**f-Biharmonic Maps Between Doubly Warped Product Manifolds**

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**Abstract.** In this paper, by applying the first variation formula of f-bi-energy given in [OND], we study f-biharmonic maps between doubly warped product manifolds $M \times_{(\mu,\lambda)} N$. Under imposing existence condition concerning proper f-biharmonic maps, we derive f-biharmonicity’s characteristic equations for the inclusion maps: $i_{\mu} : (M, g) \rightarrow (M \times_{(\mu,\lambda)} N, \bar{g})$, $i_{\lambda} : (N, h) \rightarrow (M \times_{(\mu,\lambda)} N, \bar{g})$ and the product maps: $\mathcal{V} = \mathcal{R}_M \times \varphi_N : M \times_{(\mu,\lambda)} N \rightarrow M \times N$, $\mathcal{V} = \varphi_M \times \text{Id}_N : M \times_{(\mu,\lambda)} N \rightarrow M \times N$ with $\varphi_M / \varphi_N$ being a harmonic map.

1. **Introduction and Preliminaries**

f-Biharmonic maps between Riemannian manifolds, as a generalization of biharmonic maps and f-harmonic maps, were first introduced by Ouakkas etc. in 2010 [OND]. On the one hand, f-harmonic maps between Riemannian manifolds (as a generalization of harmonic maps) were first introduced and studied by Lichnerowicz in [L] (see also Section 10.20 in Eells-Lemaires report [EL]). The study of f-harmonic maps comes from a physical motivation, since in physics, f-harmonic map can be viewed as stationary solution of the inhomogeneous Heisenberg spin system (see [LW]). On the other hand, the research in the field of Riemannian geometry in the last 10-15 years was partly characterized by the study of some fourth order partial differential equations. Whether the origin of these equations is analytic, as in the case of the Paneitz operator, or geometric, as in the case of Willmore surfaces, they represent a generalization of the concept of harmonic map. A natural extension of the harmonic maps was suggested by Eells and Sampson themselves in [ES]. The first results in this field were obtained by Jiang in 1986 (see [J]). He derived the Euler-Lagrange equation of the bienergy functional given by the vanishing of the bitension field

$$\tau^2(\phi) = \text{trace}(\nabla^g \nabla^g \tau(\phi) - \nabla^g_{\nabla^g \tau(\phi)} - \text{trace}_g(R^g(d\phi, \tau(\phi))d\phi)) = 0,$$

where $\phi : (M, g) \rightarrow (N, h)$ is a smooth map between two Riemannian manifolds. The essential difference between both f-harmonic maps and biharmonic maps is that the energy functional $\int_{\mathcal{M}} e(\phi)dv_g$, where in former case $e(\phi) = \frac{1}{2} f|d\phi|^2$ with positive function $f \in C^\infty(M)$, while in the latter $e(\phi) = \frac{1}{2} |\tau(\phi)|^2$ with the tension field $\tau(\phi) = \text{trace}_g \nabla d\phi$. f-Biharmonic maps consider to integrate the above two generalized harmonic maps and use the energy density $e(\phi) = \frac{1}{2} |\tau_f(\phi)|^2$ with the f-tension field $\tau_f(\phi) = f\tau(\phi) + d\phi(\text{grad } f)$. Since any harmonic map is f-biharmonic, we are interested in non-harmonic f-biharmonic maps which are called proper biharmonic.

The concept of warped product plays an important role in differential geometry as well as in theoretical physics. The notion of warped metric was first introduced by Bishop and O’Neill in [BO]. Since then, in Riemannian geometry, warped products have offered...
new examples of Riemannian manifolds with special curvature properties. Beem, Ehrlich and Powell pointed out that many exact solutions to Einstein’s field equation can be expressed in terms of Lorentzian warped products in [BEP]. Furthermore, Beem and Ehrlich concluded that causality and completeness of warped products can be related to causality and completeness of components of warped products in [BE]. O’Neill discussed warped products and explored curvature formulas of warped products in terms of curvatures of components of warped products in [ON]. He also examined Robertson-Walker, static, Schwarschild and Kruskal space-times as warped products.

In general, doubly warped products \((M \times N, g + \lambda^2 h)\) can be considered as a generalization of singly warped products \((M \times N, g + \lambda^2 h)\) (see Definitions [3.1]). Beem and Powell considered these products for Lorentzian manifolds in [BP]. Allison considered causality and global hyperbolicity of doubly warped products in [A1] and null pseudocovexity of Lorentzian doubly warped products in [A2]. Conformal properties of doubly warped products are studied by Gebarowski (cf. [G] or references therein).

As an application of biharmonic maps, the biharmonic maps were studied from between single warped product manifolds \([SE],[BMU]\). The authors of \([SE]\) investigated biharmonicity of the inclusion \(i_0 : M \rightarrow M \times_{\mu, \lambda} N\) r.s.p. \(i_0 : N \rightarrow M \times_{\mu, \lambda} N\) of a Riemannian manifold \(M\) r.s.p. \(N\) into the doubly warped product manifold \(M \times_{\mu, \lambda} N\) and of the projection from \(M \times_{\mu, \lambda} N\) into the first factor \(M\) r.s.p. the second factor \(N\). Also, they gave two new classes of proper biharmonic maps by using product of harmonic maps and doubly warping the metric in the domain or codomain \(\Psi : (M \times N, g \oplus h) \rightarrow (M \times N, g \oplus h)\).

Motivated by the mentioned works, we have extended these ideas to study \(f\)-biharmonic by considering the situation of doubly warped products manifolds \([LE]\). In this paper, we shall continue to study \(f\)-biharmonic in the same setting based on our previous work [Lu] and the work [SE]. Since the the \(f\)-bitension field \(\tau_f^2(\phi)\) (see (2.3)) is rather complex when \(\phi\) is any one of the cases such as the inclusion map, the projection, the special product map, the general product map so on, we only obtain some distinct characterization results. Especially, it is a bit pity that we cannot yet find a concrete example in terms of our present understanding. However, we conjecture that the study of \(f\)-biharmonic map/morphism will become an inevitable trend in the future because it inherits each merits from biharmonic and \(f\)-harmonic map. This is a strong encouragement for the study of \(f\)-biharmonic maps.

The structure of this paper is as follows. In the second section we give some basic definitions on biharmonic maps, \(f\)-harmonic maps and \(f\)-biharmonic maps. In Section 3, we give the definition of doubly warped product manifolds and the more explicit expressions of the connection \(\nabla\) on doubly warped product, as well as give the expression of curvature tensor \(\bar{R}\) on doubly warped product \(M \times_{\mu, \lambda} N\) by referring to [PR]. Section 4 is devoted to analyzing the conditions for both of the leaves \(\{x_0\} \times N\) and \(M \times \{y_0\}\) to be \(f\)-biharmonic as a submanifold and we can’t show that both of the leaves \(\{x_0\} \times N\) and \(M \times \{y_0\}\) can not be proper \(f\)-biharmonic as a submanifold of the doubly warped product manifold \(M \times_{\mu, \lambda} N\). In Section 5, we characterize a doubly warped product according to its projection map \(\tilde{r}_1 : M \times_{\mu, \lambda} N \rightarrow M\) and \(\tilde{r}_2 : M \times_{\mu, \lambda} N \rightarrow N\) being a \(f\)-biharmonic maps for some particular functions. We consider the product of two harmonic maps \(\Psi : (M \times N, g \oplus h) \rightarrow (M \times N, g \oplus h)\). By doubly warping the metric on the domain or codomain we lose the harmonicity; nevertheless, under certain conditions on the two warping functions, the product maps \(\Psi = \tilde{I}d_M \times \phi_N\), \(\Psi = \phi_M \times \tilde{I}d_N\) and \(\Psi = \tilde{I}d_M \times \phi_N\) remain \(f\)-harmonic. In last section, we character the case under general product maps.
2. f-BIHARMONIC AND f-BIHARMONIC

Recall that the energy of a smooth map \( \phi : (M, g) \to (N, h) \) between two Riemannian manifolds is defined by integral \( E(\phi) = \int_{\Omega} e(\phi) dv_g \), for every compact domain \( \Omega \subset M \) where \( e(\phi) = \frac{1}{2} |\phi|^2 \) is energy density and \( \phi \) is called harmonic if it’s a critical point of energy. From the first variation formula for the energy, the Euler-Lagrange equation is given by the vanishing of the tension field \( \tau(\phi) = \text{trace}_g \nabla^2 \phi (\phi) \) (see [ES]). As the generalizations of harmonic maps, we now introduce the concepts of biharmonic maps and \( f \)-harmonic maps.

**Definition 2.1.** (i) Biharmonic maps \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds are critical points of the bienergy functional

\[
E^2(\phi) = \frac{1}{2} \int_{\Omega} |\tau(\phi)|^2 dv_g,
\]

for any compact domain \( \Omega \subset M \).

