HODGE THEORY AND DEFORMATIONS OF KÄHLER MANIFOLDS

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ABSTRACT. We prove several formulas related to Hodge theory and the Kodaira-Spencer-Kuranishi deformation theory of Kähler manifolds. As applications, we present a construction of globally convergent power series of integrable Beltrami differentials on Calabi-Yau manifolds, and use an iteration method to construct global canonical families of holomorphic $(n, 0)$-forms on the deformation spaces of Kähler manifolds.

1. INTRODUCTION

In this paper, we will present several results about Hodge theory and the deformation theory of Kodaira-Spencer-Kuranishi on compact Kähler manifolds. Our main observations include a simple L^2-quasi-isometry result for bundle valued differential forms, an explicit formula for the deformed $\overline{\partial}$-operator, and an iteration method to construct global Beltrami differentials on Calabi-Yau (CY) manifolds and holomorphic $(n, 0)$-forms on the deformation of compact Kähler manifolds of dimension n. We will present an alternative simple method to solve the $\overline{\partial}$-equation, prove global convergence of the formal power series of the Beltrami differentials and the holomorphic $(n, 0)$-forms constructed from the Kodaira-Spencer-Kuranishi theory. These series previously were only proved to converge in an arbitrarily small neighborhood. We will discuss more applications to the Torelli problem and the extension of twisted pluricanonical sections in a sequel to this paper.

Let us first fix some notations to be used throughout this paper. All the manifolds in this paper are assumed to be compact, though some results still hold for complete Kähler manifolds. In this paper a Calabi-Yau, or CY manifold X, is a compact projective manifold with trivial canonical line bundle. By Yau's solution of Calabi conjecture, there is a CY metric on X of dimension n such that the holomorphic $(n, 0)$-form Ω_0 on X is parallel with respect to the metric connection. For a complex manifold (X, ω) and a Hermitian holomorphic vector bundle (E, h) on X, we denote by $A^{p,q}(X)$ the space of smooth (p, q)-forms on X and by $A^{p,q}(E) = A^{p,q}(X, E)$ the space of smooth (p, q)-forms on X with values in E. Similarly, let $\mathbb{H}^{p,q}(X)$ be the space of the harmonic (p, q)-forms and let $\mathbb{H}^{p,q}(X, E)$ be the space of the harmonic (p, q)-forms with values in E. Let ∇ be the Chern connection on (E, h) with canonical decomposition $\nabla = \nabla' + \overline{\partial}$, where ∇' is the $(1, 0)$ part of the Chern connection ∇. Let $\mathbb{G}$ and $\mathbb{H}$ denote the Green operator and harmonic projection in the Hodge decomposition with respect to the operator $\overline{\partial}$, that is $\mathbb{I} = \mathbb{H} + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\mathbb{G}$. A Beltrami differential is an element in $A^{0,1}(X, T^{1,0}_X)$, where $T^{1,0}_X$ denotes the holomorphic tangent bundle of X. The L^2-norm $\| \cdot \| = \| \cdot \|^{\frac{1}{2}}_{L^2}$ is induced by the metrics ω and h. The Hölder $C^{k,\alpha}$-norm $\| \cdot \|_{k,\alpha}$ will be used on the Beltrami differentials.

Now we briefly describe the main results in this paper. The following quasi-isometry on compact Kähler manifolds is obtained in Section 2.

Date: July 6, 2012.

2010 *Mathematics Subject Classification.* Primary 32G05; Secondary 58A14, 53C55, 14J32.

Key words and phrases. Deformations of complex structures, Hodge theory, Hermitian and Kählerian manifolds, Calabi-Yau manifolds.

Theorem 1.1 (Quasi-isometry). *Let (E,h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X,ω).*

(1) *For any $g\in A^{n,*}(X,E)$, we have the following estimate*

$$\|\overline{\partial}^*\mathbb{G}g\|^2\leq\langle g,\mathbb{G}g\rangle.$$

(2) *If (E,h) is a strictly positive line bundle and $\omega=\sqrt{-1}\Theta^E$, then for any $g\in A^{n-1,*}(X,E)$, we obtain*

$$\|\overline{\partial}^*\mathbb{G}\nabla' g\|\leq\|g\|.$$

(3) *If E is the trivial line bundle, for any smooth $g\in A^{*,*}(X)$,*

$$\|\overline{\partial}^*\mathbb{G}\partial g\|\leq\|g\|.$$

In particular, if $\overline{\partial}\partial g=0$ and g is ∂^*-exact, we obtain the isometry

$$\|\overline{\partial}^*\mathbb{G}\partial g\|=\|g\|.$$

Here the operator $\overline{\partial}^*\mathbb{G}$ can be viewed as the "inverse operator" of $\overline{\partial}$. More precisely, we can write down the explicit solutions of some $\overline{\partial}$-equations by using $\overline{\partial}^*\mathbb{G}$, which can also be considered as a bundle-valued version of the very useful $\partial\overline{\partial}$-lemma in complex geometry.

Proposition 1.2. *Let (E,h) be a Hermitian holomorphic vector bundle with semi-Nakano positive curvature tensor Θ^E over the compact Kähler manifold (X,ω). Then, for any $g\in A^{n-1,*}(X,E)$ with $\overline{\partial}\nabla' g=0$, the $\overline{\partial}$-equation $\overline{\partial}s=\nabla' g$ admits a solution*

$$s=\overline{\partial}^*\mathbb{G}\nabla' g,$$

such that

$$\|s\|^2\leq\langle\nabla' g,\mathbb{G}\nabla' g\rangle.$$

Moreover, this solution is unique if we require $\mathbb{H}(s)=0$ and $\overline{\partial}^*s=0$.

Note that the proofs of the above results only need very basic properties of Hodge theory, so they still hold on general Kähler manifolds as long as Hodge theory can be applied.

Next we let i_ϕ and $\phi\lrcorner$ both denote the contraction operator with $\phi\in A^{0,1}(X,T^{1,0}_X)$. The Lie derivative can be lifted to bundle valued forms by

$$\mathcal{L}_\phi=-\nabla\circ i_\phi+i_\phi\circ\nabla.$$

The following theorem in Section 3 gives explicit formulas for the deformed differential operators on the deformation spaces of complex structures.

Theorem 1.3. *Let $\phi\in A^{0,1}(X,T^{1,0}_X)$. Then on the space $A^{*,*}(X,E)$, we have*

$$e^{-i_\phi}\circ\nabla\circ e^{i_\phi}=\nabla-\mathcal{L}_\phi-\ i_{\frac{1}{2}[\phi,\phi]}=\nabla-\mathcal{L}^{1,0}_\phi+i_{\overline{\partial}\phi-\frac{1}{2}[\phi,\phi]}.$$

In particular, if $\sigma\in A^{n,*}(X,E)$ and ϕ is integrable, i.e., $\overline{\partial}\phi-\frac{1}{2}[\phi,\phi]=0$, then

$$\left(e^{-i_\phi}\circ\nabla\circ e^{i_\phi}\right)(\sigma)=\overline{\partial}\sigma+\nabla'(\phi\lrcorner\sigma).$$

As an application of Theorem 1.1 and Theorem 1.2, we develop a recursive method in Section 4 to construct Beltrami differentials in Kodaira-Spencer-Kuranishi deformation theory, which are globally convergent in L^2-norm. More precisely we will obtain the following global convergence result on the deformation space of CY manifolds.

Theorem 1.4. *Let X be a CY manifold and $\varphi_1 \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ with norm $\|\varphi_1\| = \frac{1}{2}$. Then there exits a smooth globally convergent power series in L^2-norm for $|t| \leq 1$,*

$$\Phi(t) = \varphi_1 t^1 + \varphi_2 t^2 + \cdots + \varphi_k t^k + \cdots \in A^{0,1}(X, T_X^{1,0}),$$

which satisfies:

(1) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$;
(2) $\overline{\partial}^* \varphi_k = 0$ *for each* $k \geq 1$;
(3) $\varphi_k \lrcorner \Omega_0$ *is* ∂*-exact for each* $k \geq 2$*, where* Ω_0 *is a holomorphic* $(n, 0)$*-form on* X;
(4) $\|\Phi(t) \lrcorner \Omega_0\|_{L^2} < \infty$ *as long as* $|t| \leq 1$.

The key point in the above result is that the convergent radius of the power series is at least 1, which was previously proved to be sufficiently small. This power series thus obtained is called an L^2-global canonical family of Beltrami differentials on the CY manifold.

In Section 5, we obtain the following theorem to construct deformations of holomorphic $(n, 0)$-forms, which are globally convergent in L^2-norm for CY manifolds. Analogous results for general compact Kähler manifolds are also proved in Section 5.

Theorem 1.5. *Let $X_t = (X_t, J_{\Phi(t)})$ be the deformation of the CY manifold X induced by $\Phi(t)$ as constructed in Theorem 1.4. Then for any holomorphic $(n, 0)$-form Ω_0 on X and $|t| < 1$, $\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_0$ defines an L^2-global canonical family of holomorphic $(n, 0)$-forms on X_t.*

The proof of this theorem is based on the global construction of $\Phi(t)$ in Theorem 1.4 and an iteration procedure to construct holomorphic sections of the canonical line bundle K_{X_t} of the deformation X_t of a Kähler manifold. As a straightforward corollary of Theorem 1.5, we have the following global expansion of the canonical family of $(n, 0)$-forms on the deformation space of CY manifolds in cohomology classes. This expansion has interesting applications in studying the global Torelli problem.

Corollary 1.6. *With the same notations as in Theorem 1.5, there holds the following global expansion of $[\Omega_t^C]$ in cohomology classes for $|t| < 1$*

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega_0] t_i + O(|t|^2), \tag{1.1}$$

where $O(|t|^2) \in \bigoplus_{j=2}^{n} H^{n-j,j}(X)$ denotes the terms of orders at least 2 in t and $N =$ $\dim \mathbb{H}^{0,1}(X, T_X^{1,0})$.

