FORMAL PSEUDODIFFERENTIAL OPERATORS AND WITTEN'S r-SPIN NUMBERS

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ABSTRACT. We derive an effective recursion for Witten's r-spin intersection numbers, using Witten's conjecture relating r-spin numbers to the Gel'fand-Dikii hierarchy (Theorem 4.1). Consequences include closed-form descriptions of the intersection numbers (for example, in terms of gamma functions: Propositions 5.2 and 5.4, Corollary 5.5). We use these closed-form descriptions to prove Harer-Zagier's formula for the Euler characteristic of $\mathcal{M}_{g,1}$. Finally in §6, we extend Witten's series expansion formula for the Landau-Ginzburg potential to study r-spin numbers in the small phase space in genus zero. Our key tool is the calculus of formal pseudodifferential operators, and is partially motivated by work of Brézin and Hikami.

CONTENTS

1. INTRODUCTION

Motivated by two dimensional gravity, E. Witten proposed two influential conjectures relating integrable hierarchies to the intersection theory of moduli spaces of curves, see [30, 31].

We begin by recalling Witten's definition of r-spin intersection numbers. Witten's original papers [31, 32] remain the best introduction to the mathematical and physical background of this subject. Other excellent expositions can be found in [13, 26]. For an introduction to relevant facts about the moduli spaces of curves, see [29].

Let Σ be a Riemann surface of genus g with marked points x_1, x_2, ..., x_s. Fix an integer $r \geq 2$. Label each marked point x_i by an integer m_i, $0 \leq m_i \leq r - 1$. Consider the line bundle $\mathcal{S} = \mathcal{K} \otimes \mathcal{O}(-\sum_{i=1}^{s} m_i x_i)$ over Σ, where $\mathcal{K}$ as usual denotes the canonical line bundle. If $2g - 2 - \sum_{i=1}^{s} m_i$ is divisible by r, then there

are r^{2g} isomorphism classes of line bundles $\mathcal{T}$ such that $\mathcal{T}^{\otimes r} \cong \mathcal{S}$. The choice of an isomorphism class of $\mathcal{T}$ determines a finite étale cover $\mathcal{M}_{g,s}^{1/r}$ of $\mathcal{M}_{g,s}$, the moduli space of r-spin curves, which comes with a universal curve $\pi : \mathcal{C}_{g,n}^{1/r} \to \mathcal{M}_{g,s}^{1/r}$, on which lives a universal bundle, which we also sloppily denote T. A compactification of $\mathcal{M}_{g,s}^{1/r}$, denoted by $\overline{\mathcal{M}}_{g,s}^{1/r}$, was constructed in [1, 12].

Let $\mathcal{V}$ be a vector bundle over $\overline{\mathcal{M}}_{g,s}^{1/r}$ whose fiber is the dual space to $H^1(\Sigma, \mathcal{T})$. More precisely, $\mathcal{V} := R^1\pi_*\mathcal{T}$. The top Chern class $c_{top}(\mathcal{V})$ of this bundle has degree $(g-1)(r-2)/r + \sum_{i=1}^s m_i/r$. The algebro-geometric constructions of $c_{top}(\mathcal{V})$ can be found in [4, 24].

We associate with each marked point x_i an integer $n_i \geq 0$. Witten's r-spin intersection numbers are defined by

$$(1) \qquad \langle \tau_{n_1,m_1} \ldots \tau_{n_s,m_s} \rangle_g = \frac{1}{r^g} \int_{\overline{\mathcal{M}}_{g,s}^{1/r}} \prod_{i=1}^s \psi(x_i)^{n_i} \cdot c_{top}(\mathcal{V}),$$

which is non-zero only if

$$(2) \qquad (r+1)(2g-2) + rs = r\sum_{j=1}^s n_j + \sum_{j=1}^s m_j.$$

Fix an integer $r \geq 2$. Consider the pseudodifferential operator

$$(3) \qquad Q = D^r + \sum_{i=0}^{r-2} \gamma_i(x)D^i, \quad \text{where } D = \frac{\sqrt{-1}}{\sqrt{r}} \frac{\partial}{\partial x}.$$

It is easy to see that there is a unique pseudodifferential operator L such that $L^r = Q$ (see Lemma 3.1),which we denote

$$Q^{1/r} = D + \sum_{i>0} w_{-i}D^{-i},$$

where the coefficients $\{w_{-i}\}$ are universal differential polynomials in the $\{\gamma_i\}$.

The Gel'fand–Dikii equations read

$$i\frac{\partial Q}{\partial t_{n,m}} = [Q_+^{n+(m+1)/r}, Q] \cdot \frac{c_{n,m}}{\sqrt{r}},$$

where the constants $c_{n,m}$ are given by

$$c_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\cdots(nr+m+1)}.$$

Consider the formal series F in variables $t_{n,m}$, $n \geq 0$ and $0 \leq m \leq r-1$,

$$F(t_{0,0}, t_{0,1}, \dots) = \sum_{d_{n,m}} \langle \prod_{n,m} \tau_{n,m}^{d_{n,m}} \rangle \prod_{n,m} \frac{t_{n,m}^{d_{n,m}}}{d_{n,m}!}.$$

Witten conjectured in [31] that the above F is the string solution of the r-Gel'fand–Dikii hierarchy, namely that F satisfies

$$(4) \qquad \frac{\partial^2 F}{\partial t_{0,0}\partial t_{n,m}} = -c_{n,m}\,\text{Res}(Q^{n+\frac{m+1}{r}}),$$

where Q satisfies the Gel'fand–Dikii equations and $t_{0,0}$ is identified with x. In addition, F satisfies the string equation

$$(5) \qquad \frac{\partial F}{\partial t_{0,0}} = \frac{1}{2} \sum_{i,j=0}^{r-2} \delta_{i+j,r-2} t_{0,i} t_{0,j} + \sum_{n=0}^{\infty} \sum_{m=0}^{r-2} t_{n+1,m} \frac{\partial F}{\partial t_{n,m}}.$$

This should be regarded as a boundary condition for F.

When $r = 2$, the above assertion is the celebrated Witten-Kontsevich theorem [15], to which there are a number of enlightening proofs. Witten's conjecture for any $r \geq 2$ has been proved by Faber, Shadrin and Zvonkine [8], building on work of Givental and Lee [16]. In fact, Witten's r-spin theory corresponds to A_{r-1} singularity in the Landau-Ginzburg theory. Fan, Javis and Ruan [9] have developed a Gromov-Witten type quantum theory for all non-degenerate quasi-homogeneous singularities and proved the ADE-integrable hierarchy conjecture of Witten. Chang and Li [6] have initiated a program to give an algebro-geometric construction of Landau-Ginzburg theory.

Witten's constraints (4) (the r-Gel'fand–Dikii equation) and (5) (the string equation) uniquely determine F. There is much interest in understanding the structure of r-spin intersection numbers both in mathematics and physics (cf. [2, 3, 5, 14, 21, 27]).

The paper is organized as follows. In §2, we recall useful identities of r-spin numbers. In §3, we prove a structure theorem of formal pseudodifferential operators and use it to derive/define "universal differential polynomials" $W_r(z)$, which will play a central role in the rest of the paper. In §4, we present a recursive algorithm for computing Witten's r-spin numbers for all genera. Consequences include closed-form descriptions of the one-point r-spin numbers, which we use in §5 to prove Harer-Zagier's formula for the Euler characteristic of $\mathcal{M}_{g,1}$. In §6, we study r-spin numbers on small phase spaces in genus zero.

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2. Review: Witten's r-spin intersection numbers

In this section, we collect fundamental properties of r-spin intersection numbers that we will use in this paper. The proof of the these identities can be found in [31, 13]. The r-spin numbers satisfy the following:

    i) If $m_i = r - 1$, for some $1 \leq i \leq s$, then

$$\langle \tau_{n_1,m_1} \cdots \tau_{n_s,m_s} \rangle_g = 0.$$

    ii) *(string equation)*

$$(6) \qquad \langle \tau_{0,0} \prod_{i=1}^{s} \tau_{n_i,m_i} \rangle_g = \sum_{j=1}^{s} \langle \tau_{n_j-1,m_j} \prod_{\substack{i=1 \\ i \neq j}}^{s} \tau_{n_i,m_i} \rangle_g.$$

This, along with $\langle \tau_{0,0} \tau_{0,i} \tau_{0,j} \rangle_0 = \delta_{i+j,r-2}$, is equivalent to (5).

iii) *(dilaton equation)*

$$(7) \qquad \langle \tau_{1,0} \prod_{i=1}^{s} \tau_{n_i,m_i} \rangle_g = (2g - 2 + s)\langle \prod_{i=1}^{s} \tau_{n_i,m_s} \rangle_g.$$

iv) *(genus zero topological recursion relation)*

(8)

$$\langle \tau_{n_1+1,m_1} \tau_{n_2,m_2} \tau_{n_3,m_3} \prod_{i=4}^{s} \tau_{n_i,m_i} \rangle_0 = \sum_{\{4\cdots s\}=I \coprod J} \sum_{m',m''=0}^{r-2} \langle \tau_{n_1,m_1} \prod_{i \in I} \tau_{n_i,m_i} \tau_{0,m'} \rangle_0$$
$$\cdot \eta^{m',m''} \langle \tau_{0,m''} \tau_{n_2,m_2} \tau_{n_3,m_3} \prod_{i \in J} \tau_{n_i,m_i} \rangle_0,$$

where $\eta^{m',m''} = \delta_{m'+m'',r-2}$.

v) *(WDVV equation in genus zero)*

(9)

$$\sum_{m',m''=0}^{r-2} \prod_{\{5\cdots s\}=I \coprod J} \langle \tau_{n_1,m_1} \tau_{n_2,m_2} \prod_{i \in I} \tau_{n_i,m_i} \tau_{0,m'} \rangle_0 \eta^{m',m''} \langle \tau_{0,m''} \tau_{n_3,m_3} \tau_{n_4,m_4} \prod_{i \in J} \tau_{n_i,m_i} \rangle_0$$

$$= \sum_{m',m''=0}^{r-2} \prod_{\{5\cdots s\}=I \coprod J} \langle \tau_{n_1,m_1} \tau_{n_3,m_3} \prod_{i \in I} \tau_{n_i,m_i} \tau_{0,m'} \rangle_0 \eta^{m',m''} \langle \tau_{0,m''} \tau_{n_2,m_2} \tau_{n_4,m_4} \prod_{i \in J} \tau_{n_i,m_i} \rangle_0$$