(ii) An \( f \)-harmonic map with a positive function \( f \in C^\infty(M) \) is a critical point of \( f \)-energy

\[
E_f(\phi) = \frac{1}{2} \int_{\Omega \times M} f |\phi|^2 dv_g.
\]

The Euler-Lagrange equations give the bitension field \( \tau^2(\phi) \) (\[ II \]) and the \( f \)-tension field equation \( \tau_f(\phi) \) (see \[ CI, OND, Ou \]), respectively,

\[
\begin{align*}
\tau^2(\phi) &= \text{trace}(\nabla^2 \phi \phi (\phi) - \nabla^g \phi (\phi)) - \text{trace}(R^g(\phi, \phi(\phi))d\phi) = 0, \\
\tau_f(\phi) &= f \tau(\phi) + d\phi(\text{grad } f) = 0.
\end{align*}
\]

A more natural generalization of \( f \)-harmonic maps and biharmonic maps is given by integrating the square of the norm of the tension field, introduced recently in \[ OND \]. The authors of \[ OND \] give the first and second variations of \( f \)-bi-energy functional. But they cannot give any example about \( f \)-biharmonic maps.

**Definition 2.2.** \( f \)-bi-energy functional of smooth map \( \phi : (M, g) \to (N, h) \) is defined by

\[
E^2_f(\phi) = \frac{1}{2} \int_{\Omega} |\tau_f(\phi)|^2 dV
\]

for every compact domain \( \Omega \subset M \). A map \( \phi \) is called \( f \)-biharmonic map if it the critical point of \( f \)-bi-energy functional.

From the first variation, we obtain the Euler-Lagrange equation gives the \( f \)-biharmonic map equation \( \text{OND} \)

\[
\Delta_f^2 \tau_f(\phi) = -\Delta_f \tau_f(\phi) - f \text{trace}_g R^N(\tau_f(\phi), d\phi) d\phi = 0
\]

where

\[
\Delta_f^2 \tau_f(\phi) = -\text{trace}_g (\nabla^g f \nabla^g \tau_f(\phi) - f \nabla^g \nabla^g \tau_f(\phi)).
\]

For an orthonormal frame \( \{e_i\}_{i=1}^m \), we have

\[
\begin{align*}
\text{trace}_g (\nabla^g f \nabla^g \tau_f(\phi) - f \nabla^g \nabla^g \tau_f(\phi)) &= \sum_{i=1}^m (\nabla^g \tau_f(\phi) - f \nabla^g \nabla^g \tau_f(\phi)) \\
&= \sum_{i=1}^m \left( f \nabla^g_{e_i} \nabla^g_{e_i} \tau_f(\phi) - f \nabla^g_{e_i} \tau_f(\phi) - \nabla^g_{\text{grad}_f} \tau_f(\phi) \right)
\end{align*}
\]

\( \tau^2_f(\phi) \) is called the \( f \)-bi-tension field of \( \phi \).
Remark 2.3. From (2.5), the $f$-bi-tension field of $\phi$ can be simplified the following expression

$$\begin{align*}
\tau_f^2(\phi) &= f[T_{\phi}(\nabla^N_\phi)^2\tau_f(\phi) + T_{\phi}R^N(\phi, \tau_f(\phi))d\phi] - \nabla^N_{\nabla^M_\phi}\tau_f(\phi),
\end{align*}$$

where $(\nabla^N_\phi)^2 = \nabla^N_\phi \nabla^N_\phi - \nabla^N_\phi \nabla^N_\phi$.

3. Riemannian structure of doubly warped product

First we refer to [U] and give the definition of doubly warped product manifold.

Definition 3.1. Let $(M, g)$ and $(N, h)$ be Riemannian manifolds of dimensions $m$ and $n$, respectively and let $\lambda : M \to (0, +\infty)$ and $\mu : N \to (0, +\infty)$ be smooth functions. A doubly warped product $(G, \tilde{g})$ with warping functions $\lambda$ and $\mu$ is a product manifold which is of the form $G = M \times_{(\mu, \lambda)} N$ with the metric

$$\tilde{g} = \mu^2 g + \lambda^2 h.$$

When $\mu \equiv 1$ we have a (single) warped product $M \times_{\lambda} N$ with the warping function $\lambda(x)$.

For the connection of doubly warped product manifold $G = (M \times_{(\mu, \lambda)} N, \tilde{g})$, by referring to [SE] or [PR], we have

Proposition 3.2. Let $(M, g)$ and $(N, h)$ be Riemannian manifolds with Levi-Civita connections $\nabla^M$ and $\nabla^N$, respectively and let $\nabla$ and $\tilde{\nabla}$ denote the Levi-Civita connections of the product manifold $M \times N$ and doubly warped product manifold $G := M \times_{(\mu, \lambda)} N$, respectively, where $\lambda : M \to (0, \infty)$ and $\mu : N \to (0, \infty)$ are smooth maps, $\tilde{g} = \mu^2 g + \lambda^2 h$. Then we get the Levi-Civita connection of doubly warped product manifold $G$ as follows:

$$\begin{align*}
\tilde{\nabla}_{(X_1, Y_1)}(X_2, Y_2) &= \nabla_{(X_1, Y_1)}(X_2, Y_2) + \frac{1}{2\lambda^2}X_1(\lambda^2)(0_1, Y_2) + \frac{1}{2\lambda^2}X_2(\lambda^2)(0_1, Y_1) \\
&\quad + \frac{1}{2\mu^2}Y_1(\mu^2)(X_2, 0_2) + \frac{1}{2\mu^2}Y_2(\mu^2)(X_1, 0_2) \\
&\quad - \frac{1}{2}{g}(X_1, X_2)(0_1, \text{grad } \mu^2) - \frac{1}{2}{h}(Y_1, Y_2)(\text{grad } \lambda^2, 0_2) \\
&= \nabla^M_{X_1}X_2 + \frac{1}{2\lambda^2}Y_1(\mu^2)X_2 + \frac{1}{2\mu^2}Y_2(\mu^2)X_1 - \frac{1}{2}{h}(Y_1, Y_2)(\text{grad } \lambda^2, 0_2) \\
&\quad + (0_1, \nabla^N_{Y_1}Y_2) + \frac{1}{2\lambda^2}X_1(\lambda^2)Y_2 + \frac{1}{2\lambda^2}X_2(\lambda^2)Y_1 - \frac{1}{2}{g}(X_1, X_2)(\text{grad } \lambda^2)
\end{align*}$$

for any $(X_1, Y_1), (X_2, Y_2) \in \Gamma(TG), X_1, X_2 \in \Gamma(TM)$ and $Y_1, Y_2 \in \Gamma(TN)$.

Proof. See the proof of Proposition 2.1 in [LU].

Now we look at the curvature tensor of the warped product, we shall make optimal and give a rewritten expression different from that in [PR, SE] by organizing all the terms after the order. Since apparently no explicit version is given in the literature as far I know, we give our independent proof in the following.

Proposition 3.3. Let $(M, g)$ and $(N, h)$ be Riemannian manifolds with Levi-Civita connections $\nabla^M$ and $\nabla^N$, respectively and let $\nabla$ and $\tilde{\nabla}$ denote the Levi-Civita connections of the product manifold $M \times N$ and doubly warped product manifold $G := M \times_{(\mu, \lambda)} N$, respectively. If $R$ and $\tilde{R}$ denote the curvature tensors of $M \times N$ and $G$, respectively, then for all
$X = (X_1, Y_1), Y = (X_2, Y_2), Z = (X_3, Y_3) \in \Gamma(TM \times N)$, we have the following relation

$$\tilde{R}(X, Y)Z = R(X, Y)Z$$

$$+ \sum_{\sigma \in \Sigma(1, 2)} \text{sign}(\sigma) \left[ \frac{1}{4\mu} h(Y_{\sigma(1)}(\mu^2) \text{grad}_{\mu^2}X_1, X_3) + \frac{1}{4\mu} g(Y_{\sigma(1)}(\mu^2) \text{grad}_{\mu^2}X_1, X_3) \right]$$

$$- \frac{1}{4\mu} g(Y_{\sigma(1)}(\mu^2) \text{grad}_{\mu^2}X_1, X_3)$$

$$+ \frac{1}{2\mu} \text{grad}_{\mu^2}X_1, X_3 \right]$$

$$- \frac{1}{4\mu} [h(Y_{\sigma(1)}(\mu^2) \text{grad}_{\mu^2}X_1, X_3)] 01, Y_{\sigma(2)}]$$

$$+ \frac{1}{4\mu} g(Y_{\sigma(1)}(\mu^2) \text{grad}_{\mu^2}X_1, X_3) [01, Y_{\sigma(2)}]$$

$$+ \frac{1}{4\mu} \sigma(S(1, 2)) \text{grad}_{\mu^2}X_1, X_3 \right]$$

$$- \frac{1}{4\mu} g(Y_{\sigma(1)}(\mu^2) \text{grad}_{\mu^2}X_1, X_3)$$

$$+ \frac{1}{2\mu} \text{grad}_{\mu^2}X_1, X_3 \right]$$

$$- \frac{1}{4\mu} \sigma(S(1, 2)) \text{grad}_{\mu^2}X_1, X_3 \right]$$

where the doubly wedge product $(X \wedge_Z Y) \wedge_Z Z$ denotes $\tilde{g}(X, Z)Y - \tilde{g}(Y, Z)X$, and $S(1, 2)$ denotes a permutation group generated by elements 1, 2.

