M(x), \varphi_N)$.

Under this case, although we finally expect that the operators $\bar\nabla$ and $\bar R$ will be fully applied more than previous cases, it is pity that $\tau_{f,2}(\widehat{\Psi})$ is hard to work out. More precisely, we hardly find a simple form for like the previous cases. For instance, let $\varphi_M = Id_M$, then we can quickly get

$$\tau(\widehat{\Psi}) = -e(\varphi_N)(\mathrm{grad}_{\mathrm g}\lambda^2, 0_2),$$

and

$$\tau_f(\widehat{\Psi}) = (-\ e(\varphi_N) f\ \mathrm{grad}_g \lambda^2 + \mathrm{grad_g f}, \mathrm{d}\varphi_\mathrm{N}(\mathrm{grad_h f}),$$

where $e(\varphi_N)$ is the energy density of φ_N, $e(Id_N) = \frac{n}{2}$. Next, we involve to tackle the terms such as

$$\sum_{j=1}^{m} (\bar{\nabla}_{(e_j,0_2)} \bar{\nabla}_{(e_j,0_2)} - \bar{\nabla}_{^M\nabla_{e_j} e_j}) \tau_f(\widehat{\Psi}),$$

$$\sum_{\alpha=1}^{n} (\bar{\nabla}_{(0_1,d\varphi_N(\bar{e}_\alpha)} \bar{\nabla}_{(0_1,d\varphi_N(\bar{e}_\alpha)} - \bar{\nabla}_{^N\nabla_{d\varphi_N(\bar{e}_\alpha)} d\varphi_N(\bar{e}_\alpha)}) \tau_f(\widehat{\Psi}),$$

$$\sum_{j=1}^{m} \bar{R}((0_1, e_j), \tau_f(\widehat{\Psi}))(0_1, e_j)$$

$$\sum_{\alpha=1}^{n} \bar{R}((0_1, d\varphi_N(\bar{e}_\alpha)), \tau_f(\widehat{\Psi}))(0_1, d\varphi_N(\bar{e}_\alpha).$$

This produces much more sub-terms which are not good to integral. Based on the disadvantage, we omit investigating the product map $\widehat{\Psi}$.

6. The behaviors of f-bi-harmonic maps from or into singly WPM

In this section, we will discuss the behavior of f-bi-harmonicity combining with singly WPM like the previous section.

6.1. f-Bi-harmonicity of the inclusion maps.

The goal of this subsection is to characterize the f-bi-harmonicity of the inclusion map $i_{x_0} : (N, h) \to M \times_\lambda N$ $(x_0 \in M)$ in terms of warping function λ. As for the inclusion $i_{y_0} : (M, g) \to (M \times_\lambda N$ $(y_0 \in N)$, since it is always a totally geodesic map, it is harmonic and f-bi-harmonic for any warping function λ. We have

Proposition 6.1. *Let $f : N \to (0, +\infty)$ be a smooth function. For the inclusion map $i_{x_0} : (N, h) \to M \times_\lambda N$, its f-bi-tension fields is given by*

$$\begin{aligned} \tau_{f,2}(i_{x_0}) &= (-\tfrac{n^2}{8} f(y)(\mathrm{grad}_g(|\mathrm{grad}_g \lambda^2|^2) + \tfrac{n}{2}(\Delta_N f(y))\mathrm{grad}_g \lambda^2, 0_2) \\ &\quad +(0_1, 2n|\mathrm{grad_g}\lambda|^2 \mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}}. \end{aligned} \tag{6.1}$$

Proof. Refer to [BMO]. Let $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be an orthonormal frame N. From (5.9), we have

$$\tau(i_{x_0}) = -\tfrac{n}{2}(\mathrm{grad}_g \lambda^2, 0_2) \ |_{i_{x_0}} .$$

It is clear that i_{x_0} is harmonic if and only if $\mathrm{grad}_h \mu^2 \ |_{i_{x_0}} = 0$.

Further similar to (5.11), (5.12) and (5.13), we have

$$(\nabla^{i_{x_0}}_{\mathrm{grad_h f(y)}} \tau_f(i_{x_0}) = -n|\mathrm{grad_g}\lambda|^2 (0_1, \mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}}, \tag{6.2}$$

$$\mathrm{Tr_h}(\nabla^{\mathrm{i_{x_0}}})^2 \tau_\mathrm{f}(\mathrm{i_{x_0}}) = \mathrm{fracn^2 2}|\mathrm{grad_g}\lambda|^2 (\mathrm{grad_g}\lambda^2, 0_2)\Big|_{\mathrm{i_{x_0}}}, \tag{6.3}$$

$$\begin{aligned} &\sum_{\alpha=1}^{n} \bar{R}(\tau(i_{x_0}), (0_1, \bar{e}_\alpha))(0_1, \bar{e}_\alpha) \\ &= \tfrac{n^2}{4} \lambda \left({}^M\nabla_{\mathrm{grad}_g \lambda^2} \mathrm{grad}_g \lambda^2 - \tfrac{1}{2\lambda^2} \mathrm{grad}_g \lambda^2 (\lambda^2) \mathrm{grad}\ \lambda^2, 0_2 \right) \\ &= \tfrac{n^2}{8} (\mathrm{grad}_g(|\mathrm{grad}_g \lambda^2|^2), 0_2) - \tfrac{n^2}{2} |\mathrm{grad}_g \lambda|^2 (\mathrm{grad}_g \lambda^2, 0_2). \end{aligned} \tag{6.4}$$

