CURVATURES OF DIRECT IMAGE SHEAVES OF VECTOR BUNDLES AND APPLICATIONS[*]

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ABSTRACT. Let $p : \mathcal{X} \to S$ be a smooth Kähler fibration and $\mathcal{E} \to \mathcal{X}$ a Hermitian holomorphic vector bundle. As motivated by the work of Berdtsson([Bern09]), by using basic Hodge theory, we derive several general curvature formulas for the direct image $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ for general Hermitian holomorphic vector bundle $\mathcal{E}$ in a very simple way. A straightforward application is that, if the Hermitian vector bundle $\mathcal{E}$ is Nakano-negative along the base S, then the direct image $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-negative. We also use these curvature formulas to study the moduli space of projectively flat vector bundles with positive first Chern classes and obtain that, if the Chern curvature of direct image $p_*(K_X \otimes E)$–of a positive projectively flat family $(E, h(t))_{t \in \mathbb{D}} \to X$–vanishes, then the curvature forms of this family are connected by holomorphic automorphisms of the pair (X, E).

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1. INTRODUCTION

Let $\mathcal{X}$ be a Kähler manifold with dimension $m+n$ and S a Kähler manifold with dimension m. Let $p : \mathcal{X} \to S$ be a smooth Kähler fibration. That means, for each $s \in S$,

$$X_s := p^{-1}(\{s\})$$

is a compact Kähler manifold with dimension n. Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be a Hermitian holomorphic vector bundle. Consider the space of holomorphic $\mathcal{E}$-valued $(n,0)$-forms on X_s,

$$E_s := H^0(X_s, \mathcal{E}_s \otimes K_{X_s}) \cong H^{n,0}(X_s, \mathcal{E}_s)$$

where $\mathcal{E}_s = \mathcal{E}|_{X_s}$. It is well-known that

$$E = \bigcup_{s \in S} \{s\} \times E_s$$

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is isomorphic to the direct image sheaf $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. Using the canonical isomorphism

$$K_{\mathcal{X}}|_{X_s} \cong K_{X_s},$$

a local smooth section u of E over S can be identified as a holomorphic $\mathcal{E}$-valued $(n, 0)$ form on X_s. Moreover, u is a local holomorphic section of E over S if u is holomorphic on the total space $\mathcal{X}$.

By the identification above, there is a natural metric on E. For any local smooth section u of E, one can define a Hermitian metric on E by

$$(1.1) \qquad h(u, u) = c_n \int_{X_s} \{u, u\}$$

where $c_n = (\sqrt{-1})^{n^2}$. Here, we only use the Hermitian metric of $\mathcal{E}_s$ on each fiber X_s and we do not specify background Kähler metrics on the fibers. Berndtsson defined in [Bern09, Lemma 4.1], a natural Chern connection D on (E, h), and computed the curvature tensor of direct image $p_*(K_{\mathcal{X}/S} \otimes \mathcal{L})$ of (semi-)positive line bundle $\mathcal{L} \to \mathcal{X}$.

Next we would like to describe our results in this paper briefly. As motivated by the work of Berndtsson([Bern09]), we compute the curvature tensor of the direct images $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ for arbitrary Hermitian vector bundles $\mathcal{E} \to \mathcal{X}$ by using basic Hodge theory which also simplify Berndtsson's original proofs mildly. In the following formulations, if not otherwise stated, we do not make any positivity or negativity assumption on the curvature tensors of $\mathcal{E}$ or $\mathcal{L}$. Hence, a Hermitian metric for a sheaf is defined by going to the linear space associated to the sheaf and the curvature form is defined only on the set of points where the sheaf is locally free.

Let (X, ω_g) be a compact Kähler manifold with complex dimension n and $F \to X$ a Hermitian vector bundle with Chern connection $\nabla = \nabla' + \nabla''$. At first, by Hodge theory on vector bundles(Lemma 2.1), we see that if $\alpha \in \Omega^{n,0}(X, F)$, and it has no harmonic part, then $v = \nabla'^* \mathbb{G}' \alpha$ is a solution to $\nabla' v = \alpha$. Moreover, $\nabla'' v$ is a primitive $(n-1, 1)$ form. We can apply this observation to the Kähler fibration $p : \mathcal{X} \to S$. Let $(t^1, \cdots, t^m)$ be local holomorphic coordinates on the base S centered at some point $s \in S$. Let $\nabla^{\mathcal{E}} = \nabla' + \nabla''$ be the Chern connection of the Hermitian vector bundle $(\mathcal{E}, h^{\mathcal{E}})$ over $\mathcal{X}$ and $\nabla_X = \nabla'_X + \nabla''_X$ be the restriction of $\nabla^{\mathcal{E}}$ on the fiber $\mathcal{E}_s \to X_s$. For any local holomorphic section u of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$, we set

$$v_i = -\nabla'_X \mathbb{G}' \pi_\perp \left(\nabla'_{\frac{\partial}{\partial t^i}} u \right)$$

where $\pi_\perp = \mathbb{I} - \pi$ and $\pi : \Omega^{n,0}(X_s, \mathcal{E}_s) \to H^{n,0}(X_s, \mathcal{E}_s)$ is the orthogonal projection on each fiber. At first, we derive a curvature formula for $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ by a very simple method (see Theorem 3.3).

Theorem 1.1. *Let Θ^E be the Chern curvature of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. For any local holomorphic section u of E, the curvature Θ^E has the following "negative form":*

$$(1.2) \qquad (\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} u, u \right\} - (\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

We shall explain this curvature formula in details in the following sections and, also make a simple example in Section 4 to explain why this "negative form" is "natural".

By a decomposition for the second term on the right hand side of (1.2),

$$(\Delta'_X v_i, v_j) = (\Delta'_X v_i, \Delta'_X v_j) + (\Delta'_X v_i, v_j - \Delta'_X v_j)$$

we obtain a curvature form with significant geometric interpretations and it is related to deformation theory of vector bundles. Let $[\alpha_i] \in H^1(X_s, End(\mathcal{E}_s))$ be the Kodaira-Spencer class ([SchTo92, Proposition 1]) of the deformation $\mathcal{E} \to \mathcal{X} \to S$ in the direction of $\frac{\partial}{\partial t^i} \in T_s S$, i.e.

(1.3)
$$\alpha_i = \Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)\big|_{X_s} \in \Omega^{0,1}(X_s, End(\mathcal{E}_s)).$$

We observe that

$$\Delta'_X v_i = -\sqrt{-1}\Lambda_g \left(\alpha_i \cup u\right)$$

where Λ_g is the contraction operator with respect to the Kähler metric ω on the fiber X_s.

Theorem 1.2. *The curvature Θ^E of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ has the following "geodesic form":*

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\alpha_i \cup u, \alpha_j \cup u) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

(1.4)
$$+ (\Delta'_X v_i, \Delta'_X v_j - v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).$$

Rewriting each line on the right hand side of (1.4) a little bit, we reach the following special case, which is also of particular interest, since the first line in (1.4) is exactly in the geodesic form.

Corollary 1.3. *Let $(\mathcal{L}, h^{\mathcal{L}} = e^{-\varphi})$ be a Hermitian line bundle over $\mathcal{X}$ such that $(\mathcal{L}|_{X_s}, h^{\mathcal{L}}_{X_s})$ is positive on each fiber X_s. The curvature Θ^{E_k} of $E_k = p_*(K_{\mathcal{X}/S} \otimes \mathcal{L}^k)$ has the form:*

$$(\sqrt{-1}\Theta^{E_k} u, u) = c_n \int_{X_s} k c_{i\bar{j}}(\varphi)\{u, u\}(\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

(1.5)
$$+ \frac{1}{k}\left((\Delta'_X + k)^{-1}\left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j),$$

where $c_{i\bar{j}}(\varphi)$ is given by

(1.6)
$$c_{i\bar{j}}(\varphi) = \frac{\partial^2 \varphi}{\partial t^i \partial \bar{t}^j} - \left\langle \bar{\partial}_X\left(\frac{\partial \varphi}{\partial t^i}\right), \bar{\partial}_X\left(\frac{\partial \varphi}{\partial t^j}\right)\right\rangle_g$$

Remark 1.4. (1) The curvature formula (1.5) is derived implicitly in some special cases by different authors (c.f. [Bern09a], [LSYau09], [Sch13].)

