An alternative stochastic proof for the Atiyah-Singer index theorem

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Abstract

According to Watanabe [6], by generalized Wiener functional we could give a simple stochastic proof of the index theorem for twisted Dirac operator (the Atiyah-Singer index theorem) on Clifford module.

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1. Introduction

The purpose of this work is to provide a simple stochastic proof of the index theorem for the twisted Dirac operator by generalized Wiener functional. In 1950s and 1960s, Atiyah, Singer, Hirzebruch, Bott have established the index theorem([1], [12]), which is an interesting theorem in geomery and analysis. Later on, the refined local index theorems have been established by several authors Atiyah-Bott-Patodi [2], Patodi [19], [20], Gilkey [13], [14]. The proofs of the local index theorems are complicated, which need knowledge from analysis, geometry and algebra. On the classical method proofs of the local index theorem, we could refer to Atiyah-Bott-Patodi [2], Getlzer [4], [5], Berline-Getzler-Vergne [11], Yu [24], [25], Roe [17], Gilkey [14], Ponge [30] etc, where Berline-Getzler-Vergne [11] is a good reference on heat equation method for index theorem. In 1984, Bismut [7] has introduced stochastic analysis method to prove the local index theorem. Bismut's work is significant. After that, several proofs by stochastic method have been provided, for example, Watanabe [6], Baudoin [21], Hsu [28], [29]. In [6], Watanabe has proved the Gauss-Bonnet-Chern theorem, the Signature index theorem by generalized Wiener

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functional which is included in stochastic analysis. In Shigekawa-Ueki [22], Riemann-Roch theorem has been proved by this method. In this work, we prove the twist Dirac index theorem (the Atiyah-Singer theorem) by Watanabe's method.

For the convenience of readers, we state the twisted Dirac index theorem. Let $p(t, x, y)$ is the heat kernel for Dirac heat equation on Clifford module $\mathscr{E}$ of M, $F^{\mathscr{E}/S}$ is the twisting curvature of $\mathscr{E}$ (see [11] p117), then

$$\lim_{t\to 0} Str(p(t, x, x))dx = (\frac{1}{2\pi i})^{\frac{d}{2}}\{\widehat{A}(M)ch(F^{\mathscr{E}/S})\}_d(x),$$

where M is a d-dimensional Riemannian manifold, d is even, $\mathscr{E}$ is a Clifford module on M, $\widehat{A}(M)$ is A genus of M, $ch(F^{\mathscr{E}/S}) = Str_{\mathscr{E}/S}[exp(-F^{\mathscr{E}/S})]$ is relative Chern character of $\mathscr{E}$.

Getzler [5] has used the rescale to get the limit of supertrace of the heat kernel, where an important estimate was obtained by Stroock's Feynman-Kac formula. Most of his methods are classical. Bismut [7]'s stochastic method contains the splitting of Wiener space and the implicit function theorem. Our method is from Watanabe [18], [38], different from Bismut [7]'s. Of course our idea is motivated by many works which contain Bismut [7], Hsu [29], Baudoin [21] etc. By virtue of the Malliavin calculus and generalized expectation of generalized Wiener functional, we have the probabilistic representation of the heat kernel:

$$p(\varepsilon^2, x, y) = E[M^\varepsilon e^\varepsilon \delta_y(X^\varepsilon)],$$

where δ is a generalized function: Dirac delta function, M, e, X are random variables, $\delta(X)$ is generalized Wiener functional got by the composition of δ with the non-degenerate smooth Wiener functional X, E is generalized expectation (On precise definition of these notation, please read the following context or [18], [38].), then we could compute the limit of supertrace by the asymptotic expansion with respect to ε in Sobolev space of Wiener functional. Our proof doesn't involve the principal bundle and stochastic differential geometry.

Our paper is organized as follows. In the first part, we simply introduce Clifford algebra and Dirac operator. In the second part, we give Feynman-Kac formula for parabolic systems. In the third part, we state some basic definitions and theorems on the Malliavin calculus. In the fourth part, we give the proof by generalized Wiener functional and its asymptotic expansion.

