NEW PROOFS OF THE TORELLI THEOREMS FOR RIEMANN SURFACES

KEFENG LIU, QUANTING ZHAO, AND SHENG RAO

Abstract. In this paper, by using the Kuranishi coordinates on the Teichmüller space and the explicit deformation formula of holomorphic one-forms on Riemann surface, we give an explicit expression of the period map and derive new differential geometric proofs of the Torelli theorems, both local and global, for Riemann surfaces.

1. Introduction

The theme of this paper is to present a new differential geometric understanding of the Torelli problems of Riemann surfaces, which is a central topic in the study of the complex structures of Riemann surface. The Torelli problems are usually divided into two types: local Torelli and global Torelli. These two problems are about the immersion and injectivity of the period map from the moduli space of Riemann surfaces to the moduli space of principally polarized abelian varieties, respectively.

Two key points of this paper are the use of the Kuranishi coordinates on the Teichmüler space $\mathcal{T}_g$ of Riemann surface of genus g and the explicit deformation formula of holomorphic one-forms in Section 2. Roughly speaking, the Kuranishi coordinate chart of $\mathcal{T}_g$ is given by

$$\begin{array}{ccc}(B, b_0) & \to & \mathcal{T}_g \\ t & \to & [X_t, [F_t]],\end{array}$$

where the triple (ϖ, φ, F) is the Kuranishi family of Riemann surface with the Teichmüller structure of $(X_0, [F_0])$. Let us write (B, b_0) as $\Delta_{p,\epsilon}$, where p denotes the point $[X_0, [F_0]] \in \mathcal{T}_g$. Then, given a global holomorphic one-form $\theta \in H^0(X_p, \Omega^1_{X_p})$ on X_p, we have the following deformation formula $\theta(t)$ of θ for small t on X_t:

$$\theta(t) = \theta + \sum_{i=1}^{n} t_i \left(\mathbb{H}(\mu_i \lrcorner \theta) + df_i\right) + \sum_{|I|\geq 2} t^I \left(\sum_{j=1}^{n} \mathbb{H}(\mu_j \lrcorner \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right),$$

where $\eta_{(i_1,\cdots,i_n)}$ is a sequence of $(1,0)$-forms on X_p and $f_{j,(i_1,\cdots,i_j-1,\cdots,i_n)} \in C^\infty(X_p)$. Here and henceforth $\mathbb{H}$ denotes the harmonic projection on (X_p, ω_p), where ω_p is the Poincáre metric on X_p, and $n = 3g-3$. An application of this to the canonical basis $\{\theta_p^\alpha\}_{\alpha=1}^g$ of $H^0(X_p, \Omega^1_{X_p})$ with respect to the symplectic basis $\{A_\gamma, B_\gamma\}_{\gamma=1}^g$ for $\Delta_{p,\epsilon}$ tells us that

$$\theta_p^\alpha(t) = \theta_p^\alpha + \sum_{i=1}^{n} t_i \left(\mathbb{H}(\mu_i \lrcorner \theta_p^\alpha) + df_i^\alpha\right) + \sum_{|I|\geq 2} t^I \left(\sum_{j=1}^{n} \mathbb{H}(\mu_j \lrcorner \eta^\alpha_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df^\alpha_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right).$$

Date: July 25, 2012.

2010 *Mathematics Subject Classification.* Primary 14C34; Secondary 32G15, 32G05.

Key words and phrases. Torelli problem, Moduli space of Riemann surfaces, Teichmüller theory, Deformations of complex structures.

The $g\times g$ matrix $A(t)$ is naturally defined as:

$$\sum_{|I|\geq 1} t^I\left(\sum_{j=1}^n \mathbb{H}(\mu_j\lrcorner\eta^\alpha_{(i_1,\cdots,i_j-1,\cdots,i_n)}\right)=A(t)^\alpha_\beta\bar\theta^\beta_p.$$

Meanwhile, let π_p be the B period matrix of $\{\theta^\alpha_p\}^g_{\alpha=1}$. Then the period map $\Pi:\ \mathcal{T}_g\to\mathcal{H}_g$ to the Siegel upper half space can be written down explicitly:

$$\Pi(t)=\begin{pmatrix}\bar\pi_p & \pi_p\\ \mathbb{1}_g & \mathbb{1}_g\end{pmatrix}\curvearrowright A(t)^T,$$

where the action $\curvearrowright$ is given by

$$\begin{pmatrix}C_1 & C_2\\ C_3 & C_4\end{pmatrix}_{2g\times 2g}\curvearrowright Z=(C_1Z+C_2)(C_3Z+C_4)^{-1};$$

The transition formula between $\Pi(t)$ and $\Pi(\tau)$ of two adjacent Kuranishi coordinates is

$$\Pi(t)=L_{pq}\curvearrowright\Pi(\tau),$$

where L_{pq} is defined at the end of the proof of Theorem 2.5.

Let Γ_g be the mapping class group of Riemann surface of genus g, which has a natural representation in the symplectic group $\mathrm{Sp}(g,\mathbb{Z})$ with integral coefficients, written as $\rho:\Gamma_g\to\mathrm{Sp}(g,\mathbb{Z})$. The moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g is the quotient space of $\mathcal{T}_g$ by Γ_g, while $\mathcal{A}_g=\mathcal{H}_g/\mathrm{Sp}(g,\mathbb{Z})$ is known as the moduli space of principally polarized abelian varieties. In Section 3, we first give a proof of the following two well-known local Torelli theorems by our deformation method.

Theorem 1.1. *a*) (Local Torelli Theorem 1) *The period map $\Pi:\mathcal{T}_g\to\mathcal{H}_g$ is an immersion on the non-hyperelliptic locus and also when restricted to the hyperelliptic locus for $g\geq 3$; while for $g=2$, Π is an immersion on the whole $\mathcal{T}_g$.*

b) (Local Torelli Theorem 2) *For $g\geq 2$, the period map $\mathcal{J}:\mathcal{M}_g\to\mathcal{A}_g$ is an immersion.*

Write the quotient space of the Teichmüller space $\mathcal{T}_g$ by the Torelli group T_g as $\mathcal{T}or_g$, which has a natural $\mathbb{Z}_2$ action. Recall that the Torelli group T_g is the kernel of the representation $\rho:\ \Gamma_g\to\mathrm{Sp}(g,\mathbb{Z})$. Then we will present a new proof of the following global Torelli theorem in Section 4:

Theorem 1.2. *$\mathcal{J}^{tor}:\mathcal{T}or_g/\mathbb{Z}_2\to\mathcal{H}_g$ is an embedding for $g\geq 3$.*

We also prove that the period map Π maps the Γ_g orbit of $\Delta_{p,\epsilon}$ onto the $\mathrm{Sp}(g,\mathbb{Z})$ orbit of its image in $\mathcal{H}_g$. More precisely, let $\Delta_{p,\epsilon}$ be a Kuranishi coordinate chart on $\mathcal{T}_g$ and $\Delta^{[\phi]}_{p,\epsilon}:=[\phi]\Delta_{p,\epsilon}$ for $[\phi]\in\Gamma_g$. Set

$$\rho([\phi])=\begin{pmatrix}U & V\\ R & S\end{pmatrix}\in\mathrm{Sp}(g,\mathbb{Z}).$$

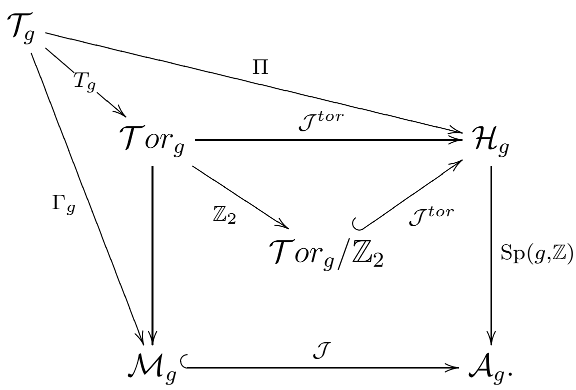
Then on $\Delta^{[\phi]}_{p,\epsilon}$, the period map $\widetilde\Pi(t)$ has the following relation with $\Pi(t)$:

$$\widetilde\Pi(t)=\begin{pmatrix}S & R\\ V & U\end{pmatrix}\curvearrowright\Pi(t).$$

Based on these, we prove that two Γ_g orbits of $\mathcal{M}_g$, if mapped to the same $\mathrm{Sp}(g,\mathbb{Z})$ orbit by $\mathcal{J}$, must coincide, and thus prove the main result of this paper:

Theorem 1.3 (Torelli Theorem)**.** *The period map $\mathcal{J}:\mathcal{M}_g\to\mathcal{A}_g$ is injective for $g\geq 2$.*

The maps considered in this paper can be summarized in the following diagram:



It is well-known that global Torelli theorem holds by R. Torelli's result [22] and also the modern proofs [1, 23] while the local Torelli holds due to the work of [19]. A more complete list of the history about Torelli problems are contained in the bibliographical notes on Page 261 of [4].

Acknowledgement The authors dedicate this paper to Prof. Andrey Todorov, who unexpectedly passed away in March 2012 during his visit of Jerusalem. He had taught graduate courses at the Center of Mathematical Sciences of Zhejiang University on deformation theory and Hodge structures during every summer of the recent years. The last two authors would also like to express their gratitude to Dr. Fangliang Yin, Prof. Fangyang Zheng, and Dr. Shengmao Zhu for many inspirational discussions at CMS of Zhejiang University, and also to Prof. Richard Hain for communications.

2. Kuranishi Coordinates On $\mathcal{T}_g$

We first recall some basics of the construction of Kuranishi coordinate charts, which is based on [2]. Fix a compact topological surface Σ of genus g with $g \geq 2$. The pair $(C, [f])$ is a Riemann surface C with the Teichmüller structure $[f]$, where f is an orientation-preserving homeomorphism from C to Σ and $[f]$ denotes the isotopic class represented by f. An isomorphism between Riemann surfaces with the Teichmüller structures, $(C, [f])$ and $(C', [f'])$, is a biholomorphic map ϕ from C to C' such that $[f] = [f'\phi]$. The equivalence classes of all compact Riemann surfaces of genus g with the Teichmüller structure, modulo the isomorphism equivalences, is actually $\mathcal{T}_g$, the Teichmüler space of Riemann surfaces of genus g. Thus an isomorphism class of $[C, [f]]$ is a point in $\mathcal{T}_g$.

From the construction of Hilbert scheme, the existence of the Kuranishi family of Riemann surfaces follows. To be more precise, for every Riemann surface C, there exists a holomorphic deformation (ϖ, φ)

$$\varpi : \mathcal{X} \to B, \quad \varphi : C \overset{\simeq}{\to} X_{b_0}$$

of C parametrized by a pointed base (B, b_0), a complex manifold with $\dim_{\mathbb{C}} B = 3g - 3$, and this deformation is universal at b_0, actually universal at every point b of B. The pair (ϖ, φ) is called the Kuranishi family of C. For any other deformation (ι, ψ)

$$\iota : \mathcal{X}' \to B', \; \psi : C \overset{\simeq}{\to} X'_{b'_0}$$

of C, there exists a unique map (ϕ, Φ) in a small neighborhood of b_0' such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\Phi} & \mathcal{X} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \varpi} \\ (B', b_0') & \xrightarrow{\phi} & (B, b_0), \end{array}$$

where $\varphi^{-1}\Phi_{b_0'}\psi = \mathbb{1}_C$ and $\mathcal{X}'$ is isomorphic to the pullback family $\Phi^*\mathcal{X}$ on the small neighborhood of b_0'. Accordingly, we also have a family of Riemann surfaces with the Teichmüller structure $(X_b, [f_b])$, i.e., $\varpi : \mathcal{X} \to B$ together with local topological trivialization $F^\alpha : \mathcal{X}|_{U_\alpha} \to \Sigma \times U_\alpha$, where $\bigcup_\alpha U_\alpha$ is an open covering of B such that $[F_b^\alpha] = [f_b]$ with $b \in U_\alpha$. For any Riemann surface with the Teichmüller structure $(C, [f])$, Kuranishi family also exists and satisfies exactly analogous universal properties to the one without the Teichmüller structure above. Possibly shrinking B, we can describe the Kuranishi family of $(C, [f])$ as a triple (ϖ, φ, F) given by

$$\varpi : \mathcal{X} \to B, \quad \varphi : C \xrightarrow{\simeq} C_{b_0}, \quad F : \mathcal{X} \to \Sigma \times B,$$

where F is a topological trivialization such that $F_{b_0}\varphi = f$.