(2) In the real parameter case,

$$c(\varphi) = \ddot{\varphi} - |\bar{\partial}_X \dot{\varphi}|^2_g.$$

When $c(\varphi) = 0$, it is the geodesic equation in the space of Kähler potentials. For this comprehensive topic, we just refer the reader to [Semmes92], [Donald99], [Chen00], [PhoStu06], [Bern09a], [Bern11a] and references therein.

(3) For the vector bundle case, the authors also expect that the first line on the right hand side of (1.4), i.e.

$$c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} u, u \right\} - (\alpha_i \cup u, \alpha_j \cup u) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

can be written into certain geodesic form in the space of Hermitian metrics on $\mathcal{E}$ when $\mathcal{E}$ has some stability property (see formula (3.9) for the line bundle case).

(4) If $p : \mathcal{X} \to S$ is the universal curve with geneus $g \geq 2$, i.e. $p : \mathcal{T}_g \to \mathcal{M}_g$. If $\mathcal{L} = K_{\mathcal{T}_g/\mathcal{M}_g}$, one can deduce Wolpert's curvature formula ([Wolp86]) for the (dual) Weil-Petersson metric on $p_*(K^{\otimes 2}_{\mathcal{T}_g/\mathcal{M}_g})$ easily from (1.6) (see also [Siu86], [LSYau09] [Bern11] and [Sch13]).

(5) When $k = 1$, one can use (1.5) to study the convex and concave property of the logarithm volume functional on a Fano manifold ([Bern11a], see also Theorem 4.5, Proposition 4.6). Intrinsically, it amounts to the standard $\bar{\partial}$-estimate $\|\psi\| \leq \|\bar{\partial}\psi\|$ on functions if the Fano manifold is polarized by its anti-canonical class.

As a first application of Theorem 1.1, we obtain

Theorem 1.5. *If there exists a Hermitian metric on $\mathcal{E}$ which is Nakano-negative along the base, then $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-negative.*

Next, we follow Berndtsson's ideas in his remarkable papers [Bern09], [Bern09a], [Bern11], [Bern11a] and set

$$\widetilde{u} = u - dt^i \wedge v_i$$

By using "Berndtsson's magic formula"

$$c_n \int_{X_s} \{u, u\} = c_n \int_{X_s} \{\widetilde{u}, \widetilde{u}\},$$

we obtain

Proposition 1.6. *The curvature Θ^E of $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ has the following "positive form":*

$$(1.7) \qquad (\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} \widetilde{u}, \widetilde{u} \right\} + \left(\nabla''_X v_i, \nabla''_X v_j \right) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

When $\mathcal{E}$ is a line bundle, the curvature formula (1.7) is firstly obtained by Berndtsson in [Bern09], [Bern09a], [Bern11] and [Bern11a]. When $\mathcal{E}$ is a Nakano-positive vector bundle, a similar formulation seems to be obtained in [MouTak08] by using Berndtsson's idea, but v_i are not given explicitly. As it is shown, these v_i play the key role in curvature formulas and their applications.

Let $c_{i\bar{j}}$ be the $\mathcal{E}$-valued $(n, 0)$-form coefficient of $dt^i \wedge d\bar{t}^j$ in the expression

$$\Theta^{\mathcal{E}} (u - dt^i \wedge v_i).$$

Theorem 1.7. *The curvature Θ^E of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ has the following "compact form":*

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \{c_{i\bar{j}}, u\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

As applications, we use it to study the degeneracy of the curvature tensor of $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ under the assumption that $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ is Nakano semi-positive.

Theorem 1.8. *Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be Nakano semi-positive. Then*

$$(\sqrt{-1}\Theta^E u, u) = 0$$

if and only if $c_{ij} = 0$.

Moreover, $c_{i\bar{j}}$ is closely related to the geometry of the family $\mathcal{E} \to \mathcal{X} \to S$. When $(\mathcal{E}, h^{\mathcal{E}} = e^{-\varphi})$ is a relatively positive line bundle, $c_{i\bar{j}}$ is the same as the geodesic term $c_{i\bar{j}}(\varphi)(u)$ defined in (1.6) when the curvature degenerates. Furthermore, when $H^{n,1}(X_s, \mathcal{E}_s) = 0$, we show that v_i are all holomorphic over the total space $\mathcal{X}$ and we can use it to construct holomorphic automorphisms of the family $\mathcal{E} \to \mathcal{X} \to S$ and study the moduli space of projectively flat vector bundles.

We consider a smooth family of projectively flat vector bundles $(\mathcal{E}_s, h^{\mathcal{E}_s})_{s \in S}$ with polarization

$$(1.8) \qquad \sqrt{-1}\Theta^{\mathcal{E}_s} = \omega_g \otimes h^{\mathcal{E}_s}.$$

Let W_i be the dual vector of the Kodaira-Spencer form α_i defined in (1.3), i.e. W_i is an $End(\mathcal{E}_s)$-valued $(1,0)$ vector field. Then v_i, u and the Kodaira-Spencer vectors W_i are related by

$$(1.9) \qquad i_{W_i} u = -v_i$$

when the curvature of $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is degenerated. In this case, W_i is an $End(\mathcal{E}_s)$-valued holomorphic vector field on the fiber. We also see that, the horizontal lift of $\frac{\partial}{\partial t^i}$,

$$V_i = \frac{\partial}{\partial t^i} - W_i$$

is a (local) $End(\mathcal{E})$-valued holomorphic vector field over the total space $\mathcal{X}$. Moreover, the Lie derivatives of the curvature tensor of $\mathcal{E}_s$ with respect to V_i are all zero, i.e.

$$\mathscr{L}_{V_i} \omega_g = 0$$

That means, if the curvature of $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ degenerates at some point $s \in S$, then the family $\mathcal{E} \to \mathcal{X} \to S$ moves by an infinitesimal automorphism of $\mathcal{E}$ when the base point varies.

We can formulate it into a global version. Let $\mathcal{X} = X \times \mathbb{D}$, where $\mathbb{D}$ is a unit disk. Let $\mathbb{E}_0 \to X$ be a holomorphic vector bundle. If $(\mathbb{E}_0, h(t))_{t \in \mathbb{D}} \to X$ is a smooth family of projectively flat vector bundles with polarization (1.8). We denote by $\mathcal{E}$, the pullback family $p_2^*(\mathbb{E}_0)$ over $p_2 : \mathcal{X} \to X$.

Theorem 1.9. *If the curvature Θ^E of $E = p_*(K_{\mathcal{X}/\mathbb{D}} \otimes \mathcal{E})$ vanishes in a small neighborhood of $0 \in \mathbb{D}$, then there exists a holomorphic vector field V on X with flows $\Phi_t \in Aut_H(X, \mathbb{E}_0)$ such that*

$$\Phi_t^*(\omega_t) = \omega_0$$

for small t.

Remark 1.10. We can also use the holomorphic vector field V to study the uniqueness of Hermitian-Einstein metrics on stable bundles, the stability of the direct image $p_*(\mathcal{E})$ and the asymptotic stability of $p_*(\mathcal{E} \otimes \mathcal{L}^k)$ for large k. We shall carry it out in the sequel to this paper.

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2. Background materials

2.1. **Hodge theory on vector bundles.** Let (E, h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X, ω) and $\nabla = \nabla' + \nabla''$ be the Chern connection on it. Here, we have aslo $\nabla'' = \overline{\partial}$. With respect to metrics on E and X, we set

$$\Delta'' = \nabla'' \nabla''^* + \nabla''^* \nabla'',$$

$$\Delta' = \nabla' \nabla'^* + \nabla'^* \nabla'.$$

Accordingly, we associate the Green operators and harmonic projections $\mathbb{G}$, $\mathbb{H}$ and $\mathbb{G}'$, $\mathbb{H}'$ in Hodge decomposition to them, respectively. More precisely,

$$\mathbb{I} = \mathbb{H} + \Delta'' \circ \mathbb{G}, \ \mathbb{I} = \mathbb{H}' + \Delta' \circ \mathbb{G}'.$$

For any $\varphi, \psi \in \Omega^{\bullet, \bullet}(X, E)$, there is a *sesquilinear pairing*

$$\{\varphi, \psi\} = \varphi^\alpha \wedge \overline{\psi^\beta} \langle e_\alpha, e_\beta \rangle \tag{2.1}$$

if $\varphi = \varphi^\alpha e_\alpha$ and $\psi = \psi^\beta e_\beta$ in the local frame $\{e_\alpha\}$ of E. By the metric compatible property,

$$\partial\{\varphi, \psi\} = \{\nabla'\varphi, \psi\} + (-1)^{p+q}\{\varphi, \nabla''\psi\} \tag{2.2}$$

if $\varphi \in \Omega^{p,q}(X, E)$.