$$sign(\sigma) = \begin{cases} 1, & (\sigma(1), \sigma(2)) = (1, 2); \\ -1, & (\sigma(1), \sigma(2)) = (2, 1). \end{cases}$$

4. $f$-Harmonicity of the inclusion maps

In this section, we consider the inclusion map of $M$

$$i_{y_0} : (M, g) \to (M \times_{(\mu, \lambda)} N, \tilde{g})$$

at the point $y_0 \in N$ level in $M \times_{(\mu, \lambda)} N$ and the inclusion map of $N$

$$i_{x_0} : (N, h) \to (M \times_{(\mu, \lambda)} N, \tilde{g})$$

at the point $x_0$ level in $M \times_{(\mu, \lambda)} N$. By generalizing the methods in [SE] and [Lu], we obtain some non-existence results for the $f$-harmonicity of inclusion maps $i_{y_0}$ of $M$ and $i_{x_0}$ of $N$ under the doubly warped product case.

**Theorem 4.1.** Let $f : M \to (0, +\infty)$ be a smooth function. The inclusion map of the manifold $(M, g)$ into the nontrivial (proper) doubly warped product manifold $G = M \times_{(\mu, \lambda)} N$ is a proper $f$-harmonic map if and only if $\lambda, \mu, f$ satisfy

$$f(T_{\tilde{g}}[\tilde{g}] \text{grad}_{\tilde{g}}^2 f - \frac{1}{8\mu} f[\text{grad}_{\tilde{g}}^2 f]^2 \text{grad}_{\tilde{g}}^2 f$$

$$+ \frac{1}{2\mu} [\text{grad}_{\tilde{g}}^2 f]^2 \text{grad}_{\tilde{g}}^2 f + \text{Ric}_{\tilde{g}}(\text{grad}_{\tilde{g}} f)$$

$$- \frac{1}{4\mu} [\text{grad}_{\tilde{g}}^2 f]^2 \text{grad}_{\tilde{g}}^2 f + \frac{1}{4\mu} [\text{grad}_{\tilde{g}}^2 f]^2 \text{grad}_{\tilde{g}} f - \tilde{g} \nabla_{\text{grad}_{\tilde{g}} f} \text{grad}_{\tilde{g}} f = 0$$

(4.1)
\[ -\frac{1}{4}g(\nabla f, \nabla i) \nabla^2 f \nabla_s \nabla^2 f + f(-\frac{1}{4} \Delta_s f - \frac{1}{2} \nabla \nabla_s \nabla^2 f) = 0. \]

**Proof.** Let \( \{e_j\}_{j=1}^m \) be an orthonormal frame on \( (M, g) \). By using the tension field of \( i_{y_0} \)

\[
\tau(i_{y_0}) = Tr_g \nabla di_{y_0} = \sum_{j=1}^m (\nabla_e \tau_{G}) \nabla e_j \]

\[ = -\frac{m}{2} (0_1, f \cdot \nabla_g \nabla_i \mu^2) + (\nabla_g f, 0_2) \]

and (2.2), the f-tension field of \( i_{y_0} \) is

\[
\tau_f(i_{y_0}) = -\frac{m}{2} (0_1, f \cdot \nabla_g \nabla_i \mu^2) + (\nabla_g f, 0_2) \]

Since from (2.6) the f-biharmonic map of \( i_{y_0} \) is

\[
\tau_f^2(i_{y_0}) = f(Tr_g(\nabla^2 f)^2 \tau_f(i_{y_0}) + Tr_g \bar{R}(di_{y_0}, \tau_f(i_{y_0}))di_{y_0}) - \nabla^\mu_{(\nabla_g f, 0_2)} \tau_f(i_{y_0}) \]

\[ = f \sum_{j=1}^m (\nabla_{e_j} \nabla^\mu_{e_j} \tau_f(i_{y_0}) + \bar{R}(e_j, \tau_f(i_{y_0}))(e_j, 0_2)) - \nabla^\mu_{(\nabla_g f, 0_2)}(\tau_f(i_{y_0})) \]

we only need to calculate the following terms:

\[
\nabla_{e_j} \tau_f(i_{y_0}) = \nabla_{(\nabla_g f, 0_2)}(\nabla_g f, 0_2) \]

\[ = (\nabla \nabla_g \nabla f, 0_2) - \frac{1}{2} g(e_j, \nabla_g f)(0_1, \nabla_g \nabla_i \mu^2) \]

\[
\nabla^\mu_{(\nabla_g f, 0_2)}(\tau_f(i_{y_0})) = \nabla_{(\nabla_g f, 0_2)}(\nabla_g f, 0_2) \]

\[ = (\nabla \nabla_g \nabla f, 0_2) - \frac{1}{2} (\nabla \nabla_g f)(0_1, \nabla_g \nabla_i \mu^2) \]

so

\[
\tau_f(i_{y_0}) = f \sum_{j=1}^m (\nabla_e \nabla_{e_j} \nabla^\mu_{(\nabla_g f, 0_2)}(\nabla_g f, 0_2) \]

\[ - \frac{1}{2} \sum_{j=1}^m f(e_j, 0_1)(\Delta_s f - \nabla \nabla_g \nabla^2 f) \]

\[ = (Tr_g(\nabla^2 f)^2 \nabla_g f, 0_2) - \frac{1}{2} (0_1, (\Delta_s f - \nabla \nabla_g \nabla^2 f)) \]
On the other hand, by (3.2)
\[
\sum_{j=1}^{m} R((e_j, 0_2), \tau_f(i_0))(e_j, 0_2) = \sum_{j=1}^{m} \left[ -\frac{m}{2} R((e_j, 0_2), (0, f \cdot \text{grad}_f \mu^2)) + R((e_j, 0_2), (\text{grad}_f f, 0_2)) \right](e_j, 0_2)
\]
\[
= -\frac{m}{8}s^2 f(\text{grad}_f \mu^2)(\mu^2) \sum_{j=1}^{m} g(\text{grad}_f \mu^2, e_j)(e_j, 0_2)
\]
\[
- \frac{m}{4 s^2} \sum_{j=1}^{m} \left( \frac{1}{\bar{R}_e} e_j(\lambda^2) \text{grad}_{\bar{g}} e_j(e_j, \lambda^2, e_j)(0_1, f \cdot \text{grad}_f \mu^2) \right) \right)
\]
\[
- \frac{m}{4 s^2} (0_1, \nabla_{\text{grad}_f \mu^2} \text{grad}_f \mu^2 - \frac{1}{\bar{R}_e} f \cdot \text{grad}_f \mu^2)(\mu^2) \text{grad}_f \mu^2 \right)
\]
\[
+ \frac{m}{4 s^2} \text{grad}_f \mu^2(\mu^2) \text{grad}_f \mu^2, 0_2)
\]
\[
+ (\text{Ric}^M(\text{grad}_f f), 0_2) - \frac{m}{4 s^2} |\text{grad}_f \mu|^2 \sum_{j=1}^{m} (\text{grad}_f f, 0_2)
\]
\[
+ \frac{1}{4 s^2} |\text{grad}_f \mu|^2 \sum_{j=1}^{m} g(\text{grad}_f f, e_j)(e_j, 0_2)
\]
\[
= -\frac{m}{8 s^2} f|\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2)
\]
\[
- \frac{m}{4 s^2} f|\Lambda_0(f) - \frac{1}{\bar{R}_e} \sum_{j=1}^{m} (e_j(\lambda^2))^2)|0_1, \cdot \text{grad}_f \mu^2)
\]
\[
- \frac{m}{4 s^2} (0_1, \nabla_{\text{grad}_f \mu^2} \text{grad}_f \mu^2 - \frac{1}{\bar{R}_e} f \cdot \text{grad}_f \mu^2)(\mu^2) \text{grad}_f \mu^2 \right)
\]
\[
+ \frac{m}{4 s^2} \text{grad}_f \mu^2(\mu^2) \text{grad}_f \mu^2, 0_2)
\]
\[
+ (\text{Ric}^M(\text{grad}_f f), 0_2) - \frac{m}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f f, 0_2)
\]
\[
+ \frac{1}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f f, 0_2)
\]