Witten gives a detailed study of r-spin numbers in genus zero in [31]. As he points out, the genus zero topological recursion relation can be used to eliminate all descendent indices (those $\tau_{i,j}$ with $i > 0$), so we only need to consider primary intersection numbers $\langle \tau_{0,m_1}, \cdots \tau_{0,m_s} \rangle$ on the small phase space. Witten proves that the WDVV equation uniquely determines primary r-spin intersection numbers in genus zero. For the reader's convenience, we record Witten's work below in a more explicit form. We will denote $\langle \tau_{0,a_1}, \cdots, \tau_{0,a_s} \rangle_0$ by either $\langle \tau_{a_1}, \cdots, \tau_{a_s} \rangle$ or $\langle a_1, \cdots, a_s \rangle$. Witten proves that

$$\langle \tau_{a_1} \tau_{a_2} \tau_{a_3} \rangle = \delta_{a_1+a_2+a_3,r-2},$$
$$\langle \tau_{a_1} \tau_{a_2} \tau_{a_3} \tau_{a_4} \rangle = \frac{1}{r} \cdot \min(a_i, r-1-a_i).$$

Theorem 2.1 (Witten, [31]). *Let $s \geq 5$, $a_1 \geq \cdots \geq a_s$ and $\sum_{j=1}^{s} a_j = r(s-2)-2$. Define $z = a_1, y = a_2, x = a_3$ and*

$$m_1 = x + z - (r-1), \quad m_2 = r - 1 - z, \quad m_3 = y, \quad m_4 = z.$$

Then Witten's formula can be written as

$$(10) \quad \langle a_1, \cdots, a_s \rangle = \left\langle x + y + z - (r-1), r - 1 - z, z, \prod_{i=4}^{s} a_i \right\rangle$$

$$+ \sum_{\substack{I \coprod J=\{4,\ldots,s\} \\ I,J \neq \emptyset}} \sum_{j=0}^{r-2} \left(\left\langle j, m_1, m_3, \prod_{i \in I} a_i \right\rangle \left\langle r - 2 - j, m_2, m_4, \prod_{i \in J} a_i \right\rangle \right.$$

$$\left. - \left\langle j, m_1, m_2, \prod_{i \in I} a_i \right\rangle \left\langle r - 2 - j, m_3, m_4, \prod_{i \in J} a_i \right\rangle \right).$$

This formula recursively computes all primary r-spin numbers.

Proof. The argument is due to Witten. From $s \geq 5$, and $0 \leq a_i \leq r - 2$, it is not difficult to check that $0 \leq m_i \leq r - 2$. By the WDVV equation (9), we have

$$\sum_{I \coprod J = \{4,\ldots,s\}} \sum_{j=0}^{r-2} \left\langle j, m_1, m_3, \prod_{i \in I} a_i \right\rangle \left\langle r - 2 - j, m_2, m_4, \prod_{i \in J} a_i \right\rangle$$

$$= \sum_{I \coprod J = \{4,\ldots,s\}} \sum_{j=0}^{r-2} \left\langle j, m_1, m_2, \prod_{i \in I} a_i \right\rangle \left\langle r - 2 - j, m_3, m_4, \prod_{i \in J} a_i \right\rangle.$$

Then Witten's formula follows from the inequalities $m_3 + m_4 > r - 2$ and $m_2 + m_4 > r - 2$.

For the effectiveness of Witten's formula (10), it is not difficult to prove that if $z' \geq y' \geq x'$ are the three largest numbers in the index set $\{x + y + z - (r - 1), r - 1 - z, z, a_4, \ldots, a_s\}$, then $r - 1 - z$ is not one of x', y', z' as long as $s \geq 5$. On the other hand, each bracket in the quadratic terms in the right hand side of (10) has strictly less than s points. □

3. Formal pseudodifferential operators

A formal pseudodifferential operator is an expression of the form

$$L = \sum_{i=-\infty}^{N} u_i(x) \partial^i, \quad \text{where } \partial = \frac{\partial}{\partial x}.$$

Its positive and negative parts are defined to be

$$L_+ = \sum_{i=0}^{N} u_i(x) \partial^i, \quad L_- = \sum_{i=-\infty}^{-1} u_i(x) \partial^i.$$

For $k \in \mathbb{Z}$, we define

$$\partial^k \cdot f = \sum_{j \geq 0} \binom{k}{j} f^{(j)} \partial^{k-j}, \quad \text{where } f^{(j)} = \frac{\partial^j f}{\partial x^j}.$$

We follow the usual convention that

$$\binom{-a-1}{b} = \binom{a+b}{b}(-1)^b, \quad a, b \geq 0.$$

In particular, $\partial \cdot f = f' + f \partial$. Note that we reserve the notation ∂f for the derivative of f. It is straightforward to check that the set of all formal pseudodifferential operators forms an associative algebra, denoted by ΨDO.

The idea of fractional powers appeared in the work of Gel'fand and Dikii [10]. It plays an important role in integrable systems (cf. [25]). The following lemma is well-known.

Lemma 3.1. *Recall the pseudodifferential operator Q defined in* (3)

$$Q = D^r + \sum_{i=0}^{r-2} \gamma_i(x) D^i.$$

There exists a unique pseudodifferential operator of the form

$$Q^{1/r} = D + \sum_{i \geq 0} w_{-i} D^{-i},$$

whose r-th power is Q; and $w_0 = 0$.

Proof. Let $Q^{\frac{1}{r}} = D + w_0 + w_{-1} D^{-1} + \cdots$. Then $(Q^{\frac{1}{r}})^r = D^r + r w_0 D^{r-1} + \cdots$. Since there is no D^{r-1} term on Q, we have $w_0 = 0$. Thus we may write

$$(Q^{\frac{1}{r}})^r = D^r + r w_{-1} D^{r-2} + (r w_{-2} + \frac{r(r-1) D w_{-1}}{2}) D^{r-3} + \cdots.$$

In general, we have

$$r w_{-i} + p_i(w_{-1}, \cdots w_{-i+1}) = \gamma_{r-1-i},$$

where p_i is a differential polynomial of its argument. So w_{-i} can be uniquely determined recursively as differential polynomials of γ_i.   □

Fix $k \geq 1$. Write

$$Q^{k/r} = D^k + \sum_{i=0}^{k-2} \gamma_i^k D^i + \sum_{i=1}^{\infty} \gamma_{-i}^k D^{-i}.$$

Here we emphasize that throughout this paper, the superscript k in γ_i^k never denotes a power. In particular, we have $\gamma_i^r = \gamma_i$.

Since $Q^{(k+1)/r} = Q^{1/r} \cdot Q^{k/r}$, for $\ell \leq k - 1$ we have

$$(11) \qquad \gamma_\ell^{k+1} = w_{\ell-k} + D\gamma_\ell^k + \gamma_{\ell-1}^k + \sum_{j=1}^{k-2-\ell} w_{-j} \sum_{i=j+\ell}^{k-2} \binom{-j}{i-j-\ell} D^{i-j-\ell} \gamma_i^k.$$

This identity can be used to determine γ_ℓ^{k+1} recursively as differential polynomials of $\{w_{-i}\}$.

Lemma 3.2. *With the notation above, if we assign $w_{-i}^{(j)} = D^j w_{-i}$ the weight $i + j + 1$, then γ_ℓ^k is homogeneous of weight $k - \ell$.*

Proof. Since $\gamma_\ell^1 = w_\ell$ is of weight $1 - \ell$, the general statement follows from the equation (11).   □

Lemma 3.3. *Let $[w_{-i}^{(j)}]\gamma_\ell^k$ denote the coefficient of $w_{-i}^{(j)}$ in γ_ℓ^k. If $k \geq 1, \ell \leq k - 2$ and $1 \leq i \leq k - \ell - 1$, then we have*

$$(12) \qquad\qquad [D^{k-\ell-i-1} w_{-i}]\gamma_\ell^k = \binom{k}{k-\ell-i}.$$

In particular, $[w_{\ell-k+1}]\gamma_\ell^k = k$ and $[Dw_{\ell-k+2}]\gamma_\ell^k = \frac{k(k-1)}{2}$.

Proof. When $k = 1$, by definition, $\gamma_\ell^1 = w_\ell$ for $\ell < 0$. The identity (12) obviously holds in this case. So we apply the recursive equation (11) and use induction on k.

When $i = k - \ell - 1$, we have

$$\begin{aligned} [w_{\ell-k+1}]\gamma_\ell^k &= 1 + [w_{\ell-k+1}]\gamma_{\ell-1}^{k-1} \\ &= 1 + k - 1 \\ &= k \end{aligned}$$

and similarly when $i < k - \ell - 1$, we have

$$[D^{k-\ell-i-1}w_{-i}]\gamma_\ell^k = [D^{k-\ell-i-2}w_{-i}]\gamma_\ell^{k-1} + [D^{k-\ell-i-1}w_{-i}]\gamma_{\ell-1}^{k-1}$$
$$= \binom{k-1}{k-\ell-i-1} + \binom{k-1}{k-\ell-i}$$
$$= \binom{k}{k-\ell-i}$$

as desired. □

Lemma 3.4 (Witten, [31]). *With the above notation, $\gamma_{-1}^{i+1} = \mathrm{Res}(Q^{(i+1)/r})$, we can express coefficients γ_i of Q as differential polynomials in γ_{-1}^{i+1}, $0 \le i \le r - 2$.*

Proof. By Lemmas 3.2 and 3.3, we have

$$\tag{13} \gamma_{-1}^{i+1} = (i+1)w_{-1-i} + p_i(w_{-1}, \cdots w_{-i})$$
$$= \frac{(i+1)\gamma_{r-2-i}}{r} + p_i'(\gamma_{r-2}, \cdots \gamma_{r-1-i}),$$

where p_i and p_i' are differential polynomials of their arguments. Thus we can recursively express γ_i as differential polynomials in γ_{-1}^{i+1}, $0 \le i \le r - 2$. □

Denote by $P(\gamma_\ell^k)$ the sum of monomials in γ_ℓ^k that does not contain derivatives of w_{-i}. Then we have

$$\tag{14} P(\gamma_\ell^k) = [p^{k-\ell}]\left(1 + \sum_{i>0} w_{-i}p^{i+1}\right)^k$$
$$= \mathrm{Res}_{p=0}\frac{(1 + \sum_{i>0} w_{-i}p^{i+1})^k}{p^{k-\ell+1}}.$$

Fix an integer $r \ge 2$. From the Gel'fand-Dikii equation (4), we have

$$\gamma_{-1}^{m+1} = \mathrm{Res}(Q^{\frac{m+1}{r}}) = -\frac{m+1}{r}\langle\langle\tau_{0,0}\tau_{0,m}\rangle\rangle, \qquad \text{for } 0 \le m \le r - 2$$
$$= (m+1)w_{-m-1} + \cdots$$

and

$$\tag{15} \gamma_{-1}^{r+1} = \mathrm{Res}(Q^{1+\frac{1}{r}}) = \frac{r+1}{r^2}\langle\langle\tau_{0,0}\tau_{1,0}\rangle\rangle$$
$$= (r+1)w_{-r-1} + \frac{r(r+1)}{2}Dw_{-r} + \cdots.$$

For the first time, we use the fact that Q is a differential operator (i.e. $Q_- = 0$), which implies that

$$\tag{16} 0 = \gamma_{-1}^r = r \cdot w_{-r} + \cdots \qquad \text{and}$$

$$\tag{17} 0 = \gamma_{-2}^r = r \cdot w_{-r-1} + \cdots.$$

The leading coefficients of the above equations come from Lemma 3.3.