Let Θ^E be the Chern curvature of (E, h). It is well-known

$$\Delta'' = \Delta' + [\sqrt{-1}\Theta, \Lambda_g] \tag{2.3}$$

where Λ_g is the contraction operator with respect to the Kähler metric ω. The following observation plays an important role in our computations.

Lemma 2.1. *Let E be any Hermitian vector bundle over a compact Kähler manifold (X, ω). For any $\alpha \in \Omega^{n,0}(X, E)$ with no harmonic part with respect to Δ'', i.e. $\mathbb{H}(\alpha) = 0$, then*

(1) $\mathbb{H}'(\alpha) = 0$;
(2) *The $(n-1, 0)$ form $v = \nabla'^*\mathbb{G}'\alpha$ is a solution to the equation*

$$\nabla'v = \alpha;$$

(3) $\nabla''v$ *is primitive.*

Proof. The first statement follows from the Bochner identity on E-valued $(n, 0)$-forms. More precisely, by (2.3)

$$\Delta''\beta = \Delta'\beta$$

for any $\beta \in \Omega^{n,0}(X, E)$. Hence $\mathbb{H}'(\beta) = \mathbb{H}(\beta)$. For (2), by Hodge decomposition, we have

$$
\begin{aligned}
\nabla' v &= \nabla' \nabla'^* \mathbb{G}'(\alpha) \\
&= \alpha - \mathbb{H}'(\alpha) - \nabla'^* \nabla' \mathbb{G}'(\alpha) \\
&= \alpha - \mathbb{H}'(\alpha) = \alpha.
\end{aligned}
$$

For (3), let $L_g = \omega \wedge$. By Hodge identity $[\nabla'^*, L_g] = -\sqrt{-1} \nabla''$,

$$
\begin{aligned}
\omega \wedge \nabla'' v = L_g \nabla'' v &= L_g \nabla'' \nabla'^* \mathbb{G}' \alpha \\
&= -L_g \nabla'^* \nabla'' \mathbb{G}' \alpha \\
&= \left(-\sqrt{-1} \nabla'' - \nabla'^* L_g \right) \nabla'' \mathbb{G}' \alpha \\
&= 0
\end{aligned}
$$

since $L_g \nabla'' \mathbb{G}' \alpha$ is an $(n+1, 2)$ form. $\qquad \square$

The following Rieman-Hodge bilinear relation will be used frequently, and the proof of it can be found in [Huyb05, Corollary 1.2.36] or [Voisin02, Proposition 6.29].

Lemma 2.2. *If* $\varphi, \psi \in \Omega^{p,q}(X, E) \subset \Omega^n(X, E)$ *are primitive, then*

$$
(2.4) \qquad (\varphi, \psi) = \left(\sqrt{-1} \right)^{n(n-1)+(p-q)} \int_X \{\varphi, \psi\}
$$

where $(\bullet, \bullet)$ *is the standard inner product (norm) induced by metrics on* X *and* E.

2.2. **Positivity of vector bundles.** Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $\Theta^E \in \Gamma(X, \Lambda^2 T^* X \otimes E^* \otimes E)$ has the form

$$
(2.5) \qquad \Theta^E = R^\gamma_{i\bar{j}\alpha} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma,
$$

where $R^\gamma_{i\bar{j}\alpha} = h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}$ and

$$
(2.6) \qquad R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}.
$$

Here and henceforth we adopt the Einstein convention for summation.

Definition 2.3. *A Hermitian vector bundle* (E, h) *is said to be* Griffiths-positive, *if for any nonzero vectors* $u = u^i \frac{\partial}{\partial z^i}$ *and* $v = v^\alpha e_\alpha$,

$$
(2.7) \qquad \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0
$$

(E, h) *is said to be* Nakano-positive, *if for any nonzero vector* $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$
(2.8) \qquad \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0
$$

(E, h) *is said to be* dual-Nakano-positive, *if for any nonzero vector* $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$
(2.9) \qquad \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^{i\beta} \overline{u}^{j\alpha} > 0
$$

It is easy to see that (E, h) is dual-Nakano-positive if and only if (E^*, h^*) is Nakano-negative. The notions of semi-positivity, negativity and semi-negativity can be defined similarly. We say E is Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, $\cdots$), if it admits a Nakano-positive(resp. Griffiths-positive, dual-Nakano-positive, $\cdots$) metric.

2.3. **Direct image sheaves of vector bundles.** Let $\mathcal{X}$ be a Kähler manifold with dimension $m + n$ and S a Kähler manifold with dimension m. Let $p : \mathcal{X} \to S$ be a smooth Kähler fibration. That means, for each $s \in S$,

$$X_s := p^{-1}(\{s\})$$

is a compact Kähler manifold with dimension n. Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be a Hermitian holomorphic vector bundle. In the following, we adopt the setting in [Bern09, Section 4]. Consider the space of holomorphic $\mathcal{E}$-valued $(n, 0)$-forms on X_s,

$$E_s := H^0(X_s, \mathcal{E}_s \otimes K_{X_s}) \cong H^{n,0}(X_s, \mathcal{E}_s)$$

where $\mathcal{E}_s = \mathcal{E}|_{X_s}$ and then

$$E = \bigcup_{s \in S} \{s\} \times E_s$$

is isomorphic to the direct image sheaf $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. By the canonical isomorphism

$$K_{\mathcal{X}}|_{X_s} \cong K_{X_s}$$

we make the following identification which will be used frequently in the sequel:

(1) a local smooth section u of E over S is a holomorphic $\mathcal{E}$-valued $(n, 0)$ form on X_s. In the local holomorphic coordinates, $(z, t) := (z^1, \cdots, z^n, t^1, \cdots, t^m)$ on $\mathcal{X}$,

$$u = u^\alpha(z, t) dz^1 \wedge \cdots \wedge dz^n \otimes \widetilde{e}_\alpha,$$

where $u^\alpha(z, t)$ are local holomorphic functions on X_s, i.e. they are holomorphic in z.

(2) u is a local holomorphic section of E over S, if $u^\alpha(z, t)$ are holomorphic in both z and t.

By the identification above, there is a natural metric on the E induced by metrics $h^{\mathcal{E}_s}$ on $\mathcal{E}_s$. For any local smooth section u of E, we define a Hermitian metric h on E by

(2.10) $$h(u, u) = c_n \int_{X_s} \{u, u\}$$

where $c_n = (\sqrt{-1})^{n^2}$.

Next, we want to define the Chern connection for the Hermitian holomorphic vector bundle $(E, h) \to S$. Let $\nabla^{\mathcal{E}} = \nabla' + \nabla''$ be the Chern connection of $(\mathcal{E}, h^{\mathcal{E}})$ over the total space $\mathcal{X}$. For any local smooth section u of E, it is also a $\mathcal{E}$-valued holomorphic $(n, 0)$-form on $\mathcal{X}$. It is obvious that

(2.11) $$\nabla'' u = d\bar{t}^j \wedge \tau_{\bar{j}}$$

where $\tau_{\bar{j}} = \frac{\partial u}{\partial \bar{t}^j}$ since u is holomorphic on each fiber. Similarly,

(2.12) $$\nabla' u = dt^i \wedge \nu_i$$

where $\nu_i = \nabla'_{\frac{\partial}{\partial t^i}} u$ since u is top $(n,0)$ form on each fiber. The following lemma is given in [Bern09, Lemma 4.1].

Lemma 2.4. *Let $D = D' + D''$ be the Chern connection of the Hermitian holomorphic vector bundle $(E, h) \to S$, then for any local smooth section u of E,*

(2.13)
$$D'' u = \tau_{\bar{j}} d\bar{t}^j, \qquad D' u = \pi(\nu_i) dt^i$$

where π is the orthogonal projection

(2.14)
$$\pi : \Omega^{n,0}(X_s, \mathcal{E}_s) \to H^{n,0}(X_s, \mathcal{E}_s)$$

Let Θ^E be the Chern curvature of $(E, h) \to S$. The following formula is obvious.