Combining these equations with (4.9), we obtain

$$
\begin{aligned}
\tau_{f,2}(i_{x_0}) &= -\tfrac{n^2}{8} f(y)(\mathrm{grad}_g(|\mathrm{grad}_g \lambda^2|^2), 0_2) + \tfrac{n}{2}(\Delta_N f(y))(\mathrm{grad}_g \lambda^2, 0_2) \\
&\quad +2n|\mathrm{grad_g}\lambda|^2(0_1, \mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}} \\
&= \Big(-\tfrac{n^2}{8} f(y)(\mathrm{grad}_g(|\mathrm{grad}_g \lambda^2|^2) + \tfrac{n}{2}(\Delta_N f(y))\mathrm{grad}_g \lambda^2, 0_2) \\
&\quad +(0_1, 2n|\mathrm{grad_g}\lambda|^2\mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}},
\end{aligned} \tag{6.5}
$$

as claimed. □

Corollary 6.2. *Let $f : (N, h) \to (0, +\infty)$ be a smooth function. The inclusion map i_{x_0} : $(N, h) \to M \times_\lambda N$ is a f-bi-harmonic map if and only if λ and f satisfy*

$$
\begin{cases}
n f(y)(\mathrm{grad}_g(|\mathrm{grad}_g \lambda^2|^2) - 8(\Delta_N f(y))\mathrm{grad}_g \lambda^2\Big|_{i_{x_0}} = 0, \\
|\mathrm{grad_g}\lambda|^2\mathrm{grad_h f(y)})\Big|_{\mathrm{i_{x_0}}} = 0.
\end{cases} \tag{6.6}
$$

From the second equation in (6.6), we easily observe that it holds if and only if either $\lambda \circ i_{x_0}$ or $f \circ i_{x_0}$ is constat. This implies that

Corollary 6.3. *Let $f : N \to (0, +\infty)$ be a smooth function. The inclusion map i_{x_0} : $(N, h) \to M \times_\lambda N$ admits no a non-trivial f-bi-harmonic map.*

Remark 6.4. If in (6.1), $\tau_{2,f}(i_{x_0})$ only contains the first term on the right-hand side, then when x_0 is a critical point of $|\mathrm{grad}_g \lambda^2|^2$ but not a critical point of λ^2, the inclusion map i_{x_0} is a non-trivial f-bi-harmonic map. For example (c.f. [CMO1, CMO2]), let S^n be a unit Euclidean sphere with dimension n. Then for $p \in S^{n+1}$, the space $S^{n+1} - \{\pm p\}$ can be viewed as the SWPM

$$(0, \pi) \times_{\sin t} S^n.$$

Consider the inclusion map

$$i_{\pi/4}(resp.\ i_{3\pi/4}) : S^n \to 0, \pi) \times_{\sin t} S^n.$$

Since $\mathrm{grad_t} \sin^2 \mathrm{t}\Big|_{\pi/4} = \mathrm{grad_t} \sin^2 \mathrm{t}\Big|_{3\pi/4} = 0$ but $\sin^2(\pi/4) = 1/2 = \sin^2(3\pi/4)$, by Corollary 6.3, we know that $i_{\pi/4}$ and $i_{3\pi/4}$ are non-trivial f-bi-harmonic map with non-constant positive function $f(y)|_{S^n}$.

6.2. **f-Bi-harmonicity of the projection maps.** From Subsection 5.2, we have known that the second factor projection map $\bar{\pi}_2 : M \times_\lambda N \to N$ satisfies $\tau(\bar{\pi}_2) = 0$, whereas the first factor projection map $\bar{\pi}_1 : M \times_\lambda N \to M$ satisfies $\tau(\bar{\pi}_1) \neq 0$ with non-constant positive function f. Thus $\bar{\pi}_2$ is a trivial case since it automatically becomes a f-bi-harmonic map. Now, we only need to consider the projection map $\bar{\pi}_1$.

Lemma 6.5. *Let $f : M \times_\lambda N \to (0, +\infty)$ be smooth function. The f-bi-tension field of $\bar{\pi}_1 : M \times_\lambda N \to M$ is given by*

$$
\begin{aligned}
\tau_{f,2}(\bar{\pi}_1) &= -n f(x, y)\mathrm{Tr_g}({}^{\mathrm{M}}\nabla)^2\mathrm{grad_g} \log \lambda) - \tfrac{\mathrm{n}^2}{2}\mathrm{f(x, y)}\, {}^{\mathrm{M}}\mathrm{grad_g}(|\mathrm{grad_g} \log \lambda|^2) \\
&\quad -n f(x, y)^M\mathrm{Ric}(\mathrm{grad_g} \log \lambda) - \mathrm{n}(\Delta_{\mathrm{M}\times_\lambda \mathrm{N}}\mathrm{f(x, y)})\mathrm{grad_g} \log \lambda \\
&\qquad -2n\, {}^M\nabla_{\mathrm{grad_g f}}\mathrm{grad_g} \log \lambda)\Big|\bar{\pi}_1.
\end{aligned} \tag{6.7}
$$