2. Clifford algebra and Spin bundle

For the convenience of the readers, we would introduce the knowledge on Clifford algebra simply. On details the readers may consult Berline-Getzler-Vergne [11], Jost [26]. Let V be an oriented d dimensional Euclidean space. We assume that $d = 2l$ is even. The tensor algebra is

$$T(V) = R \oplus V \oplus (V \otimes V) + ... \quad .$$

By the equivalent relations $uv + vu = -2 < u, v >$ in tensor algebra, we could define Clifford algebra $Cl(V)$. Let $\{e_i\}$ be an oriented basis of V and c_i be the image of the e_i in the $Cl(V)$. Then

$$c_{i_1}...c_{i_k}, \qquad 0 \leq k \leq d, 1 \leq i_1 < ... < i_k \leq d,$$

forms a basis of $Cl(V)$. Let $Cl^+(V)$ be the subspace of elements of even degrees, $Cl^-(V)$ be the subspace of the elements of odd degree. There is a unique graded complex Clifford module $S = S^+ \oplus S^-$, called the spinor module, such that

$$Cl(V) \otimes C \cong End(S),$$

wiith

$$Cl^+(V) \cdot S^+ \subset S^+, Cl^+(V) \cdot S^- \subset S^-,$$

$$Cl^-(V) \cdot S^- \subset S^+, Cl^-(V)S^+ \subset S^-.$$

The supertrace on $Cl(V)$ is defined by

$$Str(a) = tr_{S^+}a - tr_{S^-}a,$$

where a is regarded as an element of $End(S)$. If

$$a = \sum_{0 \leq k \leq d} \sum_{1 \leq i_1 < ... < i_k \leq d} a_{i_1...i_k} c_{i_1}...c_{i_k},$$

then

$$Str(a) = (-2i)^l a_{12...d}. \qquad (1.1)$$

Let $E = E^+ \oplus E^-$ is a superspace, we could define $Str(b) = tr_+(b) - tr_-(b)$ for $b \in End(E)$. Then we could define supertrace on $End(S \otimes E)$, such that $Str(a \otimes b) = Str(a) \cdot Str(b)$. There is a linear isomorphism (not algebra isomorphism) between the wedge product $\bigwedge(V)$ and $Cl(V)$(see [11] Pro 3.5) which is denoted by c. The group $Spin(V)$ is the group obtained by exponentiating the Lie algebra $C^2(V)$ inside the Clifford algebra $C(V)$.

Let us assume that M be a Riemannian manifold which has a spin structure, the spin principal bundle $Spin(M)$ has a connection which is induced from $SO(M)$. The spinor bundle $\mathcal{L}$ is defined to be the associated bundle $Spin(M)\times_{Spin(d)} S$, which also has a linear induced connection. The Clifford bundle $Cl(M)$ has a action on $\mathcal{L}$, and $C\otimes Cl(M)\cong End(\mathcal{L})$. A Clifford module $\mathscr{E}$ on an even diemnsional Riemannian manifold M is $\mathbb{Z}_2$-graded bundle on M with a graded action of the bundle $C(M)$ on it. Dirac operator D is defined to be

$$D=\sum c(e^i)\nabla_{e_i},$$

where $\{e_i\}$ is a local orthonormal frame, $\{e^i\}$ is its dual base.

Lemma 1 (Lichnerowicz formula): Let $\mathscr{E}$ be a Clifford module on M, s be the scalar curvature of M, $F^{\mathscr{E}/S}$ be the twisting curvature of $\mathscr{E}$ (see [11]), then

$$-D^2=\triangle-c(F^{\mathscr{E}/S})-\frac{s}{4},$$

where s is scalar curvature on M.

3. Stroock's version of Feynman-Kac formula for the parabolic systems

Stroock has given Feynman-Kac formula for parabolic systems (see [34]). According to his technique, we consider a slightly different parabolic system in form. Let f,φ are C^n valued functions, $\{g_{ij}\}$, $1\leq i,j\leq d$ is a real positive symmetric matrix, Γ_i, $1\leq i\leq d$, c are bounded complex $n\times n$ matrixs valued function defined in R^d whose all partial derivatives are bounded, b^i ,$1\leq i\leq d$ are bounded scalar functions defined in R^d whose all partial derivatives are bounded.

$$\begin{cases} \dfrac{\partial f}{\partial t}=\dfrac{1}{2}g^{ij}(\dfrac{\partial}{\partial x^i}+\Gamma_i)(\dfrac{\partial}{\partial x^j}+\Gamma_j)f+\dfrac{1}{2}b^i(\dfrac{\partial}{\partial x^i}+\Gamma_i)f+\dfrac{1}{2}cf, \qquad (t,x)\in(0,\infty)\times R^d\\ f(0,x)=\varphi(x), \qquad x\in R^d \end{cases}$$