Finally we briefly explain the backgrounds and motivations. The main motivation is the classical Teichmüller theory for Riemann surfaces in [1], where the main result is the proof of the existence of a solution of the Beltrami differential equation in $\mathbb{C}$

$$\frac{\partial}{\partial \overline{z}} w(\tau; z, \overline{z}) = \tau \mu(z, \overline{z}) \frac{\partial}{\partial z} w(\tau; z, \overline{z}),$$

where $\mu(z, \overline{z})$ is a Beltrami differential with $\|\mu(z, \overline{z})\|_{L^\infty} \leq c < 1$. The solution of Beltrami differential equation is based on an iteration method due to Bojairski [2], while it was Morrey [9] who first proved the existence of the solution of Beltrami equation. One of the main ingredients in the proof of the convergence of the Bojairski iteration method is the L^2-isometry of the inverse $\overline{\partial}^{-1}$ of the $\overline{\partial}$ operator in one complex variable. The iteration method of Bojairski was generalized by Kuranishi, who considered a compact

Hermitian manifold (X,ω) without global holomorphic vector fields and used an analogue of the Bojairski method to construct the Kuranishi map

$$\kappa:\ \mathbb{H}^{0,1}\left(X,T_X^{1,0}\right)\to\mathbb{H}^{0,2}\left(X,T_X^{1,0}\right)$$

as follows. Let $\{\phi_i\}_{i=1}^N$ be a basis of $\mathbb{H}^{0,1}\left(X,T_X^{1,0}\right)$ and let us consider a power series $\phi_X\left(\tau\right)$ with coefficients in $A^{0,1}\left(X,T_X^{1,0}\right)$ defined by the recursive relations:

$$\phi_X\left(\tau\right)=\sum_{i=1}^N\phi_i\tau^i+\frac{1}{2}\overline{\partial}^*\circ\mathbb{G}\left[\phi_X(\tau),\phi_X(\tau)\right].$$

Kuranishi showed that there exists $\varepsilon>0$ such that $\phi_X(\tau)\in A^{0,1}\left(X,T_X^{1,0}\right)$ for $|\tau|<\varepsilon$, and defined the map:

$$\kappa\left(\sum_{i=1}^N\phi_i\tau^i\right)=\mathbb{H}\left(\left[\phi_X(\tau),\phi_X(\tau)\right]\right)\in\mathbb{H}^{0,2}\left(X,T_X^{1,0}\right).\tag{1.2}$$

He proved that there exists a locally complete family of complex manifold $\pi:\ \mathcal{X}\to\kappa^{-1}(0)$. The Kuranishi map (1.2) is the most basic technical tool in various aspects of deformation theory. For details see [10].

Acknowledgement This paper originated from many discussions with Prof. Andrey Todorov, who unexpectedly passed away in March 2012 during his visit of Jerusalem. We dedicate this paper to his memory. The second author would also like to express his gratitude to Weijun Lu, Quanting Zhao and Shengmao Zhu for their interest and useful comments.

2. $\overline{\partial}$-equations on non-negative vector bundles

In this section, we will prove a quasi-isometry result in L^2-norm with respect to the operator $\overline{\partial}^*\circ\mathbb{G}$ on a compact Kähler manifold. This gives a rather simple and explicit way to solve vector bundle valued $\overline{\partial}$-equations with L^2-estimate.

Let (E,h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X,ω) and $\nabla=\nabla'+\overline{\partial}$ be the Chern connection on it. With respect to metrics on E and X, we set

$$\overline{\Box}=\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial},$$
$$\Box'=\nabla'\nabla'^*+\nabla'^*\nabla'.$$

Accordingly, we associate the Green operators and harmonic projections $\mathbb{G}$, $\mathbb{H}$ and $\mathbb{G}'$, $\mathbb{H}'$ in Hodge decomposition to them, respectively. More precisely,

$$\mathbb{I}=\mathbb{H}+\overline{\Box}\circ\mathbb{G},\ \ \mathbb{I}=\mathbb{H}'+\Box'\circ\mathbb{G}'.$$

Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $\Theta^E\in\Gamma(X,\Lambda^2T^*X\otimes E^*\otimes E)$ has the form

$$\Theta^E=R_{i\bar{j}\alpha}^\gamma dz^i\wedge d\bar{z}^j\otimes e^\alpha\otimes e_\gamma,\tag{2.1}$$

where $R_{i\bar{j}\alpha}^\gamma=h^{\gamma\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}$ and

$$R_{i\bar{j}\alpha\bar{\beta}}=-\frac{\partial^2h_{\alpha\bar{\beta}}}{\partial z^i\partial\bar{z}^j}+h^{\gamma\bar{\delta}}\frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\bar{\beta}}}{\partial\bar{z}^j}.\tag{2.2}$$

Definition 2.1. *A Hermitian vector bundle (E,h) is said to be* semi-Nakano-positive *(resp.* Nakano-positive*), if for any nozero vector $u=u^{i\alpha}\frac{\partial}{\partial z^i}\otimes e_\alpha$,*

$$\sum_{i,j,\alpha,\beta}R_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\bar{u}^{j\beta}\geq 0,\ \ (resp. >0).\tag{2.3}$$

For a line bundle, it is strictly positive if and only if it is Nakano-positive.

Theorem 2.2 (Quasi-isometry)**.** *Let (E,h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X,ω).*

(1) *For any $g \in A^{n,*}(X,E)$, we have the following estimate*

$$\|\overline{\partial}^*\mathbb{G}g\|^2 \leq \langle g, \mathbb{G}g\rangle.$$

(2) *If (E,h) is a strictly positive line bundle and $\omega = \sqrt{-1}\Theta^E$, for any $g \in A^{n-1,*}(X,E)$,*

$$\|\overline{\partial}^*\mathbb{G}\nabla' g\| \leq \|g\|.$$

(3) *If E is the trivial line bundle, for any smooth $g \in A^{*,*}(X)$,*

$$\|\overline{\partial}^*\mathbb{G}\partial g\|^2 = \|g\|^2 - \|\mathbb{H}(g)\|^2 - \langle \partial^* g, \mathbb{G}(\partial^* g)\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \leq \|g\|^2.$$

In particular, if $\overline{\partial}\partial g = 0$ and g is ∂^*-exact, we obtain the isometry

$$\|\overline{\partial}^*\mathbb{G}\partial g\| = \|g\|.$$

Proof. (1). For $g \in A^{n,*}(X,E)$,

$$\begin{aligned}
\|\overline{\partial}^*\mathbb{G}g\|^2 &= \langle \overline{\partial}\overline{\partial}^*\mathbb{G}g, \mathbb{G}g\rangle \\
&= \langle g, \mathbb{G}g\rangle - \langle \overline{\partial}^*\overline{\partial}\mathbb{G}g, \mathbb{G}g\rangle - \langle \mathbb{H}g, \mathbb{G}g\rangle \\
&= \langle g, \mathbb{G}\rangle - \langle \overline{\partial}\mathbb{G}g, \overline{\partial}\mathbb{G}g\rangle \\
&\leq \langle g, \mathbb{G}g\rangle
\end{aligned}$$

since the Green operator is self-adjoint and zero on the kernel of Laplacian by definition.

(2). For any $g \in A^{n-1,*}(X,E)$, by the well-known Bochner-Kodaira-Nakano identity $\overline{\Box} = \Box' + [\sqrt{-1}\Theta^E, \Lambda_\omega]$, we have

$$\overline{\Box}(\nabla' g) = \Box'(\nabla' g) + q(\nabla' g) = (\Box' + q)(\nabla' g).$$

Hence $\mathbb{H}(\nabla' g) = 0$ and $\overline{\Box}\mathbb{G}(\nabla' g) = \nabla' g = \Box'\mathbb{G}'(\nabla' g)$ since obviously $\mathbb{H}'(\nabla' g) = 0$ by Hodge decomposition. Moreover,

$$\begin{aligned}
\langle \nabla' g, \mathbb{G}(\nabla' g)\rangle &= \langle \nabla' g, \overline{\Box}^{-1}(\nabla' g)\rangle \\
&= \langle \nabla' g, (\Box' + q)^{-1}(\nabla' g)\rangle \\
&\leq \langle \nabla' g, \Box'^{-1}(\nabla' g)\rangle \\
&= \langle \nabla' g, \mathbb{G}'(\nabla' g)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\overline{\partial}^*\mathbb{G}\nabla' g\|^2 &\leq \langle \nabla' g, \mathbb{G}\nabla' g\rangle \\
&\leq \langle \nabla' g, \mathbb{G}'\nabla' g\rangle \\
&= \langle g, \nabla'^*\nabla'\mathbb{G}' g\rangle \\
&= \langle g, g - \mathbb{H}'(g) - \nabla'\nabla'^*\mathbb{G}' g\rangle \\
&= \|g\|^2 - \|\mathbb{H}'(g)\|^2 - \langle \nabla'^* g, \mathbb{G}'\nabla'^* g\rangle \\
&\leq \|g\|^2.
\end{aligned}$$

(3). If E is the trivial line bundle, for any $g \in A^{*,*}(X)$, we have

$$\begin{aligned}
\|\overline{\partial}^*\mathbb{G}\partial g\|^2 &= \left\langle \overline{\partial}^*\mathbb{G}\partial g, \overline{\partial}^*\mathbb{G}\partial g \right\rangle = \left\langle \overline{\partial}\overline{\partial}^*\mathbb{G}\partial g, \mathbb{G}\partial g \right\rangle \\
&= \left\langle \overline{\Box}\mathbb{G}\partial g - \overline{\partial}^*\overline{\partial}\mathbb{G}\partial g, \mathbb{G}\partial g \right\rangle \\
&= \langle \partial g, \mathbb{G}\partial g\rangle - \left\langle \overline{\partial}^*\overline{\partial}\mathbb{G}\partial g, \mathbb{G}\partial g \right\rangle \\
&= \langle g, \partial^*\partial\mathbb{G}g\rangle - \langle \mathbb{G}\overline{\partial}\partial g, \mathbb{G}\overline{\partial}\partial g\rangle \\
&= \langle g, \Box'\mathbb{G}g - \partial\partial^*\mathbb{G}g\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \\
&= \langle g, g - \mathbb{H}(g) - \partial\partial^*\mathbb{G}g\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \\
&= \|g\|^2 - \|\mathbb{H}(g)\|^2 - \langle \partial^* g, \mathbb{G}(\partial^* g)\rangle - \|\mathbb{G}(\overline{\partial}\partial g)\|^2 \\
&\leq \|g\|^2,
\end{aligned}$$

since the Green operator is nonnegative. In particular, if $\overline{\partial}\partial g = 0$ and g is ∂^*-exact, we have $\mathbb{H}(g) = 0$ and $\partial^* g = 0$. Hence, we obtain the isometry $\|\overline{\partial}^* G\partial g\| = \|g\|$. $\Box$

Proposition 2.3 ($\overline{\partial}$-Inverse formula)**.** *Let (E, h) be a Hermitian holomorphic vector bundle with semi-Nakano positive curvature Θ^E over the compact Kähler manifold (X, ω). Then, for any $g \in A^{n-1,*}(X, E)$,*

$$s = \overline{\partial}^*\mathbb{G}\nabla' g$$

is a solution to the equation $\overline{\partial}s = \nabla' g$ with $\overline{\partial}\nabla' g = 0$, such that

$$\|s\|^2 \leq \langle \nabla' g, \mathbb{G}\nabla' g\rangle.$$

This solution is unique as long as it satisfies $\mathbb{H}(s) = 0$ and $\overline{\partial}^* s = 0$.