Proof. Similar to the proof of Lemma 5.6, we have

$$\tau(\bar{\pi}_1) = n\mathrm{grad_g} \log \lambda \mid \bar{\pi}_1,$$

$$\nabla^{\bar{\pi}_1}_{\mathrm{grad_g f(x,y)}}\tau(\bar{\pi}_1) = n\, {}^M\nabla_{\mathrm{grad_g f}}\mathrm{grad_g} \log \lambda) \circ \bar{\pi}_1,$$

$$\begin{aligned} Tr_{\bar g}(\nabla^{\bar\pi_1})^2\tau(\bar\pi_1) &= nTr_g(\,^M\nabla)^2\mathrm{grad}_g\log(\lambda) \\ &\quad+\tfrac{n^2}{2}\,^M\mathrm{grad}_g(|\mathrm{grad}_g\log\lambda|^2)\circ\bar\pi_1, \end{aligned}$$

$$Tr_{\bar g}R^M(\tau(\bar\pi_1),d\bar\pi_1)d\bar\pi_1 = n^M\mathrm{Ric}(\mathrm{grad}_g\log\lambda)\circ\bar\pi_1,$$

from which, we get

$$\begin{aligned} \tau_{f,2}(\bar\pi_1) &= -f\big(\mathrm{Tr}_{\bar g}(\nabla^{\bar\pi_1})^2\tau(\bar\pi_1) - \mathrm{f}\mathrm{Tr}_{\bar g}\mathrm{R}^{\mathrm M}(\tau(\bar\pi_1),\mathrm d\bar\pi_1)\mathrm d\bar\pi_1\big) \\ &\quad -(\Delta_{M\times_\lambda N}f(x,y))\tau(\bar\pi_1) - 2\nabla^{\bar\pi_1}_{\mathrm{grad}_{\bar g}\mathrm f(x,y)}\tau(\bar\pi_1) \\ &= -nf(x,y)\mathrm{Tr}_g(\,^{\mathrm M}\nabla)^2\mathrm{grad}_g\log\lambda) - \tfrac{n^2}{2}\mathrm f(x,y)\,^{\mathrm M}\mathrm{grad}_g(|\mathrm{grad}_g\log\lambda|^2) \\ &\quad -nf(x,y)^M\mathrm{Ric}(\mathrm{grad}_g\log\lambda) - \mathrm n(\Delta_{\mathrm M\times_\lambda\mathrm N}\mathrm f(x,y))\mathrm{grad}_g\log\lambda \\ &\quad -2n\,^M\nabla_{\mathrm{grad}_g\mathrm f}\mathrm{grad}_g\log\lambda)\big|\bar\pi_1 \end{aligned}$$

Thus we complete the proof. ◻

As a consequence, we have

Proposition 6.6. *The projection map $\bar\pi_1$ is a non-trivial f-bi-harmonic map if and only if λ and f satisfy*

$$\begin{aligned} & f(x,y)\mathrm{Tr}_g(\,^{\mathrm M}\nabla)^2\mathrm{grad}_g\log\lambda) + \tfrac{\mathrm n}{2}\mathrm f(x,y)\,^{\mathrm M}\mathrm{grad}_g(|\mathrm{grad}_g\log\lambda|^2) \\ & +f(x,y)^M\mathrm{Ric}(\mathrm{grad}_g\log\lambda) + (\Delta_{\mathrm M\times_\lambda\mathrm N}\mathrm f(x,y))\mathrm{grad}_g\log\lambda \\ & \quad +2\,^M\nabla_{\mathrm{grad}_g\mathrm f}\mathrm{grad}_g\log\lambda)\big|\bar\pi_1 = 0. \end{aligned} \tag{6.8}$$

6.3. **f-Bi-harmonicity of the product maps with harmonic factor.** Now, we turn to consider two types of product map such as

$$\begin{aligned} \Psi = Id_M\times\psi : M\times_\lambda N &\to (M\times N, g\oplus h) \\ \Psi(x,y) &= (x,\psi(y)) \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \overline{\Psi} = \overline{Id_M\times\psi} : (M\times N, g\oplus h) &\to M\times_\lambda N \\ \overline{\Psi}(x,y) &= (x,\psi(y)) \end{aligned} \tag{6.10}$$

where $\psi : N\to N$ is a harmonic map.

For the first case (6.9), we have

Proposition 6.7. *Suppose that $\psi : N\to N$ is a harmonic map and $f\in C^\infty(M\times_\lambda N)$ is a positive function. For the product map $\Psi = Id_M\times\psi : M\times_\lambda N\to(M\times N, g\oplus h)$, we have*

$$\tau_{2,f}(\Psi) = (\tau_{2,f}(\bar\pi_1), 0). \tag{6.11}$$

Proof. Similar to the proof of Proposition 5.9, by using $\tau(\psi) = 0$ we have

$$\begin{aligned} \tau(\overline{\Psi}) &= \mathrm{Tr}_{\bar g}\nabla\mathrm d\Psi \\ &= \tfrac{n}{2\lambda^2}(\mathrm{grad}_g\lambda^2, 0_2) + \tfrac{1}{\lambda^2}(0_1,\tau(\psi)) \\ &= \tfrac{n}{2\lambda^2}(\mathrm{grad}_g\lambda^2, 0_2). \end{aligned} \tag{6.12}$$