We first substitute (17) and then (16) into (15) to eliminate w_{-r-1} and w_{-r} respectively. Next we substitute $\gamma_{-1}^{r-1}, \gamma_{-1}^{r-2}, \ldots, \gamma_{-1}^1$ consecutively into (15) to eliminate $w_{-r+1}, w_{-r+2}, \ldots, w_{-1}$ successively. Then it is easy to see that γ_{-1}^{r+1} is now expressed in terms of differential polynomials of γ_{-1}^{m+1}, $0 \le m \le r-2$. From now we on will use $S(\gamma_{-1}^{r+1})$ to denote this differential polynomial in γ_{-1}^{m+1}, $0 \le m \le r-2$

resulting from substitutions in γ_{-1}^{r+1}. We will keep the notation γ_{-1}^{r+1} for the differential polynomial (15) in w_{-i}.

If we use the notation

$$(18) \qquad z_m^{(j)} = -\frac{r}{m+1} \cdot \frac{\partial^j \gamma_{-1}^{m+1}}{\partial x^j} = \langle\langle \tau_{0,0}^{j+1} \tau_{0,m} \rangle\rangle,$$

then we have the following structure theorem of formal pseudodifferential operators.

Theorem 3.5. *(As discussed above, we may regard $S(\gamma_{-1}^{r+1})$ as a differential polynomial in z_m.) We have*

$$\frac{r^2}{r+1} S(\gamma_{-1}^{r+1}) = \frac{1}{2} \sum_{j=0}^{r-2} z_j z_{r-2-j} + W_r(z),$$

where $W_r(z)$ represents the terms containing derivatives of some z_m.

Proof. Since (16) is used to eliminate w_{-r-1} in γ_{-1}^{r+1}, it is not difficult to see that the identity of Theorem 3.5 is equivalent to

$$\frac{r^2}{r+1} P(\gamma_{-1}^{r+1}) - r P(\gamma_{-2}^r) = \frac{1}{2} \sum_{j=0}^{r-2} \frac{-r}{j+1} P(\gamma_{-1}^{j+1}) \frac{-r}{r-1-j} P(\gamma_{-1}^{r-1-j}).$$

From equation (14), this is precisely the combinatorial identity shown in the next proposition. $\qquad\square$

Proposition 3.6. *Let a_j be formal variables and*

$$f(x) = 1 + \sum_{j=2}^{\infty} a_j x^j \in \mathbb{C}[[x]]$$

be a formal series satisfying $f(0) = 1$ and $f'(0) = 0$. Then for any $n \geq 1$,

$$(19) \qquad \frac{[x^{n+2}]f^{n+1}}{n+1} = \frac{1}{2} \sum_{j=1}^{n-1} \frac{[x^{j+1}]f^j}{j} \cdot \frac{[x^{n-j+1}]f^{n-j}}{n-j} + \frac{[x^{n+2}]f^n}{n},$$

where $[x^n]f^k$ denotes the coefficient of x^n in the series expansion of f^k.

The proof of Proposition 3.6 along with other interesting equivalent formulations can be found in Appendix A.

Example 3.7. We illustrate the above procedure explicitly for $r = 4$. Let $Q^{1/4} = D + \sum_{i>0} w_{-i} D^{-i}$. Then

$$-\frac{1}{4} \langle\langle \tau_{0,0} \tau_{0,0} \rangle\rangle = \mathrm{Res}(Q^{1/4}) = w_{-1},$$

$$-\frac{1}{2} \langle\langle \tau_{0,0} \tau_{0,1} \rangle\rangle = \mathrm{Res}(Q^{2/4}) = 2w_{-2} + Dw_{-1},$$

$$-\frac{3}{4} \langle\langle \tau_{0,0} \tau_{0,2} \rangle\rangle = \mathrm{Res}(Q^{3/4}) = 3w_{-3} + D^2 w_{-1} + 3Dw_{-2} + 3w_{-1}^2.$$

We also have

$$0 = \mathrm{Res}(Q) = 4w_{-4} + D^3 w_{-1} + 4D^2 w_{-2} + 6Dw_{-3} + 6w_{-1}Dw_{-1} + 12w_{-1}w_{-2},$$

$$0 = \gamma_{-2}^4 = 4w_{-5} + 6Dw_{-4} + 4D^2 w_{-3} + D^3 w_{-2} + 6w_{-1}Dw_{-2} - (Dw_{-1})^2$$
$$+ 12w_{-1}w_{-3} + 6w_{-2}^2 + 4w_{-1}^3 + 2w_{-1}D^2 w_{-1}.$$

Substituting the above two groups of identities into

$$\gamma_{-1}^5 = \frac{5}{16}\langle\langle\tau_{0,0}\tau_{1,0}\rangle\rangle = \mathrm{Res}(Q^{5/4})$$
$$= 5w_{-5} + D^4 w_{-1} + 5D^3 w_{-2} + 10D^2 w_{-3} + 10Dw_{-4} + 5(Dw_{-1})^2 + 10w_{-1}D^2 w_{-1}$$
$$+ 10w_{-2}^2 + 10w_{-1}^3 + 20w_{-1}Dw_{-2} + 10w_{-2}Dw_{-1} + 20w_{-1}w_{-3}$$

and using $D = \frac{\sqrt{-1}}{2}\frac{\partial}{\partial x}$, we get

$$\frac{16}{5}\gamma_{-1}^5 = z_0 z_2 + \frac{1}{2}z_1^2 + \frac{1}{4}z_2^{(2)} + \frac{1}{48}z_0 z_0^{(2)} + \frac{1}{32}z_0' z_0' + \frac{1}{480}z_0^{(4)}.$$

If we substitute the z_m using equation (18), we get exactly the recursion formula (25).

The universal differential polynomial $W_r(z)$ in $z_0, \ldots, z_{r-2}$ is particularly interesting in view of Theorem 3.5. We present $W_r(z)$ for $2 \le r \le 6$ below:

$$W_2(z) = \frac{1}{12}z_0^{(2)}, \qquad W_3(z) = \frac{1}{6}z_1^{(2)},$$

$$W_4(z) = \frac{1}{4}z_2^{(2)} + \frac{1}{48}z_0 z_0^{(2)} + \frac{1}{32}z_0' z_0' + \frac{1}{480}z_0^{(4)},$$

$$W_5(z) = \frac{1}{10}z_0' z_1' + \frac{1}{30}z_0 z_1^{(2)} + \frac{1}{30}z_0^{(2)} z_1 + \frac{1}{3}z_3^{(2)} + \frac{1}{150}z_1^{(4)},$$

$$W_6(z) = \frac{5}{864}z_0^{(3)} z_0' + \frac{1}{144}z_0(z_0')^2 + \frac{1}{8}z_2' z_0' + \frac{1}{24}z_0 z_2^{(2)} + \frac{1}{432}z_0^2 z_0^{(2)} + \frac{1}{24}z_2 z_0^{(2)}$$

$$+ \frac{1}{72}z_2^{(4)} + \frac{1}{9072}z_0^{(6)} + \frac{11}{2592}(z_0^{(2)})^2 + \frac{1}{12}(z_1')^2 + \frac{1}{18}z_1 z_1^{(2)} + \frac{1}{720}z_0 z_0^{(4)} + \frac{5}{12}z_4^{(2)}.$$

We now study their coefficients.

Proposition 3.8. *We have* $[z_{r-2}^{(2)}]W_r(z) = \frac{r-1}{12}$.

Proof. From equation (18) and $D = \frac{\sqrt{-1}}{\sqrt{r}}\frac{\partial}{\partial x}$, we have

$$\frac{\partial^2 z_{r-2}}{\partial x^2} = -\frac{r}{r-1}\cdot\frac{\partial^2 \gamma_{-1}^{r-1}}{\partial x^2} = \frac{r^2}{r-1}D^2\gamma_{-1}^{r-1}.$$

So from Theorem 3.5, we get

$$(20) \qquad\qquad [z_{r-2}^{(2)}]W_r(z) = \frac{r-1}{r+1}[D^2\gamma_{-1}^{r-1}]S(\gamma_{-1}^{r+1}).$$

Recall that in $\gamma_{-1}^{r+1} = (r+1)w_{-r-1} + \frac{r(r+1)}{2}Dw_{-r} + \cdots$, we first substitute w_{-r-1} using γ_{-2}^r and then substitute w_{-r} using γ_{-1}^r, see equations (16), (17). Then γ_{-1}^{r+1} becomes a differential polynomial in $w_{-1}, \ldots, w_{-r+1}$. We need to take care that when substituting w_{-r-1} by γ_{-2}^r, a new term of Dw_{-r} will appear. With the above substitutions in mind and note that $\gamma_{-1}^{r-1} = (r-1)w_{-r+1} + \cdots$, we may apply Lemma 3.3 to get

$$\frac{r-1}{r+1}[D^2\gamma_{-1}^{r-1}]S(\gamma_{-1}^{r+1}) = \frac{1}{r+1}[D^2 w_{-r+1}]\left(\gamma_{-1}^{r+1} - (r+1)\cdot\frac{1}{r}\gamma_{-2}^r\right)$$

$$+ \frac{1}{r+1}[D^2 w_{-r+1}]\left(\left(\frac{r+1}{r}[Dw_{-r}]\gamma_{-2}^r - \frac{(r+1)r}{2}\right)\cdot\frac{1}{r}D\gamma_{-1}^r\right)$$

$$= \frac{1}{r+1}\left(\frac{(r+1)r(r-1)}{3!} - \frac{r+1}{r} \cdot \frac{r(r-1)(r-2)}{3!}\right)$$

$$+ \frac{1}{r+1}\left(\frac{r+1}{r}[Dw_{-r}]\gamma_{-2}^r - \frac{(r+1)r}{2}\right) \cdot \frac{1}{r}[Dw_{-r+1}]\gamma_{-1}^r$$

$$= \frac{r-1}{3} + \frac{-1}{2r}[Dw_{-r+1}]\gamma_{-1}^r = \frac{r-1}{12}.$$

From (20), we get the desired result. $\qquad\square$

Corollary 3.9 (Witten, [31])**.** *We have the following identity for r-spin numbers:*

$$\langle\tau_{1,0}\rangle_1 = \frac{r-1}{24}.$$

Proof. From the dilaton equation, $\langle\tau_{1,0}\tau_{0,0}\tau_{2,0}\rangle_1 = 2\langle\tau_{1,0}\rangle_1$. On the other hand, from Theorems 4.1 and 3.5, we have

$$\langle\tau_{1,0}\tau_{0,0}\tau_{2,0}\rangle_1 = [z_{r-2}^{(2)}]W_r(z)\langle\tau_{0,0}^3\tau_{0,r-2}\tau_{2,0}\rangle_0$$

$$= \frac{r-1}{12}.$$

In the right hand side of the first equation, all other terms vanish for dimensional reason (see (22), (23)). Hence $\langle\tau_{1,0}\rangle_1 = \frac{r-1}{24}$. $\qquad\square$

Proposition 3.8 generalizes as follows.