It is worth noting that unlike Hörmander's L^2-estimate to solve $\overline{\partial}$-equation, we do not need the a priori L^2-estimate condition here.

Proof. By the well-known Bochner-Kodaira-Nakano identity $\overline{\Box} = \Box' + [\sqrt{-1}\Theta^E, \Lambda_\omega]$, one can see that for any $\phi \in A^{n,*}(X, E)$,

$$\langle \sqrt{-1}[\Theta^E, \Lambda_\omega]\phi, \phi\rangle \geq 0$$

if E is semi-Nakano positive (e.g. [4]). It implies that, for any $\phi \in A^{n,*}(X, E)$,

$$\langle \overline{\Box}\phi, \phi\rangle \geq \langle \Box'\phi, \phi\rangle.$$

Thus, on the space $A^{n,*}(X, E)$,

$$\ker \overline{\Box} \subseteq \ker \Box'. \tag{2.4}$$

By Hodge decomposition, we have

$$\overline{\partial}s = \overline{\partial}\overline{\partial}^*\mathbb{G}\nabla' g = \nabla' g - \mathbb{H}\nabla' g - \overline{\partial}^*\overline{\partial}\mathbb{G}\nabla' g = \nabla' g - \mathbb{H}\nabla' g = \nabla' g,$$

where the identity $\mathbb{H}\nabla' g = 0$ is used. Actually, we know $\nabla' g \perp \ker \Box'$ and obviously $\nabla' g \perp \ker \overline{\Box}$ by (2.4). The uniqueness of this solution follows easily: If $\mathbb{H}(s) = 0$, $\overline{\partial}^* s = 0$ and $\overline{\partial}s = 0$, then

$$s = \mathbb{H}(s) \oplus \overline{\Box}\mathbb{G}(s) = \mathbb{H}(s) \oplus (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\mathbb{G}(s) = 0.$$

$\Box$

3. Beltrami differentials and deformation theory

In this section we prove several new formulas to construct explicit deformed differential operators for bundle valued differential forms on the deformation spaces of Kähler manifolds. These formulas will be applied to the deformation space of CY manifolds in later sections while more applications to the deformation theory of Kähler manifolds and holomorphic line bundles will be discussed in the sequel to this paper. Throughout this section, X is always assumed to be a complex manifold.

For $X_0 \in \Gamma(X, T_X^{1,0})$, the contraction operator is defined as

$$i_{X_0} : A^{p,q}(X) \to A^{p-1,q}(X)$$

by

$$(i_{X_0}\omega)(X_1, \cdots, X_{p-1}, Y_1, \cdots, Y_q) = \omega(X_0, X_1, \cdots, X_{p-1}, Y_1, \cdots Y_q)$$

for $X_1, \cdots, X_{p-1} \in \Gamma(X, T_X^{1,0})$ and $Y_1, \cdots, Y_q \in \Gamma(X, T_X^{0,1})$. We will also use the notation '$\lrcorner$' to represent the contraction operator in the sequel, that is, $i_{X_0}(\omega) = X_0 \lrcorner \omega$.

For $\phi \in A^{0,s}(X, T_X^{1,0})$, the contraction operator can be extended to

$$i_\phi : A^{p,q}(X) \to A^{p-1,q+s}(X).$$

For example, if $\phi = \eta \otimes Y$ with $\eta \in A^{0,q}(X)$ and $Y \in \Gamma(X, T_X^{1,0})$, then for any $\omega \in A^{p,q}(X)$,

$$(i_\phi)(\omega) = \eta \wedge (i_Y \omega).$$

The following result follows easily.

Lemma 3.1. *Let $\phi \in A^{0,q}(X, T_X^{1,0})$ and $\psi \in A^{0,s}(X, T_X^{1,0})$. Then*

$$i_\phi \circ i_\psi = (-1)^{(q+1)(s+1)} i_\psi \circ i_\phi. \tag{3.1}$$

For $Y \in \Gamma(X, T_X)$, the Lie derivative $\mathcal{L}_Y$ is defined as

$$\mathcal{L}_Y = d \circ i_Y + i_Y \circ d : A^s(X) \to A^s(X). \tag{3.2}$$

For any $\phi \in A^{0,q}(X, T_X^{1,0})$, we can define i_ϕ as (3) and thus extend $\mathcal{L}_\phi$ to be

$$\mathcal{L}_\phi = (-1)^q d \circ i_\phi + i_\phi \circ d. \tag{3.3}$$

According to the types, we can decompose

$$\mathcal{L}_\phi = \mathcal{L}_\phi^{1,0} + \mathcal{L}_\phi^{0,1},$$

where

$$\mathcal{L}_\phi^{1,0} = (-1)^q \partial \circ i_\phi + i_\phi \circ \partial$$

and

$$\mathcal{L}_\phi^{0,1} = (-1)^q \overline{\partial} \circ i_\phi + i_\phi \circ \overline{\partial}.$$

Let

$$\varphi^i = \frac{1}{p!} \sum \varphi^i_{\bar{j}_1, \cdots, \bar{j}_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p} \otimes \partial_i \text{ and } \psi^i = \frac{1}{q!} \sum \psi^i_{\bar{k}_1, \cdots, \bar{k}_q} d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q} \otimes \partial_i.$$

Then, we write

$$[\varphi, \psi] = \sum_{i,j=1}^n (\varphi^i \wedge \partial_i \psi^j - (-1)^{pq} \psi^i \wedge \partial_i \varphi^j) \otimes \partial_j, \tag{3.4}$$

where

$$\partial_i \varphi^j = \frac{1}{p!} \sum \partial_i \varphi^j_{\bar{j}_1, \cdots, \bar{j}_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p}$$

and similar for $\partial_i\psi^j$. In particular, if $\varphi,\psi\in A^{0,1}(X,T^{1,0}_X)$,

$$[\varphi,\psi]=\sum_{i,j=1}^{n}(\varphi^i\wedge\partial_i\psi^j+\psi^i\wedge\partial_i\varphi^j)\otimes\partial_j.$$

Let (E,h) be a Hermitian holomorphic vector bundle over X and ∇ be the Chern connection on (E,h). Then the operators $i_\bullet,\mathcal{L}_\bullet$, $[\bullet,\bullet]$ can be extended to the E-valued (p,q) forms in the canonical ways. For example, for any $\phi\in A^{0,k}(X,T^{1,0}_X)$, we can define

$$\mathcal{L}_\phi=(-1)^k\nabla\circ i_\phi+i_\phi\circ\nabla. \tag{3.5}$$

Then we have the following general commutator formula.

Lemma 3.2 ([7]). *For $\varphi\in A^{0,k}(X,T^{1,0}_X)$, $\varphi'\in A^{0,k'}(X,T^{1,0}_X)$ and $\alpha\in A^{*,*}(X,E)$,*

$$(-1)^{k'}\varphi\lrcorner\mathcal{L}_{\varphi'}\alpha+(-1)^{k'k+1}\mathcal{L}_{\varphi'}(\varphi\lrcorner\alpha)=[\varphi,\varphi']\lrcorner\alpha,$$

or equivalently,

$$[\mathcal{L}_{\varphi'},i_\varphi]=i_{[\varphi',\varphi]}.$$

In particular, if $\varphi,\varphi'\in A^{0,1}(X,T^{1,0}_X)$, then

$$[\varphi,\varphi']\lrcorner\alpha=-\nabla'(\varphi'\lrcorner(\varphi\lrcorner\alpha))-\varphi\lrcorner(\varphi'\lrcorner\nabla'\alpha)+\varphi\lrcorner\nabla'(\varphi'\lrcorner\alpha)+\varphi'\lrcorner\nabla'(\varphi\lrcorner\alpha) \tag{3.6}$$

and

$$0=-\overline{\partial}(\varphi'\lrcorner(\varphi\lrcorner\alpha))-\varphi\lrcorner(\varphi'\lrcorner\overline{\partial}\alpha)+\varphi\lrcorner\overline{\partial}(\varphi'\lrcorner\alpha)+\varphi'\lrcorner\overline{\partial}(\varphi\lrcorner\alpha). \tag{3.7}$$

Proof. Since the formulas are all local and $\mathbb{C}$-linear, without loss of generality, we can assume that

$$\varphi=\eta\otimes\chi,\ \ \varphi'=\eta'\otimes\chi',$$

where $\eta\in A^{0,k}(X)$, $\eta'\in A^{0,k'}(X)$, $\chi,\chi'\in\Gamma(X,T^{1,0}_X)$ and $d\eta=d\eta'=0$. Since $d\eta=d\eta'=0$, we have $\chi'(\eta)=\chi(\eta')=0$. Hence, we obtain

$$[\varphi,\varphi']=\eta\wedge\eta'[\chi,\chi'].$$

On the other hand, for any $\alpha\in A^{*,*}(X,E)$,

$$\begin{aligned}\mathcal{L}_\varphi\alpha&=\eta\wedge(\chi\lrcorner\nabla\alpha)+(-1)^k\nabla(\eta\wedge(\chi\lrcorner\alpha))\\&=\eta\wedge(\chi\lrcorner\nabla\alpha)+(-1)^k(d\eta\wedge(\chi\lrcorner\alpha)+(-1)^k\eta\wedge\nabla(\chi\lrcorner\alpha))\\&=\eta\wedge(\chi\lrcorner\nabla\alpha+\nabla(\chi\lrcorner\alpha))\\&=\eta\wedge\mathcal{L}_\chi\alpha.\end{aligned}$$