Further, we get

$$\begin{aligned} \nabla^\Psi_{\mathrm{grad}_g\mathrm f}\tau(\bar\Phi) &= n\,\nabla_{(\mathrm{grad}_g\mathrm f,\frac{1}{\lambda^2}\mathrm{grad}_h\mathrm f)}(\mathrm{grad}_g\log\lambda, 0_2) \\ &= n\,(^M\nabla_{\mathrm{grad}_g\mathrm f}\mathrm{grad}_g\log\lambda, 0_2), \end{aligned} \tag{6.13}$$

$$\begin{aligned}\mathrm{Tr}_{\bar g}(\nabla^{\bar\Phi})^2\tau_{\mathrm f}(\bar\Phi) &= \sum_{j=1}^{m}\Big(\nabla_{(e_i,0_2)}\nabla_{(e_i,0_2)}\tau(\Psi)-\nabla_{d\Psi\left(\bar\nabla_{(e_i,0_2)}(e_i,0_2)\right)}\tau(\Psi)\Big)\\ &+\tfrac{1}{\lambda^2}\sum_{\alpha=1}^{n}\Big(\nabla_{(0_1,\bar e_\alpha)}\nabla_{(0_1,\bar e_\alpha)}\tau(\Psi)-\nabla_{d\Psi\left(\bar\nabla_{(0_1,\bar e_\alpha)}(0_1,\bar e_\alpha)\right)}\tau(\Psi)\Big)\\ &= n\mathrm{Tr_g}(\,^{\mathrm M}\nabla)^2\mathrm{grad_g}\log(\lambda)+\tfrac{\mathrm n^2}{2}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2,0_2),\end{aligned}\tag{6.14}$$

$$\begin{aligned}\mathrm{Tr}_{\bar g}\mathrm R(\tau_{\mathrm f}(\Psi,\mathrm d\Psi)\mathrm d\Psi &= n\sum_{j=1}^{m}R(\mathrm{grad_g}\log(\lambda),0_2),(\mathrm e_i,0_2))(\mathrm e_i,0_2)\\ &+\tfrac{n}{\lambda^2}\sum_{\alpha=1}^{n}R(\mathrm{grad_g}\log\lambda,0_2),(0_1,\mathrm d\psi(\bar{\mathrm e}_\alpha)))(0_1,\mathrm d\psi(\bar{\mathrm e}_\alpha))\\ &= n(^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda)),0_2).\end{aligned}\tag{6.15}$$

Putting (6.13), (6.14) and (6.15) together, we have

$$\begin{aligned}\tau_{2,f}(\Psi) &= -f(\mathrm{Tr}_{\bar g}(\nabla^{\bar\pi_1})^2\tau(\Psi)-\mathrm{f}\mathrm{Tr}_{\bar g}\mathrm R^{\mathrm M}(\tau(\Psi),\mathrm d\bar\pi_1)\mathrm d\Psi)\\ &\quad-(\Delta_{M\times_\lambda N}f(x,y))\tau(\Psi)-2\nabla^{\Psi}_{\mathrm{grad}_{\bar g}\mathrm f(x,y)}\tau(\Psi)\\ &= -n\Big(f(x,y)\mathrm{Tr_g}(\,^{\mathrm M}\nabla)^2\mathrm{grad_g}\log\lambda)+\tfrac{\mathrm n}{2}\mathrm{f(x,y)}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)\\ &\quad+f(x,y)^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda)+(\Delta_{\mathrm{M}\times_\lambda\mathrm N}\mathrm{f(x,y)})\mathrm{grad_g}\log\lambda\\ &\qquad+\,^M\nabla_{\mathrm{grad_g f}}\mathrm{grad_g}\log\lambda),0_2\Big).\end{aligned}\tag{6.16}$$

Finally, by using Lemma 6.5, (6.25) can be written as

$$\tau_{2,f}(\Psi)=(\tau_{2,f}(\overline{\pi}_1),0_2),$$

as claimed. □

As a consequence, we have

Proposition 6.8. *Suppose that $\psi : N \to N$ is a harmonic map and $f \in C^\infty(M\times_\lambda N)$ is a positive function. The product map $\Psi = Id_M\times\psi : M\times_\lambda N \to (M\times N, g\oplus h)$ is a non-trivial f-bi-harmonic if and only if the warping function λ is a non-constant solution to*

$$\begin{aligned}0 &= f(x,y)\mathrm{Tr_g}(\,^{\mathrm M}\nabla)^2\mathrm{grad_g}\log\lambda+\tfrac{\mathrm n}{2}\mathrm{f(x,y)}\mathrm{grad_g}(|\mathrm{grad_g}\log\lambda|^2)\\ &\quad+f(x,y)^M\mathrm{Ric}(\mathrm{grad_g}\log\lambda)+(\Delta_{\mathrm M\times_\lambda\mathrm N}\mathrm{f(x,y)})\mathrm{grad_g}\log\lambda\\ &\qquad+\,^M\nabla_{\mathrm{grad_g f}}\mathrm{grad_g}\log\lambda.\end{aligned}\tag{6.17}$$

Remark 6.9. It is easy to see that besides its harmonicity, the map ψ gives no other contribution to the f-bi-tension field of Ψ. This enables us to construct a wide range of examples of non-trivial f-bi-harmonic maps of product type (6.9).