Proposition 3.10. *Suppose $2 \le i \le r$. If i is odd, then $[z_{r-i}^{(i)}]W_r(z) = 0$. If i is even ($2k$, say), then*

$$[z_{r-2k}^{(2k)}]W_r(z) = \frac{(-1)^{k+1}(r+1-2k)}{r^k(r+1)}\binom{r+1}{2k}B_{2k},$$

where B_{2k} are Bernoulli numbers.

See Appendix B for a proof. By similar arguments, we have the following fact, which means that the genera in the right-hand side of (21) are integers (see (23)). We omit the details.

Proposition 3.11. *The order of derivatives in each monomial of $W_r(z)$ is an even number.*

4. An algorithm for computing Witten's r-spin numbers

Let $\eta^{ij} = \delta_{i+j,r-2}$ and

$$\langle\langle\tau_{n_1,m_1}\ldots\tau_{n_s,m_s}\rangle\rangle = \frac{\partial}{\partial t_{n_1,m_1}}\ldots\frac{\partial}{\partial t_{n_s,m_s}}F(t_{0,0}, t_{0,1},\ldots).$$

The main result of this paper is the following simple and effective recursion formula for computing all r-spin intersection numbers.

Theorem 4.1. *For fixed $r \ge 2$, we have*

$$(21) \qquad \langle\langle\tau_{1,0}\tau_{0,0}\rangle\rangle_g = \frac{1}{2}\langle\langle\tau_{0,0}\tau_{0,m'}\rangle\rangle_{g'}\eta^{m'm''}\langle\langle\tau_{0,m''}\tau_{0,0}\rangle\rangle_{g-g'} + \mathrm{Lower}(r),$$

where $\mathrm{Lower}(r)$ is a explicit sum of products of $\langle\langle\ldots\rangle\rangle$ with genera strictly lower than g.

Proof. Since $\frac{r^2}{r+1}\gamma_{-1}^{r+1} = \langle\langle\tau_{0,0}\tau_{1,0}\rangle\rangle$, from Theorem 3.5, we need only prove that those monomials in $W_r(z)$ must have genera strictly less than the left hand side.

Let us compare $\langle\langle\tau_{0,0}\tau_{1,0}\rangle\rangle_g$ and

$$\prod_k z_{i_k}^{(j_k)} = \prod_k \langle\langle\tau_{0,0}^{j_k+1}\tau_{0,i_k}\rangle\rangle_{g_k}.$$

Since the weight of γ_{-1}^{r+1} is $r+2$ and the weight of $z_m^{(j)}$ is $m+j+2$, we have

$$(22) \qquad \sum_k (i_k + j_k + 2) = r + 2.$$

Combining with the dimensional constraints (2), we have

$$(23) \qquad (2r+2)\left(g - \sum_k g_k\right) = (r+1)\sum_k j_k.$$

So $g = \sum_k g_k$ if and only if all $j_k = 0$. $\qquad\square$

Remark 4.2. Following a suggestion of Witten [31, p.248], Shadrin [26] derived an expansion of $\langle\langle\tau_{n,m}\tau_{0,0}^2\rangle\rangle$ when $r = 3$ and used it to compute some special r-spin numbers. Because of a lack of an elegant structural description, Shadrin's formula (and its generalization to higher r) results in a much more complicated algorithm than (21).

For example, when $r = 3$, (21) gives

$$(24) \qquad \langle\langle\tau_{1,0}\tau_{0,0}\rangle\rangle_g = \langle\langle\tau_{0,0}\tau_{0,1}\rangle\rangle_{g'}\langle\langle\tau_{0,0}^2\rangle\rangle_{g-g'} + \frac{1}{6}\langle\langle\tau_{0,0}^3\tau_{0,1}\rangle\rangle_{g-1}.$$

When $r = 4$, we have

$$(25) \quad \langle\langle\tau_{1,0}\tau_{0,0}\rangle\rangle_g = \langle\langle\tau_{0,0}\tau_{0,2}\rangle\rangle_{g'}\langle\langle\tau_{0,0}^2\rangle\rangle_{g-g'} + \frac{1}{2}\langle\langle\tau_{0,0}\tau_{0,1}\rangle\rangle_{g'}\langle\langle\tau_{0,0}\tau_{0,1}\rangle\rangle_{g-g'}$$

$$+ \frac{1}{4}\langle\langle\tau_{0,0}^3\tau_{0,2}\rangle\rangle_{g-1} + \frac{1}{48}\langle\langle\tau_{0,0}^2\rangle\rangle_{g'}\langle\langle\tau_{0,0}^4\rangle\rangle_{g-1-g'} + \frac{1}{32}\langle\langle\tau_{0,0}^3\rangle\rangle_{g'}\langle\langle\tau_{0,0}^3\rangle\rangle_{g-1-g'}$$

$$+ \frac{1}{480}\langle\langle\tau_{0,0}^6\rangle\rangle_{g-2}.$$

Now we show how to use Theorem 1.1 to compute intersection numbers. It consists of three steps.

(i) When $g = 0$, these intersection numbers can be computed by WDVV equations, using Witten's algorithm [31], as discussed in §2.

(ii) Assume now that $g \geq 1$. For an intersection number containing a puncture operator $\langle\tau_{0,0}\tau_{n_1,m_1}\ldots\tau_{n_s,m_s}\rangle_g$, we have from Theorem 1.1 and the dilaton equation

$$(26) \quad (2g - 1 + s - a)\langle\tau_{0,0}\tau_{n_1,m_1}\ldots\tau_{n_s,m_s}\rangle_g$$

$$= \frac{1}{2}\sum_{\underset{s=I\bigsqcup J}{\widetilde{}}}\langle\tau_{0,0}\tau_{0,m'}\prod_{i\in I}\tau_{n_i,m_i}\rangle_{g'}\eta^{m'm''}\langle\tau_{0,m''}\tau_{0,0}\prod_{i\in J}\tau_{n_i,m_i}\rangle_{g-g'} + \text{Lower}(r)$$

where $a = \#\{i \mid n_i = 0\}$. Note that in the summation of the right-hand side, we rule out the cases $I = \{i_1\}$ and $n_{i_1} = 0$ or $J = \{i_1\}$ and $n_{i_1} = 0$. Then the right hand side follows by induction on genera or numbers of marked points.

(iii) For any intersection number $\langle \tau_{n_1,m_1} \ldots \tau_{n_s,m_s} \rangle_g$ with $n_1 \geq n_2 \geq \cdots \geq n_s$, we apply the string equation first:

$$\langle \tau_{n_1,m_1} \ldots \tau_{n_s,m_s} \rangle_g = \langle \tau_{0,0} \tau_{n_1+1,m_1} \ldots \tau_{n_s,m_s} \rangle_g - \sum_{j=2}^{s} \langle \tau_{n_1+1,m_1} \tau_{n_j-1,m_j} \prod_{i \neq 1,j} \tau_{n_i,m_i} \rangle_g$$

The first term in the right hand side follows from step (ii) and the second term follows by induction on the maximum descendent index. This ends the algorithm.

The results of the above algorithm agree with the table of r-spin numbers when $r = 3$ and 4 given in [18]. Some r-spin numbers when $r = 5$ are presented in Table 1.

<div align="center">TABLE 1. Witten's r-spin numbers ($r = 5$)</div>

$\langle \tau_{1,0} \rangle_1$	$\frac{1}{6}$	$\langle \tau_{0,2}\tau_{1,3} \rangle_1$	$\frac{1}{60}$	$\langle \tau_{0,1}\tau_{0,1}\tau_{2,3} \rangle_1$	$\frac{1}{30}$
$\langle \tau_{3,2} \rangle_2$	$\frac{11}{3600}$	$\langle \tau_{0,3}\tau_{1,2} \rangle_1$	$\frac{1}{60}$	$\langle \tau_{0,1}\tau_{0,2}\tau_{2,2} \rangle_1$	$\frac{1}{20}$
$\langle \tau_{8,1} \rangle_4$	$\frac{341}{25920000}$	$\langle \tau_{0,1}\tau_{4,1} \rangle_2$	$\frac{7}{1200}$	$\langle \tau_{0,1}\tau_{0,3}\tau_{2,1} \rangle_1$	$\frac{1}{20}$
$\langle \tau_{10,3} \rangle_5$	$\frac{161}{777600000}$	$\langle \tau_{0,2}\tau_{4,0} \rangle_2$	$\frac{7}{1200}$	$\langle \tau_{0,2}\tau_{0,2}\tau_{2,1} \rangle_1$	$\frac{1}{15}$
$\langle \tau_{13,0} \rangle_6$	$\frac{3397 \times 10^{-6}}{93312}$	$\langle \tau_{1,1}\tau_{3,1} \rangle_2$	$\frac{17}{1200}$	$\langle \tau_{0,2}\tau_{0,3}\tau_{2,0} \rangle_1$	$\frac{1}{30}$
$\langle \tau_{15,2} \rangle_7$	$\frac{3421 \times 10^{-7}}{419904}$	$\langle \tau_{1,2}\tau_{3,0} \rangle_2$	$\frac{47}{3600}$	$\langle \tau_{0,1}\tau_{0,1}\tau_{5,0} \rangle_2$	$\frac{31}{3600}$
$\langle \tau_{20,1} \rangle_9$	$\frac{1670581 \times 10^{-9}}{846526464}$	$\langle \tau_{2,0}\tau_{2,2} \rangle_2$	$\frac{59}{3600}$	$\langle \tau_{0,1}\tau_{0,3}\tau_{4,3} \rangle_2$	$\frac{7}{6000}$
$\langle \tau_{22,3} \rangle_{10}$	$\frac{2660573 \times 10^{-12}}{1088391168}$	$\langle \tau_{2,1}\tau_{2,1} \rangle_2$	$\frac{9}{400}$	$\langle \tau_{0,2}\tau_{0,2}\tau_{4,3} \rangle_2$	$\frac{1}{500}$
$\langle \tau_{25,0} \rangle_{11}$	$\frac{21324511 \times 10^{-12}}{5986151424}$	$\langle \tau_{3,2}\tau_{3,2} \rangle_3$	$\frac{697}{324000}$	$\langle \tau_{0,2}\tau_{0,3}\tau_{4,2} \rangle_2$	$\frac{23}{9000}$
$\langle \tau_{27,2} \rangle_{12}$	$\frac{87572287 \times 10^{-14}}{13060694016}$	$\langle \tau_{3,1}\tau_{3,3} \rangle_3$	$\frac{1111}{756000}$	$\langle \tau_{0,3}\tau_{0,3}\tau_{4,1} \rangle_2$	$\frac{1}{500}$
$\langle \tau_{32,1} \rangle_{14}$	$\frac{7787064791 \times 10^{-16}}{6582589784064}$	$\langle \tau_{2,1}\tau_{4,3} \rangle_3$	$\frac{803}{756000}$	$\langle \tau_{0,1}\tau_{1,1}\tau_{4,0} \rangle_2$	$\frac{17}{600}$
$\langle \tau_{34,3} \rangle_{15}$	$\frac{538156369 \times 10^{-17}}{4231664861184}$	$\langle \tau_{2,2}\tau_{4,2} \rangle_3$	$\frac{557}{324000}$	$\langle \tau_{1,1}\tau_{1,1}\tau_{3,0} \rangle_2$	$\frac{41}{600}$

The Boussinesq hierarchy ($r = 3$).