A Kuranishi coordinate chart of $\mathcal{T}_g$ is given by

$$\begin{array}{ccc} (B, b_0) & \to & \mathcal{T}_g \\ t & \to & [X_t, [F_t]], \end{array}$$

where the triple (ϖ, φ, F) is the Kuranishi family of $(C, [f])$. From the classical Ehresmann's theorem, there is a natural diffeomorphism $\Psi : X_{b_0} \times B \to \mathcal{X}$; all the fibers of $\varpi : \mathcal{X} \to B$

$$\begin{array}{ccc} \Sigma \times B & & \\ \uparrow{\scriptstyle F} & & \\ \mathcal{X} & \xrightarrow{\Psi} & X_{b_0} \times B \end{array}$$

share the same differential structure as X_{b_0}. From this point of view, for every $b \in B$, the map $F_b\Psi_b^{-1}$ can be deformed to $F_{b_0}\Psi_{b_0}^{-1}$, i.e., $[F_b\Psi_b^{-1}] = [F_{b_0}\Psi_{b_0}^{-1}]$. Let $\omega : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$ be the intersection pairing on Σ. The symplectic basis of $H_1(\Sigma, \mathbb{Z})$ on (Σ, ω) gives, from the map ΨF^{-1}, one such basis on X_{b_0}, which is enjoyed by the whole Kuranishi family $\mathcal{X}$ over the Kuranishi coordinate chart B. Later on we will write (B, b_0) as $\Delta_{p,\epsilon}$, where p denotes the point $[C, [f]]$ in $\mathcal{T}_g$, and $\Delta_{p,\epsilon} = \{t \in \mathbb{C}^n \big| \|t\| < \epsilon,\ t(p) = 0\}$ with $n = 3g - 3$.

Fix the representation $\rho : \Gamma_g \to \mathrm{Sp}(g, \mathbb{Z})$, where Γ_g is the mapping class group, namely the isotopic classes of orientation preserving homeomorphisms of Σ and $\mathrm{Sp}(g, \mathbb{Z})$ is actually $\mathrm{Aut}(\Sigma, \omega)$. Now we have two Kuranishi coordinate charts $\Delta_{p,\epsilon}$ and $\Delta_{q,\epsilon'}$ with $\Delta_{p,\epsilon} \cap \Delta_{q,\epsilon'} \neq \emptyset$. Let $(\mathcal{X}, F)$ and $(\mathcal{Y}, G)$ denote the two Kuranishi families with Teichmüler structure over $\Delta_{p,\epsilon}$ and $\Delta_{q,\epsilon'}$, respectively. Let $r \in \Delta_{p,\epsilon} \cap \Delta_{q,\epsilon'}$. The definition of Kuranishi coordinates tells us that $[X_{t(r)}, F_{t(r)}] = [Y_{\tau(r)}, G_{\tau(r)}]$. Then we have a biholomorphic map $\phi : X_{t(r)} \to Y_{\tau(r)}$ such that $[F_{t(r)}] = [G_{\tau(r)}\phi]$. It is described in the following picture

$$\begin{array}{ccccccc} \Sigma & & & & & & \Sigma \\ \uparrow{\scriptstyle F_0} & & & & & & \uparrow{\scriptstyle G_0} \\ X_p & \xrightarrow{\mathrm{diffeo}\Psi_X} & X_{t(r)} & \xrightarrow{\phi} & Y_{\tau(r)} & \xrightarrow{\mathrm{diffeo}\Psi_Y} & Y_q \end{array}$$

that $[G_0\Psi_Y\phi\Psi_X F_0^{-1}]$ gives us an element of Γ_g, which is obtained by a matrix in $\mathrm{Sp}(g,\mathbb{Z})$ from the representation ρ, linking the two symplectic bases of the two Kuranishi coordinates.

2.1. **Small Deformation Of Holomorphic One-Forms.** Let $\Delta_{p,\epsilon}$ be a Kuranishi coordinate chart centered at $p\in\mathcal{T}_g$ as above. Denote the corresponding Kuranishi family on $\Delta_{p,\epsilon}$ by $\varpi:\mathcal{X}\to\Delta_{p,\epsilon}$ with the central fiber $\varpi^{-1}(p)=X_p$. Let $\theta\in H^0(X_p,\Omega^1_{X_p})$ be a global holomorphic one-form on X_p. We will construct an explicit formula $\theta(t)\in H^0(X_t,\Omega^1_{X_t})$, the holomorphic deformation of θ.

Denote the Poincaré metric on X_p by ω_p. Fix $\{\mu_i\}_{i=1}^n$ as a basis of harmonic $T^{(1,0)}_{X_p}$-valued $(0,1)$-form on (X_p,ω_p), namely $\mathbb{H}^{0,1}_{\overline{\partial}}(X_p,T^{(1,0)}_{X_p})$. And let $\mu(t)=\sum_{i=1}^n t_i\mu_i$ denote the Beltrami differential of the Kuranishi family $\varpi:\mathcal{X}\to\Delta_{p,\epsilon}$.

Theorem 2.1. *Given $\theta\in H^0(X_p,\Omega^1_{X_p})$, there exists a unique $(1,0)$-form $\eta(t)$ on X_p, which is holomorphic in t for sufficiently small t, satisfying*

(1) $\mathbb{H}(\eta(t))=\theta$, where $\mathbb{H}$ is the harmonic projection on (X_p,ω_p),
(2) $\theta(t)=(\mathbb{1}+\mu(t))\lrcorner\eta(t)\in H^0(X_t,\Omega^1_{X_t})$

and $\theta(t)$ is the desired deformation of θ.

Proof. The formal power series of $\eta(t)\in A^{1,0}(X_p)$ can be written out as

$$\eta(t)=\theta+\sum_{i=1}^n t_i\eta_i+\sum_{|I|\geq 2}t^I\eta_I,$$

where $I=(i_1,\cdots,i_n)$, $t^I=t_1^{i_1}t_2^{i_2}\cdots t_n^{i_n}$ and $|I|=\sum_{j=1}^n i_j$.

Condition (1) implies

$$\begin{cases}\mathbb{H}(\eta_i)=0,\\ \mathbb{H}(\eta_I)=0,\ |I|\geq 2.\end{cases}\tag{2.1}$$

Then, one has

$$\begin{aligned}\theta(t)&=(\mathbb{1}+\mu(t))\lrcorner\eta(t)\\ &=\left(\mathbb{1}+\sum_{i=1}^n t_i\mu_i\right)\lrcorner\left(\theta+\sum_{j=1}^n t_j\eta_j+\sum_{|J|\geq 2}t^J\eta_J\right)\\ &=\theta+\sum_{i=1}^n t_i(\eta_i+\mu_i\lrcorner\theta)+\sum_{|I|\geq 2}t^I\left(\eta_{(i_1,\cdots,i_n)}+\sum_{k=1}^n\mu_k\lrcorner\eta_{(i_1,\cdots,i_k-1,\cdots,i_n)}\right).\end{aligned}$$

Since $\theta(t)$ is a holomorphic one-form on X_t from condition (2), i.e., $d\theta(t)=0$, which implies

$$\begin{cases}d(\eta_i+\mu_i\lrcorner\theta)=0,\\ d(\eta_{(i_1,\cdots,i_n)}+\sum_{k=1}^n\mu_k\lrcorner\eta_{(i_1,\cdots,i_k-1,\cdots,i_n)})=0,\end{cases}$$

we see that

$$\begin{cases}\overline{\partial}\eta_i+\partial(\mu_i\lrcorner\theta)=0,\\ \overline{\partial}\eta_{(i_1,\cdots,i_n)}+\partial(\sum_{k=1}^n\mu_k\lrcorner\eta_{(i_1,\cdots,i_k-1,\cdots,i_n)})=0.\end{cases}\tag{2.2}$$

Combining with (2.1) and solving the $\overline{\partial}$-equation, we get

$$\begin{cases}\eta_i=-\mathbb{G}\overline{\partial}^*\partial(\mu_i\lrcorner\theta),\\ \eta_{(i_1,\cdots,i_n)}=-\mathbb{G}\overline{\partial}^*\partial(\sum_{k=1}^n\mu_k\lrcorner\eta_{(i_1,\cdots,i_k-1,\cdots,i_n)}).\end{cases}\tag{2.3}$$

Here $\mathbb{G}$ denotes the Green operator in the Hodge decomposition with respect to the operator $\overline{\partial}$, and $\mathbb{1} = \mathbb{H} + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\mathbb{G}$. Thus we have proved the uniqueness of $\eta(t)$, which is fixed by conditions (1) and (2).

Now let us discuss the convergence of the power series constructed above. By the standard estimates of elliptic operators $\mathbb{G}$, $\overline{\partial}^*$ and ∂, such as in [17], we easily have

$$\|\eta_I\|_{m+\alpha} \le C^{|I|}\|\theta\|_{m+\alpha},$$

where the constant C depends on m, α and X_p, and $\|\cdot\|_{m+\alpha}$ is the Hölder norm. Consequently the estimates of $\eta(t)$ yield

$$\begin{aligned}\|\eta(t)\|_{m+\alpha} &\le \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha}\sum_{\|I\|\ge 1} C^{|I|}\epsilon^{|I|}\\ &= \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha}\sum_{k\ge 1}\sum_{\|I\|\ge k} C^{|I|}\epsilon^{|I|}\\ &= \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha}\sum_{k\ge 1} C^k\epsilon^k C^k_{n+k-1}\\ &\le \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha}\sum_{k\ge 1} C^k\epsilon^k n^k,\end{aligned}$$

where C^k_{n+k-1} is the common combinatorial number. By taking ϵ smaller than $\frac{1}{2nC}$, we are done. □

Corollary 2.2. *The deformation formula of θ, with t small, is given by*

$$\theta(t) = \theta + \sum_{i=1}^n t_i\left(\mathbb{H}(\mu_i \lrcorner \theta) + df_i\right) + \sum_{|I|\ge 2} t^I\left(\sum_{j=1}^n \mathbb{H}(\mu_j \lrcorner \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right)$$

where $f_{j,(i_1,\cdots,i_j-1,\cdots,i_n)} \in C^\infty(X_p)$.