Lemma 2: We assume that $\sum\sigma^i_k\sigma^j_k=g^{ij}$,

$$\begin{cases} du^i_x(t)=\sigma^i_k(u_x(t))dw^k+\dfrac{1}{2}b^i dt,\\ de(t)=e_t\Gamma_i(u_x(t))\circ du^i_x(t),\\ dM(t)=\dfrac{1}{2}M(t)e(t)c(u(t))e^{-1}(t)dt,\\ e_0=1, M_t=1, u_x(0)=x, \end{cases}$$

where $\{\omega^k\}$ is a Brownian motion on R^d, $e_t, M(t)$ is a $n \times n$ dimensional matrix valued process, " $\circ$ " stands for Stratonovich integral. Then $f(t,x) = E(M_t e_t \varphi(u_x(t)))$.
Proof: we could apply the Ito formula to $M_s e_s \varphi(t-s, u_x(s))$.

4. The Malliavin calculus

The materials in this section are from ([6], [18], [38]). Due to limited space ,we couldn't state too many definitions and theorems on the Malliavin calculus. For more details we could refer to the above works. Let $(W, \mathscr{F}, P)$ be the d-dimensional Wiener space: $W = \{W : C[0,1] \to R^d; w(0) = 0\}$. With the supremum norm, W is a Banach space. Let H be the subspace of W consisting of all h which is absolutely continuous and has square integrable derivatives. If we define the inner product:

$$< h, g >= \sum_i \int_0^1 \dot{h^i}(t)\dot{g^i}(t)dt,$$

then H is a Hilbert space. We define $h[w]$ by

$$h[w] = \sum_i \int_0^1 \frac{dh^i}{dt}(s)dw^i(s)$$

for $h \in H$. $h[w]$ is a Gaussian random variable, $E(h_1[w]h_2[w]) =< h_1, h_2 >$.

For a measurable function $F(w)$ on W, if there is a polynomial g such that $F(w) = g(h_1[w], h_2[w], h_3[w], ..., h_n[w])$, then we call it polynomial functional, the total space of polynomial functional is denoted by $\mathbf{S}$. Let $L_2(P)$ be the linear space of square integrable random variables on W. There is a decomposition called Wiener's homogeneous chaos:

$$L_2(P) = C_0 \oplus C_1 \oplus ... \oplus C_n \oplus ...$$

We denote the progection of $L_2(P)$ onto C_n by J_n. Let operators $L, (I-L)^s$ be defined on $\mathbf{S}$ by

$$LF = \sum_n (-n)J_n F,$$

$$(I-L)^s F = \sum_n (1+n)^s J_n F,$$

where for polynomial functional $F \in \mathbf{S}$, the above sum has only finite items. Let $\| \|_{p,s}$ be defined by

$$\|F\|_{p,s} = \|(I-L)^{\frac{s}{2}} F\|_p,$$

where $\|\|_p$ is the usual L_p norm. Let D_p^s is the completion of $\mathbf{S}$ under the norm $\|\|_{p,s}$. Then

$$D_{p'}^{s'} \subseteq D_p^s, p \le p', s \le s',$$

$$(D_p^s)' = D_q^{-s}, p^{-1} + q^{-1} = 1.$$

We define

$$D^{\infty} = \cap_{s>0} \cap_{1<p<\infty} D_p^s, D^{-\infty} = \cup_{s>0} \cup_{1<p<\infty} D_p^{-s},$$

$$\widetilde{D}^{-\infty} = \cup_{k=1}^{\infty} \cap_{1<p<\infty} D_p^{-k}, \widetilde{D}^{\infty} = \cap_{k=1}^{\infty} \cup_{1<p<\infty} D_p^{k}.$$

Similarly, for a separable Hilbert space $E, D^{\infty}(E), D^{-\infty}(E)$, we may define the functional space $\mathbf{S}(E)$. For a polynomial functional F, the derivative $DF \in \mathbf{S}(H)$ is given by:

$$DF = \sum_{i}^{n} \partial_i g(h_1[w], h_2[w], ..., h_n[w]) \int_0^{\cdot} h_i'(s) ds.$$

By computation, we could have:

$$< DF, h >= \frac{d}{d\epsilon} F(\omega + \epsilon h)|_{\epsilon=0}.$$

The operator D could be uniquely extended to a linear operator by Meyer's work

$$D : D^{-\infty}(E) \to D^{-\infty}(H \otimes E).$$

The following theorem plays an important role in Malliavin calculus.