thus (4.4) gives
\[
\tau_f^2(i_0) = f(\text{Tr}_e(\nabla^2) f \text{grad}_f f - \frac{m}{8 s^2} f|\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2)
\]
\[
+ \frac{m}{s x^2} |\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2) + \text{Ric}^M(\text{grad}_f f)
\]
\[
- \frac{m}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2)
\]
\[
- (\nabla_{\text{grad}_f \mu^2} \text{grad}_f \mu^2, 0_2) + \frac{1}{4} |\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2)
\)
\[
+ f(0_1, -\frac{1}{\bar{R}_e} \Lambda_0(f) - \frac{1}{\bar{R}_e} \sum_{a=1}^{m} (\nabla_{\text{grad}_f \mu^2} \text{grad}_f \mu^2)
\]
\[
- \frac{m}{4 s^2} (\nabla_{\text{grad}_f \mu^2} \text{grad}_f \mu^2 - \frac{1}{\bar{R}_e} f \cdot \text{grad}_f \mu^2)(\mu^2) \text{grad}_f \mu^2 \right)
\]
\[
+ \frac{m}{4 s^2} \text{grad}_f \mu^2(\mu^2) \text{grad}_f \mu^2, 0_2)
\]
\[
+ (\text{Ric}^M(\text{grad}_f f), 0_2) - \frac{m}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f f, 0_2)
\]
\[
+ \frac{1}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f f, 0_2)
\]
\[
\text{Therefore, the inclusion map } i_0 : (M, g) \rightarrow (M \times (\mu, g), \bar{g}), \forall y \in M \text{ is a proper } f\text{-harmonic map } (\tau_f^2(i_0) = 0) \text{ if and only if}
\]
\[
f(\text{Tr}_e(\nabla^2) f \text{grad}_f f - \frac{m}{8 s^2} f|\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2)
\]
\[
+ \frac{m}{s x^2} |\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2) + \text{Ric}^M(\text{grad}_f f)
\]
\[
- \frac{m}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f \mu^2, 0_2)
\]
\[
= 0
\]
\[
\text{and}
\]
\[
\frac{1}{4} |\text{grad}_f \mu|^2(\text{grad}_f \mu^2 - \frac{1}{\bar{R}_e} f \cdot \text{grad}_f \mu^2)(\mu^2) \text{grad}_f \mu^2 \right)
\]
\[
- \frac{m}{4 s^2} (\nabla_{\text{grad}_f \mu^2} \text{grad}_f \mu^2 - \frac{1}{\bar{R}_e} f \cdot \text{grad}_f \mu^2)(\mu^2) \text{grad}_f \mu^2 \right)
\]
\[
+ \frac{m}{4 s^2} \text{grad}_f \mu^2(\mu^2) \text{grad}_f \mu^2, 0_2)
\]
\[
+ (\text{Ric}^M(\text{grad}_f f), 0_2) - \frac{m}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f f, 0_2)
\]
\[
+ \frac{1}{4 s^2} |\text{grad}_f \mu|^2(\text{grad}_f f, 0_2)
\]
\[
\text{For inclusion map } i_{\mu_0}, \text{ in the same way, since}
\]
\[
\tau_f(i_{\mu_0}) = -\frac{n}{2} (f \cdot \text{grad}_f \mu^2, 0_2) + (0_1, \text{grad}_f f)
\]
Remark 4.3. We don’t observe that whether $f$-tension field $\tau_f^2(i_{y_0})$ and $\tau_f^2(i_{i_0})$ have solution. If when $f = \text{const.}$ or $\lambda = \text{const.}$ or $\mu = \text{const.}$ can ensure $\tau_f^2(i_{y_0}) = 0$ or $\tau_f^2(i_{i_0}) = 0$, then we can conclude that there never exist proper $f$-biharmonic maps for the inclusion case as same as the result in [SE] and [Lu].

5. $f$-HARMONICITY OF THE SPECIAL PRODUCT MAPS

In this section we shall give a method to construct proper $f$-biharmonic maps of product type.

An obvious verification shows that $\Psi = \text{Id}_M \times \varphi_N : (M \times N, g \oplus h) \rightarrow (M \times N, g \oplus h)$ is a harmonic map if $\text{Id}_M : (M, g) \rightarrow M$ is an identity map and $\varphi_N : (N, h) \rightarrow N$ is a harmonic map. However, with somewhat modification, one can replace the product metric $g \oplus h$ on $M \times N$ (either as domain or codomain) by the doubly warped product metric tensor $\bar{g} = \mu^2g \oplus \lambda^2h$, then the product map may fail to remain harmonic. For this reason, we can find out what condition on $\lambda$ and $\mu$ so that the product map preserves its $f$-biharmonicity.

We first take into account the product map

$$\bar{\Psi} = \text{Id}_M \times \varphi_N : (M^n \times_{(\mu, \lambda)} N^m, \bar{g}) \rightarrow (M \times N, g \oplus h), \bar{\Psi}(x, y) = (x, \varphi_N(y)).$$

We obtain
Theorem 5.1. Let \((M^m, g)\) and \((N^n, h)\) be Riemannian manifolds, and let \(\lambda \in C^\infty(M)\), \(\mu \in C^\infty(M \times N)\) be three positive functions. Suppose that \(\varphi_N : N \rightarrow N\) is a harmonic map. Then the product map \(\overline{\Psi} = \overline{I}_M \times \varphi_N\) defined by (5.1) is a proper \(f\)-biharmonic map if and only if \(\lambda, \mu\) satisfy

\[
\frac{f}{\mu} Tr_{g}(\nabla \varphi_N) \cdot \nabla f - \frac{f}{\mu} \nabla \varphi_N \cdot \nabla f = 0
\]

and

\[
\frac{f}{\mu} Tr_{h}(\nabla \varphi_N) \cdot \nabla f - \frac{f}{\mu} \nabla \varphi_N \cdot \nabla f = 0.
\]

Proof. Let \(\{e_j\}_{j=1}^m\) be an orthonormal frame on \((M, g)\) and \(\{\tilde{e}_a\}_{a=1}^n\) be an orthonormal frame on \((N, h)\). Then \(\{(\frac{1}{2} e_j, 0_2), (0_1, \frac{1}{2} \tilde{e}_a)\}_{j=1,...,m,a=1,...,n}\) is an orthonormal frame on the doubly warped product manifold \(M^m \times_{(\mu, \lambda)} N^n\). By (3.1), together with the assumption \(\varphi_N\) is harmonic, i.e. \(\tau(\varphi_N) = 0\), we have

\[
\tau(\overline{\Psi}) = \sum_{j=1}^m \left( \frac{1}{\mu} \nabla \varphi_N (e_j, 0_2) \cdot \nabla f \right) + \sum_{a=1}^n \left( \frac{1}{\mu} \nabla \varphi_N (\tilde{e}_a, 0_2) \cdot \nabla f \right) + \frac{1}{\mu} \nabla \varphi_N \cdot \nabla f
\]

and

\[
\tau_f(\overline{\Psi}) = \frac{f}{\mu} \nabla \varphi_N (e_j, 0_2) \cdot \nabla f + \frac{f}{\mu} \nabla \varphi_N \cdot \nabla f
\]

Since from (2.6) the \(f\)-biharmonic map of \(\overline{\Psi}\) is

\[
\tau^2_f(\overline{\Psi}) = f \left| T_{\overline{\Psi}}(\nabla \overline{\Psi}) \right|^2 + Tr_{g}(\nabla \varphi_N) \cdot \nabla f + Tr_{h}(\nabla \varphi_N) \cdot \nabla f - \nabla f \varphi_N \cdot \nabla f
\]

we in the position computer the following terms:

\[
\nabla \overline{\Psi} (\overline{\partial}_j, 0_2) \cdot \nabla f = \frac{1}{\mu} \nabla \varphi_N (\overline{\partial}_j, 0_2) \cdot \nabla f
\]

and

\[
\nabla \overline{\Psi} (0_1, \overline{\partial}_j, 0_2) \cdot \nabla f = \frac{1}{\mu} \nabla \varphi_N (0_1, \overline{\partial}_j, 0_2) \cdot \nabla f
\]
From (5.7), (5.9)–(5.11), we have

\[ \frac{1}{\lambda} \nabla_{\lambda_i} \nabla_{\lambda_j} \phi \nabla_{\lambda_k} \phi \nabla_{\lambda_l} \phi = 0, \]

and similarly

\[ \sum_{j=1}^{m} R((\frac{1}{\lambda} e_j, 0), \nabla_{\Lambda_j} \nabla_{\Lambda_k} \phi \nabla_{\Lambda_l} \phi e_j, 0) = \frac{1}{\lambda} \sum_{j=1}^{m} \nabla_{\Lambda_j} \nabla_{\Lambda_k} \phi \nabla_{\Lambda_l} \phi e_j, 0) + R(e_j, \nu) f \nabla_{\Lambda_j} \phi \nabla_{\Lambda_k} \phi \nabla_{\Lambda_l} \phi e_j, 0) \]

and similarly

\[ \sum_{a=1}^{m} R((0, d\phi_N(e_{\nu_a}), \nabla_{\Lambda_j} \nabla_{\Lambda_k} \phi \nabla_{\Lambda_l} \phi e_{\nu_a}) = \frac{1}{\lambda} \sum_{a=1}^{m} R^N(d\phi_N(e_{\nu_a}), f \nabla_{\Lambda_j} \phi \nabla_{\Lambda_k} \phi \nabla_{\Lambda_l} \phi e_{\nu_a}) + R((0, d\phi_N(e_{\nu_a}), f \nabla_{\Lambda_j} \phi \nabla_{\Lambda_k} \phi \nabla_{\Lambda_l} \phi e_{\nu_a}) \]
Therefore, by $\tau_{\mathcal{J}}^2(\Psi) = 0$ we conclude the statement. \hfill \Box

In order to find the relation between product map and the projection map, we first investigate the $f$-bitemension field of the projection and give the following lemmas.