Proof. From Theorem 2.1, we can easily write out

$$\begin{aligned}\theta(t) &= (\mathbb{1} + \mu(t))\lrcorner\eta(t)\\ &= \left(\mathbb{1} + \sum_{i=1}^n t_i\eta_i\right)\lrcorner\left(\theta - \sum_{j=1}^n t_j\left(\mathbb{G}\overline{\partial}^*\partial(\mu_j\lrcorner\theta)\right) + \sum_{|J|\ge 2} t^J\eta_J\right)\\ &= \theta + \sum_{|I|\ge 1} t^I\left(\mathbb{1} - \mathbb{G}\overline{\partial}^*\partial\right)\left(\sum_{j=1}^n \mu_j\lrcorner\eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}\right)\\ &= \theta + \sum_{|I|\ge 1} t^I\left(\mathbb{1} - \mathbb{G}\overline{\partial}^*\partial\right)\left(\sum_{j=1}^n \mathbb{H}(\mu_j\lrcorner\eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + \overline{\partial}f_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right)\\ &= \theta + \sum_{|I|\ge 1} t^I\left(\sum_{j=1}^n \mathbb{H}(\mu_j\lrcorner\eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + \overline{\partial}f_{j,(i_1,\cdots,i_j-1,\cdots,i_n)} + \partial\mathbb{G}\square_{\overline{\partial}}f_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right)\\ &= \theta + \sum_{|I|\ge 1} t^I\left(\sum_{j=1}^n \mathbb{H}(\mu_j\lrcorner\eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right).\end{aligned}$$

The convergence follows from Theorem 2.1. □

Remark 2.3. *The iteration method to construct canonical forms on the deformation space of Riemann surfaces is essentially contained in* [16] *and* [24, Theorem 2.1]. *For* *more generalization to Kähler manifolds, see* [15]*, while our proof emphasizes on the uniqueness of the construction.*

Denote the canonical basis [1] of $H^0(X_p, \Omega^1_{X_p})$ by $\{\theta_p^\alpha\}_{\alpha=1}^g$ with respect to the symplectic basis $\{A_\gamma, B_\gamma\}_{\gamma=1}^g$ on the Kuranishi coordinate chart $\Delta_{p,\epsilon}$. Let σ_p and π_p be the A and B period matrices of $\{\theta_p^\alpha\}_{\alpha=1}^g$, respectively, and $M_p = \mathrm{Im}(\pi_p)$. Applying the deformation formula above, we get the holomorphic one-forms $\theta_p^\alpha(t)$ on X_t, starting with θ_p^α, given by

(2.4)

$$\theta_p^\alpha(t) = \theta_p^\alpha + \sum_{i=1}^n t_i \left(\mathbb{H}(\mu_i \lrcorner \theta_p^\alpha) + df_i^\alpha\right) + \sum_{|I|\geq 2} t^I \left(\sum_{j=1}^n \mathbb{H}(\mu_j \lrcorner \eta^\alpha_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df^\alpha_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}\right).$$

Definition 2.4. *Let $A(t)$ be a $g\times g$ matrix and $E(t)$ a $g \times 1$ vector given by:*

$$\begin{cases} \sum_{|I|\geq 1} t^I \left(\sum_{j=1}^n \mathbb{H}(\mu_j \lrcorner \eta^\alpha_{(i_1,\cdots,i_j-1,\cdots,i_n)})\right) = A(t)^\alpha_\beta \bar\theta^\beta_p, \\ \sum_{|I|\geq 1} t^I (\sum_{j=1}^n df^\alpha_{j,(i_1,\cdots,i_j-1,\cdots,i_n)}) = E^\alpha(t). \end{cases}$$

Also the homogeneous part of order N of $A(t)$ is written as $A_N(t) = \sum_{|I|=N} t^I A_I$,

$$\sum_{j=1}^n \left(\mathbb{H}(\mu_j \lrcorner \eta^\alpha_{(i_1,\cdots,i_j-1,\cdots,i_n)})\right) = A_{I,}{}^\alpha_\beta\, \bar\theta^\beta_p.$$

In particular, $\mathbb{H}(\mu_i \lrcorner \theta_p^\alpha) = A_{i,}{}^\alpha_\beta\, \bar\theta^\beta_p$.

Set

$$\Theta_p(t) = \begin{pmatrix} \theta_p^1(t) \\ \vdots \\ \theta_p^g(t) \end{pmatrix} \text{ and } \Theta_p = \begin{pmatrix} \theta_p^1 \\ \vdots \\ \theta_p^g \end{pmatrix}.$$

Thus by use of $A(t)$ and $E(t)$, we rewrite (2.4) as

$$\Theta_p(t) = \begin{pmatrix} \mathbb{1}_g & A(t) \end{pmatrix} \begin{pmatrix} \Theta_p \\ \bar\Theta_p \end{pmatrix} + E(t). \tag{2.5}$$

Since a holomorphic one-form on Riemann surfaces is uniquely determined by its integration on A cycles, it is clear that $\{\theta_p^\alpha(t)\}_{\alpha=1}^g$ being a frame of $H^0(X_t, \Omega^1_{X_t})$ on X_t, is equivalent to non-degeneration of the A period matrix $\sigma_{\alpha\beta}(t)$ on X_t , i.e.,

$$\det\left(\sigma_{\alpha\beta}(t)\right) = \det\left(\int_{A_\alpha} \theta_p^\beta(t)\right) \neq 0 \iff \det\left(\mathbb{1}_g + A(t)^T\right) \neq 0, \tag{2.6}$$

where $A(t)^T$ is the transpose of $A(t)$. And when $\{\theta_p^\alpha(t)\}_{\alpha=1}^g$ becomes a frame, we have the Hodge-Riemann bilinear relations on X_t

$$\begin{cases} 0 = \frac{\sqrt{-1}}{2} \int_{X_t} \theta_p^\alpha(t) \wedge \theta_p^\beta(t), \\ 0 < \frac{\sqrt{-1}}{2} \int_{X_t} \theta_p^\alpha(t) \wedge \bar\theta_p^\beta(t), \end{cases}$$

which and also (2.5) imply that

$$\begin{cases} 0 = \frac{\sqrt{-1}}{2} \int_{X_p} \left(\theta_p^\alpha + A(t)^\alpha_\gamma \bar\theta^\gamma + E^\alpha(t)\right) \wedge \left(\theta_p^\beta + A(t)^\beta_\lambda \bar\theta^\lambda + E^\beta(t)\right), \\ 0 < \frac{\sqrt{-1}}{2} \int_{X_p} \left(\theta_p^\alpha + A(t)^\alpha_\gamma \bar\theta_p^\gamma + E^\alpha(t)\right) \wedge \left(\bar\theta_p^\beta + \overline{A(t)}^\beta_\lambda \theta_p^\lambda + \bar E^\beta(t)\right), \end{cases}$$

[1] A canonical basis means the unique basis of $H^0(X_p, \Omega^1_{X_p})$ such that its A period matrix is $\mathbb{1}_g$.

and thus

$$\begin{cases} 0 = M_{p,\alpha\gamma}\, A(t)^{\beta}_{\gamma} - M_{p,\beta\gamma}\, A(t)^{\alpha}_{\gamma}, \\ 0 < M_{p,\alpha\beta} - M_{p,\lambda\gamma}\, A(t)^{\alpha}_{\gamma} \overline{A(t)}^{\beta}_{\lambda}. \end{cases}$$

The matrix forms of these are given by

$$\begin{cases} A(t)M_p = (A(t)M_p)^T, \\ M_p - A(t)M_p\overline{A(t)}^T > 0. \end{cases} \tag{2.7}$$

As our deformation formula is local, $\{\theta^{\alpha}_{p}(t)\}^{g}_{\alpha=1}$ is always a frame when $t \in \Delta_{p,\epsilon}$ with ϵ sufficiently small. Therefore, (2.6) and (2.7) hold.

2.2. Transition Formulas Between the Kuranishi Coordinates.

Theorem 2.5. *Assume that the two Kuranishi coordinate charts $\Delta_{p,\epsilon}$ and $\Delta_{q,\epsilon'}$ have a non-empty intersection containing those two centers p and q, and let t and τ denote the corresponding Kuranishi coordinates. Then $A(t)$ and $A(\tau)$ are related by the following equality:*

$$A(t)^T = \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix}^{-1} L_{pq} \begin{pmatrix} \bar{\pi}_q & \pi_q \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \curvearrowright A(\tau)^T, \tag{2.8}$$

where $L_{pq} \in \mathrm{Sp}(g,\mathbb{Z})$ denotes the transition matrix between the symplectic bases of the two Kuranishi coordinates in terms of transformations in $\mathrm{Sp}(g,\mathbb{Z})$ of $\mathcal{H}_g$, and the action $\curvearrowright$ is given by

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \curvearrowright Z = (C_1 Z + C_2)(C_3 Z + C_4)^{-1}.$$

Observe that the transition matrix linking $A(t)$ and $A(\tau)$ depends only on p and q, but not the coordinates t and τ.

Proof. From (2.5), it yields

$$[\Theta_p(q)] = \begin{pmatrix} \mathbb{1}_g & A(t(q)) \end{pmatrix} \begin{pmatrix} [\Theta_p] \\ [\bar{\Theta}_p] \end{pmatrix}, \tag{2.9}$$

where $[\Theta_p(q)]$ denotes the cohomology class represented by $\Theta_p(q)$. The frames given by the deformation formula $[\Theta_p(q)]$ and the canonical one $[\Theta_q]$ at q are different by a multiple of a nonsingular matrix C:

$$[\Theta_q] = C\,[\Theta_p(q)]\,. \tag{2.10}$$

Let $\{A_\gamma, B_\gamma\}^{g}_{\gamma=1}$ and $\{A'_\gamma, B'_\gamma\}^{g}_{\gamma=1}$ be the symplectic bases on $\Delta_{p,\epsilon}$ and $\Delta_{q,\epsilon'}$, respectively. Set

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A_1 \\ \vdots \\ A_g \\ B_1 \\ \vdots \\ B_g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} A'_1 \\ \vdots \\ A'_g \\ B'_1 \\ \vdots \\ B'_g \end{pmatrix}.$$

Denote the matrix linking these two bases by $\begin{pmatrix} U & V \\ R & S \end{pmatrix} \in \mathrm{Sp}(g,\mathbb{Z})$, i.e.,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} U & V \\ R & S \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix}. \tag{2.11}$$

By (2.11), we integrate over A cycles and B cycles on (2.10) to get

$$\begin{cases} U+V\pi_q=\left(\mathbb{1}_g+A\left(t(q)\right)^T\right)C^T, \\ R+S\pi_q=\left(\pi_p+\bar{\pi}_p A\left(t(q)\right)^T\right)C^T, \end{cases}$$

which imply that

$$\begin{cases} C^T=\left(\mathbb{1}_g+A\left(t(q)\right)^T\right)^{-1}\left(U+V\pi_q\right), \\ \left(\pi_p+\bar{\pi}_p A\left(t(q)\right)^T\right)\left(\mathbb{1}_g+A\left(t(q)\right)^T\right)^{-1}=\left(R+S\pi_q\right)\left(U+V\pi_q\right)^{-1}. \end{cases} \tag{2.12}$$

By (2.9) and (2.10), we have

$$\begin{aligned} [\Theta_q] &= C\begin{pmatrix}\mathbb{1}_g & A\left(t(q)\right)\end{pmatrix}\begin{pmatrix}[\Theta_p] \\ [\bar{\Theta}_p]\end{pmatrix} \\ &= \begin{pmatrix}C & CA\left(t(q)\right)\end{pmatrix}\begin{pmatrix}[\Theta_p] \\ [\bar{\Theta}_p]\end{pmatrix}. \end{aligned} \tag{2.13}$$

Let $r\in\Delta_{p,\epsilon}\cap\Delta_{q,\epsilon'}$. Then one has

$$[\Theta_p(r)]=\begin{pmatrix}\mathbb{1}_g & A(t)\end{pmatrix}\begin{pmatrix}[\Theta_p] \\ [\bar{\Theta}_p]\end{pmatrix};$$

while by (2.13), one also has

$$\begin{aligned} [\Theta_p(r)] = C_r\left[\Theta_q(r)\right] &= C_r\begin{pmatrix}\mathbb{1}_g & A(\tau)\end{pmatrix}\begin{pmatrix}[\Theta_q] \\ [\bar{\Theta}_q]\end{pmatrix} \\ &= C_r\begin{pmatrix}\mathbb{1}_g & A(\tau)\end{pmatrix}\begin{pmatrix}C & CA\left(t(q)\right) \\ \overline{CA\left(t(q)\right)} & \overline{C}\end{pmatrix}\begin{pmatrix}[\Theta_p] \\ [\bar{\Theta}_p]\end{pmatrix}, \end{aligned}$$

where the two frames $[\Theta_p(r)]$ and $[\Theta_q(r)]$ at the point r are related by a nonsingular matrix C_r. These give us the following identities:

$$\begin{cases} \mathbb{1}_g=C_r\left(C+A(\tau)\overline{CA\left(t(q)\right)}\right), \\ A(t)=C_r\left(CA\left(t(q)\right)+A(\tau)\overline{C}\right). \end{cases}$$