Next, we consider the second case (6.10), which just interchanges the domain and codomain in the first case (6.9). In this case, as we shall see, the information of ψ increases, involving its energy density $e(\phi)$. The difficulties are due to the contribution of the curvature tensor field $\bar R$ (see previous (2.5) or (2.6)) of $M\times_\lambda N$ in the expression of the f-bi-tension field. We have

Proposition 6.10. *Suppose that $\psi : N \to N$ is a harmonic map and $f \in C^\infty(M\times N)$ is a positive function. For the product map $\overline{\Psi} = \overline{Id_M\times\psi} : (M\times N, g\oplus h) \to M\times_\lambda N$, its*

f-bi-tension field is given by

$$\begin{aligned}\tau_{2,f}(\overline{\Psi}) &= \Big(e(\psi)f(x,y)(\,\mathrm{Tr_g}({}^{\mathrm{M}}\nabla)^2\mathrm{grad_g}\lambda^2 - \tfrac{1}{2}\mathrm{e}^2(\psi)\mathrm{f(x,y)grad_g|grad_g}\lambda^2|^2\\ &+2e(\psi)f(x,y)\,{}^{M}\mathrm{Ric}(\mathrm{grad_g}\lambda^2) + \mathrm{d}\psi(\Delta_{\mathrm{N}}\mathrm{e}(\psi))\mathrm{f(x,y)grad_g}\lambda^2\\ &+(\Delta_{M\times N}f(x,y))e(\psi)\mathrm{grad_g}\lambda^2 + \mathrm{e}(\psi)\,{}^{\mathrm{M}}\nabla_{\mathrm{grad_g f(x,y)}}\mathrm{grad_g}\lambda^2\\ &\quad+2\mathrm{grad_h f(x,y)(\,e}(\psi)\,)\mathrm{grad_g}\lambda^2, 0_2\Big)\\ &+(0_1, \tfrac{e(\psi)}{\lambda}|\mathrm{grad_g}\lambda^2|^2\mathrm{grad_h f(x,y)} + \tfrac{1}{\lambda^2}|\mathrm{grad_g}\lambda^2|^2\mathrm{f(x,y)d}\psi(\mathrm{grad_g e}(\psi)\,),\end{aligned} \tag{6.18}$$

where $d\psi(\Delta_N e(\psi))$ and $d\psi(\mathrm{grad_g e}(\psi)\,)$ are defined by (6.22) and (6.23), respectively.

Proof. Let $\{e_j\}_{j=1}^m$ and $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be orthonormal basises on (M,g) on (N,h). Then $\{(e_j,0_2),(0_1,\bar{e}_\alpha)\}_{\substack{j=1,\dots,m,\\ \alpha=1,\dots,n}}$ is a local orthonormal basis on the direct product manifold $M\times N$.

Note that $\tau(\psi)=0$ and $e(\psi)=\frac{1}{2}\frac{1}{\lambda^2}\sum_{\alpha=1}^n h(d\psi(\bar{e}_\alpha), d\psi(\bar{e}_\alpha))$, we have

$$\begin{aligned}\tau(\overline{\Psi}) &= \mathrm{Tr_{g\oplus h}}\nabla\mathrm{d}\overline{\Psi}\\ &= \sum_{j=1}^m(\bar{\nabla}_{(e_j,0_2)}(e_j,0_2) - (\nabla_{(e_j,0_2)}(e_j,0_2))\\ &+\sum_{\alpha=1}^n(\bar{\nabla}_{(0_1,d\bar{\psi}(\bar{e}_\alpha)\,)}(0_1,d\bar{\psi}(\bar{e}_\alpha)\,) - (0_1,d\psi(\bar{\nabla}_{e_\alpha}\bar{e}_\alpha\,)\,)\\ &= (0_1,\tau(\psi)\,) - \tfrac{1}{2}\sum_{\alpha=1}^n h(d\psi(\bar{e}_\alpha),d\psi(\bar{e}_\alpha)\,)(\mathrm{grad_g}\lambda^2,\ 0_2) + (0_1,\bar{\nabla}_{\bar{e}_\alpha}\bar{e}_\alpha\,))\\ &= -e(\psi)(y)\,(\mathrm{grad_g}\lambda^2,0_2).\end{aligned} \tag{6.19}$$

Further, we get

$$\begin{aligned}\nabla^{\overline{\Psi}}_{\mathrm{grad_{g\oplus h}f(x,y)}}\tau(\overline{\Psi}) &= \bar{\nabla}_{(\mathrm{grad_g f,grad_h f})} - e(\psi)(y)(\mathrm{grad_g}\lambda^2,0_2)\\ &= -\ \mathrm{grad_h f(\,e}(\psi)\mathrm{(y)}\,)(\mathrm{grad_g}\lambda^2,0_2)\\ &\quad -e(\psi)(y)\bar{\nabla}_{(\mathrm{grad_g f,grad_g f})}(\mathrm{grad_g}\lambda^2,0_2)\\ &= -\ \mathrm{grad_h f(\,e}(\psi)\,)(\mathrm{grad_g}\lambda^2,0_2)\\ &\quad -e(\psi)(\,{}^{M}\nabla_{(\mathrm{grad_g f}}\mathrm{grad_g}\lambda^2,0_2) - \tfrac{\mathrm{e}(\psi)}{2\lambda}|\mathrm{grad_g}\lambda^2|^2(\,0_1,\mathrm{grad_h f}\,)\end{aligned} \tag{6.20}$$