Lemma 2.5. *Let u be a local holomorphic section of E over S, then*

(2.15)
$$\overline{\partial}\partial(u, u) = (D'' D' u, u) - (D' u, D' u) = (\Theta^E u, u) - (D' u, D' u)$$

To end this section, we list the notations we shall use in the sequel:

- $D = D' + D''$ the Chern connection on the Hermitian vector bundle $(E, h) \to S$;

- $\nabla^{\mathcal{E}} = \nabla' + \nabla''$ the Chern connection of $\mathcal{E}$ over the total space $\mathcal{X}$; We will also use $\overline{\partial}_{\mathcal{X}}$ for ∇'' if there is no confusion;

- To simplify notations, we will denote the Chern connection $\nabla^{\mathcal{E}}|_{X_s}$ of the Hermitian vector bundle $(\mathcal{E}_s, h^{\mathcal{E}_s}) \to X_s$ by $\nabla_X = \nabla'_X + \nabla''_X$ although it depends on $s \in S$;

- $d = \partial + \overline{\partial}$ the natural decomposition of d on the base S;

- $\{\omega_s\}_{s \in S}$ a smooth family of Kähler metrics on $\{X_s\}_{s \in S}$;

- $\mathbb{G}'$ the Green's operator for $\Delta'_X = \nabla'_X \nabla'^*_X + \nabla'^*_X \nabla'_X$;

- $\pi : \Omega^{n,0}(X_s, \mathcal{E}_s) \to H^{n,0}(X_s, \mathcal{E}_s)$ the orthogonal projection on the fiber;

- $\pi_\perp = \mathbb{I} - \pi$.

3. Curvature formulas of direct images of vector bundles

3.1. **A straightforward computation.** In this section, we will derive several general curvature formulas for direct image $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ by using Lemma 2.1.

For base vectors $\frac{\partial}{\partial t_i}$ and $\frac{\partial}{\partial \bar{t}^j}$, we also use the following conventions:

$$\nabla'_i u = \nabla'_{\frac{\partial}{\partial t^i}} u = (\nabla' u)\left(\frac{\partial}{\partial t^i}\right)$$

$$\frac{\partial u}{\partial \bar{t}^j} = \nabla''_{\bar{j}} u = \nabla''_{\frac{\partial}{\partial \bar{t}^j}} u = (\nabla'' u)\left(\frac{\partial}{\partial \bar{t}^j}\right)$$

The following corollary is a special case of Lemma 2.1.

Corollary 3.1. *For any local section u of E, we set*

(3.1) $$v_i = -\nabla'^*_X \mathbb{G}' \pi_\perp \left(\nabla'_i u\right).$$

Then

   (1) $\nabla'_X v_i = -\pi_\perp \left(\nabla'_i u\right)$;
   (2) $\nabla''_X v_i$ *is primitive.*

Note that, in this paper, v_i is fixed to be $-\nabla'^*_X \mathbb{G}' \pi_\perp \left(\nabla'_i u\right)$, and we do not change it anymore. Before computing the curvature tensors of the direct images, we need a well-known result:

Lemma 3.2. ∂ *and $\overline{\partial}$ commute with the fiber integration. More precisely,*

$$\overline{\partial} \int_{X_s} \alpha = \int_{X_s} \overline{\partial}_{\mathcal{X}} \alpha, \qquad \partial \int_{X_s} \alpha = \int_{X_s} \partial_{\mathcal{X}} \alpha$$

for any smooth $\alpha \in \Omega^{\bullet,\bullet}(\mathcal{X})$.

Theorem 3.3. *Let Θ^E be the Chern curvature of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. For any local holomorphic section u of E, the curvature Θ^E has the following "negative form":*

(3.2) $$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

Proof. By curvature formula (2.15), we obtain

$$\begin{aligned}
(\sqrt{-1}\Theta^E u, u) &= \sqrt{-1}(D'u, D'u) - \sqrt{-1}\partial\overline{\partial}\|u\|^2 \\
&= \sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'u, \nabla'u\right\} - c_n \int_{X_s} \sqrt{-1}\left\{u, \nabla''\nabla'u\right\} \\
&= \sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'u, \nabla'u\right\} + c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\}
\end{aligned}$$

since $\nabla'' u = 0$ and $\sqrt{-1}\Theta^{\mathcal{E}}$ is Hermitian. By definition, we have $D'u = \pi(\nabla'_i u) \wedge dt^i$. From the orthogonal decomposition $\nabla'_i u = \pi(\nabla'_i u) + \pi_\perp\left(\nabla'_i u\right)$, we see that

$$\sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'u, \nabla'u\right\} = -c_n \int_{X_s} \{\nabla'_X v_i, \nabla'_X v_j\}(\sqrt{-1} dt^i \wedge d\bar{t}^j).$$

where we use the fact that $-\pi_\perp(\nabla'_i u) = \nabla'_X v_i$. Since $\nabla' v_i$ are primitive and $\nabla''^*_X v_i = 0$, by Riemann-Hodge bilinear relation (2.4), we obtain

$$\begin{aligned}
\sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'u, \nabla'u\right\} &= -(\nabla'_X v_i, \nabla'_X v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\
&= -(\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)
\end{aligned}$$

Now the curvature formula (3.2) follows. $\qquad\square$

Remark 3.4. It is easy to see that, if $\Theta^{\mathcal{E}}$ is negative along the base S, then the first term in the curvature formula (3.2) is negative, and so (E, h) is a negative vector bundle. We shall make this statement more precisely in the next section.

As a straightforward consequence of Theorem 3.3, we obtain

Corollary 3.5. *The curvature Θ^E has the following form:*

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, u\right\} - (\Delta'_X v_i, \Delta'_X v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

(3.3)
$$+ (\Delta'_X v_i, \Delta'_X v_j - v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).$$

Let $[\alpha_i] \in H^{0,1}(X_s, End(\mathcal{E}_s))$ be the Kodaira-Spencer class ([SchTo92, Proposition 1]) of the deformation $\mathcal{E} \to \mathcal{X} \to S$ in the direction of $\frac{\partial}{\partial t^i} \in T_s S$, i.e.

(3.4)
$$\alpha_i = \Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)|_{X_s} \in \Omega^{0,1}(X_s, End(\mathcal{E}_s)).$$

By Hodge identity $[\Lambda_g, \nabla''_X] = -\sqrt{-1}\nabla'^*_X$, $\nabla''_X u = 0$ and $\nabla''_X(\pi(\nabla'_i u)) = 0$, we get

$$\begin{aligned}
\Delta'_X v_i &= \nabla'^*_X \nabla'_X v_i = \sqrt{-1}\Lambda_g \nabla''_X \nabla'_X v_i \\
&= -\sqrt{-1}\Lambda_g \nabla''_X \nabla'_i u \\
&= -\sqrt{-1}\Lambda_g \left(\nabla''_X \nabla'_i + \nabla'_i \nabla''_X\right) u \\
&= -\sqrt{-1}\Lambda_g(\alpha_i \cup u).
\end{aligned}$$

(3.5)

We obtain the following by Corollary 3.5:

Theorem 3.6. *The curvature Θ^E has the following "geodesic form":*

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, u\right\} - (\alpha_i \cup u, \alpha_j \cup u) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

(3.6)
$$+ (\Delta'_X v_i, \Delta'_X v_j - v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).$$

Next we want to explain why (3.3) is called a "geodesic form" by a little bit more computations in some special cases. Let $(\mathcal{E}, e^{-\varphi})$ be a relative positive line bundle over $\mathcal{X}$, and we set $\omega_g = \sqrt{-1}\partial_X \overline{\partial}_X \varphi$ on each fiber.