Combine with (2.12) to simplify the computation as follows:

$$\begin{aligned} A(t)^T &= \begin{pmatrix}\bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g\end{pmatrix}^{-1}\begin{pmatrix}\left(\bar{\pi}_p+\pi_p\overline{A\left(t(q)\right)}^T\right)\bar{C}^T & \left(\pi_p+\bar{\pi}_p A\left(t(q)\right)^T\right)C^T \\ \left(\mathbb{1}_g+\overline{A\left(t(q)\right)}^T\right)\bar{C}^T & \left(\mathbb{1}_g+A\left(t(q)\right)^T\right)C^T\end{pmatrix}\curvearrowright A(\tau)^T \\ &= \begin{pmatrix}\bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g\end{pmatrix}^{-1}\begin{pmatrix}S & R \\ V & U\end{pmatrix}\begin{pmatrix}\bar{\pi}_q & \pi_q \\ \mathbb{1}_g & \mathbb{1}_g\end{pmatrix}\curvearrowright A(\tau)^T, \end{aligned}$$

where $\begin{pmatrix}S & R \\ V & U\end{pmatrix}$ also belongs to $\mathrm{Sp}(g,\mathbb{Z})$, denoted by L_{pq}. □

On our Kuranishi coordinate chart $\Delta_{p,\epsilon}$, the period map $\Pi:\mathcal{T}_g\to\mathcal{H}_g$ can be written out quite explicitly:

$$\begin{aligned} \Pi(t)_{\alpha\beta} &= \int_{B_\alpha}\sigma(t)^{\gamma\beta}\theta_p^\gamma(t) \\ &= \int_{B_\alpha}\sigma(t)^{\gamma\beta}\left(\theta_p^\gamma+A(t)_\delta^\gamma\bar{\theta}_p^\delta\right) \\ &= \pi_{p,\alpha\gamma}\,\sigma(t)^{\gamma\beta}+\bar{\pi}_{p,\alpha\delta}\,A(t)_\delta^\gamma\sigma(t)^{\gamma\beta}, \end{aligned}$$

where $\sigma(t)^{\alpha\beta}$ is the inverse matrix of $\sigma(t)_{\alpha\beta}$. (2.6) gives us

$$\sigma_{\alpha\beta}(t) = \int_{A_\alpha} \theta_p^\beta(t) = \left(\mathbb{1}_g + A(t)^T\right)_{\alpha\beta}.$$

Now we can formulate these into the matrix form:

$$\begin{aligned} \Pi(t) &= \left(\pi_p + \bar{\pi}_p A(t)^T\right)\left(\mathbb{1}_g + A(t)^T\right)^{-1} \\ &= \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \curvearrowright A(t)^T. \end{aligned} \tag{2.14}$$

Corollary 2.6. *The period maps $\Pi(t)$ and $\Pi(\tau)$ on the intersection of the two Kuranishi coordinate charts $\Delta_{p,\epsilon}$ and $\Delta_{q,\epsilon'}$ have the following transition formula*

$$\Pi(t) = L_{pq} \curvearrowright \Pi(\tau). \tag{2.15}$$

Proof. By (2.14) and Theorem 2.5, we have

$$\begin{aligned} \Pi(t) &= \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \curvearrowright A(t)^T \\ &= \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix}\begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix}^{-1} L_{pq} \begin{pmatrix} \bar{\pi}_q & \pi_q \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix}\begin{pmatrix} \bar{\pi}_q & \pi_q \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix}^{-1} \curvearrowright \Pi(\tau) \\ &= L_{pq} \curvearrowright \Pi(\tau). \end{aligned}$$

□

3. Local Torelli Theorems and Matrix Model

Theorem 3.1. (Local Torelli Theorem 1) *For $g \geq 3$, the period map $\Pi : \mathcal{T}_g \to \mathcal{H}_g$ is an immersion on the non-hyperelliptic locus $\mathcal{T}_g - \mathcal{HET}_g$ and also on the hyperelliptic locus $\mathcal{HET}_g$. In the case $g = 2$, Π is an immersion on the whole $\mathcal{T}_g$.*

Proof. From (2.14), the period map can be written as $\Pi(t) = \left(\bar{\pi}_p A(t)^T + \pi_p\right)\left(A(t)^T + \mathbb{1}_g\right)^{-1}$ via Kuranishi coordinates. By use of $A(t)$, we expand it to obtain the first order part $\Pi^{(1)}(t)$ of $\Pi(t)$:

$$\begin{aligned} \Pi^{(1)}(t) &= \bar{\pi}_p A_1^T(t) - \pi_p A_1^T(t) \\ &= -2\sqrt{-1} M_p A_1^T(t) \\ &= -2\sqrt{-1} \sum_{i=1}^n t_i M_{p,\alpha\gamma}\, A_{i,\gamma}^{\ \beta} \\ &= \sum_{i=1}^n t_i \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}(\mu_i \lrcorner \theta_p^\beta) \\ &= \sum_{i=1}^n t_i \int_{X_p} \theta_p^\alpha \wedge (\mu_i \lrcorner \theta_p^\beta). \end{aligned} \tag{3.1}$$

It is a well-known fact that the pairing $\mathbb{H}_{\bar{\partial}}^{(0,1)}(X_p, T_{X_p}^{(1,0)}) \times H^0(X_p, 2K_{X_p}) \to \mathbb{C}$ is non-degenerate. The matrices $\{\int_{X_p} \theta_p^\alpha \wedge (\mu_i \lrcorner \theta_p^\beta)\}_{i=1}^n$ are linearly dependent if and only if there exists a nonzero vector $t = (t_1, \cdots, t_n)$ such that the matrix

$$\int_{X_p} \theta^\alpha \wedge \left(\sum_{i=1}^n t_i \mu_i\right) \lrcorner \theta^\beta \equiv 0.$$

This is equivalent to that the multiplication map $H^0(X_p, K_{X_p}) \times H^0(X_p, K_{X_p}) \to H^0(X_p, 2K_{X_p})$ is not surjective.

A well-known theorem by Max Noether in [4, P. 117] tells us the multiplication map $H^0(X_p, K_{X_p}) \times H^0(X_p, K_{X_p}) \to H^0(X_p, 2K_{X_p})$ is always surjective when X_p is non-hyperelliptic. Thus the period map Π is an immersion when restricted to $\mathcal{T}_g - \mathcal{HET}_g$ for $g \geq 3$. As to the hyperelliptic case which is described in [7, P. 104], the dimension of the image of the multiplication map is exactly the vector space $\left(H^0(X_p, 2K_{X_p})\right)^J$, namely, the elements in $H^0(X_p, 2K_{X_p})$ invariant under the action by the hyperelliptic involution J with $\dim_{\mathbb{C}} \left(H^0(X_p, 2K_{X_p})\right)^J = 2g-1$. Also the tangent direction of the hyperelliptic locus can be identified with $\left(H^1(X_p, T_{X_p})\right)^J$. Hence these directions can not be degenerate and thus $\Pi|_{\mathcal{HET}_g}$ is still an immersion for $g \geq 3$. As we know, any Riemann surface of genus 2 is hyperelliptic and the above multiplication map is surjective since $2g-1 = 3g-3$ when $g = 2$. Consequently, Π is an immersion on $\mathcal{T}_g$ for $g = 2$. □

Definition 3.2. *T_g, $\widetilde{T}_g$ and $\mathcal{T}or_g$.*

T_g, called the Torelli group, is the kernel of the representation $\rho : \Gamma_g \to \mathrm{Sp}(g, \mathbb{Z})$ while the extended Torelli group $\widetilde{T}_g$ is defined to be $\rho^{-1}(\langle -\mathbb{1}_{2g} \rangle)$ where $\langle -\mathbb{1}_{2g} \rangle$ is the subgroup of $\mathrm{Sp}(g, \mathbb{Z})$ generated by $-\mathbb{1}_{2g}$. The Torelli space $\mathcal{T}or_g$ is the quotient space of the Teichmüller space $\mathcal{T}_g$ by T_g.

Definition 3.3. *$\Gamma_g(n)$ and $\mathcal{M}_g^{(n)}$.*

$\Gamma_g(n)$, the level n subgroup of the mapping class group Γ_g, is the kernel of the representation $\Gamma_g \xrightarrow{\rho} \mathrm{Sp}(g, \mathbb{Z}) \xrightarrow{\pi} \mathrm{Sp}(g, \mathbb{Z}_n)$. $\mathcal{M}_g^{(n)}$ is the moduli space of Riemann surfaces of genus g with level n structure, which is defined as the quotient space of the Teichmüller space $\mathcal{T}_g$ by the group action of $\Gamma_g(n)$. And we identify $\Gamma_g(1)$ with T_g.

As we know, the action of the mapping class group Γ_g on the Teichmüller space $\mathcal{T}_g$ is properly discontinuous. From the construction of the Kuranishi coordinate of $\mathcal{T}_g$ in [2], we know that the isotropy group Γ_g^p of Γ_g at $p = [X_p, [f_p]]$ on $\mathcal{T}_g$ is $\mathrm{Aut}(X_p)$ if we fix the injective homomorphism

$$\begin{array}{ccc} \mathrm{Aut}(X_p) & \to & \Gamma_g \\ h & \to & [f_p h f_p^{-1}]. \end{array}$$

Moreover, we can choose ϵ and ϵ' sufficiently small such that the points p and p' in different Γ_g orbits have disjoint Kuranishi coordinates, i.e., $\Delta_{p,\epsilon} \cap \Delta_{p',\epsilon'} = \emptyset$, and

$$\{\gamma \in \Gamma_g \big| \gamma\Delta_{p,\epsilon} \cap \Delta_{p,\epsilon} \neq \emptyset\} = \Gamma_g^p.$$

Proposition 3.4. *The action of T_g and $\Gamma_g(n)$ with $n \geq 3$ on $\mathcal{T}_g$ is fixed point free.*

This proposition implies that $\mathcal{T}or_g$ and $\mathcal{M}_g^{(n)}$ with $n \geq 3$ are complex manifolds of complex dimension $3g-3$.

Proof. We just need to show that $T_g \cap \Gamma_g^p = \{1\}$ and $\Gamma_g(n) \cap \Gamma_g^p = \{1\}$. But we can identify Γ_g^p with $\mathrm{Aut}(X_p)$. It follows from the theory of automorphism groups of Riemann surfaces in [7, Chapter V] that the representation of $\mathrm{Aut}(X_p)$ in $H_1(X_p, \mathbb{Z})$ and $H_1(X_p, \mathbb{Z}_n)$ with $n \geq 3$ are faithful, i.e., the homomorphisms $\mathrm{Aut}(X_p) \to \mathrm{Sp}(g, \mathbb{Z})$ and $\mathrm{Aut}(X_p) \to \mathrm{Sp}(g, \mathbb{Z}_n)$ are injective. Now the isotropy group Γ_g^p embeds into $\mathrm{Sp}(g, \mathbb{Z})$ by the representation $\rho : \Gamma_g \to \mathrm{Sp}(g, \mathbb{Z})$ if we view Γ_g^p as $f_p^{-1}\mathrm{Aut}(X_p)f_p$ while T_g is the kernel of ρ. Thus $T_g \cap \Gamma_g^p = \{1\}$. Similarly, Γ_g^p embeds into $\mathrm{Sp}(g, \mathbb{Z}_n)$ by the representation $\Gamma_g \xrightarrow{\rho} \mathrm{Sp}(g, \mathbb{Z}) \xrightarrow{\pi} \mathrm{Sp}(g, \mathbb{Z}_n)$, and $\Gamma_g(n)$ is the kernel of the representation $\pi\rho$. Finally we have $\Gamma_g(n) \cap \Gamma_g^p = \{1\}$. □

From the discussion above, we can shrink our Kuranishi coordinate chart $\Delta_{p,\epsilon}$ on $\mathcal{T}_g$ such that $\gamma\Delta_{p,\epsilon} \cap \Delta_{p,\epsilon} = \emptyset$ for any $\gamma \in T_g$ and $\gamma \neq 1$. Naturally, the Kuranishi coordinate chart $\Delta_{p,\epsilon}$ descends to $\mathcal{T}or_g$. Let $\mathbb{Z}_2 \cong \widetilde{T}_g/T_g$ and then $\mathcal{T}or_g$ has a natural $\mathbb{Z}_2$ action. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_g & & \\ {\scriptstyle T_g}\downarrow & \searrow^{\Pi} & \\ \mathcal{T}or_g & \xrightarrow[\mathcal{J}^{tor}]{} & \mathcal{H}_g. \end{array}$$

Lemma 3.5. *Let X be a compact Riemann surface with genus $g \geq 2$ and J an involution on X, which does not fix any element of $H^0(X, K_X)$. Then X is hyperelliptic and J must be a hyperelliptic involution.*