$$\begin{aligned}\mathrm{Tr_{g\oplus h}}(\nabla^{\overline{\Psi}})^2\tau(\overline{\Psi}) &= \sum_{i=1}^m\Big(\bar{\nabla}_{(e_i,0_2)}\bar{\nabla}_{(e_i,0_2)}\tau(\overline{\Psi}) - \bar{\nabla}_{\nabla_{(e_i,0_2)}(e_i,0_2)}\Big)\tau(\overline{\Psi})\\ &- \sum_{\alpha=1}^n\Big(\bar{\nabla}_{(0_1,d\psi(\bar{e}_\alpha),)}\bar{\nabla}_{(0_1,d\psi(\bar{e}_\alpha),)} - \bar{\nabla}_{\nabla_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha)}\Big)\tau(\overline{\Psi})\\ &= -e(\phi)\left(\sum_{i=1}^m({}^{M}\nabla_{e_i}{}^{M}\nabla_{e_i} - {}^{M}\nabla_{{}^{M}\nabla_{e_i}e_i})\mathrm{grad_g}\lambda^2, 0_2\right)\\ &- \sum_{\alpha=1}^n\Big(d\psi(\bar{e}_\alpha),)d\psi(\bar{e}_\alpha) - d\psi({}^{N}\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)(e(\psi)\Big)(\mathrm{grad_g}\lambda^2,0_2)\\ &-\tfrac{e(\psi)}{2\lambda^2}|\mathrm{grad_g}\lambda^2|^2(\,0_1,\sum_{\alpha=1}^{\mathrm{n}}({}^{\mathrm{N}}\nabla_{\mathrm{d}\bar{\psi}(\bar{\mathrm{e}}_\alpha)}\mathrm{d}\bar{\psi}(\bar{\mathrm{e}}_\alpha) - \mathrm{d}\psi({}^{\mathrm{N}}\nabla_{\mathrm{e}_\alpha}\bar{\mathrm{e}}_\alpha)\,)\\ &-\tfrac{1}{\lambda^2}|\mathrm{grad_g}\lambda^2|^2(\,0_1,\sum_{\alpha=1}^{\mathrm{n}}\mathrm{d}\psi(\bar{\mathrm{e}}_\alpha)(\,\mathrm{e}(\psi)\,)\mathrm{d}\psi(\bar{\mathrm{e}}_\alpha)\,)\\ &= -e(\psi)(\,\mathrm{Tr_g}({}^{\mathrm{M}})^2\mathrm{grad_g}\lambda^2,0_2\,) - \mathrm{d}\psi(\Delta_{\mathrm{N}}\mathrm{e}(\psi))(\mathrm{grad_g}\lambda^2,0_2)\\ &-\tfrac{e(\psi)}{2\lambda^2}|\mathrm{grad_g}\lambda^2|^2(0_1,\tau(\psi)\,) - \tfrac{1}{\lambda^2}|\mathrm{grad_g}\lambda^2|^2(0_1,\mathrm{d}\psi(\mathrm{grad_g e}(\psi)\,))\\ &+\tfrac{1}{2\lambda^2}|\mathrm{grad_g}\lambda^2|^2\mathrm{e}^2(\psi)(\mathrm{grad_g}\lambda^2,0_2),\end{aligned} \tag{6.21}$$

where

$$d\psi(\Delta_N e(\psi)) = \sum_{\alpha=1}^n\Big(d\psi(\bar{e}_\alpha),)d\psi(\bar{e}_\alpha) - d\psi({}^{N}\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)(e(\psi)\Big), \tag{6.22}$$

$$d\psi(\mathrm{grad_g e}(\psi)) = \sum_{\alpha=1}^{n} \mathrm{d}\psi(\bar{\mathrm{e}}_\alpha)(\,\mathrm{e}(\psi)\,)\mathrm{d}\psi(\bar{\mathrm{e}}_\alpha), \tag{6.23}$$

$$\begin{aligned} \mathrm{Tr_{g\oplus h}R}(\tau(\overline{\Psi}, \mathrm{d}\overline{\Psi})\mathrm{d}\overline{\Psi} &= -e(\psi)\sum_{j=1}^{m}\bar{R}((\mathrm{grad_g}\lambda^2, 0_2), (\mathrm{e_i}, 0_2))(\mathrm{e_i}, 0_2) \\ &\quad -e(\psi)\sum_{\alpha=1}^{n}\bar{R}((\mathrm{grad_g}\lambda^2, 0_2), (0_1, \mathrm{d}\psi(\bar{\mathrm{e}}_\alpha))(0_1, \mathrm{d}\psi(\bar{\mathrm{e}}_\alpha),) \\ &= -e(\psi)(\,{}^{M}\mathrm{Ric}(\mathrm{grad_g}\lambda^2), 0_2\,) \\ &\quad +\tfrac{1}{2}e^2(\psi)(\mathrm{grad_g}|\mathrm{grad_g}\lambda^2|^2, 0_2) \\ &\quad -\tfrac{1}{2\lambda^2}e^2(\psi)|\mathrm{grad_g}\lambda^2|^2(\mathrm{grad_g}\lambda^2, 0_2) \end{aligned} \tag{6.24}$$