Now, we have

$$\begin{aligned}\varphi\lrcorner\mathcal{L}_{\varphi'}\alpha=&\eta\wedge\chi\lrcorner(\eta'\wedge\mathcal{L}_{\chi'}\alpha)\\=&(-1)^{k'}\eta\wedge\eta'(\chi\lrcorner\mathcal{L}_{\chi'}\alpha)\\=&(-1)^{k'}\eta\wedge\eta'\,(\mathcal{L}_{\chi'}(\chi\lrcorner\alpha)-[\chi',\chi]\lrcorner\alpha)\\=&(-1)^{k'}\,(\eta\wedge\mathcal{L}_{\varphi'}(\chi\lrcorner\alpha)-\eta\wedge\eta'\wedge([\chi',\chi]\lrcorner\alpha))\\=&(-1)^{k'}[\varphi,\varphi']\lrcorner\alpha+(-1)^{k'(1+k)}\mathcal{L}_{\varphi'}(\eta\wedge(\chi\lrcorner\alpha))\\=&(-1)^{k'}[\varphi,\varphi']\lrcorner\alpha+(-1)^{k'(1+k)}\mathcal{L}_{\varphi'}(\varphi\lrcorner\alpha),\end{aligned}$$

where we apply the formula

$$[\chi',\chi]\lrcorner\alpha=\mathcal{L}_{\chi'}(\chi\lrcorner\alpha)-\chi\lrcorner\mathcal{L}_{\chi'}\alpha,$$

which is proven in [7], and

$$\mathcal{L}_{\varphi'}(\varphi\lrcorner\alpha)=(-1)^{k'k}\eta\wedge\mathcal{L}_{\varphi'}(\chi\lrcorner\alpha).$$

In fact,

$$\begin{aligned}
&\mathcal{L}_{\varphi'}(\varphi\lrcorner\alpha)\\
=&\mathcal{L}_{\varphi'}(\eta\wedge(\chi\lrcorner\alpha))\\
=&\varphi'\lrcorner\nabla(\eta\wedge(\chi\lrcorner\alpha))+(-1)^{k'}\nabla\circ\varphi\lrcorner(\eta\wedge(\chi\lrcorner\alpha))\\
=&\varphi'\lrcorner(d\eta\wedge(\chi\lrcorner\alpha))+(-1)^{k}\varphi'\lrcorner(\eta\wedge\nabla(\chi\lrcorner\alpha))+(-1)^{k'+k(k'-1)}\nabla(\eta\wedge(\varphi'\lrcorner(\chi\lrcorner\alpha)))\\
=&(-1)^{k+k(k'-1)}\eta\wedge(\varphi'\lrcorner(\nabla(\chi\lrcorner\alpha)))+(-1)^{k'+k(k'-1)+k}\eta\wedge\nabla(\varphi'\lrcorner(\chi\lrcorner\alpha))\\
=&(-1)^{k'k}\eta\wedge\mathcal{L}_{\varphi'}(\chi\lrcorner\alpha).
\end{aligned}$$

□

As an easy corollary, we have the following result which was known as Tian-Todorov lemma.

Lemma 3.3 ([16, 15])**.** *If $\varphi,\psi\in A^{0,1}(X,T_X^{1,0})$ and $\Omega\in A^{n,0}(X)$, then one has*

$$[\varphi,\psi]\lrcorner\Omega=-\partial(\psi\lrcorner(\varphi\lrcorner\Omega))+\varphi\lrcorner\partial(\psi\lrcorner\Omega)+\psi\lrcorner\partial(\varphi\lrcorner\Omega). \tag{3.8}$$

In particular, if X is a CY manifold and $\varphi,\psi\in\mathbb{H}^{0,1}(X,T_X^{1,0})$, then

$$[\varphi,\psi]\lrcorner\Omega_0=-\partial(\psi\lrcorner(\varphi\lrcorner\Omega_0)). \tag{3.9}$$

Note that here $\varphi\lrcorner\Omega_0$ and $\psi\lrcorner\Omega_0$ are both harmonic.

Let $\phi\in A^{0,1}(X,T_X^{1,0})$ and i_ϕ be the contraction operator. Define an operator

$$e^{i_\phi}=\sum_{k=0}^{\infty}\frac{1}{k!}i_\phi^k,$$

where $i_\phi^k=\underbrace{i_\phi\circ\cdots\circ i_\phi}_{k\text{ copies}}$. Since the dimension of X is finite, the summation in the above formulation is also finite.

The following theorem gives explicit formulas for the deformed differential operators on the deformation spaces of complex structures. It also explains why it is relatively easy to construct extension of sections of the bundle K_X+E where K_X is the canonical bundle of X. We remark that this result is motivated by [3] where a special case was proved.

Theorem 3.4. *Let $\phi\in A^{0,1}(X,T_X^{1,0})$. Then on the space $A^{*,*}(E)$, we have*

$$e^{-i_\phi}\circ\nabla\circ e^{i_\phi}=\nabla-\mathcal{L}_\phi-i_{\frac{1}{2}[\phi,\phi]}, \tag{3.10}$$

or equivalently

$$e^{-i_\phi}\circ\overline{\partial}\circ e^{i_\phi}=\overline{\partial}-\mathcal{L}_\phi^{0,1} \tag{3.11}$$

and

$$e^{-i_\phi}\circ\nabla'\circ e^{i_\phi}=\nabla'-\mathcal{L}_\phi^{1,0}-i_{\frac{1}{2}[\phi,\phi]}. \tag{3.12}$$

Moreover, if $\overline{\partial}\phi=\frac{1}{2}[\phi,\phi]$, then

$$\overline{\partial}-\mathcal{L}_\phi^{1,0}=e^{-i_\phi}\circ(\overline{\partial}-\mathcal{L}_\phi)\circ e^{i_\phi}. \tag{3.13}$$

Proof. (3.11) follows from (3.6) and formula

$$[\overline{\partial},i_\phi^k]=ki_\phi^{k-1}\circ[\overline{\partial},i_\phi],$$

which can be proved by induction by using (3.6). Similarly, (3.12) follows from (3.7) and

$$[\nabla',i_\phi^k]=ki_\phi^{k-1}\circ[\nabla',i_\phi]-\frac{k(k-1)}{2}i_\phi^{k-2}\circ i_{[\phi,\phi]},\quad k\geq 2. \tag{3.14}$$

Now we prove (3.14) by induction. It is obvious that (3.14) is equivalent to for any $k \geq 2$,

$$F_k := -k i_\phi^{k-1} \circ \nabla' \circ i_\phi + (k-1) i_\phi^k \circ \nabla' + \nabla' \circ i_\phi^k + \frac{k(k-1)}{2} i_\phi^{k-2} i_{[\phi,\phi]} = 0. \tag{3.15}$$

If $k = 2$, it is (3.7). As for $k = 3$,

$$\begin{aligned} 0 &= i_{[\phi,\phi]} \circ i_\phi - i_\phi \circ i_{[\phi,\phi]} \\ &= 3 i_\phi \circ \nabla' \circ i_\phi^2 - \nabla' \circ i_\phi^3 - 3 i_\phi^2 \circ \nabla' \circ i_\phi + i_\phi^3 \circ \nabla' \\ &= 3 i_\phi^2 \circ \nabla' \circ i_\phi - 2 i_\phi^3 \circ \nabla' - \nabla' \circ i_\phi^3 - 3 i_\phi \circ i_{[\phi,\phi]} \\ &= -F_3, \end{aligned}$$

where Lemma 3.1 is applied.

Now we assume that (3.15) is right for all integers less than $k \geq 3$. That is,

$$F_2 = F_3 = \cdots = F_{k-1} = 0.$$

We will show $F_k = 0$. Now we set

$$\begin{aligned} G_k &= F_k - i_\phi \circ F_{k-1} \\ &= -i_\phi^{k-1} \circ \nabla' \circ i_\phi + i_\phi^k \circ \nabla' + \nabla' \circ i_\phi^k - i_\phi \circ \nabla' \circ i_\phi^{k-1} + (k-1) i_\phi^{k-2} i_{[\phi,\phi]}. \end{aligned}$$

So, by induction, we have

$$\begin{aligned} & G_k - i_\phi \circ G_{k-1} \\ = \;& \nabla' \circ i_\phi^k - 2 i_\phi \circ \nabla' \circ i_\phi^{k-1} + i_\phi^2 \circ \nabla' \circ i_\phi^{k-2} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ = \;& (\nabla' \circ i_\phi^2 + i_\phi^2 \circ \nabla' - 2 i_\phi \circ \nabla' \circ i_\phi) \circ i_\phi^{k-2} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ = \;& -i_{[\phi,\phi]} \circ i_\phi^{k-2} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ = \;& -i_\phi \circ i_{[\phi,\phi]} \circ i_\phi^{k-3} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ = \;& -i_\phi^2 \circ i_{[\phi,\phi]} \circ i_\phi^{k-4} + i_\phi^{k-2} \circ i_{[\phi,\phi]} \\ = \;& -i_\phi^{k-3} \circ i_{[\phi,\phi]} \circ i_\phi + i_\phi^{k-3} \circ i_\phi \circ i_{[\phi,\phi]} \\ = \;& -i_\phi^{k-3} \circ (i_{[\phi,\phi]} \circ i_\phi - i_\phi \circ i_{[\phi,\phi]}) \\ = \;& 0 \end{aligned}$$

since $i_{[\phi,\phi]} i_\phi - i_\phi i_{[\phi,\phi]} = 0$. Alternatively, we can also approach this equality directly by induction on the term $G_k - i_\phi \circ G_{k-1}$, i.e., $0 = G_{k-1} - i_\phi \circ G_{k-2} = -i_{[\phi,\phi]} \circ i_\phi^{k-3} + i_\phi^{k-3} \circ i_{[\phi,\phi]}$. Now we finish the proof of (3.14). From (3.14), it follows that

$$[\nabla', e^{i_\phi}] = e^{i_\phi} \circ [\nabla', i_\phi] - e^{i_\phi} \circ \frac{1}{2} i_{[\phi,\phi]}$$

by comparing the degrees. Then, we have

$$\begin{aligned} e^{-i_\phi} \circ \nabla' \circ e^{i_\phi} &= e^{-i_\phi} \circ [\nabla', e^{i_\phi}] + \nabla' \\ &= [\nabla', i_\phi] + \nabla' - i_{\frac{1}{2}[\phi,\phi]} \\ &= \nabla' - \mathcal{L}_\phi^{1,0} - i_{\frac{1}{2}[\phi,\phi]}. \end{aligned}$$

Now we finish the proof of (3.12) while the proof of (3.11) is similar.