Proof. Since $J^2 = \mathbb{1}$, the automorphism $J^* : H^0(X, K_X) \to H^0(X, K_X)$ has two eigenvalue ± 1. As J^* fixes no element of $H^0(X, K_X)$, $J^* = -\mathbb{1}_g$ on $H^0(X, K_X)$. Consider the quotient map $\pi : X \to X/J$, a $2:1$ branched covering map, and $\pi = J\pi$. We claim that $g(X/J) = 0$. If not, there exists a nonzero holomorphic one-form $\theta \in H^0(X/J, K_X)$. Pulling it back, we derive a nonzero holomorphic one-form $\pi^*\theta \in H^0(X, K_X)$. But $\pi^*\theta$ is invariant under J^*, which is a contradiction. Thus X/J is the Riemann sphere and π is a degree 2 meromorphic function on X, which implies that X is hyperelliptic and J is a hyperelliptic involution. □

Proposition 3.6. *$\mathbb{Z}_2$ acts freely on the non-hyperelliptic locus $\mathcal{T}or_g - \mathcal{HET}or_g$ of $\mathcal{T}or_g$ and fixes every point in the hyperelliptic locus $\mathcal{HET}or_g$ for $g \geq 3$. In the case $g = 2$, $\mathbb{Z}_2$ acts trivially on $\mathcal{T}or_g$.*

Proof. Let $\{[\phi]\}$ be the non-unit element in $\mathbb{Z}_2$, where $[\phi] \in \Gamma_g$ is a representative of the class $\{[\phi]\}$ and $\rho([\phi]) = -\mathbb{1}_{2g}$. Then that $\{[\phi]\}$ fixes a point $\{[X_p, [f_p]]\}$ in $\mathcal{T}or_g$ is equivalent to that there exists some element $[\psi] \in T_g$ such that $[X_p, [\phi f_p]] = [X_p, [\psi f_p]]$. We have a commutative diagram up to isotopy

$$\begin{array}{ccc} & \Sigma & \\ {\scriptstyle \phi f_p}\nearrow & & \nwarrow{\scriptstyle \psi f_p} \\ X_p & \xrightarrow{\ h\ } & X_p \end{array} ,$$

where h is an automorphism of X_p. Hence $h \simeq f_p^{-1}\psi^{-1}\phi f_p$. As $\rho([\psi]) = \mathbb{1}_{2g}$ and $\rho([\phi]) = -\mathbb{1}_{2g}$, $h_* : H_1(X_p, \mathbb{Z}) \to H_1(X_p, \mathbb{Z})$ is nothing but $-\mathbb{1}_{2g}$. Since a holomorphic one-form is uniquely determined by its integration on A cycles and

$$\int_{A_\alpha} h^*\theta = \int_{h_* A_\alpha} \theta = -\int_{A_\alpha} \theta,$$

$h^* : H^0(X_p, K_{X_p}) \to H^0(X_p, K_{X_p})$ is $-\mathbb{1}_g$. Also the representation of $\mathrm{Aut}(X_p)$ to $H_1(X_p, \mathbb{Z})$ is faithful and hence h is an involution. From Lemma 3.5, h is a hyperelliptic involution and X_p is hyperelliptic. □

It is easy to check that the $\mathbb{Z}_2$ orbit of $\mathcal{T}or_g$ has the same image under $\mathcal{J}^{tor}$, since we also have Kuranishi coordinate on $\mathcal{T}or_g$ by using (2.14). Consequently, $\mathcal{J}^{tor}$ factors through $\mathcal{T}or_g/\mathbb{Z}_2$:

$$\begin{array}{ccc} \mathcal{T}or_g & \xrightarrow{\mathcal{J}^{tor}} & \mathcal{H}_g \, . \\ & {\scriptstyle \mathbb{Z}_2} \searrow \quad \nearrow {\scriptstyle \mathcal{J}^{tor}} & \\ & \mathcal{T}or_g/\mathbb{Z}_2 & \end{array}$$

From Proposition 3.6, $\mathcal{T}or_g \to \mathcal{T}or_g/\mathbb{Z}_2$ is a $2:1$ branched covering map branching over the hyperelliptic locus $\mathcal{HET}or_g$ for $g \geq 3$. Meanwhile, the Kuranishi coordinate chart $\Delta_{p,\epsilon}$, $p \in \mathcal{T}or_g - \mathcal{HET}or_g$, also descends to $\mathcal{T}or_g/\mathbb{Z}_2$. When $p \in \mathcal{HET}or_g$, we can view the Kuranishi coordinate $\Delta_{p,\epsilon}$ on $\mathcal{T}or_g$ as follows: Δ^{3g-3} decomposes into $\Delta^{2g-1} \times \Delta^{g-2}$ where Δ^{2g-1} indicates the direction of $T_p(\mathcal{HET}or_g)$ and Δ^{g-2} is the normal direction in which the period map $\mathcal{J}^{tor}_*$ vanishes. The $\mathbb{Z}_2$ action fixes Δ^{2g-1} but acts as the multiplication of -1 on Δ^{g-2}. Thus $\mathcal{T}or_g/\mathbb{Z}_2$ locally looks like $\Delta^{2g-1} \times (\Delta^{g-2}/\mathbb{Z}_2)$ around the hyperelliptic locus.

Theorem 3.7. (Local Torelli Theorem 2) *$\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ is an immersion for $g \geq 2$.*

This local Torelli Theorem was first proved by F. Oort and J. Steenbrink [19] and then by Y. Karpishpan [13] under his framework of understanding higher order derivatives of period map in terms of Čech cohomology. We approach it by our deformation method.

Proof. From the local Torelli Theorem 3.1, the tangent map Π_*, restricted to $\mathcal{T}_g - \mathcal{HET}_g$, is injective for $g \geq 3$ and everywhere injective for $g = 2$. Thus it suffices to show that the tangent map of $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ at hyperelliptic locus $\mathcal{HE}_g$ is injective for $g \geq 3$. To this end, we lift the period map to $\mathcal{J}^{tor} : \mathcal{T}or_g/\mathbb{Z}_2 \to \mathcal{H}_g$. Fix $p \in \mathcal{HET}or_g$ which descends to $\tilde{p}$ in $\mathcal{T}or_g/\mathbb{Z}_2$. From Proposition 3.7, $\tilde{p}$ is a double point. Moreover, the dimension of the Zariski tangent space at $\tilde{p}$ is $\frac{g(g+1)}{2}$. In fact, as $\mathcal{T}or_g$ is a complex manifold of complex dimension $3g-3$ and p is a smooth point, we can choose local parameters $(t_1, t_2, \cdots, t_{3g-3})$ such that $\widehat{\mathcal{O}}_{\mathcal{T}or_g,p} = \mathbb{C}[[t_1, t_2, \cdots, t_{3g-3}]]$ and $\mathbb{Z}_2$ action is given by

$$\begin{cases} \mathbb{Z}_2^* t_i = t_i, & 1 \leq i \leq 2g-1, \\ \mathbb{Z}_2^* t_i = -t_i, & 2g \leq i \leq 3g-3, \end{cases}$$

where $\{t_i\}_{i=1}^{2g-1}$ indicates the tangent directions of $\mathcal{HET}or_g$ and $\{t_i\}_{i=2g}^{3g-3}$ is the normal directions in which $\mathcal{J}^{tor}_*$ vanishes. Clearly,

$$\widehat{\mathcal{O}}_{\mathcal{T}or_g/\mathbb{Z}_2,\tilde{p}} = \left(\widehat{\mathcal{O}}_{\mathcal{T}or_g,p}\right)^{\mathbb{Z}_2} = \mathbb{C}[[t_1, \cdots, t_{2g-1}, t_{2g}^2, t_{2g}t_{2g+1}, \cdots, t_{3g-3}^2]].$$

It is exactly the $\frac{g(g+1)}{2}$ parameters that give the basis of the Zariski tangent space at $\tilde{p}$. We denote these directions by $\{D_k, D_{ij}\}_{1 \leq k \leq 2g-1, 2g \leq i \leq j \leq 3g-3}$, respectively. Also by (2.14), we know that $\mathcal{J}^{tor}$ can also be written as $\left(\bar{\pi}_p A(t)^T + \pi_p\right)\left(A(t)^T + \mathbb{1}_g\right)^{-1}$. The first and second order parts of $\mathcal{J}^{tor}$ are given by

$$\begin{aligned} &\left(\mathcal{J}^{tor}\right)^{(1)} + \left(\mathcal{J}^{tor}\right)^{(2)} \\ =& -2\sqrt{-1}M_p A_1(t)^T + 2\sqrt{-1}M_p\left(A_1(t)^T\right)^2 - 2\sqrt{-1}M_p A_2(t)^T \\ =& \sum_{i=1}^n t_i \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}(\mu_i \lrcorner \theta_p^\beta) - \sum_{i,j=1}^n \frac{\sqrt{-1}}{2} t_i t_j \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}(\mu_i \lrcorner \theta_p^\delta) M_p^{\delta\gamma} \int_{X_p} \theta_p^\gamma \wedge \mathbb{H}(\mu_j \lrcorner \theta_p^\beta) \\ &+ \sum_{i,j=1}^n t_i t_j \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}(\mu_i \lrcorner \eta_j^\beta), \end{aligned}$$

where $\eta_i^\alpha = -\mathbb{G}\overline{\partial}^*\partial(\mu_i \lrcorner \theta_p^\alpha)$ and $M_p^{\alpha\beta}$ is the inverse matrix of $M_{p,\alpha\beta}$. From the choice of t_i above, for any $1 \leq \alpha, \beta \leq g$, we have

$$\int_{X_p} \theta_p^\alpha \wedge \mathbb{H}(\mu_i \lrcorner \theta_p^\beta) = 0,$$

where $2g \leq i \leq 3g-3$. Hence we can write out the image of $\{D_k, D_{ij}\}_{1\leq k\leq 2g-1, 2g\leq i\leq j\leq 3g-3}$ under $\mathcal{J}_*^{tor}$ by using the expansion formula of $\mathcal{J}^{tor}$:

$$\begin{cases} \mathcal{J}_*^{tor}(D_k) = \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}(\mu_k \lrcorner \theta_p^\beta), & 1 \leq k \leq 2g-1, \\ \mathcal{J}_*^{tor}(D_{ij}) = \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}\left(\mu_i \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_j \lrcorner \theta^\beta)\right), & 2g \leq i = j \leq 3g-3, \\ \mathcal{J}_*^{tor}(D_{ij}) = \int_{X_p} \theta_p^\alpha \wedge \mathbb{H}\left(\mu_i \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_j \lrcorner \theta^\beta) + \mu_j \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_i \lrcorner \theta^\beta)\right), & 2g \leq i < j \leq 3g-3. \end{cases}$$

Finally we need to show that $\{\mathcal{J}_*^{tor}(D_k), \mathcal{J}_*^{tor}(D_{ij})\}$ are linearly independent. Since X_p is a hyperelliptic Riemann surface, these Čech cohomology groups, such as $H^0(\Omega^1_{X_p})$, $\check{H}^1(\mathcal{O}_{X_p})$ and $\check{H}^1(T_{X_p})$, have explicit bases just as described in [13, 19]. Moreover, these papers have showed that these directions are linearly independent in terms of Čech cohomology. We give a proof in Appendix 5 that our directions are actually the same as theirs, which completes the proof of this theorem. □

Local Torelli Theorems 3.1 and 3.7 tell us that the period map gives a local embedding of the Kuranishi coordinate chart $\Delta_{p,\epsilon}$ when p lies in the nonhyperelliptic locus, and of $\Delta_{p,\epsilon}/\mathbb{Z}_2$ when p lies in the hyperelliptic locus. This local embedding induces a matrix model for the local Kuranishi coordinates.

Definition 3.8. *Matrix Model for the Kuranishi coordinate charts.*

The image of the Kuranishi coordinate chart under the period map is called the matrix model when the local Torelli theorems hold. Here we identify the Kuranishi coordinate chart with its matrix model, which lies in $\mathcal{H}_g \subset \mathbb{C}^{\frac{g(g+1)}{2}}$.

4. Proof of the global Torelli Theorems

This section is devoted to the proof of global Torelli theorem for Riemann surfaces.