Putting (6.20), (6.21) and (6.24) together, we have

$$\begin{aligned} \tau_{f,2}(\overline{\Psi}) &= -f\big(\mathrm{Tr_{g\oplus h}}(\nabla^{\overline{\Psi}})^2\tau(\overline{\Psi}) - \mathrm{fTr_{g\oplus h}}\bar{\mathrm{R}}(\tau(\overline{\Psi}), \mathrm{d}\overline{\Psi})\mathrm{d}\overline{\Psi}) \\ &\quad -(\Delta_{M\times N}f(x,y))\tau(\overline{\Psi}) - 2\nabla^{\overline{\Psi}}_{\mathrm{grad_{g\oplus h}f(x,y)}}\tau(\overline{\Psi}) \\ &= e(\psi)f(x,y)(\,\mathrm{Tr_g}({}^{\mathrm{M}}\nabla)^2\mathrm{grad_g}\lambda^2, 0_2\,) + \mathrm{d}\psi(\Delta_{\mathrm{N}}\mathrm{e}(\psi))\mathrm{f(x,y)}(\mathrm{grad_g}\lambda^2, 0_2) \\ &\quad +\tfrac{1}{\lambda^2}|\mathrm{grad_g}\lambda^2|^2\mathrm{f(x,y)}(0_1, \mathrm{d}\psi(\mathrm{grad_g e}(\psi)\,)) \\ &\quad +e(\psi)f(x,y)(\,{}^{M}\mathrm{Ric}(\mathrm{grad_g}\lambda^2), 0_2\,) \\ &\quad -\tfrac{1}{2}e^2(\psi)f(x,y)(\mathrm{grad_g}|\mathrm{grad_g}\lambda^2|^2, 0_2) \\ &\quad +(\Delta_{M\times N}f(x,y))e(\psi)(\mathrm{grad_g}\lambda^2, 0_2\,) \\ &\quad +\, 2\mathrm{grad_h f}(\,\mathrm{e}(\psi)\,)(\mathrm{grad_g}\lambda^2, 0_2) \\ &\qquad +2e(\psi)(\,{}^{M}\nabla_{\mathrm{grad_g f}}\mathrm{grad_g}\lambda^2, 0_2) + \tfrac{\mathrm{e}(\psi)}{\lambda}|\mathrm{grad_g}\lambda^2|^2(\,0_1, \mathrm{grad_h f}\,) \\ &= \Big(e(\psi)f(x,y)(\,\mathrm{Tr_g}({}^{\mathrm{M}}\nabla)^2\mathrm{grad_g}\lambda^2 - \tfrac{1}{2}\mathrm{e}^2(\psi)\mathrm{f(x,y)grad_g}|\mathrm{grad_g}\lambda^2|^2 \\ &\quad +2e(\psi)f(x,y)\,{}^{M}\mathrm{Ric}(\mathrm{grad_g}\lambda^2) + \mathrm{d}\psi(\Delta_{\mathrm{N}}\mathrm{e}(\psi))\mathrm{f(x,y)grad_g}\lambda^2 \\ &\quad +(\Delta_{M\times N}f(x,y))e(\psi)\mathrm{grad_g}\lambda^2 + \mathrm{e}(\psi)\,{}^{\mathrm{M}}\nabla_{\mathrm{grad_g f}}\mathrm{grad_g}\lambda^2 \\ &\qquad +2\mathrm{grad_h f}(\,\mathrm{e}(\psi)\,)\mathrm{grad_g}\lambda^2, 0_2\Big) \\ &\quad +(0_1, \tfrac{e(\psi)}{\lambda}|\mathrm{grad_g}\lambda^2|^2\mathrm{grad_h f(x,y)} + \tfrac{1}{\lambda^2}|\mathrm{grad_g}\lambda^2|^2\mathrm{f(x,y)d}\psi(\mathrm{grad_g e}(\psi)\,), \end{aligned} \tag{6.25}$$

as claimed. □

As a consequence, we have

Proposition 6.11. *Suppose that $\psi : N \to N$ is a harmonic map and $f \in C^\infty(M\times N)$ is a positive function. The product map $\overline{\Psi} = \overline{Id_M \times \psi} : (M\times N, g\oplus h) \to M\times_\lambda N$ is a non-trivial f-bi-harmonic if and only if the warping function λ is a non-constant solution to*

$$\begin{aligned} &e(\psi)f(x,y)(\,\mathrm{Tr_g}({}^{\mathrm{M}}\nabla)^2\mathrm{grad_g}\lambda^2 - \tfrac{1}{2}\mathrm{e}^2(\psi)\mathrm{f(x,y)grad_g}|\mathrm{grad_g}\lambda^2|^2 \\ &+2e(\psi)f(x,y)\,{}^{M}\mathrm{Ric}(\mathrm{grad_g}\lambda^2) + \mathrm{d}\psi(\Delta_{\mathrm{N}}\mathrm{e}(\psi))\mathrm{f(x,y)grad_g}\lambda^2 \\ &+(\Delta_{M\times N}f(x,y))e(\psi)\mathrm{grad_g}\lambda^2 + \mathrm{e}(\psi)\,{}^{\mathrm{M}}\nabla_{\mathrm{grad_g f(x,y)}}\mathrm{grad_g}\lambda^2 \\ &\qquad +2\mathrm{grad_h f(x,y)}(\,\mathrm{e}(\psi)\,)\mathrm{grad_g}\lambda^2 = 0, \end{aligned} \tag{6.26}$$

and

$$e(\psi)\lambda\mathrm{grad_h f(x,y)} + \mathrm{f(x,y)d}\psi(\mathrm{grad_g e}(\psi)) = 0. \tag{6.27}$$

7. Comparisons on bi-f-harmonic maps and f-bi-harmonic maps

From the previous sections, we know that bi-f-harmonic map and f-bi-harmonic map are two wider generalizations via mixing bi-harmonic map and f-harmonic map. More precisely, according to their tension fields, there are some following relations:
(1) f-harmonic map must be bi-f-harmonic map, but the usual harmonic map is not bi-f-harmonic map. In particular, bi-f-harmonic has nothing to do with bi-harmonic map.