Theorem 1 (Malliavin[36], Ikeda-Watanabe [18] etc): If $F \in D^{\infty}(R^d)$, let

$$\sigma^{ij} =< DF^i(\omega), DF^j(\omega) >\in D^{\infty}, i, j = 1, ..., d.$$

(σ^{ij}) is called the Malliavin covariance of F. If

$$[det(\sigma^{ij})]^{-1} \in \cap_{1<p<\infty} L_p,$$

F is called to be nondegenerate in the sense of Malliavin, then the law of F has the $C^{\infty}-$density.

5. The probabilistic proof

In this section we would use generalized Wiener functional and its asymptotic expansion, which are introduced in [6], [18], [38]. The most notations in this section are the same as the above works. The asymptotic expansion of the fundamental solution (heat kernel) of heat equation is determined by local conditions ([11], [18]). We consider the linear space R^d, equipped with $\{g_{ij}\}$, which is the same as $\{\delta_{ij}\}$ except a bounded set containing the original point. We may choose a normal coordinate around the original point, $\{e_i\}$, $1 \leq i \leq d$, is a basis of R^d. Let $\mathscr{E} = \mathcal{L} \bigotimes \mathcal{W}$ be a Clifford module on R^d, $\mathcal{W}$ is a s-dimensional graded vector bundle. Parallel transport along the rays from the original point gives a trivialization of $\mathcal{W}$. Let ω be the connection of $\mathcal{W}$ in this frame, $\frac{1}{2}F(\partial_i, \partial_j)dx^i \bigwedge dx^j$ be the curvature of $\mathcal{W}$. We have (see[6], [11])

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{imkj}(0)x^m x^k + O(|x|^3),$$

and

$$\Gamma^i_{jk}(x) = \frac{1}{3}R_{ijkm}(0)x^m + \frac{1}{3}R_{ikjm}(0)x^m + O(|x|^2),$$

and

$$\sigma^i_k = \delta^i_k + \frac{1}{6}R_{imnk}(0)x^m x^n + O(|x|^3),$$

and

$$\omega_i(x) = -\frac{1}{2}\sum_j F(\partial_i, \partial_j)(0)x^j + O(|x|^2).$$

Let $c_i = \sigma^k_i e_k$, then $\{c_i\}$ is an oriented orthonormal frame. Let $\{\widetilde{\sigma}^i_k\}$ be the inverse of $\{\sigma^i_k\}$, define $\widetilde{\Gamma}^i_{jk} = (\nabla_{e_j} c_k, c^i)$, then

$$\widetilde{\Gamma}^i_{jk} = \frac{\partial}{\partial x^j}(\sigma^l_k)\widetilde{\sigma}^i_l + \sigma^l_k \widetilde{\sigma}^i_s \Gamma^s_{jl},$$

$$\begin{aligned}\widetilde{\Gamma}^i_{jk}(x) &= (\frac{1}{6}R_{imjk}(0) + \frac{1}{6}R_{ijmk}(0))x^m + \frac{1}{3}R_{ijkm}(0)x^m + \frac{1}{3}R_{ikjm}(0)x^m + O(|x|)^2 \\ &= \frac{1}{2}R_{ikjm}(0)x^m + O(|x|^2)\end{aligned}$$

Let $Cl(R^d)$ denote Clifford algebra of Euclid space R^d. Because of the scaling property of Brownian motion, we consider the following stochastic differential equation on $R^d \times (Cl(R^d) \otimes End(R^s))$, with a parameter ε

$$\begin{cases} dX^i(t) = \varepsilon\sigma^i_k(X(t))dw^k(t) - \frac{\varepsilon^2}{2}g^{jk}(X(t))\Gamma^i_{jk}(X(t))dt, \\ de(t) = e(t)\widetilde{\Gamma}_i \circ dX^i(t), \qquad i,j,k = 1,2,...d. \\ (X(0), e(0) = (0,1), \end{cases} \tag{1}$$

where $e(t)$ is a Clifford algebra valued process, $\widetilde{\Gamma}_i = \frac{1}{4}\widetilde{\Gamma}_{ij}^k c_j c_k \otimes 1 + 1 \otimes \omega_i$. Denote this solution by $r^\varepsilon(t) = (X^\varepsilon(t), e^\varepsilon(t))$, then $r^\varepsilon(t) \approx r(\varepsilon^2 t)$, where "$\approx$" stands for the laws are the same, $r(t)$ is the solution when $\varepsilon = 1$.