Theorem 4.1. $\mathcal{J}^{tor} : \mathcal{T}or_g/\mathbb{Z}_2 \to \mathcal{H}_g$ *is an embedding for* $g \geq 3$.

Proof. From the discussion of Section 3, $\mathcal{T}or_g/\mathbb{Z}_2$ is a complex orbifold of complex dimension $3g-3$. For every point p in the non-hyperelliptic locus, we have the Kuranishi coordinate chart $\Delta_{p,\epsilon}$ centered at p, which descends from $\mathcal{T}or_g$. As to the hyperelliptic locus, we denote by $\Delta_{p,\epsilon}/\mathbb{Z}_2$ the local coordinate chart around the hyperelliptic point according to the local behavior of the hyperelliptic locus. From the local Torelli theorems 3.1 and 3.7, $\mathcal{J}^{tor}$ gives a local embedding on both of these two kinds of coordinate charts. All we need to show is that $\mathcal{J}^{tor}$ is injective. It is easy to see that $\mathcal{T}or_g/\mathbb{Z}_2 \cong \mathcal{T}_g/\widetilde{T}_g$. Thus the proof of the one-to-one correspondence between $\widetilde{T}_g$ orbit and its Jacobian is our ultimate, which is equivalent to say that two points in $\mathcal{T}_g$ with the same Jacobian must be related by some element in $\widetilde{T}_g$.

According to [11] and [12], $\mathcal{H}_g$ can be viewed as the isomorphism classes of principally polarized abelian varieties together with a symplectic basis (A, γ), where $\gamma : H_1(\Sigma, \mathbb{Z}) \to H_1(A, \mathbb{Z})$ preserves the intersection paring on Σ and the principally polarized form on A. And the identification is given from (A, γ) to its period matrix with respect to this symplectic basis. By changing the symplectic basis, we have the natural $\mathrm{Sp}(g, \mathbb{Z})$ action on $\mathcal{H}_g$. However, the kernel of the $\mathrm{Sp}(g, \mathbb{Z})$ action is $\pm\mathbb{1}_{2g}$. That is to say

$$(A, \gamma) \cong (A, -\gamma).$$

Also $\mathcal{T}or_g$ can be identified with the isomorphism classes of Riemann surfaces together with a symplectic basis (C, γ), where $\gamma: H_1(\Sigma, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ preserves the intersection paring on Σ and C, since $\Gamma_g/T_g = \mathrm{Sp}(g, \mathbb{Z})$. Moreover, the period map $\mathcal{J}^{tor}: \mathcal{T}or_g \to \mathcal{H}_g$ is given by

$$\begin{array}{ccc} \mathcal{T}or_g & \longrightarrow & \mathcal{H}_g \\ (C,\gamma) & \longrightarrow & (\mathrm{Jac}C, \gamma), \end{array}$$

where we have the natural isomorphism $H_1(C, \mathbb{Z}) \cong H_1(\mathrm{Jac}C, \mathbb{Z})$.

Now assume that two points $[C, [f]]$ and $[C', [f']]$ on $\mathcal{T}_g$ are mapped to the same Jacobian, namely (A, γ). Write (C, γ) and (C', γ') on $\mathcal{T}or_g$ as the corresponding two points descended from $[C, [f]]$ and $[C', [f']]$, respectively. As (C, γ) and (C', γ') are mapped to the same Jacobian (A, γ), their symplectic bases will be the same up to a change of the sign. Without loss of generality, we may assume that (C, γ) and (C', γ') share the same symplectic basis after changing the sign (If γ and γ' are different by a sign, pick $[\psi] \in \rho^{-1}(-\mathbb{1}_{2g})$ where $\rho: \Gamma_g \to \mathrm{Sp}(g, \mathbb{Z})$ is surjective. Then $[C, [f]]$ and $[C', [\psi f']]$ have the same symplectic basis). Going back to the two corresponding points on $\mathcal{T}_g$, the following picture appears since we can see the construction of Kuranishi coordinate charts of $\mathcal{T}_g$ from deformation theoretic point of view:

$$\begin{array}{ccc} \Sigma & & \Sigma \\ f \uparrow & & \uparrow f' \\ C & \xrightarrow{\ \phi\ } & C' \end{array},$$

where ϕ is a diffeomorphism obtained from the deformation of the complex structures between C and C'. What is more, we have $[f\phi^{-1}f'^{-1}] \in T_g$. Denote F by $f\phi^{-1}f'^{-1}$. Then the commutative diagram follows

$$\begin{array}{ccccc} & & \Sigma & & \\ & {}^{f}\nearrow & & \nwarrow^{Ff'} & \\ C & & \xrightarrow{\ \ \phi\ \ } & & C' \end{array}.$$

In fact we will prove that ϕ is holomorphic and thus obviously biholomorphic. Hence $[C, [f]]$ and $[C', [f']]$ are related by $\widetilde{T}_g$.

To see ϕ is holomorphic, we first recall the definition of the Jacobian. The Jacobian of a Riemann surface X is nothing but $\mathbb{C}^g/\Lambda$ and

$$\Lambda = \mathbb{Z}\left\{ \int_{A_1} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix}, \cdots, \int_{A_g} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix}, \int_{B_1} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix}, \cdots, \int_{B_g} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix} \right\},$$

where $\{\theta^\alpha\}_{\alpha=1}^g$ is a basis of $H^0(X, K_X)$. $[C, [f]]$ and $[C', [f']]$ are mapped to the same Jacobian by the period map, then their symplectic basis (A_α, B_α) and (A'_α, B'_α) are related by ϕ together with

$$\int_{A_\alpha} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix} = \int_{A'_\alpha} \begin{pmatrix} \theta'^1 \\ \vdots \\ \theta'^g \end{pmatrix}, \int_{B_\alpha} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix} = \int_{B'_\alpha} \begin{pmatrix} \theta'^1 \\ \vdots \\ \theta'^g \end{pmatrix}. \tag{4.1}$$

Let

$$\Theta = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^g \end{pmatrix}, \ \Theta' = \begin{pmatrix} \theta'^1 \\ \vdots \\ \theta'^g \end{pmatrix}.$$

Set $\phi^*\theta'^\alpha = \sum_{\beta=1}^g x_{\alpha\beta}\theta^\beta + \sum_{\beta=1}^g y_{\alpha\beta}\bar{\theta}^\beta$. Thus

$$\phi^*\Theta' = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} \Theta \\ \bar{\Theta} \end{pmatrix}. \tag{4.2}$$

Put (4.1) and (4.2) together to get

$$\int_{A_\alpha} \Theta = \int_{A'_\alpha} \Theta' = \int_{\phi_* A_\alpha} \Theta' = \int_{A_\alpha} \phi^*\Theta' = \begin{pmatrix} X & Y \end{pmatrix} \int_{A_\alpha} \begin{pmatrix} \Theta \\ \bar{\Theta} \end{pmatrix},$$

$$\int_{B_\alpha} \Theta = \int_{B'_\alpha} \Theta' = \int_{\phi_* B_\alpha} \Theta' = \int_{B_\alpha} \phi^*\Theta' = \begin{pmatrix} X & Y \end{pmatrix} \int_{B_\alpha} \begin{pmatrix} \Theta \\ \bar{\Theta} \end{pmatrix}.$$

Reformulating these two equalities into matrix form, we get

$$\begin{pmatrix} \int_{A_\alpha} \Theta & \int_{B_\alpha} \Theta \\ \int_{A_\alpha} \bar{\Theta} & \int_{B_\alpha} \bar{\Theta} \end{pmatrix} = \begin{pmatrix} X & Y \\ \bar{Y} & \bar{X} \end{pmatrix} \begin{pmatrix} \int_{A_\alpha} \begin{pmatrix} \Theta \\ \bar{\Theta} \end{pmatrix} & \int_{B_\alpha} \begin{pmatrix} \Theta \\ \bar{\Theta} \end{pmatrix} \end{pmatrix}. \tag{4.3}$$

Observe that $\det \begin{pmatrix} \int_{A_\alpha} \Theta & \int_{B_\alpha} \Theta \\ \int_{A_\alpha} \bar{\Theta} & \int_{B_\alpha} \bar{\Theta} \end{pmatrix} \neq 0$. In fact, it is well-known that (A, B) period matrix of canonical basis Θ_c is $\begin{pmatrix} \mathbb{1}_g & M \end{pmatrix}$ with $\mathrm{Im} M > 0$. This matrix can be written as

$$\begin{pmatrix} D & DM \\ \bar{D} & \bar{D}\bar{M} \end{pmatrix},$$

where nonsingular matrix D is given by the equality $\Theta = D\Theta_c$. Also it is easy to check that

$$\begin{pmatrix} D & DM \\ \bar{D} & \bar{D}\bar{M} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_g & 0 \\ 0 & \bar{D} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & \mathbb{1}_g \end{pmatrix} \begin{pmatrix} \mathbb{1}_g & M \\ \mathbb{1}_g & \bar{M} \end{pmatrix},$$

and then its determinant is nonzero since

$$\det \begin{pmatrix} \mathbb{1}_g & M \\ \mathbb{1}_g & \bar{M} \end{pmatrix} = \det \begin{pmatrix} \mathbb{1}_g & M \\ 0 & 2\sqrt{-1}\mathrm{Im} M \end{pmatrix} \neq 0.$$

These imply that

$$X = \mathbb{1}_g, \ Y = 0.$$

Hence the diffeomorphism ϕ preserves holomorphic one-forms. Now choose coordinates centered at p and $\phi(p)$, which are denoted by (z, p) and $(w, \phi(p))$, respectively. Pick $\Omega \in H^0(C', K_{C'})$ with $\Omega(\phi(p)) \neq 0$. Locally Ω can be written as

$$\Omega = g(w)dw.$$

Pulling Ω back by ϕ, we get a holomorphic one-form on C. However,

$$\phi^*\Omega = g(\phi(z))\frac{\partial \phi}{\partial z}dz + g(\phi(z))\frac{\partial \phi}{\partial \bar{z}}d\bar{z}.$$

Then $g(\phi(z))\frac{\partial \phi}{\partial \bar{z}} = 0$. At the point p, $g(\phi(p)) \neq 0$, we have $\frac{\partial \phi}{\partial \bar{z}}\big|_{z=0} = 0$. Hence, ϕ is holomorphic, finishing the proof of the theorem.