Finally, when $\overline{\partial}\phi = \frac{1}{2}[\phi, \phi]$, we have $[2\overline{\partial} - \mathcal{L}_\phi, i_\phi] = 0$ and thus

$$[2\overline{\partial} - \mathcal{L}_\phi, e^{i_\phi}] = 0,$$

which implies that

$$e^{-i_\phi} \circ (\overline{\partial} - \mathcal{L}_\phi) \circ e^{i_\phi} = 2\overline{\partial} - \mathcal{L}_\phi - e^{-i_\phi} \circ \overline{\partial} \circ e^{i_\phi} = \overline{\partial} - \mathcal{L}_\phi^{1,0}.$$

□

Corollary 3.5. *If $\sigma \in A^{n,*}(X,E)$, we have*

$$\begin{aligned}\left(e^{-i_\phi}\circ\nabla\circ e^{i_\phi}\right)(\sigma) &= \overline{\partial}\sigma - \mathcal{L}^{1,0}_\phi(\sigma) + i_{\overline{\partial}\phi-\frac{1}{2}[\phi,\phi]}(\sigma)\\ &= \overline{\partial}\sigma + \nabla'(\phi\lrcorner\sigma) + \left(\overline{\partial}\phi - \frac{1}{2}[\phi,\phi]\right)\lrcorner\sigma.\end{aligned}$$

In particular, if ϕ is integrable, i.e., $\overline{\partial}\phi - \frac{1}{2}[\phi,\phi] = 0$, then

$$\left(e^{-i_\phi}\circ\nabla\circ e^{i_\phi}\right)(\sigma) = \overline{\partial}\sigma + \nabla'(\phi\lrcorner\sigma). \tag{3.16}$$

The above formula gives an explicit recursive formula to construct deformed cohomology classes for deformation of Kähler manifolds. When E is a trivial bundle, the above formula was used in [6] to prove the global Torelli theorem.

4. Global canonical family of Beltrami differentials

In this section, based on the techniques developed in Sections 2 and 3, we shall construct the following globally convergent power series of Beltrami differentials in L^2-norm on a CY manifold. To avoid the bewildering notations, we just present the details on the one-parameter case and only give a sketch of the multi-parameter case.

The convergence of the power series in the following lemma is crucial in our proof of the global convergence and regularity results.

Lemma 4.1. *Let $\{x_i\}_{i=1}^{+\infty}$ be a series given by $x_k = c\sum_{i=1}^{k-1} x_i\cdot x_{k-i}$ with real initial value x_1. Then the power series $S(\tau) = \sum_{i=1}^{\infty} x_i\tau^i$ converges as long as $|\tau| \le \frac{1}{|4cx_1|}$.*

Proof. Setting $S := S(\tau) = \sum_{i=1}^{\infty} x_i\tau^i$, we have

$$cS^2 = c\left(\sum_{i=1}^{\infty} x_i\tau^i\right)\left(\sum_{j=1}^{\infty} x_j\tau^j\right) = \sum_{k\ge 1}^{+\infty} x_k\tau^k - x_1\tau = S - x_1\tau. \tag{4.1}$$

It follows from (4.1) that

$$S = \frac{1\pm\sqrt{1-4cx_1\tau}}{2c}.$$

Here we take $S(\tau) = \frac{1-\sqrt{1-4cx_1\tau}}{2c}$, since we have $S(0) = 0$ according to the assumption. We therefore have the following expansion for S

$$\begin{aligned}S &= \frac{1}{2c}\left(1-\left(1+\sum_{n\ge 1}\frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4cx_1\tau)^n\right)\right)\\ &= \sum_{n\ge 1}\frac{1}{2c}\left(\frac{\frac{1}{2}(1-\frac{1}{2})\cdots((n-1)-\frac{1}{2})}{n!}\right)(4cx_1)^n\tau^n,\end{aligned}$$

which implies that

$$x_n = \frac{\frac{1}{2}(1-\frac{1}{2})\cdots((n-1)-\frac{1}{2})}{2cn!}(4cx_1)^n.$$

This is the explicit expression for each x_n. Now it is easy to check that the convergence radius of the power series $S = \sum_{i=1}^{\infty} x_i\tau^i$ is $4|cx_1|$, and that this power series still converges when $\tau = \pm\frac{1}{4|cx_1|}$. □

Lemma 4.2. *Let $\{x_i\}_{i=1}^{+\infty}$ be a series given by*

$$x_1 = \frac{1}{2},\ x_2 = \frac{1}{2}x_1 \cdot x_1,\ \cdots,\ x_k = \frac{1}{2}\sum_{i=1}^{k-1} x_i \cdot x_{k-i},\ \cdots. \tag{4.2}$$

Then the power series $S(\tau) = \sum_{i=1}^{\infty} x_i \tau^i$ converges as long as $|\tau| \le 1$.

Now we prove the global convergence of the Beltrami differential from the Kodaira-Spencer-Kuranishi theory. All sub-indices of the Beltrami differentials are at least 1.

The following result is contained in [16, 15], we briefly recall here for reader's convenience.

Lemma 4.3. *Assume that for $\varphi_\nu \in A^{0,1}(X, T_X^{1,0})$, $\nu = 2, \cdots, K$,*

$$\overline{\partial}\varphi_\nu = \frac{1}{2}\sum_{\alpha+\beta=\nu} [\varphi_\alpha, \varphi_\beta] \quad \text{and} \quad \overline{\partial}\varphi_1 = 0. \tag{4.3}$$

Then one has

$$\overline{\partial}\left(\sum_{\nu+\gamma=K+1} [\varphi_\nu, \varphi_\gamma]\right) = 0. \tag{4.4}$$

Proof. By definition formula (3.4), one has

$$[\overline{\partial}\varphi, \varphi'] = -[\varphi', \overline{\partial}\varphi]. \tag{4.5}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\overline{\partial}\left(\sum_{\nu+\gamma=K+1} [\varphi_\nu, \varphi_\gamma]\right) =& \frac{1}{2}\sum_{\nu+\gamma=K+1} \left([\overline{\partial}\varphi_\nu, \varphi_\gamma] - [\varphi_\nu, \overline{\partial}\varphi_\gamma]\right) \\
=& \sum_{\nu+\gamma=K+1} \left[\overline{\partial}\varphi_\nu, \varphi_\gamma\right] \\
=& \frac{1}{2}\sum_{\nu+\gamma=K+1} \left[\sum_{\alpha+\beta=\nu} [\varphi_\alpha, \varphi_\beta], \varphi_\gamma\right] \\
=& \frac{1}{2}\sum_{\alpha+\beta+\gamma=K+1} \left[[\varphi_\alpha, \varphi_\beta], \varphi_\gamma\right],
\end{aligned}$$

where the second equality is implied by (4.5) and the third one follows from the assumption (4.3). When $\alpha = \beta = \gamma$, by Jacobi identity one has

$$3\left[[\varphi_\alpha, \varphi_\beta], \varphi_\gamma\right] = 0.$$

Otherwise, Jacobi identity implies that

$$\left[[\varphi_\alpha, \varphi_\beta], \varphi_\gamma\right] + \left[[\varphi_\beta, \varphi_\gamma], \varphi_\alpha\right] + \left[[\varphi_\gamma, \varphi_\alpha], \varphi_\beta\right] = 0.$$

□

Theorem 4.4. *Let X be a CY manifold and $\varphi_1 \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ with norm $\|\varphi_1\| = \frac{1}{2}$. Then there exits a smooth globally convergent power series in L^2-norm for $|t| \le 1$,*

$$\Phi(t) = \varphi_1 t^1 + \varphi_2 t^2 + \cdots + \varphi_k t^k + \cdots \in A^{0,1}(X, T_X^{1,0}), \tag{4.6}$$

which satisfies:

a) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$;

b) $\overline{\partial}^*\varphi_k = 0$ *for each* $k \ge 1$;

c) $\varphi_k \lrcorner \Omega_0$ *is* ∂*-exact for each* $k \geq 2$*;*
d) $\|\Phi(t) \lrcorner \Omega_0\|_{L^2} < \infty$ *as long as* $|t| \leq 1$*.*

Proof. Let us first review the construction of the power series $\Phi(t)$ by induction from [15] and [16]. Suppose that we have constructed φ_k for $2 \leq k \leq j$ such that:
a) $\bar{\partial}\varphi_k = \frac{1}{2}\sum_{i=1}^{k-1}[\varphi_{k-i}, \varphi_i]$;
b) $\bar{\partial}^*\varphi_k = 0$;
c) $\varphi_k \lrcorner \Omega_0$ is ∂-exact and thus $\partial(\varphi_k \lrcorner \Omega_0) = 0$.
Then we need to construct φ_{j+1} such that:
a′) $\bar{\partial}\varphi_{j+1} = \frac{1}{2}\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i]$;
b′) $\bar{\partial}^*\varphi_{j+1} = 0$;
c′) $\varphi_{j+1} \lrcorner \Omega_0$ is ∂-exact and thus $\partial(\varphi_{j+1} \lrcorner \Omega_0) = 0$.
Actually, it follows from Lemma 3.3 and the assumption *c*) that

$$\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i] \lrcorner \Omega_0 = -\partial \left(\sum_{i+k=j+1} \varphi_i \lrcorner \varphi_k \lrcorner \Omega_0 \right). \tag{4.7}$$