**Lemma 5.2.** Let the projection $\tilde{\pi}_1 : (M^m \times_{(\varphi,\bar{\varphi})} N^n, \tilde{g}) \rightarrow (M^m, g)$, $\tilde{\pi}_1(x, y) = x$, be a doubly warped product onto its first factor and let $f : M \times N \rightarrow (0, +\infty)$ be smooth function. Then the $f$-bitemension field of $\tilde{\pi}_1$ is

$$
\tau_{\mathcal{J}}^2(\tilde{\pi}_1) = \frac{1}{\mu} \frac{\partial}{\partial \varphi} \left. \left( \mu \nabla^2 \right) f \cdot \nabla_{\mu \varphi} \log(f \lambda^n) \right| \tilde{\pi}_1
$$

Proof. By a similar calculation as (5.4) and (5.5), we have

$$
\tau(\tilde{\pi}_1) = \frac{\partial}{\partial \varphi} \left. \left( \mu \nabla^2 \right) f \cdot \nabla_{\mu \varphi} \log(f \lambda^n) \right| \tilde{\pi}_1
$$

Since

$$
\nabla_{(\varphi,\rho)}^2 \tau_f(\tilde{\pi}_1) = \frac{1}{\mu} \left. \left( \mu \nabla^2 \right) f \cdot \nabla_{\mu \varphi} \log(f \lambda^n) \right| \tilde{\pi}_1,
$$

we have

$$
\nabla_{(\varphi,\rho)} f(\tilde{\pi}_1) = \frac{1}{\mu} \left. \left( \mu \nabla^2 \right) f \cdot \nabla_{\mu \varphi} \log(f \lambda^n) \right| \tilde{\pi}_1
$$

for $\mu(\nabla^2 f) = \frac{\partial}{\partial \varphi} \left. \left( \mu \nabla^2 \right) f \cdot \nabla_{\mu \varphi} \log(f \lambda^n) \right| \tilde{\pi}_1$.\hfill \Box
We get
\begin{equation}
\text{(5.26)}
\end{equation}
On the other hand, since
\begin{equation}
\text{(5.24)}
\end{equation}
we have
\begin{equation}
\text{(5.25)}
\end{equation}
in Theorem 5.1 and Lemma 5.2, we easily obtain
\begin{equation}
\text{(5.23)}
\end{equation}

Thus we end the proof.

By a similar discussion for the projection \(\bar{\pi}_2 : M \times_{(\mu, h)} N \to N\), we have

**Lemma 5.3.** Let the projection \(\bar{\pi}_2 : (M^n \times_{(\mu, h)} N^n, \bar{g}) \to (N^n, h)\), \(\bar{\pi}_1(x, y) = y\), be a doubly warped product onto its second factor and let \(f : M \times N \to (0, +\infty)\) be smooth function. Then the \(f\)-bitension field of \(\bar{\pi}_2\) is

\begin{equation}
\text{(5.27)}
\end{equation}

Observe the \(f\)-bitensions in Theorem 5.1 and Lemma 5.2 we easily obtain

**Corollary 5.4.** The product map \(\bar{\varphi} : M^n \times_{(\mu, h)} N^n \to M \times N\), \(\bar{\varphi}(x, y) = (x, \varphi_N(y))\) with harmonic map \(\varphi\) is a proper \(f\)-harmonic map with \(f \in C^\infty(M \times N)\) if and only if the projections \(\bar{\pi}_1\) is a proper \(f\)-biharmonic map and

\begin{equation}
\text{(5.28)}
\end{equation}

Now we proceed to treat the product map \(\bar{\varphi} = \varphi_M \times h : (M^n \times_{(\mu, h)} N^n, g) \to (M \times N, g \oplus h)\), \(\bar{\varphi}(x, y) = (\varphi_M(x), y)\).
Theorem 5.5. With the same notions of $\lambda$, $\mu$, $f$ as stated in Theorem 5.1, the product map $\bar{\Psi} = \varphi_M \times \text{Id}_N$ defined by \(5.29\) is a proper $f$-harmonic map if and only $\lambda$, $\mu$ satisfy
\[
\frac{1}{\mu_0} Tr_h \left( (\nabla_{\psi_0}^2)^2 f \cdot \varphi_M (\text{grad}_{\psi_0} \log(f \lambda)) \right) + \mu_0 \nabla \varphi_M (\text{grad}_{\psi_0} \log(f \lambda)) d\varphi_M (f, \mu_0) - \frac{1}{\mu_0} Tr_h R^M (d\varphi_M, \text{grad}_{\psi_0} \log(f \lambda)) d\varphi_M (f, \mu_0) = 0
\]
and the projections $\tilde{\pi_2}$ is a proper $f$-biharmonic map, that is
\[
\tau_f^2(\tilde{\pi_2}) = \frac{1}{\mu_0} Tr_h \left( (\nabla_{\psi_0}^2)^2 f \cdot \varphi_M (\text{grad}_{\psi_0} \log(f \mu_0)) \right) + \frac{1}{\mu_0} Tr_h R^N (d\varphi_M, \text{grad}_{\psi_0} \log(f \mu_0)) d\varphi_M (f, \mu_0) = 0.
\]

In the remainder of this section, we turn to consider the case of the product map $\bar{\Psi} = \text{Id}_M \times \varphi_N : (M \times N, g \oplus h) \rightarrow (M^n \times_{(\mu, \lambda)} N^n, \bar{g})$, $\bar{\Psi}(x, y) = (x, \varphi_N(y))$; that is the case of the product metric on the codomain is doubly warped metric. We will see that the harmonic energy density $e(\varphi_N)$ has an important role for the $f$-biharmonicity of the product map $\bar{\Psi} = \text{Id}_M \times \varphi_N$.

Let $\{(e_j, 0), (0, 1), \cdots, (0, 1)\}$ is an local orthonormal frame on the usual product manifold $M^n \times N^n$, where $\{e_j\}_{j=1}^n$ and $\{\bar{e}_a\}_{a=1}^m$ respectively denote orthonormal frames on $(M, g)$ and on $(N, h)$. By $\tau(\varphi_N) = 0$, the tension field of $\bar{\Psi}$ is
\[
\tau(\bar{\Psi}) = Tr_{g_{\bar{\Psi}}} \nabla d\bar{\Psi} = \sum_{j=1}^m (\nabla_{\psi_0} (e_j, 0, 0, 0) d\bar{\Psi}(e_j, 0) - d\bar{\Psi}(\nabla_{\psi_0} (e_j, 0, 0, 0)))
\]
\[
+ \sum_{a=1}^n (\nabla_{Id_M \times \delta_{\varphi_N}} (0, \bar{e}_a) d\varphi_N (0, 1) - Id_M \times \varphi_N (\nabla_{0, \bar{e}_a} (0, 1)))
\]
\[
= -\frac{2m}{\mu_0} (0, \text{grad}_{\psi_0} \mu_0) + (0, \tau(\varphi_N)) - \frac{1}{2} \sum_{a=1}^n \varphi_N^* h(\bar{e}_a, \bar{e}_a) (\text{grad}_{\psi_0} \lambda^2, 0)
\]
\[
= (0, -\frac{1}{\mu_0} \text{grad}_{\psi_0} \mu_0^2) - (e(\varphi_N) \text{grad}_{\psi_0} \lambda^2, 0),
\]
thus we get
\[
\tau_f(\bar{\Psi}) = (0, -\frac{2}{\mu_0} \text{grad}_{\psi_0} \mu_0^2) + (- e(\varphi_N) f \text{grad}_{\psi_0} \lambda^2, 0)
\]
\[
+ Id_M \times \varphi_N (\text{grad}_{\psi_0} f, \text{grad}_h f)
\]
\[
= (0, -\frac{2}{\mu_0} \text{grad}_{\psi_0} \mu_0^2 + \varphi_N (\text{grad}_h f))
\]
\[
+ (- e(\varphi_N) f \text{grad}_{\psi_0} \lambda^2 + \text{grad}_h f, 0).
\]