Corollary 3.7. *Let $(\mathcal{L}, h^{\mathcal{L}} = e^{-\varphi})$ be a Hermitian line bundle over $\mathcal{X}$ such that $(\mathcal{L}|_{X_s}, h^{\mathcal{L}}_{X_s})$ is positive on each fiber X_s. Then the curvature Θ^E of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{L})$ has the form:*

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} c_{i\bar{j}}(\varphi)\left\{u, u\right\} \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

(3.7)
$$+ \left((\Delta'_X + 1)^{-1}\left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).$$

where $c_{i\bar{j}}(\varphi)$ is given by

(3.8)
$$c_{i\bar{j}}(\varphi) = \frac{\partial^2 \varphi}{\partial t^i \partial \bar{t}^j} - \left\langle \overline{\partial}_X\left(\frac{\partial \varphi}{\partial t^i}\right), \overline{\partial}_X\left(\frac{\partial \varphi}{\partial t^j}\right)\right\rangle_g$$

Proof. By formula (3.5), we obtain
(3.9)
$$c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{L}}u, u\right\} - (\Delta'_X v_i, \Delta'_X v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j) = c_n \int_{X_s} c_{i\bar{j}}(\varphi)\left\{u, u\right\} \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).$$

On the other hand, we see that, the second line on the right hand side of (3.3) is non-negative. In fact, by formula (2.3) with the fact $\omega_g = \sqrt{-1}\Theta^{\mathcal{L}}|_{X_s}$, we have $\Delta''_X v_i = \Delta'_X v_i - v_i$ and $\nabla''_X \Delta''_X v_i + \nabla''_X v_i = \nabla''_X \Delta'_X v_i$. Therefore,

$$
\begin{aligned}
(\Delta'_X v_i, \Delta'_X v_j - v_j) &= (\Delta''_X v_i + v_i, \Delta''_X v_j) = (\Delta''_X v_i, \Delta''_X v_j) + (v_i, \Delta''_X v_j) \\
&= (\Delta''_X v_i, \Delta''_X v_j) + (\Delta''_X v_i, v_j) = (\Delta'' v_i, \Delta' v_j) \\
&= (\nabla'' v_i, \nabla'' \Delta'_X v_j).
\end{aligned}
$$

Note also that

$$(\Delta'_X + 1)(\nabla''_X v_i) = \nabla''_X \Delta'_X v_i.$$

Hence, we obtain

$$(\Delta'_X v_i, \Delta'_X v_j - v_j) = (\nabla'' v_i, \nabla'' \Delta'_X v_j) = \left((\Delta'_X + 1)^{-1} \left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right).$$

$\square$

Similarly, we get the "quantization" version:

Proposition 3.8. *The curvature Θ^{E_k} of $E_k = p_*(K_{\mathcal{X}/S} \otimes \mathcal{L}^k)$ has the following form:*

$$
\begin{aligned}
(\sqrt{-1}\Theta^{E_k} u, u) &= c_n \int_{X_s} k c_{i\bar{j}}(\varphi)\{u, u\}(\sqrt{-1}dt^i \wedge d\bar{t}^j) \\
&\quad + \frac{1}{k}\left((\Delta'_X + k)^{-1}\left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).
\end{aligned}
$$
(3.10)

where $c_{i\bar{j}}(\varphi)$ satisfies (3.8).

Proof. By Theorem 3.3, we rewrite the curvature formula as

$$
\begin{aligned}
(\sqrt{-1}\Theta^{E_k} u, u) &= c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{L}^k} u, u\right\} - \frac{1}{k}(\Delta'_X v_i, \Delta'_X v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j) \\
&\quad + \frac{1}{k}(\Delta'_X v_i, \Delta'_X v_j - k v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).
\end{aligned}
$$
(3.11)

As similar as the arguments in Corollary 3.7, we deduce

$$c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{L}^k} u, u\right\} - \frac{1}{k}(\Delta'_X v_i, \Delta'_X v_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j) = c_n \int_{X_s} k c_{i\bar{j}}(\varphi)\{u, u\}(\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

and

$$(\Delta'_X v_i, \Delta'_X v_j - k v_j) = \left((\Delta'_X + k)^{-1}\left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right).$$

Hence (3.10) follows. $\square$

3.2. **Computations by using Berndtsson's magic formula.**

In this subsection, we will derive several curvature formulas for direct image sheaf $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ following the ideas in [Bern09], [Bern09a], [Bern11] and [Bern11a]. However, we do not make any assumption on the curvature of $\mathcal{E}$. We only make use of the following "Berndtsson's magic formula":

Lemma 3.9. *Let u be a local smooth section of E. If $\widetilde{u} = u - dt^i \wedge v_i$, then*

$$c_n \int_{X_s} \{u, u\} = c_n \int_{X_s} \{\widetilde{u}, \widetilde{u}\}$$
(3.12)

Proof. It follows by comparing the (n,n)-forms along the fiber X_s on both sides. $\square$

In the following, u will be a local holomorphic section to E, i.e. $\nabla'' u = 0$. Moreover, we set

$$\widetilde{u} = u - dt^i \wedge v_i \tag{3.13}$$

and thus fixed. Recall that, $v_i = -\nabla_X'^* \mathbb{G}' \pi_\perp \left(\nabla_i' u \right)$ as defined in (3.1). It is easy to see

$$\nabla'' \widetilde{u} = dt^i \wedge \nabla'' v_i = dt^i \wedge \nabla_X'' v_i + dt^i \wedge d\bar{t}^j \wedge \frac{\partial v_i}{\partial \bar{t}^j} \tag{3.14}$$

and

$$\begin{aligned}
\nabla' \widetilde{u} &= dt^i \wedge \nabla_i' u + dt^i \wedge \nabla' v_i \\
&= dt^i \wedge \left(\nabla_i' u + \nabla_X' v_i \right) + dt^i \wedge dt^k \wedge \nabla_k' v_i \\
&= dt^i \wedge \pi(\nabla_i' u) + dt^i \wedge dt^k \wedge \nabla_k' v_i
\end{aligned}$$

since $\nabla_X' v_i = -\pi_\perp(\nabla_i' u)$. To make the above formula into a compact form, we define

$$\mu_i := \pi(\nabla_i' u) \tag{3.15}$$

and so

$$\nabla' \widetilde{u} = dt^i \wedge \mu_i + dt^i \wedge dt^k \wedge \nabla_k' v_i \tag{3.16}$$

Therefore,

$$\begin{aligned}
\nabla'' \nabla' \widetilde{u} &= \nabla''(dt^i \wedge \mu_i + dt^i \wedge dt^k \wedge \nabla_k' v_i) \\
&= -dt^i \wedge d\bar{t}^j \wedge \frac{\partial \mu_i}{\partial \bar{t}^j} + dt^i \wedge dt^k \wedge \nabla'' \nabla_k' v_i
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\nabla' \nabla'' \widetilde{u} &= -dt^i \wedge \nabla' \nabla'' v_i \\
&= -dt^i \wedge dt^k \wedge \nabla_k' \nabla_X'' v_i - dt^i \wedge \nabla_X' \nabla_X'' v_i \\
&\quad + dt^i \wedge d\bar{t}^j \wedge \nabla_X' \nabla_{\bar{j}}'' v_i - dt^i \wedge dt^k \wedge d\bar{t}^j \wedge \nabla_k' \nabla_{\bar{j}}'' v_i
\end{aligned} \tag{3.18}$$

By curvature formula (2.15) and the magic formula (3.12), we obtain

$$\begin{aligned}
(\sqrt{-1}\Theta^E u, u) &= \sqrt{-1}(D'u, D'u) - \sqrt{-1}\partial\bar{\partial}\|u\|^2 \\
&= \sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{ \sqrt{-1}\nabla'\widetilde{u}, \nabla'\widetilde{u} \right\} \\
&\quad -c_n \int_{X_s} \sqrt{-1} \left\{ \nabla'\nabla''\widetilde{u}, \widetilde{u} \right\} - c_n(-1)^{n+1} \int_{X_s} \sqrt{-1} \left\{ \nabla''\widetilde{u}, \nabla''\widetilde{u} \right\} \\
&\quad -c_n \int_{X_s} \sqrt{-1} \left\{ \widetilde{u}, \nabla''\nabla'\widetilde{u} \right\}
\end{aligned} \tag{3.19}$$

Claim. The first line and second line on the right hand side of (3.19) are all zero, i.e.