Then, Lemma 4.3 and the assumption *a*) imply

$$\bar{\partial}\partial \left(\sum_{i+k=j+1} \varphi_i \lrcorner \varphi_k \lrcorner \Omega_0 \right) = \bar{\partial} \left(\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i] \right) \lrcorner \Omega_0 = 0. \tag{4.8}$$

So formula (4.8) and Proposition 1.2 tell us that the equation

$$\bar{\partial}\Psi_{j+1} = -\partial \left(\sum_{i+k=j+1} \varphi_i \lrcorner \varphi_k \lrcorner \Omega_0 \right)$$

has a solution $\Psi_{j+1} = -\bar{\partial}^* \mathbb{G} \partial \left(\sum_{i+k=j+1} \varphi_i \lrcorner \varphi_k \lrcorner \Omega_0 \right)$. Hence, we define

$$\varphi_{j+1} = \frac{1}{2}\Psi_{j+1} \lrcorner \Omega_0^*,$$

where $\Omega_0^* := \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}$ in local coordinates is the dual of Ω_0. It is easy to check that

$$\bar{\partial}^*(\Psi_{j+1} \lrcorner \Omega_0^*) = \bar{\partial}^*(\Psi_{j+1}) \lrcorner \Omega_0^* + \Psi_{j+1} \lrcorner \bar{\partial}^* \Omega_0^* = 0,$$

since Ω_0 is parallel, and also $\bar{\partial}\varphi_{j+1} = \frac{1}{2}\sum_{i=1}^{j}[\varphi_{j+1-i}, \varphi_i]$. See Lemma 1.2.2 and the argument on pp. 336 of [16] for more details. Now we have completed the construction of $\varphi_{j+1} = \frac{1}{2}\Psi_{j+1} \lrcorner \Omega_0^*$, which is shown to satisfy Properties *a′*), *b′*) and *c′*). To complete this induction, it suffices to work out the case $j = 2$. It is obvious that φ_2 can be constructed as

$$\varphi_2 = \frac{1}{2}\bar{\partial}^* \mathbb{G} \partial \left(\varphi_1 \lrcorner \varphi_1 \lrcorner \Omega_0 \right) \lrcorner \Omega_0^*,$$

which satisfies *a*), *b*) and *c*). Moreover, one has the following equality for each $k \geq 2$,

$$\varphi_k \lrcorner \Omega_0 = \frac{1}{2}\bar{\partial}^* \mathbb{G} \partial \sum_{i+j=k \geq 2} \varphi_i \lrcorner \varphi_j \lrcorner \Omega_0. \tag{4.9}$$

Next, let us prove the L^2-convergence and regularity. In fact, without loss of generality we can assume that $\|\varphi_1 \lrcorner \Omega_0\| = \|\varphi_1\| = \frac{1}{2}$ and thus have for $|t| \leq 1$,

$$\begin{aligned}
\|\Phi(t)\lrcorner\Omega_0\|_{L^2} &= \left\|(\varphi_1\lrcorner\Omega_0)t + (\varphi_2\lrcorner\Omega_0)t^2 + \cdots + (\varphi_k\lrcorner\Omega_0)t^k + \cdots\right\| \\
&= \left\|(\varphi_1\lrcorner\Omega_0)t + \sum_{j=2}^{\infty}\frac{1}{2}\overline{\partial}^*\mathbb{G}\partial\left(\sum_{i+k=j}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right)t^j\right\| \\
&\leq \sum_{j=2}^{\infty}\frac{1}{2}\left(\sum_{i+k=j}\|\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\|\right)|t|^j + \frac{1}{2}|t| \\
&\leq \sum_{j=2}^{\infty}\frac{1}{2}\sum_{i+k=j}\left(\|\varphi_i\|\cdot\|\varphi_k\lrcorner\Omega_0\|\right)|t|^j + \frac{1}{2}|t| \\
&\leq \sum_{j=2}^{\infty}x_j|t|^j + x_1|t| \\
&< \infty,
\end{aligned}$$

by Lemma 4.2 and quasi-isometry Theorem 1.1. Here we use the same notations for the series x_j as in Lemma 4.2.

As for the regularity of $\Phi(t)$, we can obtain it directly by the elliptic regularity theorem, which tells us that every fundamental solution of an elliptic operator is infinitely differentiable in any neighborhood not containing 0. Then it is obvious that $\partial\Phi(t) = 0$ by the ∂-exactness of $\varphi_k\lrcorner\Omega_0$ for each $k \geq 2$ and the harmonicity of $\varphi_1\lrcorner\Omega_0$, and that the operator ∂ is elliptic. For more details, see pp. 8 of [13]. □

In fact we have a more intrinsic approach to obtain the regularity by Gårding's inequality and Sobolev lemma. Gårding's inequality states that for $\omega \in A^{p,q}(X)$ on a compact n-dimensional Hermitian manifold X, there holds $\|\omega\|_1^2 \leq C(\|\omega\|^2 + \|\overline{\partial}\omega\|^2 + \|\overline{\partial}^*\omega\|^2)$, where $\|\cdot\|_s^2$ denotes the Sobolev s-norm and $C > 0$ (always) denotes the constants. Note that the Gårding's inequality also holds on Sobolev space of complex differential forms. Let $\Phi_j = \frac{1}{2}\overline{\partial}^*\mathbb{G}\partial\left(\sum_{i+k=j}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right)$ for $j \geq 2$. Then by Gårding's inequality, one has

$$\begin{aligned}
\|\Phi_j\|_1^2 &\leq C(\|\overline{\partial}^*\Phi_j\| + \|\overline{\partial}\Phi_j\| + \|\Phi_j\|) \\
&= C\left(\frac{1}{2}\left\|\partial\sum_{i+k=j}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right\| + \|\Phi_j\|\right) \\
&\leq C\left\|\sum_{i+k=j}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right\|_1^2.
\end{aligned}$$

Here we also apply Lemma 2.3 and quasi-isometry Theorem 1.1 and the definition of Sobolev norm. By the iteration construction of Φ_*'s and also φ_*'s, we know that

$$\left\|\sum_{i+k=j}\varphi_i\lrcorner\varphi_k\lrcorner\Omega_0\right\|_1^2 \leq \sum_{i+k=j}\|\varphi_i\|_1^2\cdot\|\varphi_k\lrcorner\Omega_0\|_1^2 \leq C\sum_{i+k=j}\|\Phi_i\|_1^2\cdot\|\Phi_k\|_1^2.$$

From these and Lemma 4.1, it follows that for some large valued t,

$$\|\Phi(t)\lrcorner\Omega_0\|_1^2 < \infty.$$

This argument still works for all the Sobolev norms. Here we only need a generalized Gårding's inequality such as Theorem 1.1 in Chapter *IV* of the book [13]. It is easy to choose the constant C to work uniformly for all norms and thus obtain an identical convergence radius for all $\|\Phi(t)\lrcorner\Omega_0\|_s^2$. Then from the classical global Sobolev lemma, the completion of $C^\infty(X)$ in Sobolev s-norm $\mathcal{H}_{[n/2]+1+s}(X) \subset C^s(X)$ and $\cap_s \mathcal{H}_s(X) = C^\infty(X)$, it follows that

$$\Phi(t)\lrcorner\Omega_0 \in C^\infty(X).$$

See Section 6 of Chapter 0 in [5] for more details.

Now we state the following multi-parameter result, while we just sketch its proof since it is essentially the same as the one-parameter case.

Theorem 4.5. *Let X be a CY manifold and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ be a basis with norm $\|\varphi_i\| = \frac{1}{2N}$. Then for $|t| \leq 1$, we can construct a smooth power series of Beltrami differentials on X as follows*

$$\Phi(t) = \sum_{|I|\geq 1} \varphi_I t^I = \sum_{\substack{\nu_1+\cdots+\nu_N\geq 1,\\ each\ \nu_i \geq 0, i=1,2,\cdots}} \varphi_{\nu_1\cdots\nu_N} t_1^{\nu_1}\cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0}), \tag{4.10}$$

where $\varphi_{0\cdots\nu_i\cdots 0} = \varphi_i$. This power series has the following properties:

a) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$, the integrability condition;

b) $\overline{\partial}^*\varphi_I = 0$ for each multi-index I with $|I| \geq 1$;

c) $\varphi_I \lrcorner\Omega_0$ is ∂-exact for each I with $|I| \geq 2$. Here Ω_0 is a holomorphic $(n,0)$-form on X; and more importantly,

d) global convergence: $\|\Phi(t)\lrcorner\Omega_0\| \leq \sum_I \|\varphi_I\lrcorner\Omega_0\| \cdot |t|^{|I|} < \infty$ as long as $|t| \leq 1$.

Proof. Let us construct the power series $\Phi(t)$ in multi-parameters by induction. Write

$$\mathcal{B}_{\gtreqqless K} = \{\varphi_{\nu_1\cdots\nu_N} \in A^{0,1}(M, T_M^{1,0}) \mid \text{each integer } \nu_i \geq 0 \text{ and } \nu_1 + \cdots + \nu_N \gtreqqless K,\ K \geq 1\}.$$

It is easy to see that $\Phi(t)$ should satisfy:

a) $\overline{\partial}\varphi_{\nu_1\cdots\nu_N} = \frac{1}{2} \sum\limits_{\alpha_i+\beta_i=\nu_i} [\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}]$ for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2}$;

b) $\overline{\partial}^*\varphi_{\nu_1\cdots\nu_N} = 0$ for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 1}$;

c) $\varphi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0$ is ∂-exact and thus $\partial(\varphi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0) = 0$ for each $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2}$.