For the remainder of this section, let $r = 3$. The 3-KdV hierarchy is also called the *Boussinesq hierarchy*. We see from (24), (25) that compared with the recursive formula for 3-spin intersection numbers, the recursive formula for r-spin numbers are much more complicated for $r \geq 4$.

The following closed formula holds for intersection numbers when $r = 3$. This generalizes the special case $k = 0$ obtained by Brézin and Hikami in [2].

Proposition 4.3. *Let $k \geq 0$ and $0 \leq j \leq 1$. Then*

$$\langle \tau_{0,1}^k \tau_{\frac{8g+2k-5-j}{3},j} \rangle_g = \frac{1}{12^g g!} \frac{\Gamma(\frac{g+k+1}{3})}{\Gamma(\frac{2-j}{3})},$$

where $\Gamma(z)$ is the gamma function.

Proof. We first prove the identity in $g = 0$ by induction on k, namely

$$\langle \tau_{0,1}^k \tau_{\frac{2k-5-j}{3},j} \rangle_0 = \frac{\Gamma(\frac{k+1}{3})}{\Gamma(\frac{2-j}{3})}.$$

When $k = 3, 4, 5$ respectively, we readily verify

$$\langle \tau_{0,1}^3 \tau_{0,1} \rangle_0 = \frac{1}{3}, \qquad \langle \tau_{0,1}^4 \tau_{1,0} \rangle_0 = \frac{2}{3}, \qquad \langle \tau_{0,1}^5 \tau_{1,2} \rangle_0 = 0.$$

Note the last identity is consistent with the fact that $\Gamma(z)$ has a simple pole at $z = 0$.

Thus we may assume $k \geq 6$. We apply the genus zero topological recursion relation (8) to obtain:

$$\langle \tau_{0,1}^k \tau_{\frac{2k-5-j}{3},j} \rangle_0 = \sum_{i=0}^{k-2} \binom{k-2}{i} \langle \tau_{\frac{2k-8-j}{3},j} \tau_{0,0} \tau_{0,1}^i \rangle_0 \langle \tau_{0,1}^3 \tau_{0,1}^{k-2-i} \rangle_0$$

$$= (k-2) \langle \tau_{\frac{2k-8-j}{3},j} \tau_{0,0} \tau_{0,1}^{k-3} \rangle_0 \langle \tau_{0,1}^3 \tau_{0,1} \rangle_0$$

$$= \frac{k-2}{3} \cdot \frac{\Gamma(\frac{k-2}{3})}{\Gamma(\frac{2-j}{3})}$$

$$= \frac{\Gamma(\frac{k+1}{3})}{\Gamma(\frac{2-j}{3})}.$$

The second equation comes from dimensional constraints. Thus we have proved Proposition 4.3 when $g = 0$.

We next assume $g \geq 1$ and proceed by induction on g. We have

$$\langle \tau_{1,0} \tau_{0,0} \tau_{0,1}^k \tau_{\frac{8g+2k-2-j}{3},j} \rangle_g = k \langle \tau_{0,0} \tau_{0,1}^k \tau_{\frac{8g+2k-2-j}{3},j} \rangle_g \langle \tau_{0,0}^2 \tau_{0,1} \rangle_0 + \frac{1}{6} \langle \tau_{0,0}^3 \tau_{0,1}^{k+1} \tau_{\frac{8g+2k-2-j}{3},j} \rangle_{g-1}.$$

Applying the dilaton equation (7) and the string equation (6) to the above identity and combining the first term in the right hand side with the left hand side, we get

$$\langle \tau_{0,1}^k \tau_{\frac{8g+2k-5-j}{3},j} \rangle_g = \frac{1}{12g} \langle \tau_{0,1}^{k+1} \tau_{\frac{8g+2k-11-j}{3},j} \rangle_{g-1}$$

$$= \frac{1}{12g} \cdot \frac{1}{12^{g-1}(g-1)!} \frac{\Gamma\left(\frac{(g-1)+(k+1)+1}{3}\right)}{\Gamma(\frac{2-j}{3})}$$

$$= \frac{1}{12^g g!} \frac{\Gamma(\frac{g+k+1}{3})}{\Gamma(\frac{2-j}{3})}$$

as desired. $\qquad\qquad\square$

We now show that the 3-spin numbers in genus zero in general do not have clean closed formulas in contrast to the case of $r = 2$. We will compute intersection numbers of the form $\langle \tau_{0,1}^k \tau_{2,0}^\ell \rangle_0$, which is nonzero only if $k \equiv 1 \mod 3$ and $2k - 3\ell = 8$.

We will use the temporary notation $a_m = \langle \tau_{0,1}^{3m+1} \tau_{2,0}^{2m-2} \rangle_0$, for $m \geq 1$. By applying (26) to

$$\langle \tau_{1,0} \tau_{0,0} \tau_{0,1}^{3m+1} \tau_{2,0}^{2m-1} \rangle_0 = (2m-1)(5m-3)a_m$$

and using the dilaton and string equations, it is not difficult to obtain

$$(27) \quad (2m-2)(2m-1)(5m-3)a_m$$

$$= \sum_{i=1}^{m-1} \binom{3m+1}{3i+1}\binom{2m-1}{2i} 2i(2i-1)(5i-1)(5i-3)(2m-2i-1)(5m-5i-3)a_i a_{m-i}.$$

For example, we recursively find $a_1 = \langle \tau_{0,1}^4 \rangle_0 = \frac{1}{3}$, $a_2 = \frac{80}{9}$, $a_3 = \frac{179200}{9}$, $a_4 = \frac{1281280000}{3}$.

To simplify the above equation, we substitute

$$b_m = \frac{(5m - 3)a_m}{(3m + 1)!(2m - 2)!}.$$

For example, $b_1 = \frac{1}{36}$, $b_2 = \frac{1}{126}$, $b_3 = \frac{2}{792}$, $b_4 = \frac{85}{52488}$. Then (27) becomes

$$(28) \qquad (2m - 2)b_m = \sum_{i=1}^{m-1} (5i - 1)(3m - 3i + 1)b_i b_{m-i}.$$

In terms of the generating function $y(x) = \sum_{i=1}^{\infty} b_i x^i$, we can rewrite (28) as

$$15x^2 \left(\frac{dy}{dx} \right)^2 + (2xy - 2x)\frac{dy}{dx} - y^2 + 2y = 0,$$

from which we get a first order ODE

$$\frac{dy}{dx} = \frac{1 - y - \sqrt{1 + 16y^2 - 32y}}{15x}.$$

Integrating both sides of

$$\frac{15 dy}{1 - y - \sqrt{1 + 16y^2 - 32y}} = \frac{dx}{x},$$

we get

$$x = \exp \left(\int \frac{15 dy}{1 - y - \sqrt{1 + 16y^2 - 32y}} + C \right)$$

$$= \exp \left(\ln y + \ln 36 - 8y - 32y^2 - \frac{992}{3}y^3 - 4864y^4 + O(y^5) \right)$$

$$= 36y - 288y^2 - 5760y^4 - 92160y^5 + O(y^6).$$

The constant of integration C is uniquely determined by the initial value $b_1 = \frac{1}{36}$. Thus b_i can also be computed using the Lagrange inversion formula (see Lemma A.1).

$$b_i = \frac{1}{i} \operatorname{Res}_{y=0} \left(\frac{1}{x(y)^i} \right), \quad i \geq 1.$$

We note that the above derivation becomes more difficult if we instead use the genus zero topological recursion relation (8) to compute $\langle \tau_{0,1}^k \tau_{2,0}^\ell \rangle_0$.

5. The Euler characteristic of $\mathcal{M}_{g,1}$

We now give a proof for Harer and Zagier's formula of the Euler characteristic of the moduli space of curves:

Theorem 5.1 (Harer-Zagier [11], see also [3, 15, 20, 22, 23]). *Let $g \geq 1$. Then*

$$\chi(\mathcal{M}_{g,1}) = -\frac{B_{2g}}{2g}.$$

For example, $\chi(M_{1,1}) = -\frac{1}{12}, \chi(\mathcal{M}_{2,1}) = \frac{1}{120}, \chi(\mathcal{M}_{3,1}) = -\frac{1}{252}$.

The early proofs of Harer-Zagier's formula [15, 20, 22, 23] all exploit the cell decomposition of decorated moduli space in terms of Ribbon graphs. There is an intriguing fact from Witten's construction [32] that the $r \to -1$ limit of r-spin numbers actually gives $\chi(\mathcal{M}_{g,1})$. The main difficulty is to derive an explicit formula for the one-point r-spin numbers. This was obtained recently by Brezin and Hikami [3] using rather complicated techniques from matrix integrals. We will give a proof using only properties of ΨDO. The proof will conclude just after Lemma 5.3.

We will use the case $s = 1$, and general r. Our discussion so far has assumed $r \geq 2$. However, for *any* r, there is a generalized Kontsevich (Airy) matrix model, and under the limit $r \to -1$, the model gives a logarithmic potential corresponding to the Penner matrix model, whose asymptotic expansion gives the generating function of the Euler characteristic of $\mathcal{M}_{g,1}$, [23]. (We do not understand how to make [32, (3.55-3.57)] precise, so we instead refer the reader to [2, §6] or [19] for a complete discussion.) Thus by taking $r = -1$ in our formulas (interpreted as analytic continuation), we may compute $\chi(\mathcal{M}_{g,1})$, as follows. Setting $s = 1$ in (1), we have

$$(29) \qquad \lim_{r \to -1} \langle \tau_{n,m} \rangle_g \Big|_{m=0} = \chi(\mathcal{M}_{g,1}),$$

where $\chi(\mathcal{M}_{g,1})$ is the orbifold Euler characteristic of $\mathcal{M}_{g,1}$. We now proceed to compute the left side of (29), thereby computing $\chi(\mathcal{M}_{g,1})$.