Since from \(2.6\) the f-biharmonic map of $\bar{\Psi}$ is
\[
\tau_f^2(\bar{\Psi}) = f [Tr_{g_{\bar{\Psi}}} (\nabla^2 \bar{\Psi})_f \tau_f(\bar{\Psi}) + \bar{R}(d\bar{\Psi}, \tau_f(\bar{\Psi})) d\bar{\Psi}] - \nabla^2 \bar{\Psi} \tau_f(\bar{\Psi})
\]
\[
= f \sum_{j=1}^m \left( [\nabla^2_{(0, e_j)} \nabla^2 \bar{\Psi}(0, e_j) - \nabla^2 \bar{\Psi}(0, e_j)] \tau_f(\bar{\Psi}) + \bar{R}(e_j, 0) \tau_f(\bar{\Psi})(e_j, 0) \right)
\]
\[
+ f \sum_{a=1}^n \left( [\nabla^2_{(0, \bar{e}_a)} \nabla^2 \bar{\Psi}(0, \bar{e}_a) - \nabla^2 \bar{\Psi}(0, \bar{e}_a)] \tau_f(\bar{\Psi})(0, \bar{e}_a) + \bar{R}(0, 1) \tau_f(\bar{\Psi})(0, 1) \right)
\]
\[
= -1/\mu_0 \text{grad}_{\psi_0} \mu_0^2 + \varphi_N (\text{grad}_h f) + (- e(\varphi_N) f \text{grad}_{\psi_0} \lambda^2 + \text{grad}_h f, 0).
\]
we in the position computer the following terms:

\[
\nabla_{(e,f)}^\Phi \frac{\tau_f}{(\varphi_f)} = \begin{align*}
\n\nabla_{(e,f)}^{\Phi} & (0_1, f \varphi_N(\text{grad}^\lambda \lambda^2)) + \nabla_{(e,f)}^{\Phi} (0_1, \text{grad}^\lambda \lambda^2) \\
& - \nabla_{(e,f)}^{\Phi} (e(\varphi_N)(\text{grad}^\lambda \lambda^2), \omega_2) + \nabla_{(e,f)}^{\Phi} (\text{grad} \lambda^2, \omega_2) \\
& = -\frac{mf}{4\mu^2} e^2(\varphi_N) \text{grad}^\lambda \lambda^2, \omega_2) + \frac{mf}{4\mu^2} (\varphi_N(\text{grad}^\lambda \lambda^2)(\mu^2)) \langle e, \omega_2 \rangle \\
& + \frac{1}{2\mu^2} \nabla_\omega \nabla_\varphi \nabla^\lambda \varphi (\mu^2) (\mu_2) \langle e, \omega_2 \rangle \\
& - \frac{1}{2} g(\varphi_N, \omega_2) \text{grad}^\lambda \lambda^2, \omega_2) \\
& + \frac{1}{2} g(\varphi_N, \omega_2) \text{grad}^\lambda \lambda^2, \omega_2) \langle e, \omega_2 \rangle.
\end{align*}
\]

\[(5.36)\]

\[
\n\nabla_{(e,f)}^{\Phi} (\tau_f(\varphi_f)) = \nabla_{(e,f)}^{\Phi} (\omega_2, \omega_2) \frac{\tau_f}{(\varphi_f)} \nabla_{(e,f)}^{\Phi} (\omega_2, \omega_2) + \nabla_{(e,f)}^{\Phi} (\omega_2, \omega_2) \\
& - \frac{mf}{4\mu^2} (\varphi_N(\text{grad}^\lambda \lambda^2)) (\mu^2) (\mu_2) \langle e, \omega_2 \rangle \\
& + \frac{1}{2\mu^2} \nabla_\omega \nabla_\varphi \nabla^\lambda \varphi (\mu^2) (\mu_2) \langle e, \omega_2 \rangle \\
& - \frac{1}{2} g(\varphi_N, \omega_2) \text{grad}^\lambda \lambda^2, \omega_2) \\
& + \frac{1}{2} g(\varphi_N, \omega_2) \text{grad}^\lambda \lambda^2, \omega_2) \langle e, \omega_2 \rangle.
\]

\[\]
(5.37) 
\[
\nabla^\nabla_\epsilon(e_j, e_2) \nabla^\nabla_\epsilon(e_j, e_2) j_f(\nabla) = -\frac{mf}{8\mu^2} (e_j, e_2)(0, d\varphi_N(\nabla^{\mu^2} e_2)) \\
- \frac{mf}{8\mu^2} e_j(e_2)(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) \\
- \frac{m}{4\mu^2} (\nabla^\epsilon e_j(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) + \frac{mf}{8\mu^2} (d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) \\
+ \frac{1}{4\mu^2} (e_j(e_2))((0, d\varphi_N(\nabla^{\mu^2} e_2)) + \frac{1}{4\mu^2} e_j(e_2)((d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) \\
+ \frac{1}{2\mu^2} (\nabla^\epsilon e_j(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) - \frac{1}{2\mu^2} (d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) \\
- (\nabla^\epsilon e_j(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) + \frac{1}{2\mu^2} g(e_j, \nabla^\epsilon e_j(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) \\
+ \frac{1}{4\mu^2} g(e_j, d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) \\
+ (\nabla^\epsilon e_j(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)) - \frac{1}{2\mu^2} g(e_j, \nabla^\epsilon e_j(d\varphi_N(\nabla^{\mu^2} e_2))((e_2, e_2)).
\]

(5.38) 
\[
\nabla^\nabla_{(\epsilon, \eta)}(\epsilon, \eta) j_f(\nabla) = -\frac{m}{2} (0, \nabla^\epsilon d\varphi_N(\nabla^{\mu^2} e_2)) \\
+ \frac{mf}{4} h(d\varphi_N(\nabla^{\mu^2} e_2), d\varphi_N(\nabla^{\mu^2} e_2))((\nabla^{\mu^2} e_2, e_2)) \\
+ (0, \nabla^\epsilon d\varphi_N(\nabla^{\mu^2} e_2))((d\varphi_N(\nabla^{\mu^2} e_2))((\nabla^{\mu^2} e_2, e_2)) - \frac{1}{2\mu^2} h(d\varphi_N(\nabla^{\mu^2} e_2), d\varphi_N(\nabla^{\mu^2} e_2))((\nabla^{\mu^2} e_2, e_2)) \\
- \frac{1}{2\mu^2} g(\nabla^{\mu^2} e_2)((\nabla^{\mu^2} e_2, e_2)) + \frac{1}{2\mu^2} g(\nabla^{\mu^2} e_2)((\nabla^{\mu^2} e_2, e_2). 
\]
On the other hand, since $m = \sum_j j$ (5.40),

$$
\begin{align*}
\frac{1}{2} & (1, |\mathbf{\nabla}_{\text{grad}} e \chi(e_0, d\varphi_N(\text{grad}_{\text{grad}} e \chi^2)) (0, 0, \text{grad}_{\text{grad}} f) ) \\
+ & \frac{1}{4} \left( (e(\varphi_N))^2 f |\mathbf{\nabla}_{\text{grad}} e \chi^2 (\text{grad}_{\text{grad}} e \chi^2, 0_2) \right) \\
- & \frac{1}{4} (\text{grad}_f f (e^2_2) \text{grad}_f \varphi_N(e_0)) \\
+ & \frac{1}{4} \left( \text{grad}_f f (e^2_2) \text{grad}_f \varphi_N(e_0) \right) (0_1, 0_2) \\
+ & \frac{1}{4} \left( \text{grad}_f f (e^2_2) \text{grad}_f \varphi_N(e_0) \right) (0_1, 0_2) \\
= & \sum_{j=1}^m \left( \mathbf{\nabla}_{\text{grad}} e \chi(e_0, d\varphi_N(\text{grad}_{\text{grad}} e \chi^2)) (e_j, 0_2) + \mathbf{\nabla}_{\text{grad}} e \chi(e_0, d\varphi_N(\text{grad}_{\text{grad}} f)) (e_j, 0_2) \\
+ & \frac{1}{4} \left( \text{grad}_f f (e^2_2) \text{grad}_f \varphi_N(e_0) \right) (e_j, 0_2) + \frac{1}{4} \left( \text{grad}_f f (e^2_2) \text{grad}_f \varphi_N(e_0) \right) (e_j, 0_2) \\
\right)
\end{align*}
$$