Assuming that the above three assumptions hold for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2} \cap \mathcal{B}_{\leq K}$, then one can construct $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{K+1}$ such that it also satisfies these three assumptions. In fact, Lemma 3.3 and the assumption *c)* for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2} \cap \mathcal{B}_{\leq K}$ imply that

$$[\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}]\lrcorner\Omega_0 = -\partial\left(\varphi_{\alpha_1\cdots\alpha_N}\lrcorner\varphi_{\beta_1\cdots\beta_N}\lrcorner\Omega_0\right), \tag{4.11}$$

where $\sum_i \alpha_i + \sum_j \beta_j = K+1$. Then, by multi-index Lemma 4.3 and the assumption *a)* for $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{\geq 2} \cap \mathcal{B}_{\leq K}$, we have

$$\overline{\partial}\partial\left(\sum_{\alpha_i+\beta_i=\nu_i} \varphi_{\alpha_1\cdots\alpha_N}\lrcorner\varphi_{\beta_1\cdots\beta_N}\lrcorner\Omega_0\right) = \overline{\partial}\left(\sum_{\alpha_i+\beta_i=\nu_i} [\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}]\right)\lrcorner\Omega_0 = 0, \tag{4.12}$$

for any $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_{K+1}$. Therefore, one can construct $\Psi_{\nu_1\cdots\nu_N}$ directly by $\overline{\partial}$-Inverse formula 2.3 and (4.12) as

$$\Psi_{\nu_1\cdots\nu_N} = -\overline{\partial}^*\mathbb{G}\partial\left(\sum_{\alpha_i+\beta_i=\nu_i} \varphi_{\alpha_1\cdots\alpha_N}\lrcorner\varphi_{\beta_1\cdots\beta_N}\lrcorner\Omega_0\right).$$

Hence we define

$$\varphi_{\nu_1\cdots\nu_N} = \frac{1}{2}\Psi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0^* \in \mathcal{B}_{K+1},$$

where $\Omega_0^* := \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}$ is the dual of Ω_0. Then it is easy to check that

$$\overline{\partial}^*(\Psi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0^*) = \overline{\partial}^*(\Psi_{\nu_1\cdots\nu_N})\lrcorner\Omega_0^* + \Psi_{\nu_1\cdots\nu_N}\lrcorner\overline{\partial}^*\Omega_0^* = 0$$

since Ω_0 is parallel, and also $\overline{\partial}\varphi_{\nu_1\cdots\nu_N} = \frac{1}{2}\sum_{\alpha_i+\beta_i=\nu_i}[\varphi_{\alpha_1\cdots\alpha_N}, \varphi_{\beta_1\cdots\beta_N}]$. To complete this induction, we construct $\varphi_{\nu_1\cdots\nu_N} \in \mathcal{B}_2$ as

$$
(4.13) \quad \varphi_{\nu_1\cdots\nu_N} = \begin{cases} -\overline{\partial}^* \mathbb{G}\partial\left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega_0\right) \lrcorner \Omega_0^*, & \text{if } \nu_i = \nu_j = 1,\ i \neq j, \\ -\frac{1}{2}\overline{\partial}^* \mathbb{G}\partial\left(\varphi_i \lrcorner \varphi_i \lrcorner \Omega_0\right) \lrcorner \Omega_0^*, & \text{if } \nu_i = 2, \text{ for some } i \in \{1, \cdots, N\}, \end{cases}
$$

which obviously satisfies $a)$, $b)$ and $c)$.

Up to now we have completed the construction of the power series $\Phi(t)$ satisfying $a)$, $b)$ and $c)$ as in Theorem 4.4. It now suffices to check the global convergence in L^2-norm and regularity of $\Phi(t)$.

We may choose $\|\varphi_i \lrcorner \Omega_0\| = \|\varphi_i\| = \frac{1}{2N}$. Thus by Lemma 4.2 and our quasi-isometry, we have the following estimates for $|t| \leq 1$,

$$
\begin{aligned}
&\|\Phi(t)\lrcorner\Omega_0\|_{L^2} \\
&= \left\| \sum_{i=1}^{N} (\varphi_i \lrcorner \Omega_0) t_i + \frac{1}{2} \sum_{K:=\Sigma_i \nu_i \geq 2} \overline{\partial}^* \mathbb{G}\partial \sum_{\substack{J+L=K, J,L\geq 1 \\ \varphi_{\alpha_1\cdots\alpha_N} \in \mathcal{B}_J, \varphi_{\beta_1\cdots\beta_N} \in \mathcal{B}_L}} \left(\varphi_{\alpha_1\cdots\alpha_N} \lrcorner \varphi_{\beta_1\cdots\beta_N} \lrcorner \Omega_0\right) t_1^{\nu_1} \cdots t_N^{\nu_N} \right\| \\
&\leq \sum_{i=1}^{N} \|\varphi_i \lrcorner \Omega_0\| \cdot |t| + \frac{1}{2} \sum_{\substack{J+L=K\geq 2, J,L\geq 1 \\ \varphi_{\alpha_1\cdots\alpha_N} \in \mathcal{B}_J, \varphi_{\beta_1\cdots\beta_N} \in \mathcal{B}_L}} \|\varphi_{\alpha_1\cdots\alpha_N}\| \cdot \|\varphi_{\beta_1\cdots\beta_N} \lrcorner \Omega_0\| \cdot |t|^K \\
&\leq \sum_{K=2}^{\infty} x_K |t|^K + x_1|t| \\
&< +\infty,
\end{aligned}
$$

where the series $\{x_j\}_{j=1}^{+\infty}$ is just the one as in the Lemma 4.2.

As for the regularity of $\Phi(t)$, we can follow the argument in the proof of Theorem 4.4 word by word. Hence the proof of this theorem is completed. □

5. Global canonical family of holomorphic $(n,0)$-forms

Based on the construction of L^2-global canonical family $\Phi(t)$ of Beltrami differentials of Theorem 4.4 in Subsection 5.1, we obtain an L^2-global canonical family of holomorphic $(n,0)$-forms on the deformation space of CY manifold in Subsection 5.2.

5.1. **Iteration on Kähler manifold.** The iteration procedure is to construct holomorphic sections of the canonical line bundle K_{X_t} of the deformation X_t of a Kähler manifold X induced by the Beltrami differential $\Phi(t)$ satisfying the integrability condition. More precisely, our goal is to find a convergent power series for any holomorphic section $\Omega_0 \in H^0(X, K_X)$,

$$
\Omega_t = \Omega_0 + \sum_{|I|\geq 1} t^I \Omega_I
$$

such that $e^{\Phi(t)} \lrcorner \Omega_t \in H^0(X_t, K_{X_t})$ is holomorphic with respect to the induced complex structure $J_{\Phi(t)}$ by $\Phi(t)$. It is easy to check that the map $e^{\Phi(t)}\lrcorner : A^0(X, K_X) \to A^0(X_t, K_{X_t})$ is a well-defined linear isomorphism. Then Proposition 5.1 is used to determine the holomorphy of the section $e^{\Phi(t)}\lrcorner\Omega_t$, and our goal of the iteration procedure is thus reduced to solving the $\overline{\partial}$-equation (5.1) by $\overline{\partial}$-Inverse Lemma 2.3. See also Lemma 10.2 of [8].

Proposition 5.1. *For any $\Omega \in A^{n,0}(X)$, the section $e^{\Phi(t)} \lrcorner \Omega \in A^{n,0}(X_t)$ is holomorphic with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t if and only if*

$$\overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega) = 0. \tag{5.1}$$

Proof. This is a direct consequence of Corollary 3.5. In fact,

$$\left(e^{-i_\Phi} \circ d \circ e^{i_\Phi}\right)(\Omega) = \overline{\partial}\Omega + \partial(\Phi \lrcorner \Omega),$$

if the vector bundle E is trivial and $\Phi(t)$ satisfies the integrability condition. The operator d, which is independent of the complex structures, can be decomposed as $d = \overline{\partial}_t + \partial_t$, where $\overline{\partial}_t$ and ∂_t denote the $(0,1)$-part and $(1,0)$-part of d, with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t. Notice that $e^{\Phi(t)} \lrcorner \Omega \in A^{n,0}(X_t)$ and thus, comparing types, we get

$$\left(e^{-i_\Phi} \circ \overline{\partial}_t \circ e^{i_\Phi}\right)(\Omega) = \overline{\partial}\Omega + \partial(\Phi \lrcorner \Omega),$$

which implies the assertion. □

Let X be an n-dimensional compact Kähler manifold and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T_X^{1,0})$ a basis with the norm $\|\varphi_i\| = C(N)$, $N = \dim \mathbb{H}^{0,1}(X, T_X^{1,0})$ for each $i = 1, 2, \cdots$. A power series of Beltrami differentials in the following form

$$\Phi(t) = \sum_{i=1}^{N} \varphi_i t_i + \sum_{|I| \geq 2} \varphi_I t^I = \sum_{\nu_1 + \cdots + \nu_N \geq 1} \varphi_{\nu_1 \cdots \nu_N} t_1^{\nu_1} \cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0})$$

with $\varphi_{0 \cdots \nu_i \cdots 0} = \varphi_i$, is called an L^2-global canonical family of Beltrami differentials on the Kähler manifold X if it satisfies:

i) the integrability condition: $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$;

ii) global convergence in the sense that

$$\|\Phi(t) \lrcorner \Omega_0\|_{L^2} \leq \sum_{|I| \geq 1} \|\varphi_I\| \|\Omega_0\| \cdot t^{|I|} < \infty$$

as long as $|t| < R$, where the convergence radius R is a constant only dependent on $C(N)$ and Ω_0 is a non-vanishing $(n,0)$-form.

Proposition 5.2. *If there exists an L^2-global canonical family $\Phi(t)$ of Beltrami differentials on the Kähler manifold X with convergence radius R, let $X_t = (X_t, J_{\Phi(t)})$ be the deformation of X induced by $\Phi(t)$. Then for any holomorphic $(n,0)$-form $\Omega \in A^{n,0}(X)$, we can construct a smooth power series*

$$\Omega_t = \Omega_0 + \sum_{|I| \geq 1}^{\infty} \Omega_I t^I \in A^{n,0}(X) \tag{5.2}$$

such that $\Omega_0 = \Omega$ with the following properties:

a) *$\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_t \in H^0(X_t, K_{X_t})$ is holomorphic with respect to $J_{\Phi(t)}$;*

b) *$\Omega_I \in A^{n,0}(X)$ is ∂-exact and also $\overline{\partial}^*$-exact for all $|I| \geq 1$;*

c) *it converges in L^2-norm with convergence radius R.*

We call Ω_t^C an L^2-global canonical family of holomorphic $(n,0)$-forms on the deformation space of Kähler manifold X and R as its convergence radius.