□

Corollary 4.2. *For the case of $g = 2$, $\mathcal{J}^{tor} : \mathcal{T}or_g \to \mathcal{H}_g$ is an open embedding.*

Proof. Theorem 3.1 tells us that $\mathcal{J}^{tor} : \mathcal{T}or_g \to \mathcal{H}_g$ is an immersion everywhere when $g = 2$. Besides, $\mathcal{J}^{tor}$ is an open map from $\dim_{\mathbb{C}} \mathcal{T}or_g = \dim_{\mathbb{C}} \mathcal{H}_g = 3$. Moreover, Proposition 3.6 implies that $\mathbb{Z}_2$ is a trivial action on $\mathcal{T}or_g$ since any Riemann surface with $g = 2$ is hyperelliptic, indicating that $\widetilde{T}_g$ orbit is the same as T_g orbit on $\mathcal{T}_g$. The proof of Theorem 4.1 implies that $\mathcal{J}^{tor}$ is an open embedding. □

Corollary 4.3. *$\mathcal{J}^{tor} : \mathcal{T}or_g \to \mathcal{H}_g$ is a $2 : 1$ branched covering map branched over $\mathcal{HET}or_g$ onto its image for $g \geq 3$.*

Proof. This is a direct consequence of Theorem 4.1. □

Proposition 4.4. *Let $\Delta_{p,\epsilon}$ be the Kuranishi coordinate chart on $\mathcal{T}_g$. The period map Π maps the Γ_g orbit of $\Delta_{p,\epsilon}$ onto the $\mathrm{Sp}(g, \mathbb{Z})$ orbit of its image in $\mathcal{H}_g$.*

Proof. Recall that the Kuranishi coordinate chart $\Delta_{p,\epsilon}$ is given by

$$\begin{array}{ccc} \Delta_{p,\epsilon} & \to & \mathcal{T}_g \\ t & \to & [X_t, [F_t]], \end{array}$$

where $(\mathcal{X}, F)$ is the Kuranishi family with the Teichmüller structure of $(X_p, [F_0])$ over $\Delta_{p,\epsilon}$, while the coordinate map of $\Delta^{[\phi]}_{p,\epsilon}$ can be written as

$$\begin{array}{ccc} \Delta^{[\phi]}_{p,\epsilon} & \to & \mathcal{T}_g \\ t & \to & [X_t, [\phi F_t]], \end{array}$$

where $\Delta^{[\phi]}_{p,\epsilon} := [\phi]\Delta_{p,\epsilon}$. Now the Kuranishi family becomes $(\mathcal{X}, (\phi \times 1)F)$, where $\phi \times 1 : \Sigma \times \Delta^{[\phi]}_{p,\epsilon} \to \Sigma \times \Delta^{[\phi]}_{p,\epsilon}$, the same family as $(\mathcal{X}, F)$ up to a different symplectic basis. Two bases are linked by $\rho([\phi])$, denoted by $\begin{pmatrix} U & V \\ R & S \end{pmatrix} \in \mathrm{Sp}(g, \mathbb{Z})$, i.e.,

$$\begin{pmatrix} \widetilde{A} \\ \widetilde{B} \end{pmatrix} = \begin{pmatrix} U & V \\ R & S \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$

where $\begin{pmatrix} \widetilde{A} \\ \widetilde{B} \end{pmatrix}$ and $\begin{pmatrix} A \\ B \end{pmatrix}$ are the symplectic bases on $\Delta^{[\phi]}_{p,\epsilon}$ and $\Delta_{p,\epsilon}$, respectively. As we have seen, the matrix model of $\Delta_{p,\epsilon}$ is

$$\Pi(t) = \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \curvearrowright A(t)^T.$$

While on $\Delta^{[\phi]}_{p,\epsilon}$, one has

$$\begin{aligned} \widetilde{\Pi}(t)_{\alpha\beta} &= \int_{\widetilde{B}_\alpha} \widetilde{\sigma}(t)^{\gamma\beta} \theta^\gamma_p(t) \\ &= \int_{\widetilde{B}_\alpha} \widetilde{\sigma}(t)^{\gamma\beta} \left(\theta^\gamma_p + A(t)^\gamma_\delta \bar{\theta}^\delta_p\right) \\ &= \int_{R_{\alpha\lambda}A_\lambda + S_{\alpha\lambda}B_\lambda} \widetilde{\sigma}(t)^{\gamma\beta} \left(\theta^\gamma_p + A(t)^\gamma_\delta \bar{\theta}^\delta_p\right) \\ &= (R_{\alpha\gamma} + S_{\alpha\lambda}\pi_{p,\lambda\gamma} + R_{\alpha\delta}A(t)^\gamma_\delta + S_{\alpha\lambda}\bar{\pi}_{p,\lambda\delta}\, A(t)^\gamma_\delta)\, \widetilde{\sigma}(t)^{\gamma\beta}, \end{aligned}$$

where $\widetilde{\sigma}(t)^{\alpha\beta}$ is the inverse matrix of $\widetilde{\sigma}(t)_{\alpha\beta}$. And $\widetilde{\sigma}(t)_{\alpha\beta}$ is given by

$$\begin{aligned}\widetilde{\sigma}(t)_{\alpha\beta} &= \int_{\widetilde{A}_\alpha} \theta_p^\beta(t) \\ &= \int_{U_{\alpha\lambda}A_\lambda + V_{\alpha\lambda}B_\lambda} \theta_p^\beta + A(t)_\gamma^\beta \bar{\theta}_p^\gamma \\ &= U_{\alpha\beta} + V_{\alpha\gamma}\pi_{p,\gamma\beta} + U_{\alpha\gamma}A(t)_\gamma^\beta + V_{\alpha\gamma}\bar{\pi}_{p,\lambda\gamma}\, A(t)_\gamma^\beta .\end{aligned}$$

Then we formulate all these into the matrix form:

$$\begin{aligned}\widetilde{\Pi}(t) &= \left(R(\,\mathbb{1}_g + A(t)^T) + S(\,\pi_p + \bar{\pi}_p A(t)^T)\right)\left(U(\,\mathbb{1}_g + A(t)^T) + V(\,\pi_p + \bar{\pi}_p A(t)^T)\right)^{-1} \\ &= \begin{pmatrix} S & R \\ V & U \end{pmatrix}\begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \curvearrowright A(t)^T \\ &= \begin{pmatrix} S & R \\ V & U \end{pmatrix} \curvearrowright \Pi(t).\end{aligned}$$

Thus the Γ_g orbit of $\Delta_{p,\epsilon}$ is mapped, by the period map, onto the $\mathrm{Sp}(g,\mathbb{Z})$ orbit of its matrix model $\Pi(t)$ in $\mathcal{H}_g$, since the representation $\rho : \Gamma \to \mathrm{Sp}(g,\mathbb{Z})$ is surjective. □

Denote by ν the transformation of $\mathrm{Sp}(g,\mathbb{Z})$

$$\begin{array}{ccc} \mathrm{Sp}(g,\mathbb{Z}) & \to & \mathrm{Sp}(g,\mathbb{Z}) \\ \begin{pmatrix} U & V \\ R & S \end{pmatrix} & \to & \begin{pmatrix} S & R \\ V & U \end{pmatrix} \end{array}$$

and it is obvious that $\nu^2 = 1$.

Theorem 4.5. (Global Torelli Theorem on moduli space) $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ *is injective for* $g \geq 2$.

Proof. As we have seen from Corollary 4.3, $\mathcal{J}^{tor} : \mathcal{T}or_g \to \mathcal{H}_g$ is a $2:1$ branched covering map onto its image, branching over $\mathcal{HET}or_g$ for $g \geq 3$. That is to say that the $\widetilde{T}_g$ orbits on $\mathcal{T}_g$ have one-to-one correspondence to their Jacobian given by the period map Π. This is also true for $g = 2$, from the proof of Corollary 4.2. From Proposition 4.4, the Γ_g orbits are mapped onto $\mathrm{Sp}(g,\mathbb{Z})$ orbits. Assume that two Γ_g orbits $[p]$ and $[q]$ of $\mathcal{M}_g$ are mapped to the same $\mathrm{Sp}(g,\mathbb{Z})$ orbit by $\mathcal{J}$. We lift these to $\Pi : \mathcal{T}_g \to \mathcal{H}_g$ and thus have $\Pi(p) = L \curvearrowright \Pi(q)$ for some $L \in \mathrm{Sp}(g,\mathbb{Z})$. There is the following exact sequence

$$1 \to T_g \to \Gamma_g \xrightarrow{\rho} \mathrm{Sp}(g,\mathbb{Z}) \to 1.$$

Pick $[\phi] \in \rho^{-1}(\nu(L))$. Then $\Pi([\phi]q) = L \curvearrowright \Pi(q)$ by Proposition 4.4. Hence p and $[\phi]q$ are in the same $\widetilde{T}_g$ orbit, which implies that p and q are in the same Γ_g orbit. □

5. Appendix

Recall that the natural isomorphism between the Čech cohomology $\check{H}^1(T_X)$ and the Dolbeault cohomology $H^{0,1}_{\bar\partial}(T_X)$, and isomorphism between $\check{H}^1(\mathcal{O}_X)$ and $H^{0,1}_{\bar\partial}$ follows similarly. Assume that there is an open covering $\bigcup_\alpha U_\alpha$ on X and then the natural isomorphism Ψ is given by

$$\begin{array}{cccc} \Psi : & \check{H}^1(T_X) & \longrightarrow & H^{0,1}_{\bar\partial}(T_X) \\ & [\theta_{\alpha\beta}] & \longrightarrow & [\bar\partial \xi^\alpha] \end{array},$$

where $\xi^\alpha \in A^{0,0}(U_\alpha, T_X)$ and $\xi^\beta - \xi^\alpha = \theta_{\alpha\beta}$.

Now we return to the proof of the Theorem 3.7, that is, X is a hyperelliptic Riemann surface, covered by two affine charts U_0 and U_1 as described in [13, P. 568]. The first derivative of the period map in the direction $D_k, 1 \leq k \leq 2g-1$ in terms of Čech cohomology is given by

$$\begin{array}{ccc} H^0(X,\Omega^1) & \longrightarrow & \check{H}^1(X,\mathcal{O}_X) \\ \omega & \longrightarrow & [\theta_k \lrcorner \omega] \end{array},$$

where $[\theta_k] \in \check{H}^1(T_X)$ corresponds to $[\mu_k] \in H^{0,1}_{\overline{\partial}}(T_X)$. It is obvious that $[\theta_k \lrcorner \omega]$ is mapped to $[\mu_k \lrcorner \omega]$ by the natural isomorphism from $\check{H}^1(\mathcal{O}_X)$ to $H^{0,1}_{\overline{\partial}}$. The second derivative of period map in the direction of $D_{ij}, 2g \leq i < j \leq 3g-3$ in terms of Čech cohomology is given by

$$\begin{array}{ccc} H^0(X,\Omega^1) & \longrightarrow & \check{H}^1(X,\mathcal{O}_X) \\ \omega & \longrightarrow & [\theta_j \lrcorner \mathcal{L}_{\theta_i}\omega] \end{array},$$

where $\mathcal{L}_{\theta_i}$ denotes Lie derivative along θ_i and $[\theta_i] \in \check{H}^1(T_X)$ corresponds to $[\mu_i] \in H^{0,1}_{\overline{\partial}}(T_X)$. It is easy to see that $\theta_j \lrcorner \mathcal{L}_{\theta_i}\omega = \theta_j \lrcorner \partial(\theta_i \lrcorner \omega)$. Hence we need to show that $[\theta_j \lrcorner \partial(\theta_i \lrcorner \omega)]$ is mapped to $[\mu_i \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_j \lrcorner \omega) + \mu_j \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_i \lrcorner \omega)]$ by the natural isomorphism from $\check{H}^1(\mathcal{O}_X)$ to $H^{0,1}_{\overline{\partial}}$. The $i=j$ case follows from almost the same method as below. By the natural isomorphism between $\check{H}^1(X,T_X)$ and $H^{0,1}_{\overline{\partial}}(X,T_X)$, we get $\xi_i^1 \in A^{0,0}(U_1,T_X)$ and $\xi_i^0 \in A^{0,0}(U_0,T_X)$ such that

$$\begin{cases} \xi_i^1 - \xi_i^0 = \theta_i, \\ \overline{\partial}\xi_i^1 = \mu_i + \overline{\partial} f_i, \end{cases} \tag{5.1}$$

where $f_i \in A^{0,0}(X,T_X)$. As μ_i can change in the Dolbeault cohomology class, we can assume that $\mu_i\big|_{U_0\cap U_1} = 0$. In fact, μ_i is $\overline{\partial}$-closed and thus locally $\overline{\partial}$-exact, i.e., $\mu_i = \overline{\partial} h_i$ on $U_0 \cap U_1$. The desired representative can be chosen as $\mu_i - \overline{\partial}(\rho h_i)$, where ρ is the suitable cut-off function. Moreover we can choose f_i such that $f_i = \xi_i^0$ on $U_0 \cap U_1$, for example $f_i := \rho \xi_i^0\big|_{U_0\cap U_1}$. By use of (5.1), on U_1, we have

$$\begin{aligned} &\mu_i \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_j \lrcorner \omega) + \mu_j \lrcorner \partial \mathbb{G}\overline{\partial}^*(\mu_i \lrcorner \omega) \\ =&\overline{\partial}(\xi_i^1 - f_i) \lrcorner \partial\Big((\xi_j^1 - f_j)\lrcorner\omega\Big) + \overline{\partial}(\xi_j^1 - f_j) \lrcorner \partial\Big((\xi_i^1 - f_i)\lrcorner\omega\Big). \end{aligned}$$

Similarly, we have an analogous equality on U_0.