(2) The usual harmonic map must be f-bi-harmonic, but bi-harmonic map is not be f-bi-harmonic except $f = const.$. In particular, f-bi-harmonic has nothing to do with f-harmonic map. Under the conformal dilation, from Corollaries 3.6 and 4.9 we find that Id_M is f-bi-harmonic map but not bi-f-harmonic map. The most difference between them is in that for the product map $\overline{\Psi} = \overline{Id_M \times \psi} : (M \times N, g \oplus h) \to M \times_\lambda N$, since the difficulties are due to the contribution of the curvature tensor field $\bar{R}$ (see previous (2.5) or (2.6)) of $M \times_\lambda N$ in the expression of the f-bi-tension field and bi-f-tension field, unlike Proposition 6.10, we have to give up acquiring the bi-f-tension field (see the analysis in the last paragraph of subsection 5.3.

So far, although bi-f-harmonic maps has some progress, see [CET, Ch1, Ch2], f-bi-harmonic map is just an up-and-coming thing. Do they contain inspired anticipations of the shape of things to come? Or are there any promising outlook? We shall look forward to their evolution in the future.

References

[BFO] P. Baird, A. Fardoun And S. Ouakkas, *Conformal and semi-conformal biharmonic maps*, annals of global analysis and geometry, 34(2008), 403-414.

[BG] J K Beem, T G Powell. *Geodesic completeness and maximality in Lorentzian warped products*, Tensor (N S), 1982, 39:31-36.

[BMO] A. Balmus, S. Montaldo and C. Oniciuc, *Biharmonic maps between warped product manifolds*, J. Geom. Phys. 57(2007), 449-466.

[BW] P. Baird and T. C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, Lond. Math. Soc. Monogr., New Series 29, Oxford University Press, 2003

[CET] A. M. Cherif, H. Elhendi AND M. Terbeche, *On generalized conformal maps*, Bulletin of Mathematical Analysis and Applications, 4(4)(2012), 99-108.

[Ch1] Yuan-Jen Chiang, *f-biharmonic maps between Riemannian manifolds*, Journal of Geometry and Symmetry in Physics, 27(2012), 45-58.

[Ch2] Yuan-Jen Chiang, *Transversally f-harmonic and transversally f-biharmonic maps between foliated manifolds*, JP Journal of Geometry and topology, 13(1)(2012), 93-117. (can't loadown)

[CMO1] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds of S^3*, Internat. J. Math., 12(2001), 867-876.

[CMO2] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel. J. Math., 130(2002), 109-123.

[Co] N. Course, *f-harmonic maps*, Ph.D.Thesis, University of Warwick, Coventry, CV4 7AL, UK, 2004.

[ES] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86(1964), 109-160.

[Ji] G. Y. Jiang, *2-harmonic maps and their first and second variation formulas*, Chinese Ann. Math. Ser. A, 7(1986), 389-402.

[L] A. Lichnerowicz, *Applications harmoniques et varietes kahleriennes*, Symposia Mathematica III, Academic Press, London, 1970, 341-402.

[Lu1] Wei-Jun Lu, *f-Harmonic maps of doubly warped product manifolds*, Applied Mathematics —A Journal of Chinese Universities, 28(2)2013, 240-252

[Lu2] Wei-Jun Lu, *Bi-f-Harmonic maps between doubly warped product manifolds*, July 2011, http://www.cms.zju.edu.cn/news.asp?id=1588&ColumnName=pdfbook&Version=english, N0.11015

[Lu3] Wei-Jun Lu, *Geometry of warped product manifolds and its five applications*, Ph.D. thesis, Zhejiang University, 2013.

[LW] Y. X. Li and Y. D. Wang, *Bubbling location for f-harmonic maps and inhomogeneous Landau-Lifshitz equations*, Comment. Math. Helv., 81(2)(2006), 433-448.

[OND] S. Ouakkas, R. Nasri, and M. Djaa, *On the f-harmonic and f-biharmonic maps*, JP J. Geom. Topol. 10(1)(2010), 11-27.

[Ou1] Ye-Lin Ou, *On f-harmonic morphisms between Riemannian manifolds*, arXiv:1103.5687v1, 2011.

[Ou2] Ye-Lin Ou, *On conformal bi-harmonic immersions*, Ann. Glob. Anal. Geom., 36(2009),133-142.

[PK] S. Y. Perktaş and E. Kiliç, *Biharmonic maps between doubly warped product manifolds*, Balkan Journal of Geometry and Its Applications, (2)15(2010), 151-162.

[RV] M. Rimoldi and G. Veronelli, *f-Harmonic maps and applications to gradient Ricci solitons*, arXiv: 1112.3637v1, 2011.

[Un] B. Unal, *Doubly Warped Products*, Differential Geometry and Its Applications, 15(3)(2001), 253-263.

Center of Mathematical Sciences, Zhejiang University,
Hangzhou, Zhejiang, 310027, P. R. China;
College of Mathematics and Computer Science, Guangxi University for Nationalities,
Naning, Guangxi, 530006, P. R. China.,
E-mail: weijunlu2008@126.com