$$\sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{ \sqrt{-1}\nabla'\widetilde{u}, \nabla'\widetilde{u} \right\} = 0 \tag{3.20}$$

and

$$(3.21) \qquad - c_n \int_{X_s} \sqrt{-1} \left\{ \nabla' \nabla'' \widetilde{u}, \widetilde{u} \right\} - c_n (-1)^{n+1} \int_{X_s} \sqrt{-1} \left\{ \nabla'' \widetilde{u}, \nabla'' \widetilde{u} \right\} = 0$$

Proof. In fact, thanks to (3.16), we have

$$
\begin{aligned}
-c_n(-1)^n \int_{X_s} \left\{ \sqrt{-1} \nabla' \widetilde{u}, \nabla' \widetilde{u} \right\} &= -c_n(-1)^n \int_{X_s} \sqrt{-1} \left\{ dt^i \wedge \mu_i, dt^j \wedge \mu_j \right\} \\
&= -c_n \int_{X_s} \left\{ \mu_i, \mu_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)
\end{aligned}
$$

On the other hand,

$$D'u = \pi(\nabla'_i u) \wedge dt^i = \mu_i \wedge dt^i.$$

Hence

$$\sqrt{-1}(D'u, D'u) = c_n \int_{X_s} \left\{ \mu_i, \mu_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j).$$

We complete the proof of (3.20). On the other hand, by (3.14), we obtain

$$
\begin{aligned}
-c_n(-1)^{n+1} \int_{X_s} \sqrt{-1} \left\{ \nabla'' \widetilde{u}, \nabla'' \widetilde{u} \right\} &= -c_n(-1)^{n+1} \int_{X_s} \sqrt{-1} \left\{ dt^i \wedge \nabla''_X v_i, dt^j \wedge \nabla''_X v_j \right\} \\
&= c_n \int_{X_s} \left\{ \nabla''_X v_i, \nabla''_X v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)
\end{aligned}
$$

Similarly, by formula (3.18)

$$
\begin{aligned}
&-c_n \int_{X_s} \sqrt{-1} \left\{ \nabla' \nabla'' \widetilde{u}, \widetilde{u} \right\} \\
&= -c_n \int_X \sqrt{-1} \left\{ -dt^i \wedge (\nabla'_X \nabla''_X v_i) + dt^i \wedge d\bar{t}^\ell \wedge \nabla'_X \nabla''_{\bar{\ell}} v_i, u - dt^j \wedge v_j \right\} \\
&= (-1)^{n+2} c_n \int_{X_s} \left\{ \nabla'_X \nabla''_X v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\
&= -c_n \int_{X_s} \left\{ \nabla''_X v_i \nabla''_X v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)
\end{aligned}
$$

where in the second identity and last identity, we have used integration by parts, e.g.,

$$-c_n \int_X \sqrt{-1} \left\{ dt^i \wedge d\bar{t}^\ell \wedge \nabla'_X \nabla''_{\bar{\ell}} v_i, u \right\} = (-1)^{n+3} c_n \int_X \sqrt{-1} \left\{ dt^i \wedge d\bar{t}^\ell \wedge \nabla''_{\bar{\ell}} v_i, \nabla''_X u \right\} = 0.$$

Hence, (3.21) follows. $\qquad \square$

By taking conjugate of the real form, the curvature formula (3.19) is written as

$$
\begin{aligned}
(\sqrt{-1}\Theta^E u, u) \;&=\; -c_n \int_{X_s} \sqrt{-1}\left\{\widetilde{u}, \nabla''\nabla'\widetilde{u}\right\} \\[2mm]
&=\; c_n \int_{X_s} \sqrt{-1}\left\{\nabla''\nabla'\widetilde{u}, \widetilde{u}\right\} \\[2mm]
&=\; c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} - c_n \int_{X_s} \sqrt{-1}\left\{\nabla'\nabla''\widetilde{u}, \widetilde{u}\right\} \\[2mm]
&=\; c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} - c_n \int_{X_s} \left\{\nabla''_X v_i, \nabla''_X v_j\right\}\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j)
\end{aligned}
\tag{3.22}
$$

Note that, since $\nabla''_X v_i$ are primitive, by Riemann-Hodge bilinear relation (Lemma 2.2),

$$
-c_n \int_{X_s} \left\{\nabla''_X v_i, \nabla''_X v_j\right\} = (\nabla''_X v_i, \nabla''_X v_j).
\tag{3.23}
$$

Now we obtain

Proposition 3.10. *The curvature Θ^E of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ has the following "positive form":*

$$
(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} + \left(\nabla''_X v_i, \nabla''_X v_j\right)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j)
\tag{3.24}
$$

Let $c_{i\bar{j}}$ be the $\mathcal{E}$-valued $(n,0)$-form coefficient of $dt^i \wedge d\bar{t}^j$ in the expression

$$
\Theta^{\mathcal{E}}(u - dt^i \wedge v_i).
$$

Theorem 3.11. *The curvature Θ^E of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ has the following "compact form":*

$$
(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \left\{\Theta^{\mathcal{E}}\widetilde{u}, u\right\} = c_n \int_{X_s} \left\{c_{i\bar{j}}, u\right\}\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j)
\tag{3.25}
$$

Proof. Since $\widetilde{u} = u - dt^j \wedge v_j$,

$$
\begin{aligned}
c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} \;&=\; c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, u\right\} + c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\left(dt^i \wedge v_i\right), dt^j \wedge v_j\right\} \\[2mm]
&\quad -c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, dt^j \wedge v_j\right\} - c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\left(dt^j \wedge v_j\right), u\right\}
\end{aligned}
\tag{3.26}
$$

At first, we claim

$$
(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, u\right\} - c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\left(dt^j \wedge v_j\right), u\right\}
\tag{3.27}
$$

By (3.22) and (3.26), it is equivalent to show

$$
\begin{aligned}
&c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\left(dt^i \wedge v_i\right), dt^j \wedge v_j\right\} - c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, dt^j \wedge v_j\right\} \\[2mm]
&-c_n \int_{X_s} \left\{\nabla''_X v_i, \nabla''_X v_j\right\}\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j) = 0
\end{aligned}
\tag{3.28}
$$

It is obvious

$$(3.29) \quad c_n \int_{X_s} \left\{ \sqrt{-1}\Theta^{\mathcal{E}} \left(dt^i \wedge v_i \right), dt^j \wedge v_j \right\} = (-1)^{n-1} c_n \int \left\{ \Theta^{\mathcal{E}} v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

On the other hand, since $\nabla'' u = 0$,

$$
\begin{aligned}
-c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} u, dt^j \wedge v_j \right\} &= -c_n \int_{X_s} \sqrt{-1} \left\{ \nabla'' \nabla' u, dt^j \wedge v_j \right\} \\
(3.30) \hspace{4cm} &= -c_n (-1)^n \int_{X_s} \left\{ \nabla''_X \nabla'_i u, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\
&= c_n (-1)^n \int_{X_s} \left\{ \nabla''_X \nabla'_X v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)
\end{aligned}
$$

since $\nabla''_X \nabla'_i u = \nabla''_X \pi_\perp (\nabla'_i u) = -\nabla''_X \nabla'_X v_i$. Since $\Theta^{\mathcal{E}}|_{X_s} = \Theta^{\mathcal{E}_s} = \nabla'_X \nabla''_X + \nabla''_X \nabla'_X$,

$$-c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} u, dt^j \wedge v_j \right\} = c_n (-1)^n \int_{X_s} \left\{ \nabla''_X \nabla'_X v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

$$= (-1)^n c_n \int \left\{ \Theta^{\mathcal{E}} v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) - c_n (-1)^n \int_{X_s} \left\{ \nabla'_X \nabla''_X v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)$$

Integration by parts yields

$$
\begin{aligned}
-c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} u, dt^j \wedge v_j \right\} &= (-1)^n c_n \int \left\{ \Theta^{\mathcal{E}} v_i, v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\
(3.31) \hspace{4cm} &+ c_n \int_{X_s} \left\{ \nabla''_X v_i, \nabla''_X v_j \right\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j)
\end{aligned}
$$

By (3.29) and (3.31), we get (3.28) and also (3.27), i.e.

$$(3.32) \qquad (\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1} \left\{ \Theta^{\mathcal{E}} (u - dt^i \wedge v_i), u \right\}.$$

By degree reason, (3.25) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. CURVATURE POSITIVITY AND NEGATIVITY FOR DIRECT IMAGES OF VECTOR BUNDLES