$$\begin{cases} dM(t) = -\frac{\varepsilon^2}{2} M(t)e(t)[c(F^{\mathscr{E}/S}) + \frac{s}{4}]e^{-1}(t)dt, \\ M(0) = 1, \end{cases} \tag{2}$$

For the Dirac heat equation on spin bundle,

$$\begin{cases} \frac{\partial f}{\partial t} = -\frac{1}{2} D^2 f, \\ f(0,x) = \varphi(x). \end{cases}$$

we can conclude that the solution of heat equation $f(t,x) = E[M(t)e(t)\varphi(X_x(t))]$, where φ is bounded initial function, the heat kernel $p(\varepsilon^2, 0, 0) = E[M^\varepsilon(1)e^\varepsilon(1)\delta_0(X^\varepsilon(1))]$, where δ_0 is the Dirac $\delta-$function at the original point which is introduced in generalized functions theory.

According to [6], [18], [38], We have

$$X^\varepsilon(1) = \varepsilon w(1) + O(\varepsilon^2)$$

in D^∞ as $\varepsilon \to 0$.

$$\delta_0(X^\varepsilon(1)) = \varepsilon^{-d}\delta_0(w(1)) + O(\varepsilon^{-d+1}) \tag{3}$$

in $\widetilde{D}^{-\infty}$as $\varepsilon \to 0$.

$$e^\varepsilon(t) = 1 + \int_0^t e^\varepsilon(s) \circ d\theta^\varepsilon(s),$$

where

$$\begin{aligned} \theta^\varepsilon(t) &= \int_0^t \widetilde{\Gamma}_i \circ dX^\varepsilon(s)^i \\ &= \varepsilon^2(C_{ij}(t)c_i c_j \otimes 1 + 1 \otimes F(t)) + O(\varepsilon^3) \end{aligned}$$

in D^∞ as $\varepsilon \to 0$, where

$$C_{ij}(t) = \frac{1}{8} R_{ijmk}(0) \int_0^t w^k(s) \circ dw^m(s),$$

$$F(t) = -\frac{1}{2}\sum_j F(\partial_i, \partial_j)(0) \int_0^t w^j(s) \circ dw^i(s).$$

$$
\begin{aligned}
e^{\varepsilon}(1) &= 1+\int_0^1 e^{\varepsilon}\circ d\theta^{\varepsilon}(s)\\
&= 1+\theta^{\varepsilon}(1)+\int_0^1\int_0^{t_1} e^{\varepsilon}(t_2)\circ d\theta^{\varepsilon}(t_1)\\
&= 1+\theta^{\epsilon}(1)+\int_0^1 \theta^{\varepsilon}(t_1)\circ d\theta^{\varepsilon}(t_1)+\int_0^1\int_0^{t_1}\int_0^{t_2} e^{\varepsilon}(t_3)\circ d\theta^{\varepsilon}(t_3)\circ d\theta^{\varepsilon}(t_2)\circ d\theta^{\varepsilon}(t_1)\\
&= 1+A_1+A_2+...+A_l+O(\varepsilon^{2l+2}) \qquad (4)
\end{aligned}
$$

in $D^{\infty}(Cl(R^d)\otimes End(R^s))$ as $\varepsilon\to 0$, where assuming $C(t)=C_{ij}(t)c_ic_j\otimes 1+1\otimes F(t)$,

$$
\begin{aligned}
A_m &= \int_0^1\int_0^{t_1}\int_0^{t_2}...\int_0^{t_{m-1}} \circ d\theta^{\varepsilon}(t_m)\circ d\theta^{\varepsilon}(t_{m-1})\circ ...\circ d\theta^{\varepsilon}(t_1)\\
&= \varepsilon^{2m}\int_0^1\int_0^{t_1}...\int_0^{t_{m-1}} \circ dC(t_m)\circ dC(t_{m-1})\circ ...\circ dC(t_1)+O(\varepsilon^{2m+1})
\end{aligned}
$$

in $D^{\infty}(Cl(R^d)\otimes End(R^s))$.