By the Gel'fand-Dikii equation (4), in order to compute $\langle \tau_{0,0} \tau_{n,m} \rangle_g$, we need to compute the coefficient of $(D\gamma_{-1}^{r-1})^{2g}$ in $\mathrm{Res}(Q^{n+(m+1)/r})$. Note that

$$(30) \qquad \gamma_{-1}^{r-1} = -\frac{r-2}{r} \langle\langle \tau_{0,0} \tau_{0,r-2} \rangle\rangle.$$

By Lemma 3.4 and (13), we know that when expressing γ_i ($0 \leq i \leq r-2$) in terms of γ_{-1}^{i+1} ($0 \leq i \leq r-2$), only γ_0 contains the term γ_{-1}^{r-1}, with

$$(31) \qquad \gamma_0 = \frac{r}{r-2} \gamma_{-1}^{r-1} + p(\gamma_{-1}^1, \ldots, \gamma_{-1}^{r-2}),$$

where p is a differential polynomial in its arguments.

If we replace γ_0 by x and denote by $L = D^r + x$, it is not difficult to see from (30) and (31) that

$$(32) \qquad \langle \tau_{0,0} \tau_{n,m} \rangle_g = \frac{(-1)^g c_{n,m}}{r^g} \times \text{ the constant term in } \mathrm{Res}(L^{n+(m+1)/r}).$$

There exists a pseudodifferential operator $K \in \Psi\mathrm{DO}$ of the form

$$K = 1 + \sum_{i=1}^{\infty} b_i(x) D^{-i},$$

such that $KLK^{-1} = D^r$.

We can determine K by comparing the coefficients at both sides of

$$(33) \qquad KL = D^r K.$$

The first few terms are

$$(34) \quad K = \frac{x^2}{2r} D^{-(r-1)} + \frac{(1-r)x}{2r} D^{-r} + \frac{x^4}{8r^2} D^{-(2r-2)} + \frac{7(1-r)x^3}{12r^2} D^{-(2r-1)}$$
$$+ \frac{(r-1)(7r-3)x^2}{8r^2} D^{-2r} + \frac{(1-r)(10r^2-3r-1)x}{24r^2} D^{-(2r+1)} + \cdots.$$

In general, we have

$$K = 1 + \sum_{u=1}^{\infty} \sum_{i=1}^{2u} b_{ur+u-i} D^{-(ur+u-i)},$$

where $b_{ur+u-i} = a_{u,i} x^i$ with $a_{u,i}$ rational functions of r. In particular, from (34), we have

$$a_{1,1} = \frac{1-r}{2r}, \quad a_{1,2} = \frac{1}{2r}.$$

Given $u \geq 1$ and $1 \leq i \leq 2u$, if we equate the coefficient of $D^{-u+i-1-(u-1)r}$ in (33), we get

$$(35) \quad a_{u-1,i-2} + \big(i - u - (u-1)r\big)a_{u-1,i-1} = \sum_{k=0}^{2u-i} \binom{r}{k+1} \prod_{j=0}^{k}(i+j) \cdot a_{u,i+k}.$$

By a tedious but straightforward calculation, we find the recursion

$$(36) \quad i!\, a_{u,i} = \sum_{j=0}^{2u-2} a_{u-1,j}\big((j+1)!\, s_{j+2-i} + (j-(u-1)(r+1))j!\, s_{j+1-i}\big),$$

where s_k is the coefficient of x^k in

$$\frac{x}{(1+x)^r - 1} = \frac{1}{r + \binom{r}{2}x + \binom{r}{3}x^2 + \cdots}.$$

For convenience, let $e_{u,i} = i!\, a_{u,i}$. Then (36) becomes

$$(37) \quad e_{u,i} = \sum_{j=1}^{2u-2} e_{u-1,j}\big((j+1)s_{j+2-i} + (j-(u-1)(r+1))s_{j+1-i}\big),$$

with initial values

$$e_{1,1} = s_1 = \frac{1-r}{2r}, \qquad e_{1,2} = s_0 = \frac{1}{r}.$$

From the recursion, we see $e_{u,i}$ is nonzero only when $1 \leq i \leq 2u$.

Proposition 5.2. *Let $g \geq 0$. We have the following formula for one-point r-spin numbers*

$$(38) \quad \langle \tau_{n,m} \rangle_g = \frac{(-1)^g \Gamma(-2g - \frac{2g-1}{r})}{r^g \Gamma(1 - \frac{m+1}{r})} E_{2g},$$

where

$$E_u = \sum_{i=1}^{2u} \binom{u(r+1)-1}{i} e_{u,i}.$$

Proof. Since $L = K^{-1}D^r K$, we have $L^{n+1+(m+1)/r} = K^{-1}D^{(n+1)r+m+1}K$. From $\langle \tau_{n,m} \rangle_g = \langle \tau_{0,0}\tau_{n+1,m} \rangle_g$ and $(n+1)r + m + 1 = 2g(r+1) - 1$, it is not difficult to see that the constant term in $\mathrm{Res}(L^{n+1+(m+1)/r})$ equals the constant term in $\mathrm{Res}\, D^{(n+1)r+m+1}K$, which is E_{2g}. Finally (38) follows from (32) and

$$c_{n+1,m} = \frac{(-1)^{n+1} r^{n+2}}{(m+1)(r+m+1)\cdots((n+1)r+m+1)}$$

$$= \frac{\Gamma(-n-1-\frac{m+1}{r})}{\Gamma(1-\frac{m+1}{r})} = \frac{\Gamma(-2g - \frac{2g-1}{r})}{\Gamma(1-\frac{m+1}{r})}.$$

□

Setting $m = 0$ and taking $r \to -1$ in the right-hand side of (38), we get

$$\lim_{r \to -1} \Gamma\left(-2g - \frac{2g-1}{r}\right) E_{2g},$$

which is computed by applying L'Hôpital's Rule to the following Lemma.

Lemma 5.3. *For any integer $u \geq 1$, we have*

$$\lim_{r \to -1} \Gamma(-u - \frac{u-1}{r}) E_u = -\frac{B_u}{u}.$$

Proof. Since the residue of $\Gamma(z)$ at $z = -1$ is -1, we have

$$(39) \qquad \lim_{r \to -1} \frac{d}{dr}\left(\frac{1}{\Gamma(-u - \frac{u-1}{r})}\right) = 1 - u.$$

We also have

$$(40) \qquad \frac{d}{dr}\bigg|_{r=-1}\left(\frac{u(r+1)-1}{i}\right) = (-1)^{i+1} u H_i,$$

where $H_i = \sum_{1 \leq k \leq i} \frac{1}{k}$ is the ith harmonic number.

Setting $i = 1$ in (35), we get

$$0 = \sum_{k=1}^{2u} \binom{r}{k} e_{u,k},$$

which, after taking derivative with respect to r, becomes

$$(41) \qquad 0 = \sum_{k=1}^{2u} \left((-1)^{k+1} H_k e_{u,k}(-1) + (-1)^k e'_{u,k}(-1)\right).$$

From (40) and (41), we have

$$(42) \qquad \lim_{r \to -1} E_u = \sum_{k=1}^{2u} \left((-1)^{k+1} u H_k e_{u,k}(-1) + (-1)^k e'_{u,k}(-1)\right)$$

$$= (u-1) \sum_{k=1}^{2u} (-1)^{k+1} u H_k e_{u,k}(-1).$$

By (39) and (42), we see that Lemma 5.3 is equivalent to

$$(43) \qquad \sum_{k=1}^{2u} (-1)^{k+1} H_k e_{u,k}(-1) = \frac{B_u}{u}, \qquad u \geq 2.$$

Setting $r = -1$ in (37), we get

$$(44) \qquad e_{u,i}(-1) = (1-i) e_{u-1,i-2}(-1) + (1-2i) e_{u-1,i-1}(-1) - i e_{u-1,i}(-1).$$

Here we use $s_0(-1) = s_1(-1) = -1$ and $s_k(-1) = 0$, $k > 1$.

If we substitute (44) into (43), we get

$$(45) \qquad \sum_{k=1}^{2u-2} \frac{(-1)^k}{(k+1)(k+2)} e_{u-1,k}(-1) = \frac{B_u}{u}, \qquad u \geq 2.$$

By substituting (44) successively into (45), we get

$$(46) \qquad \sum_{k=0}^{2(u-j)} (-1)^{k+1} f_j(k) e_{u-j,k}(-1) = \frac{B_u}{u}, \quad 1 \le j \le u,$$

where f_j, $j \ge 1$ are given by the recursion

$$(47) \qquad f_{j+1}(k) = -(k+1)f_j(k+2) + (2k+1)f_j(k+1) - kf_j(k)$$

starting with $f_1(k) = \frac{-1}{(k+1)(k+2)}$.

Let $e_{0,i} = \delta_{0i}$, which is compatible with the recursion (44). Then (45) (hence (43)) is equivalent to

$$(48) \qquad f_u(0) = \frac{-B_u}{u}, \quad u \ge 2.$$

We leave the proof to the Appendix C. $\qquad\qquad\square$

From (29), (38) and Lemma 5.3, we recover the Harer-Zagier formula (Theorem 5.1).

Now we make a connection to the matrix integral approach of Brézin and Hikami [3]. First we note that a refined argument in the proof of Lemma 5.3 will give the following identity of generating functions.

$$(49) \quad \Gamma\left(\frac{1}{r}\right) + \sum_{u=1}^{\infty} E_u \Gamma\left(-u - \frac{u-1}{r}\right) y^u$$
$$= r \int_0^{\infty} \exp\left(-\frac{1}{(r+1)y}\left[\left(x + \frac{y}{2}\right)^{r+1} - \left(x - \frac{y}{2}\right)^{r+1}\right]\right) dx.$$

The integral expression of the right-hand side appeared in [3].

Consider the semigroup $\mathbb{N}^{\infty}$ of sequences $\mathbf{d} = (d_1, d_2, \dots)$ where d_i are nonnegative integers and $d_i = 0$ for sufficiently large i. For $\mathbf{d} \in \mathbb{N}^{\infty}$, we define

$$(50) \qquad |\mathbf{d}| := \sum_{i \ge 1} i d_i, \quad ||\mathbf{d}|| := \sum_{i \ge 1} d_i, \quad \mathbf{d}! := \prod_{i \ge 1} d_i!.$$

In the following exposition, we set $t_i = -\binom{r}{2i}/((2i+1)4^i)$, $i \ge 1$.