As applications of curvature formulas derived in Section 3, at first, we obtain

Theorem 4.1. *If there exists a Hermitian metric on $\mathcal{E}$ which is Nakano-negative along the base, then $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-negative.*

Proof. It follows from Theorem 3.3. In fact, we set $u = u^{i\alpha} \frac{\partial}{\partial t^i} \otimes e_\alpha$. Naturally, e_α can be viewed as a local holomorphic section of $H^{n,0}(X_s, \mathcal{E}_s)$. We set

$$\Theta^{\mathcal{E}}_{i\bar{j}\alpha\bar{\beta}} = \Theta^{\mathcal{E}} \left(\frac{\partial}{\partial t^i} \otimes e_\alpha, \frac{\partial}{\partial t^j} \otimes e_\beta \right),$$

then by (3.2),

$$(4.1) \qquad \Theta^E_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \overline{u}^{j\beta} = c_n \int_{X_s} \Theta^{\mathcal{E}}_{i\bar{j}\alpha\bar{\beta}} \{u^{i\alpha}, u^{j\beta}\} - (\Delta'_X v, v)$$

where $v = -\sum_i \nabla_X'^* \mathbb{G}' \pi_\perp \left(\nabla_i' u^{i\alpha} \otimes e_\alpha \right)$. If $\mathcal{E}$ admits a Hermitian metric which is Nakano-negative along the base, then the first term in the formula (4.1) is negative. $\square$

Corollary 4.2 ([LSYang13]). *If (E, h) is a Griffiths-positive vector bundle, then $E \otimes \det E$ is both Nakano positive and dual Nakano-positive.*

Proof. The Nakano-positivity is well-known([Demailly], [Bern09]). Now we prove the dual Nakano-positivity. Let $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle of $\mathbb{P}(E)$. Note that $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample, but L is not. The metric on (E, h) induces a metric on L which is negative along the base([Demailly, Chapter V, formula 15.15], [LSYang13, (2.12)]). On the other hand, it is easy to see

$$E^* \otimes \det E^* = p_*(K_{\mathbb{P}(E)/S} \otimes L^{r+1})$$

where $p : \mathbb{P}(E) \to S$ is the projection. Hence, by Theorem 4.1, $E^* \otimes \det E^*$ is Nakano-negative, or equivalently, $E \otimes \det E$ is dual Nakano-positive. $\square$

Similarly, for the Nakano-positivity, it follows from Proposition 3.10 and the proof is similar to that of Theorem 4.1.

Corollary 4.3 ([MouTak08]). *$p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-positive if $\mathcal{E}$ is Nakano-positive.*

Corollary 4.4 ([Bern09]). *Let $\mathcal{L}$ be a line bundle over $\mathcal{X}$. Then $p_*(K_{\mathcal{X}/S} \otimes \mathcal{L})$ is Nakano-positive if $\mathcal{L}$ is ample.*

Let X be a compact Fano manifold and $h(t) = e^{-\varphi(t)}$ be a family of positive metrics on $L = -K_X$. Let $(z^1, \cdots, z^n)$ be the local holomorphic coordinates on X. We set the local volume form

$$dV_{\mathbb{C}} = (\sqrt{-1})^n (dz^1 \wedge d\overline{z}^1 + \cdots + dz^n \wedge d\overline{z}^n)^n.$$

It is easy to see that

$$e^{-\varphi} dV_{\mathbb{C}}$$

is a family of globally defined volume of X. Berndtsson in [Bern11a] considers the logarithm volume

(4.2) $$\mathcal{F}(t) = -\log \left(\int_X e^{-\varphi} dV_{\mathbb{C}} \right)$$

and deduces that

Theorem 4.5 ([Bern11a]). *If $e^{-\varphi(t)}$ is a subgeodesics in the Kähler cone $\mathcal{K}_L$ of the class $c_1(L)$, i.e.*

$$c(\varphi) = \ddot{\varphi} - |\overline{\partial}_X \dot{\varphi}|^2 \geq 0,$$

then $\mathcal{F}(t)$ is convex.

In fact, Theorem 4.5 can be obtained easily from Corollary 3.7, following the setting in [Bern11a]. To formulate it efficiently, we use complex parameter t in the unit disk $\mathbb{D} \subset \mathbb{C}$. When we consider the direct image bundle $E = p_*(K_X \otimes L)$, it is a trivial line bundle since $L = -K_X$ and $H^0(X, K_X \otimes L) \cong \mathbb{C}$. Since E is trivial, there is a constant section $u = 1e_E$ of E, and it is identified as a holomorphic section u of $H^{n,0}(X, L)$,

$$u = dz^1 \wedge \cdots \wedge dz^n \otimes e$$

where $e = \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}$. Hence

$$\|u\|^2 = c_n \int_X \{u, u\} = \int_X e^{-\varphi} dV_{\mathbb{C}}$$

On the other hand, it is obvious that

(4.3)
$$\|u\|^2 \sqrt{-1}\partial\bar{\partial}\mathcal{F} = (\sqrt{-1}\Theta^E u, u)$$

Hence, if $c(\varphi) \geq 0$, by Corollary 3.7, $\sqrt{-1}\partial\bar{\partial}\mathcal{F}$ is Hermitian semi-positive. In real parameters, it says that $\mathcal{F}$ is convex.

As a partial converse to Berndtsson's result, we have

Proposition 4.6. *Let $e^{-\varphi(t)}$ be a curve in the Kähler cone $\mathcal{K}_L$. If $\varphi(t)$ is concave in t, then so is $\mathcal{F}(t)$.*

Proof. It follows from Theorem 1.1. In fact, $\ddot{\varphi} \leq 0$ implies the first term on the right hand side of (1.2) is negative. By formula (4.3), we see $\mathcal{F}$ is superharmonic and in the real case, it is concave. $\square$

We can also see how Theorem 4.5 and Proposition 4.6 work by the following simple example. At first, we fix a positive metric $e^{-\varphi(0)}$ in $c_1(L)$ and set

$$\varphi(t) = f(t) + \varphi(0)$$

where t is a real parameter. It is obvious that

$$c(\varphi) = \ddot{\varphi} = \ddot{f}, \qquad \mathcal{F}(t) = f(t) + c$$

Hence $\mathcal{F}$ is concave if φ is concave and vice visa.

For the general case, it is not hard to see that both Theorem 4.5 and Proposition 4.6 amount to the basic $\bar{\partial}$-estimate

(4.4)
$$\|\dot{\varphi}\| \leq \|\bar{\partial}_X \dot{\varphi}\|$$

if the Fano manifold X is polarized by its anti-canonical class.

5. Direct images of projectively flat vector bundles

In this section we only make the assumption that the vector bundle $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ is Nakano semi-positive.

Let's recall that $c_{i\bar{j}}$ is the $\mathcal{E}$-valued $(n, 0)$-form coefficient of $dt^i \wedge d\bar{t}^j$ in the expression

$$\Theta^{\mathcal{E}}(u - dt^i \wedge v_i).$$

There are four (linearly independent) terms in the expression of $\Theta^{\mathcal{E}}(u - dt^i \wedge v_i)$. However, if $\Theta^{\mathcal{E}}$ is Nakano semi-positive, then $c_{i\bar{j}}$ dominates the degeneracy of $\Theta^{\mathcal{E}}(u - dt^i \wedge v_i)$, i.e. $c_{i\bar{j}} = 0$ implies $\Theta^{\mathcal{E}}(u - dt^i \wedge v_i) = 0$. This is the content of the next theorem.

Theorem 5.1. *Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be Nakano semi-positive. Then*

$$(\sqrt{-1}\Theta^E u, u) = 0$$

if and only if $c_{ij} = 0$.

Proof. Note that if $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ is Nakano semi-positive, then by formula (3.24),

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} + \left(\nabla_X'' v_i, \nabla_X'' v_j\right) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

is a Hermitian semi-positive $(1,1)$-form. If $(\sqrt{-1}\Theta^E u, u) = 0$, we get

$$c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} = 0$$

and so $\Theta^{\mathcal{E}}\widetilde{u} = 0$. In particular, $c_{i\bar{j}} = 0$ since $\Theta^{\mathcal{E}}\widetilde{u} = \Theta^{\mathcal{E}}(u - dt^i \wedge v_i)$.