From (5.36), (5.37) - (5.39), we get the complex expression of $\mathbf{\nabla}_{\text{grad}} e \chi(\mathbf{\nabla}_{\text{grad}} e \chi) \frac{\partial}{\partial f}(\mathbf{\nabla}_{\text{grad}} e \chi)$. On the other hand, since
\begin{align*}
&= \frac{1}{4\lambda^2} f(\varphi_N(\text{grad}_\varphi h, \mu^2)(\mu^2)) \sum_{j=1}^{m} g(\text{grad}_\mu \lambda^2, e_j)(e_j, 0_2) \\
&\quad + \mu^2 \frac{1}{2\lambda^2} \sum_{j=1}^{m} g(\mathcal{N}_{e_j} \text{grad}_\mu \lambda^2, \lambda^2 - \frac{1}{2\lambda^2} e_j(\lambda^2) \text{grad}_\mu \lambda^2, e_j)(0_1, f \varphi_N(\text{grad}_\varphi h, \mu^2)) \\
&\quad + \frac{m f}{2} (0_1, \mathcal{N}_{\text{grad}_\varphi h}(\mu^2) \text{grad}_\varphi h, \mu^2 - \frac{1}{2\mu^2} f(\varphi_N(\text{grad}_\varphi h, \mu^2)(\mu^2)) \text{grad}_\varphi h, \mu^2) \\
&\quad - \frac{m f}{4\lambda^2} \varphi_N(\text{grad}_\varphi h, \mu^2)(\mu^2)(\text{grad}_\mu \lambda^2, 0_2) \\
&\quad + \frac{1}{4\lambda^2} (\varphi_N(\text{grad}_\varphi h f)(\mu^2) \sum_{j=1}^{m} g(\text{grad}_\mu \lambda^2, e_j)(e_j, 0_2) \\
&\quad + \mu^2 \frac{1}{2\lambda^2} \sum_{j=1}^{m} g(\mathcal{N}_{e_j} \text{grad}_\mu \lambda^2, \lambda^2 - \frac{1}{2\lambda^2} e_j(\lambda^2) \text{grad}_\mu \lambda^2, e_j)(0_1, \varphi_N(\text{grad}_\varphi h f)) \\
&\quad + \frac{m}{2} (0_1, \mathcal{N}_{\text{grad}_\varphi h f}(\mu^2) \text{grad}_\varphi h f, \mu^2 - \frac{1}{2\mu^2} f(\varphi_N(\text{grad}_\varphi h f)(\mu^2) \text{grad}_\varphi h f) \\
&\quad - \frac{m}{4\lambda^2} \varphi_N(\text{grad}_\varphi h f)(\mu^2)(\text{grad}_\mu \lambda^2, 0_2) \\
&\quad + (Ric^M(e(\varphi_N) f \text{grad}_\varphi \lambda^2), 0_2) - \frac{m}{4\mu^2} e(\varphi_N) f(\text{grad}_\varphi h f)(\text{grad}_\mu \lambda^2, (\mu^2) \text{grad}_\mu \lambda^2, 0_2) \\
&\quad + \frac{1}{4\mu^2} e(\varphi_N) f(\text{grad}_\mu \lambda^2)^2 \sum_{j=1}^{m} g(\text{grad}_\mu \lambda^2, e_j)(e_j, 0_2) \\
&\quad + (Ric^M(\text{grad}_\varphi f, 0_2) - \frac{m}{4\mu^2} \text{grad}_\varphi h f)(\text{grad}_\varphi f, 0_2) \\
&\quad + \frac{1}{4\mu^2} \text{grad}_\varphi h f)(\text{grad}_\varphi f, 0_2) \end{align*}
\[
\sum_{a=1}^{n} \left[ \mathcal{R}(\{(0, \bar{d} \varphi_{N}(\bar{\xi}_{a})), (0, f \varphi_{N}(\nabla_{\xi} \bar{\mu}^2))\}(0, \bar{d} \varphi_{N}(\bar{\xi}_{a}))) \\
+ \mathcal{R}((0, \bar{d} \varphi_{N}(\bar{\xi}_{a})), (0, \bar{d} \varphi_{N}(\nabla_{\bar{f}} \bar{\mu}))))(0, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ \mathcal{R}((0, \bar{d} \varphi_{N}(\bar{\xi}_{a})), \bar{\mu} \varphi_{N}(\nabla_{\bar{\xi}_{a}}^{2}, 0_2))(0, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ \mathcal{R}((0, \bar{d} \varphi_{N}(\bar{\xi}_{a})), (\bar{\mu} \varphi_{N}(\bar{\xi}_{a})), (0, \bar{d} \varphi_{N}(\bar{\xi}_{a}))) \right]
\]

\[
= (0_1, Tr_{\bar{f}}(R^{2}(\bar{d} \varphi_{N}, \bar{d} \varphi_{N}(\nabla_{\bar{\xi}} \bar{\mu}^2))) - \frac{1}{4\mu^2} e(\varphi_{N}) \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^{2}(0_1, \bar{d} \varphi_{N}(\nabla_{\bar{f}} \bar{\mu}^2)) \\
+ \frac{1}{4\mu^2} \sum_{a=1}^{n} h(\bar{d} \varphi_{N}(\nabla_{\bar{\xi}} \bar{\mu}^2), \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ (0_1, Tr_{\bar{f}}(R^{2}(\bar{d} \varphi_{N}, \bar{d} \varphi_{N}(\nabla_{\bar{\xi}} \bar{\mu}^2))) - \frac{1}{4\mu^2} e(\varphi_{N}) \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^{2}(0_1, \bar{d} \varphi_{N}(\nabla_{\bar{f}} \bar{\mu}^2)) \\
+ \frac{1}{4\mu^2} \sum_{a=1}^{n} h(\bar{d} \varphi_{N}(\nabla_{\bar{\xi}} \bar{\mu}^2), \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ \frac{\lambda^2}{2\mu^2} \sum_{a=1}^{n} h(\nabla_{d \varphi_{N}(\bar{\xi}_{a})} \nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2 \\
- \frac{1}{2\mu^2} e(\varphi_{N}) f h(\nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^2(\bar{\mu}^2) \nabla \bar{\varphi}_{N}(\bar{\xi}_{a}), \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ \frac{1}{4\mu^2} \sum_{a=1}^{n} h(\nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2, \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ (e(\varphi_{N})^2 f \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^2(g^{2} \nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2)^2 \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})) - \frac{1}{2\mu^2} \sum_{a=1}^{n} h(\nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2, \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
- \frac{1}{2\mu^2} e(\varphi_{N}) f \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^2(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})^2 \bar{\mu}^2)) \\
+ \frac{\lambda^2}{2\mu^2} \sum_{a=1}^{n} h(\nabla_{d \varphi_{N}(\bar{\xi}_{a})} \nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2 \\
- \frac{1}{2\mu^2} e(\varphi_{N}) f h(\nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^2(\bar{\mu}^2) \nabla \bar{\varphi}_{N}(\bar{\xi}_{a}), \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ \frac{1}{4\mu^2} \sum_{a=1}^{n} h(\nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2, \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
+ e(\varphi_{N}) g^{2} \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^2(g^{2} \nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2)^2 \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})) - \frac{1}{2\mu^2} \sum_{a=1}^{n} h(\nabla \bar{\varphi}_{N}(\bar{\xi}_{a}) \bar{\xi}_{a}^2 \bar{\mu}^2, \bar{d} \varphi_{N}(\bar{\xi}_{a}))(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})) \\
- \frac{1}{2\mu^2} e(\varphi_{N}) f \nabla \bar{\varphi}_{N}(\bar{\xi}_{a})^2(0_1, \bar{d} \varphi_{N}(\bar{\xi}_{a})^2 \bar{\mu}^2)),
\]
noting that
\[
\sum_{j=1}^{m} g(\text{grad}_{\mu^2} \lambda^2, e_j)(e_j, 0_2) = \sum_{j=1}^{m} e_j(\lambda^2)(e_j, 0_2) = (\text{grad}_{\mu^2} \lambda^2, 0_2) \epsilon
\]
\[
\sum_{\alpha=1}^{n} h(d\varphi_\alpha(\text{grad}_{\mu} f, d\varphi_\alpha(\bar{\epsilon}_\alpha))(0_1, d\varphi_\alpha(\bar{\epsilon}_\alpha)) = (0_1, d\varphi_\alpha(\text{grad}_{\mu} f))??
\]
thus (5.35) gives
\[
\tau^2_j(\Phi) = (A, 0_2) + (0_1, B)
\]
However, the terms "A" and "B" in (5.42) all include many complex terms.

**Remark 5.6.** Unfortunately, we cannot further simplified \(\tau^2_j\), so we don’t obtain very interesting result when \(\tau^2_j = 0\). In fact, even though the case for biharmonic map or \(f\)-harmonic map, we still have nothing good results, see [SE] and [Lu].