Proposition 5.4. *Let $g \ge 0$. We have the following closed formula for one-point r-spin numbers*

$$\langle \tau_{n,m} \rangle_g = \frac{(-1)^g}{r^g \Gamma\left(1 - \frac{1+m}{r}\right)} \sum_{|\mathbf{d}|=g} \Gamma\left(||\mathbf{d}|| - \frac{2g-1}{r}\right) \frac{\prod_{i \ge 1} t_i^{d_i}}{\mathbf{d}!}.$$

Proof. By expanding the right-hand side of (49)

$$-\frac{1}{(r+1)y}\left(\left(x + \frac{y}{2}\right)^{r+1} - \left(x - \frac{y}{2}\right)^{r+1}\right) = -x^r - \sum_{i \ge 1} t_i y^{2i} x^{r-2i}$$

and using

$$\Gamma(z) = r \int_0^{\infty} x^{rz-1} \exp(-x^r) dx,$$

we see that Proposition 5.4 follows from (38). $\qquad\qquad\square$

Corollary 5.5. *Let $k \geq 0$ and $g \geq 0$. We have the following closed formula for r-spin numbers*

$$\langle \tau_{0,1}^k \tau_{n,m} \rangle_g = \frac{(-1)^g}{r^g \Gamma(1 - \frac{1+m}{r})} \sum_{|\mathbf{d}|=g} \Gamma\left(||\mathbf{d}|| - \frac{2g-k-1}{r} \right) \frac{\prod_{i \geq 1} t_i^{d_i}}{\mathbf{d}!}.$$

This follows from the same inductive argument as in the proof of Proposition 4.3.

Remark 5.6. For r-spin numbers, we do not have the analogue of the divisor equation as in Gromov-Witten theory. But the identity of Proposition 5.4 suggests that some form of the "divisor equation" may still exist for r-spin numbers.

Corollary 5.7. *Let $g \geq 1$. Setting $r = -1$, $m = 0$ in the right-hand side of Proposition 5.4, we get*

$$\sum_{|\mathbf{d}|=g} \Gamma(||\mathbf{d}|| + 2g - 1) \frac{(-1)^{||d||}}{\mathbf{d}! \prod_{i \geq 1}((2i+1)4^i)^{d_i}} = -\frac{B_{2g}}{2g}.$$

Proof. We follow the method used [3, §3].

Letting $r \to -1$ in the right-hand of (49) and applying L'Hôpital's Rule, we get

$$RHS = -\int_0^\infty \left(\frac{x - \frac{y}{2}}{x + \frac{y}{2}} \right)^{1/y} dx,$$

which is the generating function of the left-hand side of Corollary 5.7.

Making the change of variables

$$\frac{x - \frac{y}{2}}{x + \frac{y}{2}} = e^{-z}, \qquad \text{i.e.} \quad x = \frac{y}{2} \left(\frac{1 + e^z}{1 - e^z} \right).$$

we have

$$\tag{51} RHS = -\int_0^\infty e^{-z/y} \frac{-ye^{-z}}{(1 - e^{-z})^2} dz$$

$$\tag{52} = -\int_0^\infty e^{-z/y} \frac{1}{1 - e^{-z}} dz$$

$$\tag{53} = -y \int_0^\infty e^{-t} dt \frac{1}{1 - e^{-yt}}$$

$$\tag{54} = -\sum_{k=1}^\infty \frac{B_k}{k} y^k.$$

Here (52) follows from

$$\frac{d}{dz} \left(\frac{e^{-z/y}}{1 - e^{-z}} \right) = \frac{-e^{-z/y}}{y(1 - e^{-z})} + \frac{-e^{-z}e^{-z/y}}{(1 - e^{-z})^2}$$

and (54) follows from

$$\frac{1}{1 - e^{-t}} = \sum_{k=0}^\infty B_k \frac{t^{k-1}}{k!}.$$

This completes the proof. $\qquad\qquad\square$

6. SMALL PHASE SPACE IN GENUS ZERO

In this section, we extend Witten's exposition in [31]. We first reorganize Witten's argument and highlight important relevant results of Witten for the reader's convenience. We then prove a full series expansion formula for the Landau-Ginzburg potential $W(p, x)$ in the small phase space $(t_{n,m} = 0, n > 0)$ of genus zero.

For dimensional reasons (equation (2)), a primary intersection number $\langle \tau_{0,m_1} \cdots \tau_{0,m_s} \rangle_g$ can be nonzero only when $g = 0$. Furthermore, for each r, there are only finite number of nonzero primary intersection numbers $\langle \tau_{0,m_1} \cdots \tau_{0,m_s} \rangle_0$, since we have $r(s - 2) - 2 = m_1 + \cdots + m_s \leq (r - 2)s$ (so in particular $s \leq r + 1$).

As observed by Witten, the genus zero Gel'fand-Dikii equation is obtained by replacing the differential operator Q by a function

$$W(p, x) = p^r + \sum_{i=0}^{r-2} u_i(x)p^i$$

and replacing commutators by Poisson brackets

$$\{A, B\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p}.$$

So in genus zero, the Gel'fand-Dikii equations reduce to

$$\frac{\partial W}{\partial t_{n,m}} = \frac{c_{n,m}}{r}\{W_+^{n+(m+1)/r}, W\},$$

where $c_{n,m}$ is the same constant defined in §1

Lemma 6.1 (Witten, [31]). *On the small phase space, we have*

$$\frac{\partial F}{\partial t_m} = \frac{r^2}{(m+1)(r+m+1)} \operatorname{Res}(W^{1+(m+1)/r}).$$

Proof. A special case of Witten's conjecture is

$$\frac{\partial^2 F}{\partial t_{0,0}\partial t_{1,m}} = \frac{r^2}{(m+1)(r+m+1)} \operatorname{Res}(W^{1+(m+1)/r}).$$

The string equation implies that on small phase space, we actually have

$$\frac{\partial^2 F}{\partial t_{0.0}\partial t_{1,m}} = \frac{\partial F}{\partial t_{0,m}}.$$

The desired equation follows. $\qquad\qquad\square$

Below, all of our computations will be done entirely on the small phase space $(t_{n,m} = 0, n > 0)$ and we set $t_m = t_{0,m}$ and $\tau_m = \tau_{0,m}$.

Lemma 6.2 (Witten, [31]). *For $0 \leq m \leq r - 2$, we have*

$$\frac{\partial W}{\partial t_m} = -\frac{1}{m+1}\frac{\partial}{\partial p}W_+^{(m+1)/r}.$$

Proof. A special case of Witten's conjecture is

$$(55) \qquad\qquad \frac{\partial^2 F}{\partial t_0 \partial t_m} = \frac{-r}{m+1} \operatorname{Res}(W^{(m+1)/r}).$$

By (13) in the proof of Lemma 3.4 (the differential polynomials p, p' there should be replaced by plain polynomials of their arguments), we can use equation (55) to express the coefficients u_i of W as differential polynomials in $\partial^2 F/\partial t_0 \partial t_m$. Hence

W can be regarded as a function in $p, t_0, \ldots, t_m$. If we set all $t_m = 0$, then the left hand side of (55) is obviously zero for dimensional reasons, so all $u_i = 0$ by Lemma 3.4. Thus $W = p^r$ when all $t_m = 0$. We then get the constant term of W.

Differentiating (55) with respect to $x = t_0$, we get

$$(56) \qquad \delta_{m,r-2} = -\frac{r}{m+1}\frac{\partial}{\partial x}\mathrm{Res}(W^{(m+1)/r}).$$

Since $\partial W/\partial x$ is a polynomial in p of degree at most $r - 2$, if this polynomial is of degree k, then from (13) in the proof of Lemma 3.4, the right hand side of (56) is non-zero for $m = r - 2 - k$. Thus $k = 0$ and

$$\frac{\partial W}{\partial x} = \frac{\partial u_0}{\partial x} = \frac{r}{r-1}\frac{\partial}{\partial x}\mathrm{Res}(W^{\frac{r-1}{r}}) = -1.$$

From this and $\partial u_i/\partial x = 0$ when $1 \le i \le r - 2$, we have for $0 \le m \le r - 2$,

$$\frac{\partial}{\partial x}W_+^{(m+1)/r} = 0,$$

since the coefficients of $W_+^{(m+1)/r}$ do not contain u_0. This follows from a weight count, since by our convention (see Lemma 3.2), $W_+^{(m+1)/r}$ is homogeneous of weight $m + 1 \le r - 1$, while the weight of u_i is $r - i$. $\qquad\square$

Theorem 6.3 (Witten, [31])**.** *For $0 \le m \le r - 2$, define $\phi_m = -\frac{\partial W}{\partial t_m}$. Then*

$$\frac{\partial^3 F}{\partial t_j \partial t_m \partial t_s} = r \cdot \mathrm{Res}\left\{\frac{\phi_j \phi_m \phi_s}{\partial_p W}\right\}.$$

Now we can state our new results: the full series expansion for W in $t_0, \ldots, t_m$, extending Witten's computation up to linear terms [31].

Theorem 6.4. *We have the following series expansion for W:*

$$W = p^r + \sum_{k=0}^{r-2} p^k \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \cdot r^{n-1}} \sum_{v_1+\cdots+v_n=(n-1)r+k} \frac{(k+n-1)!}{k!}t_{v_1}\cdots t_{v_n}$$

$$= p^r + \sum_{k=0}^{r-2} p^k \left(-t_k + \frac{1}{2! \cdot r}\sum_{u+v=r+k}(k+1)t_u t_v\right.$$

$$\left. -\frac{1}{3! \cdot r^2}\sum_{u+v+w=2r+k}(k+1)(k+2)t_u t_v t_w + \cdots\right)$$

Proof. We can compute the degree n term of $W_+^{(m+1)/r}$ from terms of W up to degree n. Then we use Lemma 6.2 to compute the degree $n + 1$ term of W from the degree n term of $W_+^{(m+1)/r}$.

$$W_+^{(m+1)/r} = p^{m+1} - \frac{m+1}{r}\sum_{u \ge r-m-1}t_u p^{m+u-r+1}$$

$$+ \frac{m+1}{2! \cdot r^2}\sum_{u+v \ge 2r-m-1}(m+u+v+2-2r)t_u t_v p^{m+u+v-2r+1} + \cdots$$

$$= p^{m+1} + (m+1) \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \cdot r^n} \sum_{v_1 + \cdots + v_n \geq nr-m-1} \frac{(m + \sum\limits_{i=1}^{n} v_i + n - nr)!}{(m + \sum\limits_{i=1}^{n} v_i + 1 - nr)!}$$
$$\times t_{v_i} \cdots t_{v_n} p^{m+v_1 + \cdots + v_n - nr + 1}$$

Thus the theorem can be proved inductively. $\hfill\square$

Corollary 6.5. *Let $0 \leq m \leq r - 2$. The series expansion for ϕ_m is*

$$\phi_m = -\frac{\partial W}{\partial t_m} = p^m - \sum_{u \geq r-m} \frac{m + u + 1 - r}{r} t_u p^{m+u-r}$$

$$+ \frac{1}{2! \cdot r^2} \sum_{u+v \geq 2r-m} (u + v + m + 1 - 2r)(u + v + m + 2 - 2r) t_u t_v p^{m+u+v-2r} + \cdots$$

$$= p^m + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \cdot r^n} \sum_{v_1 + \cdots + v_n \geq nr-m} \frac{(m + \sum\limits_{i=1}^{n} v_i + n - nr)!}{(m + \sum\limits_{i=1}^{n} v_i - nr)!} t_{v_i} \cdots t_{v_n} p^{m+v_1 + \cdots + v_n - nr}$$

Proof. This follows from the definition of ϕ_m and a direct computation. $\hfill\square$

In [7], Dijkgraaf, Verlinde and Verlinde give a closed formula of ϕ_m in terms of the determinant of matrices. Presumably their formula is equivalent to ours, although we have not checked the details.