$$
M^{\varepsilon}(1)=1+B_1+B_2+...+B_l+O(\varepsilon^{2l+2}) \qquad (5)
$$

in $D^{\infty}(Cl(R^d)\otimes End(R^s))$, where

$$
\begin{aligned}
B_n &= (-1)^n\varepsilon^{2n}\int_0^1\int_0^{t_1}...\int_0^{t_{n-1}}\frac{1}{2}b^{\varepsilon}(t_n)\frac{1}{2}b^{\varepsilon}(t_{n-1})...\frac{1}{2}b^{\varepsilon}(t_1)dt_ndt_{n-1}...dt_1\\
&= \frac{\varepsilon^{2n}}{n!}[\frac{1}{2}(-c(F^{\mathscr{E}/S}(0))-\frac{s(0)}{4})]^n+O(\varepsilon^{2n+1}),
\end{aligned}
$$

where $b(t)=e(t)[-c(F^{\mathscr{E}/S})-\frac{s}{4}]e^{-1}(t)$.

Lemma 3 ([18]):

$$
E[\delta_0(w(1))\cdot\Phi(w)]=(2\pi)^{-l}E[\Phi(w)|w(1)=0], \qquad \Phi\in\widetilde{D}^{\infty}. \qquad \odot
$$

By (3), (4), (5), lemma 3,

$$
\begin{aligned}
& Str[p(\varepsilon^2,0,0)]e^1\wedge e^2\wedge...\wedge e^d\\
&= (-2i)^{\frac{d}{2}}\sum_{m+n=\frac{d}{2}}E[\frac{1}{m!}(\frac{1}{4}\Omega_{ij}\int_0^1 w^i(s)\circ dw^j(s))^m\cdot\delta_0(w(1))]\frac{Str_{\mathscr{E}/S}[-F^{\mathscr{E}/S}(0)]^n}{2^nn!}+O(\varepsilon)\\
&= \frac{1}{(2i\pi)^{\frac{d}{2}}}\sum_{m+n=\frac{d}{2}}E[\frac{1}{m!}(\frac{1}{2}\Omega_{ij}\int_0^1 w^i(s)\circ dw^j(s))^m\cdot|w(1)=0]\frac{Str_{\mathscr{E}/S}[-F^{\mathscr{E}/S}(0)]^n}{n!}+O(\varepsilon)
\end{aligned}
$$

as $\varepsilon\to 0$ where $\Omega_{ij}=\frac{1}{2}R_{ijkm}(0)e^k\wedge e^m$.

It is obvious that the computation result should be the polynomial of Ω_{ij}. Because two-forms are communicated under the product, $\{\Omega_{ij}\}$ could be regarded as a skew symmetric scalar matrix. While the Brownian bridge measure is $O(d)$-invariance, we may assume $\{\Omega_{ij}\}$ is in block diagonal form such that $\Omega_{2i-1,2i} = 2x_i, \Omega_{2i,2i-1} = -2x_i, i = 1, ..., l$, and other entries are 0. The independence in law among $\{w^i\}$ will make the computation simple.

There is a stochastic area formula which is given by P Levy ([18]):

$$E\{exp[ix(\int_0^1 w^1(s)dw^2(s) - \int_0^1 w^2(s)dw^1(s))]|w^1(1) = 0, w^2(1) = 0\} = \frac{2x}{e^x - e^{-x}} = \frac{x}{\sinh x}$$

So formally we may get that:

$$E\{exp[x(\int_0^1 w^1(s)dw^2(s) - \int_0^1 w^2(s)dw^1(s))]|w^1(1) = 0, w^2(1) = 0\} = \frac{-ix}{\sinh(-ix)} = \frac{x}{\sin x}$$

Therefore

$$E[exp(\frac{1}{2}\Omega_{ij}\int_0^1 w^i(s) \circ dw^j(s))|w(1) = 0] = \det(\frac{\Omega}{2\sinh(\frac{1}{2}\Omega)})^{\frac{1}{2}} = \widehat{A}(M).$$

This is

$$\lim_{t\to 0} Str[p(t, 0, 0)]e^1 \wedge e^2 \wedge ... \wedge e^d = \frac{1}{(2\pi i)^{\frac{d}{2}}}\{\widehat{A}(M)(0)str_{\mathscr{E}/S}[exp(F^{\mathscr{E}/S}(0))]\}_d,$$

perhaps we have completed the proof.

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