6. **f-Biharmonicity of the General Product Maps**

In this subsection, we consider the general product case, i.e., we modify the identity map \(Id_M\) to a harmonic map \(\varphi_M : (M, g) \rightarrow M\) in Theorem 5.1 then we obtain the following result.

**Theorem 6.1.** Let \((M^n, g)\) and \((N^m, h)\) be Riemannian manifolds, and let \(\lambda \in C^\infty(M), \mu \in C^\infty\) and \(f \in C^\infty(M \times N)\) be three positive functions. Suppose that \(\varphi_M : (M, g) \rightarrow M, \varphi_N : N \rightarrow N\) are two harmonic maps. Then the product map \(\Phi = \varphi_M \times \varphi_N : (M^n \times (\mu, \lambda)) \rightarrow (M \times N, g \oplus h)\) defined by \(\Phi(x, y) = (\varphi_M(x), \varphi_N(y))\) is a proper \(f\)-harmonic map if and only if \(\lambda\) and \(\mu\) satisfy

\[
\frac{\mu}{\mu} Tr_g (\nabla^M_{\mu^2} \lambda^2) f d\varphi_M (\text{grad}_{\mu^2} \log(f\lambda^m))
\]
\[
+ \frac{\mu}{\mu} Tr_g (\nabla^N_{\mu^2} \lambda^2) f d\varphi_M (\text{grad}_{\mu^2} \log(f\lambda^m))
\]
\[
+ f n (\nabla^M_{\mu^2} \lambda^2) \cdot d\varphi_M (\text{grad}_{\mu^2} \log(f\lambda^m))
\]
\[
- n (\nabla^M_{\mu^2} \lambda^2) \cdot d\varphi_M (\text{grad}_{\mu^2} \log(f\lambda^m)) = 0
\]
and

\[
\frac{\mu}{\mu} Tr_h (\nabla^N_{\mu^2} \lambda^2) f d\varphi_N (\text{grad}_{\mu^2} \log(f\mu^m))
\]
\[
+ f n (\nabla^N_{\mu^2} \lambda^2) \cdot d\varphi_N (\text{grad}_{\mu^2} \log(f\mu^m))
\]
\[
+ \frac{\mu}{\mu} Tr_h (\nabla^N_{\mu^2} \lambda^2) f d\varphi_N (\text{grad}_{\mu^2} \log(f\mu^m))
\]
\[
- n (\nabla^N_{\mu^2} \lambda^2) \cdot d\varphi_N (\text{grad}_{\mu^2} \log(f\mu^m)) = 0.
\]

**Proof.** By a similar calculation as (5.33) and (5.34), together with the assumption \(\tau(\varphi_M) = \tau(\varphi_N) = 0\), we have

\[
\tau(\Phi) = \sum_{j=1}^{m} \left( \frac{\mu}{\mu} \nabla_{\mu^2} (e_j, 0_2) - \frac{\mu}{\mu} d\Phi(\nabla_{\mu^2} (e_j, 0_2)) \right)
\]
\[
+ \sum_{\alpha=1}^{n} \left( \frac{\mu}{\mu} \nabla_{\mu^2} (\varphi_M(0_1, \bar{\epsilon}_\alpha), \bar{\epsilon}_\alpha) - \frac{\mu}{\mu} d\varphi_M(\nabla_{\mu^2} (\varphi_M(0_1, \bar{\epsilon}_\alpha), \bar{\epsilon}_\alpha)) \right)
\]
\[
= \frac{\mu}{\mu} (d\varphi_M(\tau(\varphi_M)), 0_2) + n(d\varphi_M(\text{grad}_{\mu^2} \log(\lambda)), 0_2)
\]
\[
+ m(0_1, d\varphi_N(\text{grad}_{\mu^2} \log(\mu))) + \frac{\mu}{\mu} (0_1, d\varphi_N(\tau(\varphi_N)))
\]
\[
= n(d\varphi_M(\text{grad}_{\mu^2} \log(\lambda)), 0_2) + m(0_1, d\varphi_N(\text{grad}_{\mu^2} \log(\mu))).
\]
we in the position computer the following terms:

\[ \text{(6.6)} \]

\[ \text{(6.8)} \]

\[ \text{(6.10)} \]

From (6.6), (6.8)-(6.10), we have

\[ \begin{align*}
\Phi_{\bar{g}} & = (f d\Phi_M(\text{grad}_{\mu^2} \log \lambda), 0_2) + (0_1, f d\Phi_N(\text{grad}_{\lambda} \log \mu)) \\
& + d\Phi_M \times d\Phi_N(\text{grad}_{\mu^2} \lambda \cdot \text{grad}_{\lambda} \mu) \\
& = (\text{grad}_{\mu^2} \Phi_M(\text{grad}_{\mu^2} \log \lambda) + d\Phi_M(\text{grad}_{\mu^2} \mu), 0_2) \\
& + (0_1, f d\Phi_N(\text{grad}_{\lambda} \log \mu) + d\Phi_N(\text{grad}_{\lambda} \mu)) \\
& = (f d\Phi_M(\text{grad}_{\mu^2} \log f \lambda^m), 0_2) + (0_1, f d\Phi_N(\text{grad}_{\lambda} \log f \mu^m)).
\end{align*} \]
On the other hand, since
\[ \sum_{j=1}^{m} R(d\varphi_M(e_j), 0_2), \tau_f(\bar{\Phi})) (d\varphi_M(e_j), 0_2) \]
\[ = \sum_{j=1}^{m} (\nabla_{(d\varphi_M(e_j), 0_2)} \nabla_{(f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu))}, 0_2) (d\varphi_M(e_j), 0_2) \]
\[ - \nabla_{(f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu))} \nabla_{(d\varphi_M(e_j), 0_2)} (d\varphi_M(e_j), 0_2) \]
\[ - \nabla_{(d\varphi_M(e_j), 0_2)} (f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu)) (d\varphi_M(e_j), 0_2) \]
\[ + R((d\varphi_M(e_j), 0_2), (0_1, f d\varphi_N(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_M(e_j), 0_2)) \]
\[ = (\sum_{j=1}^{m} R^M(d\varphi_M(e_j), f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_M(e_j), 0_2) \]
\[ = f(Tr_f R^M(d\varphi_M, d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_M, 0_2), \]
and similarly
\[ \sum_{a=1}^{n} R((0_1, d\varphi_N(\bar{\varphi}_a)), \tau_f(\bar{\Phi}))(0_1, d\varphi_N(\bar{\varphi}_a)) \]
\[ = (0_1, \sum_{a=1}^{n} R^N(d\varphi_N(\bar{\varphi}_a), f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_N(\bar{\varphi}_a)) \]
\[ = f(0_1, Tr_f R^N(d\varphi_N, f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_N), \]
(6.6) gives
\[ \tau_f^2(\bar{\Phi}) = \left( \frac{1}{n} Tr_f (\nabla_{d\varphi_N})^2 f d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) \right) \]
\[ + \frac{1}{n} Tr_f R^M(d\varphi_M, d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_M \]
\[ + m \nabla_{d\varphi_N(\varphi^{\mu}_M \log(f \cdot \mu^\nu))} f \cdot d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) \]
\[ - m \nabla_{(d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)))} f d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) \]
\[ + m \nabla_{d\varphi_N(\varphi^{\mu}_M \log(f \cdot \mu^\nu))} f d\varphi_M(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) \]
\[ = \left( \frac{1}{n} Tr_f (\nabla_{d\varphi_N})^2 f d\varphi_N(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) \right) \]
\[ + \frac{1}{n} Tr_f R^N(d\varphi_N, f \cdot \varphi^{\mu}_M \log(f \cdot \mu^\nu)) d\varphi_N \]
\[ - m \nabla_{d\varphi_N(\varphi^{\mu}_M \log(f \cdot \mu^\nu))} f d\varphi_N(\varphi^{\mu}_M \log(f \cdot \mu^\nu)) \]
\[ = 0 \]
Therefore, by \( \tau_f^2(\bar{\Phi}) = 0 \) we conclude the statement. \( \square \)

By using the \( f \)-tension fields (5.16) and (5.27) of the projections \( \bar{\pi}_1 \) and \( \bar{\pi}_2 \), (6.14) has the expression
\[ \tau_f(\bar{\Phi}) = f(d\varphi_M(\tau_f(\bar{\pi}_1)), d\varphi_N(\tau_f(\bar{\pi}_2))). \]
So we have

**Corollary 6.2.** The product map \( \bar{\Phi} = \bar{\varphi}_M \times \bar{\varphi}_N : M^m \times (\mu, \lambda) N^n \to M \times N, \bar{\Phi}(x, y) = (\varphi_M(x), \varphi_N(y)) \) with harmonic maps \( \varphi_M \) and \( \varphi_N \) is a proper \( f \)-harmonic map if the projections \( \bar{\pi}_1 \) and \( \bar{\pi}_2 \) are all proper \( f \)-harmonic maps.

**References**


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