Proof. By Proposition 5.1, we know that Ω_t must satisfy the equation

$$\overline{\partial}\Omega_t = -\partial(\Phi(t) \lrcorner \Omega_t). \tag{5.3}$$

By comparing the coefficients of $t_1^{\nu_1}\cdots t_N^{\nu_N}$ of both sides of (5.3), one knows that Equation (5.3) is equivalent to

$$\begin{cases} \overline{\partial}\Omega_0 = 0, \\ \overline{\partial}\Omega_{\nu_1\cdots\nu_N} = -\partial\left(\sum\limits_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0} \varphi_{\alpha_1\cdots\alpha_N}\lrcorner\Omega_{\beta_1\cdots\beta_N}\right), \end{cases} \tag{5.4}$$

where each $\nu_i \geq 0$ and $\Sigma\nu_i \geq 1$.

We first prove that the equation (5.4) has a ∂-exact solution by induction. Set

$$\eta_{\nu_1\cdots\nu_N} = -\partial\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0} \varphi_{\alpha_1\cdots\alpha_N}\lrcorner\Omega_{\beta_1\cdots\beta_N}\right),$$

which is clearly ∂-exact and thus $\mathbb{H}_{\overline{\partial}}(\eta) = 0$ by the Kähler identity $\square_\partial = \square_{\overline{\partial}}$. So by $\overline{\partial}$-Inverse Lemma 2.3 it suffices to show that $\overline{\partial}\eta_{\nu_1\cdots\nu_N} = 0$.

For the initial case $\Sigma\nu_i = 1$, one has

$$\overline{\partial}\eta_{\nu_1\cdots\nu_N} = -\overline{\partial}\partial(\varphi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0) = \partial(\overline{\partial}\varphi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0 + \varphi_{\nu_1\cdots\nu_N}\lrcorner\overline{\partial}\Omega_0) = 0$$

since $\overline{\partial}\varphi_{\nu_1\cdots\nu_N} = 0$ and $\overline{\partial}\Omega_0 = 0$. Thus we have

$$\Omega_{\nu_1\cdots\nu_N} = \overline{\partial}^*\mathbb{G}\eta_{\nu_1\cdots\nu_N} = -\overline{\partial}^*\partial\mathbb{G}(\varphi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0) = \partial\overline{\partial}^*\mathbb{G}(\varphi_{\nu_1\cdots\nu_N}\lrcorner\Omega_0) \tag{5.5}$$

by $\overline{\partial}$-Inverse Lemma 2.3 and Kähler identity.

Supposing that the $(n,0)$-forms $\Omega_{\nu_1\cdots\nu_N}$ with $\Sigma\nu_i = K$ are constructed, we can also prove

$$\overline{\partial}\eta_{\nu_1\cdots\nu_N} = 0$$

for $\Sigma\nu_i = K+1$ by induction and the commutator formula Lemma 3.3. This calculation is routine and left to the interested readers. Similar to the initial case, we can construct the $(n,0)$-forms $\Omega_{\nu_1\cdots\nu_N}$ with $\Sigma\nu_i = K+1$ as

$$\Omega_{\nu_1\cdots\nu_N} = -\overline{\partial}^*\partial\mathbb{G}\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0} \varphi_{\alpha_1\cdots\alpha_N}\lrcorner\Omega_{\beta_1\cdots\beta_N}\right) = \partial\overline{\partial}^*\mathbb{G}\left(\sum_{\alpha_i+\beta_i=\nu_i,\alpha_i\geq 0} \varphi_{\alpha_1\cdots\alpha_N}\lrcorner\Omega_{\beta_1\cdots\beta_N}\right).$$

Hence we have completed the construction of the power series Ω_t of $(n,0)$-forms.

Finally, let us prove the global convergence of the formal power series. By the global convergence of the canonical family of Beltrami differentials, we know that there exists a small constant $\xi > 0$ and a constant $R_1 \in (0, R]$ such that

$$\sum_{|I|=i} \|\varphi_I\| R_1^i \leq \xi$$

for all large $i > 0$. We may assume that this fact holds for all $i > 0$. Then we have the following estimate for each $i > 0$

$$\sum_{|I|=i} \|\Omega_I\| \leq \xi(\xi+1)^{i-1}R_1^{-i}, \tag{5.6}$$

which follows by induction and implies the convergence of power series (5.2) as long as $|t| < R_1$. We set $\|\Omega_0\| = 1$ for convenience. First for the initial case $i = 1$, one has

$$\sum_{|I|=1} \|\Omega_I\| \leq \|\Omega_0\| \sum_{|I|=1} \|\varphi_I\| \leq R_1^{-1}\xi,$$

where the quasi-isometry Theorem 1.1 is applied. Then, we assume that the estimate (5.6) is true for $l = 1, \cdots, i-1$ and try to prove the case $l = i$ as follows.

$$\begin{aligned}\sum_{|I|=i} \|\Omega_I\| &\leq \sum_{\substack{|I|=i,|I_2|\geq 1,\\ I_1+I_2=I}} \|\Omega_{I_1}\| \cdot \|\varphi_{I_2}\| \\ &\leq \xi R_1^{-1} \xi(\xi+1)^{i-2} R_1^{-(i-1)} + \cdots + \xi R_1^{-i}\xi + \xi R_1^{-i} \\ &= (\xi R_1^{-i})\xi \frac{1-(\xi+1)^{i-1}}{1-(\xi+1)} + \xi R_1^{-i} \\ &= \xi(\xi+1)^{i-1} R_1^{-i},\end{aligned}$$

where the first inequality is also due to Theorem 1.1. Yet it is easy to check that the convergence domain for $|t|$ of $\sum_{i=1} \xi(\xi+1)^{i-1} R_1^{-i}|t|^i$ is obviously $[0, R_1)$.

The regularity of Ω_t follows directly by the argument in the proof of Theorem 4.4. This completes our proof. □

5.2. **Global canonical family.** Now let us state our main result of this section in the CY case, i.e., the central fiber of the family is a CY manifold. In our case, all the fibers of this family are CY; while in infinitesimal deformation theory this is a standard result. See Lemma 1.4 of [14].

Theorem 5.3. *Let $X_t = (X_t, J_{\Phi(t)})$ be the deformation of the CY manifold X induced by the L^2-global canonical family $\Phi(t)$ of Beltrami differentials on X as constructed in Theorem 4.5. Then for any holomorphic $(n,0)$-form Ω_0 on X and $|t| < 1$, $\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_t$ defines an L^2-global canonical family of holomorphic $(n,0)$-forms on X_t and depends on t holomorphically.*

Proof. We first construct the so-called canonical family of holomorphic $(n,0)$-forms on the deformation space of CY manifolds. Actually according to the construction of the power series Ω_t, we have $\Omega_I = 0$ for each $|I| \geq 1$ by Equality (5.5) and $c)$ of Theorem 4.4, that is, Ω_t has only one term Ω_0. The holomorphic dependence of $\Phi(t)$ on t implies that Ω_t^C depends on t holomorphically. Now the proof of Theorem 5.3 is a direct corollary of Proposition 5.2 and Theorem 4.5. □

The local version of this theorem is just Proposition 3.4 in [6], first proved in [16], which states that if furthermore, Ω_0 is nowhere vanishing, then Ω_t^C gives rise to a well-defined nowhere vanishing holomorphic $(n,0)$-form on X_t as t is small.

Moreover, we can also obtain a global expansion formula of the canonical family of of holomorphic $(n,0)$-forms for general Kähler manifold case.

Proposition 5.4. *Let $\Omega_t^C := e^{\Phi(t)} \lrcorner \Omega_t$ be the L^2-global canonical family of holomorphic $(n,0)$-forms as constructed in Proposition 5.2. Then for $|t| < R$, there holds the following global expansion of the de Rham cohomology classes of it*

$$[\Omega_t^C] = [\Omega_0] + \sum_{|I|\geq 1} [\mathbb{H}(\varphi_I \lrcorner \Omega_0)] t^I + O(|t|^2),$$

where $O(|t|^2)$ denotes the terms in $\displaystyle\bigoplus_{j=2}^{n} H^{n-j,j}(X)$ of orders at least 2 in t. In particular, if X is a CY manifold, then for $|t| < 1$,

$$[\Omega_t^C] = [\Omega_0] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega_0] t_i + O(|t|^2).$$

Proof. This corollary can be regarded as a global version of Theorem 3.5 in [6]. Here note that Ω_t^C converges in L^2-norm as a formal power series, and is a holomorphic (and thus harmonic) $(n,0)$-form on X_t as long as $|t|<R$. Then from Theorem 5.3 and Hodge theory we can see that for $|t|<R$,

$$\begin{aligned}[\Omega_t^C] &= [\Omega_0] + \sum_{i=1}^{N}[\mathbb{H}(\varphi_i\lrcorner\Omega_0)]t_i + \sum_{|I|\geq 2}[\mathbb{H}(\varphi_I\lrcorner\Omega_0)]t^I + \sum_{k\geq 2}\frac{1}{k!}[\mathbb{H}\Big(\bigwedge^k \Phi(t)\lrcorner\Omega_0\Big)] \\ &= [\Omega_0] + \sum_{i=1}^{N}[\mathbb{H}(\varphi_i\lrcorner\Omega_0)]t_i + \sum_{|I|\geq 2}[\mathbb{H}(\varphi_I\lrcorner\Omega_0)]t^I + O(|t|^2).\end{aligned}$$

Note that the facts that $\varphi_i\lrcorner\Omega_0$ is harmonic and that $\varphi_I\lrcorner\Omega_0$ is ∂-exact for each $|I|\geq 2$ may not hold here. While for CY case we know that $\varphi_i\lrcorner\Omega_0$ is harmonic and that $\varphi_I\lrcorner\Omega_0$ is ∂-exact for each $|I|\geq 2$. For the first fact, see Lemma 3.3 or the proof of Proposition 3.4 of [6]. The remaining terms in the right-hand side of the expression formula are all of orders at least 2 in t and belong to $O(|t|^2)\in \bigoplus_{j=2}^{n} H^{n-j,j}(X)$. ☐

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