On the other hand, by Theorem 3.11, if $c_{ij} = 0$,

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \{c_{i\bar{j}}, u\} \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j) = 0.$$

$\square$

Remark 5.2. We can see the deformation triviality of $\mathcal{E} \to \mathcal{X} \to S$ from the degeneracy of the curvature Θ^E. In particular, we can analyze it by using $c_{i\bar{j}}$.

Now we continue to analyze the case $(\sqrt{-1}\Theta^E u, u) = 0$. In the following, we use an idea in [Bern11a].

Lemma 5.3. *If $H^{n,1}(X_s, \mathcal{E}_s) = 0$, then v_i is holomorphic on $\mathcal{X}$ for any i.*

Proof. We only need to show $\frac{\partial v_i}{\partial \bar{t}^j} = 0$ since $\nabla_X'' v_i = 0$ is obvious from curvature formula (3.24) when $(\sqrt{-1}\Theta^E u, u) = 0$.

Next we claim

(5.1) $$\nabla_X'\left(\frac{\partial v_i}{\partial \bar{t}^j}\right) = c_{i\bar{j}} + \frac{\partial}{\partial \bar{t}^j}\pi\left(\nabla_i' u\right)$$

In fact,

$$
\begin{aligned}
-\nabla'\nabla''(dt^i \wedge v_i) &= -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \nabla''\nabla'(dt^i \wedge v_i) \\
&= -\Theta^{\mathcal{E}}(dt^i \wedge v_i) - \nabla''(dt^i \wedge \nabla_X' v_i) - \nabla''(dt^i \wedge dt^k \wedge \nabla_k' v_i) \\
&= -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \nabla''\left(dt^i \wedge \left(\nabla_i' u - \pi(\nabla_i' u)\right)\right) - \nabla''(dt^i \wedge dt^k \wedge \nabla_k' v_i) \\
&= -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \Theta^{\mathcal{E}} u - \nabla''\left(dt^i \wedge \pi(\nabla_i' u)\right) - \nabla''(dt^i \wedge dt^k \wedge \nabla_k' v_i)
\end{aligned}
$$

By comparing the coefficients $dt^i \wedge d\bar{t}^j$ on both sides, we get (5.1). If the curvature is zero, we also have $c_{i\bar{j}} = 0$. According to different types in Hodge decomposition, we conclude from (5.1) that

$$\nabla_X'\left(\frac{\partial v_i}{\partial \bar{t}^j}\right) = \frac{\partial}{\partial \bar{t}^j}\pi\left(\nabla_i' u\right) = 0$$

Therefore, $\nabla_X''^*(\omega \wedge \frac{\partial v_i}{\partial \bar{t}^j}) = 0$ and $\nabla_X''(\omega \wedge \frac{\partial v_i}{\partial \bar{t}^j}) = 0$. The cohomology assumption ensures $\omega \wedge \frac{\partial v_i}{\partial \bar{t}^j} = 0$ and so $\frac{\partial v_i}{\partial \bar{t}^j} = 0$.

$\square$

In the following, we assume that $(\mathcal{E}_s, h^{\mathcal{E}_s})$ is projectively flat. Hence, the curvature tensor can be written as (c.f. [Koba87])

$$\sqrt{-1}\Theta^{\mathcal{E}_s} = \frac{1}{r} Ric(\det \mathcal{E}_s) \otimes h^{\mathcal{E}_s} \tag{5.2}$$

where r is the rank of $\mathcal{E}_s$. If $\det \mathcal{E}_s$ is positive, we set

$$\omega_g = \frac{1}{r} Ric(\mathcal{E}_s) = -\frac{\sqrt{-1}}{r}\partial_X\overline{\partial}_X \log \det(h^{\mathcal{E}_s}) \tag{5.3}$$

as the background Kähler metric on each fiber. Therefore,

$$\sqrt{-1}\Theta^{\mathcal{E}_s} = \omega_g \otimes h^{\mathcal{E}_s} \tag{5.4}$$

Recall that $[\alpha_i] \in H^{0,1}(X_s, End(\mathcal{E}_s))$ is the Kodaira-Spencer class of the deformation $\mathcal{E} \to \mathcal{X} \to S$ in the direction of $\frac{\partial}{\partial t^i}$, i.e.

$$\alpha_i = \Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)\Big|_{X_s} \in \Omega^{0,1}(X_s, End(\mathcal{E}_s)). \tag{5.5}$$

Let W_i be the dual vector of α_i, i.e. W_i is an $End(\mathcal{E}_s)$-valued $(1,0)$-vector field on the fiber X_s. Then by formulas (5.4) and (5.5), we have

$$
\begin{aligned}
(i_{W_i}\omega) \wedge u &= \sqrt{-1}\alpha_i \wedge u = \sqrt{-1}\Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t_i}, u\right) \\
&= \sqrt{-1}\nabla''_X \nabla'_i u
\end{aligned}
\tag{5.6}
$$

since $\nabla''_X u = 0$.

Proposition 5.4. *We have the relation*

$$i_{W_i} u = -v_i. \tag{5.7}$$

Moreover, W_i is an $End(\mathcal{E}_s)$-valued holomorphic vector field on the fiber X_s.

Proof. By formula (5.6), we obtain

$$
\begin{aligned}
(i_{W_i}\omega_g) \wedge u &= \sqrt{-1}\nabla''_X \nabla'_i u = -\sqrt{-1}\nabla''_X \nabla'_X v_i \\
&= -\sqrt{-1}\Theta^{\mathcal{E}_s}(v_i)
\end{aligned}
$$

since $\nabla''_X v_i = 0$. On the other hand, $(i_{W_i}\omega_g) \wedge u = (i_{W_i}u) \wedge \omega_g$. Hence we obtain (5.7) by using (5.4) again. Since v_i and u are all holomorphic on each fiber, we know W_i is also holomorphic. $\square$

We can extend the vector field $\frac{\partial}{\partial t^i}$ to an $End(\mathcal{E}_s)$-valued vector field. We still denote it by $\frac{\partial}{\partial t^i}$. Then

$$V_i = \frac{\partial}{\partial t^i} - W_i \tag{5.8}$$

is a (local) $End(\mathcal{E})$-valued holomorphic vector field over the total space $\mathcal{X}$. Let $\mathscr{L}$ be the type $(1,0)$, $End(\mathcal{E})$-valued Lie derivative, then we have

$$\mathscr{L}_{V_i}\omega_g = 0 \tag{5.9}$$

In fact, by relation (5.5), we have

$$\mathscr{L}_{W_i}\omega_g = \nabla'_X(-\sqrt{-1}\alpha) = -\sqrt{-1}\nabla'_X\left(\Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)|_{X_s}\right) = -\nabla'_{\frac{\partial}{\partial t^i}}\Theta^{\mathcal{E}_s}$$

Hence, by formula (5.4), we get (5.9). That means, if the curvature Θ^E degenerates at some point $s \in S$, then the family $\mathcal{E} \to \mathcal{X} \to S$ moves by an infinitesimal automorphism of $\mathcal{E}$.

We summarize the above into a global version. Let $\mathcal{X} = X \times \mathbb{D}$, where $\mathbb{D}$ is a unit disk. Let $\mathbb{E}_0 \to X$ be a holomorphic vector bundle. If $(\mathbb{E}_0, h(t))_{t\in\mathbb{D}} \to X$ is a smooth family of projectively flat vector bundles. We assume $Ric(\det E, h(t)) > 0$ for all t and set $\omega_t = -\frac{\sqrt{-1}}{r}\partial_X\overline{\partial}_X \log \det(h(t))$ to be a smooth family of Kähler metrics on X. We also denote by $\mathcal{E}$, the pullback family $p_2^*(\mathbb{E}_0)$ over $p_2 : \mathcal{X} \to X$.

Theorem 5.5. *If the curvature Θ^E of $E = p_*(K_{\mathcal{X}/\mathbb{D}} \otimes \mathcal{E})$ vanishes in a small neighborhood of $0 \in \mathbb{D}$, then there exists a holomorphic vector field V on X with flows $\Phi_t \in Aut_H(X, \mathbb{E}_0)$ such that*

$$\Phi_t^*(\omega_t) = \omega_0$$

for small t.

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