Now we shall identify the Čech and Dolbeault cohomology classes above. This question is equivalent to finding ϕ_{ij}^1 and ϕ_{ij}^0 belonging to $A^{0,0}(U_1)$ and $A^{0,0}(U_0)$, respectively, satisfying the following equations

$$\begin{cases} \overline{\partial}\phi_{ij}^1 = \overline{\partial}(\xi_i^1 - f_i) \lrcorner \partial\Big((\xi_j^1 - f_j)\lrcorner\omega\Big) + \overline{\partial}(\xi_j^1 - f_j) \lrcorner \partial\Big((\xi_i^1 - f_i)\lrcorner\omega\Big), \\ \overline{\partial}\phi_{ij}^0 = \overline{\partial}(\xi_i^0 - f_i) \lrcorner \partial\Big((\xi_j^0 - f_j)\lrcorner\omega\Big) + \overline{\partial}(\xi_j^0 - f_j) \lrcorner \partial\Big((\xi_i^0 - f_i)\lrcorner\omega\Big), \\ \phi_{ij}^1 - \phi_{ij}^0 = \theta_j \lrcorner \partial(\theta_i \lrcorner \omega). \end{cases} \tag{5.2}$$

It is obvious that the solutions of the first two equalities of (5.2) always exist since the right hand sides of these two equalities are $(0,1)$-forms and clearly $\overline{\partial}$-closed. As

$$\begin{aligned} &\overline{\partial}(\xi_i^1 - f_i) \lrcorner \partial\Big((\xi_j^1 - f_j)\lrcorner\omega\Big) + \overline{\partial}(\xi_j^1 - f_j) \lrcorner \partial\Big((\xi_i^1 - f_i)\lrcorner\omega\Big) \\ =\ &\overline{\partial}\Big((\xi_i^1 - f_i)\lrcorner\partial\left((\xi_j^1 - f_j)\lrcorner\omega\right)\Big) - (\xi_i^1 - f_i)\lrcorner\partial\Big(\overline{\partial}(\xi_j^1 - f_j)\lrcorner\omega\Big) \\ &+ \overline{\partial}\Big((\xi_j^1 - f_j)\lrcorner\partial\left((\xi_i^1 - f_i)\lrcorner\omega\right)\Big) - (\xi_j^1 - f_j)\lrcorner\partial\Big(\overline{\partial}(\xi_i^1 - f_i)\lrcorner\omega\Big), \end{aligned}$$

we can write ϕ^1_{ij} as

$$\begin{aligned}&(\xi^1_i - f_i)\lrcorner\partial\Big((\xi^1_j - f_j)\lrcorner\omega\Big) + (\xi^1_j - f_j)\lrcorner\partial\Big((\xi^1_i - f_i)\lrcorner\omega\Big)\\ &-\overline{\partial}^{-1}\Big((\xi^1_i - f_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)\Big) - \overline{\partial}^{-1}\Big((\xi^1_j - f_j)\lrcorner\partial\left(\overline{\partial}(\xi^1_i - f_i)\lrcorner\omega\right)\Big),\end{aligned}$$

where $\overline{\partial}^{-1}\Big((\xi^1_i - f_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)\Big)$ stands for some solution g satisfying

$$\overline{\partial}g = (\xi^1_i - f_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right).$$

This notation is reasonable as the solution always exists. Thus

$$\begin{aligned}&\phi^1_{ij} - \phi^0_{ij}\\ =&(\xi^1_i - f_i)\lrcorner\partial\Big((\xi^1_j - f_j)\lrcorner\omega\Big) - (\xi^0_i - f_i)\lrcorner\partial\Big((\xi^0_j - f_j)\lrcorner\omega\Big)\\ &+(\xi^1_j - f_j)\lrcorner\partial\Big((\xi^1_i - f_i)\lrcorner\omega\Big) - (\xi^0_j - f_j)\lrcorner\partial\Big((\xi^0_i - f_i)\lrcorner\omega\Big)\\ &-\overline{\partial}^{-1}\Big((\xi^1_i - \xi^0_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)\Big) - \overline{\partial}^{-1}\Big((\xi^1_j - \xi^0_j)\lrcorner\partial\left(\overline{\partial}(\xi^1_i - f_i)\lrcorner\omega\right)\Big)\\ =&(\xi^1_i - \xi^0_i)\lrcorner\partial\Big((\xi^1_j - f_j)\lrcorner\omega\Big) + (\xi^0_i - f_i)\lrcorner\partial\Big((\xi^1_j - \xi^0_j)\lrcorner\omega\Big)\\ &+(\xi^1_j - \xi^0_j)\lrcorner\partial\Big((\xi^1_i - f_i)\lrcorner\omega\Big) + (\xi^0_j - f_j)\lrcorner\partial\Big((\xi^1_i - \xi^0_i)\lrcorner\omega\Big)\\ &-\overline{\partial}^{-1}\Big((\xi^1_i - \xi^0_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)\Big) - \overline{\partial}^{-1}\Big((\xi^1_j - \xi^0_j)\lrcorner\partial\left(\overline{\partial}(\xi^1_i - f_i)\lrcorner\omega\right)\Big)\\ =&(\xi^1_i - \xi^0_i)\lrcorner\partial\Big((\xi^1_j - f_j)\lrcorner\omega\Big) + (\xi^0_i - f_i)\lrcorner\partial\Big((\xi^1_j - \xi^0_j)\lrcorner\omega\Big)\\ &+(\xi^1_j - \xi^0_j)\lrcorner\partial\Big((\xi^1_i - \xi^0_i)\lrcorner\omega\Big) + (\xi^1_j - \xi^0_j)\lrcorner\partial\Big((\xi^0_i - f_i)\lrcorner\omega\Big) + (\xi^0_j - f_j)\lrcorner\partial\Big((\xi^1_i - \xi^0_i)\lrcorner\omega\Big)\\ &-\overline{\partial}^{-1}\Big((\xi^1_i - \xi^0_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)\Big) - \overline{\partial}^{-1}\Big((\xi^1_j - \xi^0_j)\lrcorner\partial\left(\overline{\partial}(\xi^1_i - f_i)\lrcorner\omega\right)\Big)\\ =&(\xi^1_j - \xi^0_j)\lrcorner\partial\Big((\xi^1_i - \xi^0_i)\lrcorner\omega\Big)\\ &+(\xi^0_j - f_j)\lrcorner\partial\Big((\xi^1_i - \xi^0_i)\lrcorner\omega\Big) + (\xi^0_i - f_i)\lrcorner\partial\Big((\xi^1_j - \xi^0_j)\lrcorner\omega\Big)\\ &+(\xi^1_j - \xi^0_j)\lrcorner\partial\Big((\xi^0_i - f_i)\lrcorner\omega\Big) - \overline{\partial}^{-1}\Big((\xi^1_j - \xi^0_j)\lrcorner\partial\left(\overline{\partial}(\xi^1_i - f_i)\lrcorner\omega\right)\Big)\\ &+(\xi^1_i - \xi^0_i)\lrcorner\partial\Big((\xi^1_j - f_j)\lrcorner\omega\Big) - \overline{\partial}^{-1}\Big((\xi^1_i - \xi^0_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)\Big)\\ =&\theta_j\lrcorner\partial(\theta_i\lrcorner\omega) + (\xi^0_i - f_i)\lrcorner\partial\Big((\xi^1_j - \xi^0_j)\lrcorner\omega\Big) + (\xi^0_j - f_j)\lrcorner\partial\Big((\xi^1_i - \xi^0_i)\lrcorner\omega\Big)\\ =&\theta_j\lrcorner\partial(\theta_i\lrcorner\omega).\end{aligned}$$

The penultimate equality results from

$$\bar{\partial}\left((\xi^1_j - \xi^0_j)\lrcorner\partial\Big((\xi^0_i - f_i)\lrcorner\omega\Big)\right) = (\xi^1_j - \xi^0_j)\lrcorner\partial\left(\overline{\partial}(\xi^1_i - f_i)\lrcorner\omega\right)$$

$$\bar{\partial}\left((\xi^1_i - \xi^0_i)\lrcorner\partial\Big((\xi^1_j - f_j)\lrcorner\omega\Big)\right) = (\xi^1_i - \xi^0_i)\lrcorner\partial\left(\overline{\partial}(\xi^1_j - f_j)\lrcorner\omega\right)$$

and the last step stems from our choice of f_i.

References

[1] A. Andreotti, *On a theorem of Torelli*, Amer. J. Math. 80 (1958), 801-828.
[2] E. Arbarello, M. Cornalba, *Teichmüller space via Kuranishi families*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 1, 89-116.
[3] E. Arbarello, M. Cornalba, P. Griffiths, *Geometry of Algebraic Curves, Volume II, with a contribution by Joseph Daniel Harris*. Grundlehren der mathematischen Wissenschaften, Volume 268. Spring-Verlag, Berlin Heidelberg, 2011.
[4] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of Algebraic Curves, Volume I*. Spring-Verlag, New York, Berlin Heidelberg, Tokyo, 1985.
[5] B.Berndtsson, *Strict and non strict positivity of direct image bundles*. Mathematische Zeitschrift, Volume 269, Issue 3-4, pp 1201-1218.
[6] J. Carlson, M. Green, P. Griffiths, J. Harris, *Infinitesimal variations of Hodge structure I*. Compositio Mathematica 50 (1983) 109-205.
[7] H. M. Farkas, I. Kra, *Riemann Surface*. Graduate text in math 71. Spring-Verlag, 1991.
[8] P. Griffiths, *Topics in Transcendental Algebraic Geometry*. Annals of Mathematics Studies, Number 106. Princeton Press, Princeton, New Jersy, 1984.
[9] P. Griffiths, J. Harris, *Principles of algebraic geometry*. Wiley Interscience Publication. New York, 1994.
[10] F. Guan, K. F. Liu, A. Todorov, *A global Torelli Theorem for Calabi-Yau manifolds*. arXiv: 1112.1163v1. 6 Dec. 2011.
[11] R. Hain, *The rational cohomology ring of the moduli space of abelian 3-folds*. Math. Res. Letter. 9(2002), no. 4, 473-491.
[12] R. Hain, E. Looijenga, *Mapping class groups and moduli spaces of curves*. Algebraic geometry-Santa Cruz 1995, 97-142, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.
[13] Y. Karpishpan, *On the higher-order differentials of the period map*. Duke mathematical journal. Vol. 72. No.3 December 1993.
[14] K. Kodaira, K. Akao, *Complex manifolds and deformation of complex structures*. Grundlehren der mathematischen Wissenschaften, Volume 283, Spring-Verlag, 1986.
[15] K. F. Liu, S. Rao, X. K. Yang, *Hodge theory and deformations of Kähler manifolds*, arXiv:1207.1182. 06 July 2012.
[16] K. F. Liu, X. F. Sun, S. -T. Yau, *Recent development on the geometry of the Teichmüller and moduli spaces of Riemann surfaces*, Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces, 221-259.
[17] J. Morrow, K. Kodaira, *Complex Manifolds*. American Mathematical Society Publishing 2006.
[18] S. Nag, *The complex analytic theory of teichmüller*. Wiley Interscience Publication.New York, 1988.
[19] F. Oort, J. Steenbrink, *The local Torelli problem for algebraic curves*. In Algebraic geometry, Sijthoff, Noordhoff 1980, 157-204.
[20] S. A. Wolpert, *Chern forms and Riemann tensor for the moduli space of curves*. Invention Math 85 (1986), 119-145.
[21] S. A. Wolpert, *The hyperbolic metric and the geometry of universal curve*. J.Differential geometry. 31 (1990) 417-472.
[22] R. Torelli, *Sulle Varieta di Jacobi I, II,* Rendiconti R. Accad. ei Lincei, Vol. 22-2 (1913), pp. 98-103, 437-441.
[23] A. Weil, *Zum Beweis des Torellischen Satz*, GSttingen Nachrichten (1957), 33-53.
[24] F. L. Yin, *Geometry of the period map of Riemann Surface*. PhD Thesis, Zhejiang University 2010.

Department of Mathematics, University of California at Los Angeles, Angeles, CA 90095-1555, USA; Center of Mathematical Sciences, Zhejiang University, Hangzhou, China
E-mail address: liu@math.ucla.edu, liu@cms.zju.edu.cn

Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China
E-mail address: zhaoquanting@126.com

Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China
E-mail address: likeanyone@zju.edu.cn