In view of Lemma 6.1, it would also be interesting to have a series expansion for $W^{\frac{m+1}{r}}$. Here we write out terms up to degree 3. Let $\theta(x)$ be the Heaviside function that is 1 for $x \geq 0$ and 0 for $x < 0$.

$$(57) \quad W^{\frac{m+1}{r}} = p^{m+1} - \frac{m+1}{r} \sum_u t_u p^{m+1+u-r}$$

$$+ \frac{m+1}{2! \cdot r^2} \sum_{u,v} ((m+1-r) + (u+v-r+1)\theta(u+v-r)) t_u t_v p^{m+u+v+1-2r}$$

$$- \frac{(m+1)}{3! \cdot r^3} \sum_{u,v,w} ((m+1-r)(m+1-2r) + (m+1-r)(u+v+1-r)\theta(u+v-r)$$

$$+ (m+1-r)(u+w+1-r)\theta(u+w-r) + (m+1-r)(v+w+1-r)\theta(v+w-r)$$

$$+ (u+v+w-2r+1)(u+v+w-2r+2)\theta(u+v+w-2r)) t_u t_v t_w p^{m+u+v+w+1-3r}$$

$$+ \cdots$$

By Lemma 6.1, we can use the degree 3 term of the above expansion to get a formula for 4-point correlation functions.

Corollary 6.6.

$$\langle \tau_m \tau_u \tau_v \tau_w \rangle = \frac{1}{r}(r - m - 1 - (u + v - r + 1)\theta(u + v - r)$$
$$- (u + w - r + 1)\theta(u + w - r) - (v + w - r + 1)\theta(v + w - r)).$$

Proof. Replace m by $m + r$ in the expansion of $W^{\frac{m+1}{r}}$ and take the coefficient of p^{-1}. We get the desired result from Lemma 6.1. $\hfill\square$

The formula in Corollary 6.6 is slightly different with Witten's formula [31, (3.3.36)]

$$\langle \tau_m \tau_u \tau_v \tau_w \rangle = \frac{1}{r}(m - (m + u - r + 1)\theta(m + u - r)$$
$$- (m + v - r + 1)\theta(m + v - r) - (m + w - r + 1)\theta(m + w - r)),$$

but it is not difficult to prove that they are both equivalent to $\langle \tau_{a_1} \tau_{a_2} \tau_{a_3} \tau_{a_4} \rangle = \frac{1}{r} \cdot \min(a_i, r - 1 - a_i)$.

Our motivation in studying r-spin numbers on the small phase space is to prove the following conjectural properties of these numbers.

Conjecture 6.7. *In the small phase space, we have*

  i) (Integrality) $\frac{r^{s-3}}{(s-3)!}\langle \tau_{m_1} \cdots \tau_{m_s} \rangle \in \mathbb{Z}$.
  ii) (Vanishing) *If* $m_i < s - 3$ *for some* $1 \le i \le s$, *then* $\langle \tau_{m_1} \cdots \tau_{m_s} \rangle = 0$.
  iii) (Multinomial distribution) *If* $m_1 > m_2$, *then* $\langle \tau_{m_1-1}\tau_{m_2+1}\tau_{m_3} \cdots \tau_{m_s} \rangle \ge \langle \tau_{m_1}\tau_{m_2} \cdots \tau_{m_s} \rangle$.

We have verified this conjecture in low genus or when s is small. One can even prove $r^{s-3}\langle \tau_{m_1} \cdots \tau_{m_s} \rangle \in \mathbb{Z}$ by extending (57). However, the combinatorial difficulty for the general case is still considerable, even though we can write down explicit formulae for general s-point correlation functions $\langle \tau_{m_1} \cdots \tau_{m_s} \rangle$ via Theorem 6.4, Corollary 6.5 and Witten's Theorem 6.3.

See [17] for more on denominators and multinomial-type properties of intersection numbers.

Part of our motivation comes from Gromov-Witten invariants of $\mathbb{CP}^n$ on the small phase space. For fixed $n \ge 1$ and $d \ge 0$, consider Gromov-Witten invariants of $\mathbb{CP}^n$ on the small phase space in genus zero

$$\langle m_1, \cdots, m_s \rangle := \langle \tau_{0,m_1}, \cdots, \tau_{0,m_s} \rangle_{0,d}^{\mathbb{CP}^n},$$

which is nonzero (for dimensional reasons) only when

(58) $$\sum_{i=1}^{s} m_i = n + (n + 1)d + s - 3.$$

The following properties are analogues of corresponding statements in Conjecture 6.7:

  i) (Integrality) $\langle m_1, \cdots, m_s \rangle \in \mathbb{N}_{\ge 0}$.
  ii) (Vanishing) If $d > 1$ and $s = d + 2$, then $\langle m_1, \cdots, m_s \rangle = 0$.
  iii) (Multinomial distribution) If $m_1 > m_2$, then

$$\langle m_1 - 1, m_2 + 1, m_3, \cdots, m_s \rangle \ge \langle m_1, m_2, \cdots, m_s \rangle.$$

The integrality (i) is clear, since genus zero Gromov-Witten invariants of $\mathbb{CP}^n$ are intersections on a scheme, and hence integral. We conjecture the multinomial distribution (iii), based on numerical evidence. We now prove (ii).

Proposition 6.8. *With the notation above, we have the vanishing* $\langle m_1, \cdots, m_s \rangle = 0$ *of degree d genus 0 invariants in $\mathbb{P}^n$, where $s = d + 2$, with any m_i, and $d > 1$.*

Proof. We show that for any choice of m_i, the intersection theory problem $\langle m_1, \cdots, m_s \rangle$, interpreted as counting stable maps "meeting" generally chosen linear spaces of codimension $m_1, \ldots, m_s$, corresponds to the empty intersection. Let $c_i = n - m_i$

be the *dimension* of these linear spaces for convenience; $\sum c_i = n - 2d + 1$ from (58).

As $m_i \leq n$, we have

$$n(d+2) \geq \sum_{i=1}^{s} m_i = n + (n+1)d + (d+2) - 3 \quad \text{(using (58))},$$

from which $d \leq (n+1)/2 < n$. The image of any degree d stable map lies inside a $\mathbb{P}^d$. We show that there isn't even a $\mathbb{P}^d$ inside $\mathbb{P}^n$ meeting the (generally chosen) linear spaces of dimension c_i. The codimension of the condition (on $\mathbb{G}(d,n)$) that a $\mathbb{P}^d$ in $\mathbb{P}^n$ meet a $\mathbb{P}^{c_i}$ is $\max(0, n - d - c_i)$. Then

$$
\begin{aligned}
\sum_{i=1}^{s} \max(0, n - d - c_i) \quad &\geq \quad \sum_{i=1}^{s}(n - d - c_i) \\
&= \quad (n-d)(d+2) - (n - 2d + 1) \\
&> \quad (d+1)(n-d) \quad \text{(using } d > 1) \\
&= \quad \dim \mathbb{G}(d,n)
\end{aligned}
$$

so there is no $\mathbb{P}^d$ in $\mathbb{P}^n$ meeting the desired linear spaces, and thus no degree d stable map meeting these linear spaces. $\square$

APPENDIX A. COMBINATORIAL IDENTITIES

We prove Proposition 3.6, using the Lagrange inversion formula. The following form of Lagrange inversion formula can be found in [28, p. 38].

Lemma A.1 (Lagrange inversion formula)**.** *Let* $F(x) = a_1 x + a_2 x^2 + \cdots \in \mathbb{C}[[x]]$ *be a power series with* $a_1 \neq 0$ *and* $F^{-1}(x) \in \mathbb{C}[[x]]$ *be its inverse (defined by* $F^{-1}(F(x)) = x$*). For* $k, n \in \mathbb{Z}$ *we have*

$$(59) \qquad \frac{1}{n}[x^{n-k}]\left(\frac{x}{F(x)}\right)^n = \frac{1}{k}[x^n]F^{-1}(x)^k.$$

We now prove Proposition 3.6.

Let $f(x) = 1 + \sum_{j=2}^{\infty} a_j x^j \in \mathbb{C}[[x]]$ be as in Proposition 3.6. Then $F(x) = x/f(x)$ is a power series with $a_1 \neq 0$, so we can apply Lemma A.1. Taking $k = 1$ in equation (59), we see that $[x^2]F^{-1}(x) = \frac{1}{2}[x]f(x)^2 = 0$, so we have

$$
\begin{aligned}
(60) \qquad \frac{1}{F^{-1}(x)} &= \frac{1}{x + c_3 x^3 + c_4 x^4 + \cdots} \\
&= \frac{1}{x} - c_3 x - c_4 x^2 - \cdots.
\end{aligned}
$$

Taking $k = 1$ and $k = 2$ in equation (59) respectively, we get

$$
\begin{aligned}
\frac{[x^{n+1}]f(x)^n}{n} &= -[x^n]\frac{1}{F^{-1}}, \\
\frac{[x^{n+2}]f(x)^n}{n} &= -\frac{1}{2}[x^n]\frac{1}{F^{-1}}.
\end{aligned}
$$

Substituting the above two identities into equation (19) and then applying equation (60) to the summation term on the right hand side, equation (19) becomes

$$-[x^{n+1}]\frac{1}{F^{-1}} = \frac{1}{2}\sum_{j=1}^{n-1}[x^j]\frac{1}{F^{-1}}[x^{n-j}]\frac{1}{F^{-1}} - \frac{1}{2}[x^n]\frac{1}{(F^{-1}(x)^2}$$

$$= \left(\frac{1}{2}[x^n]\frac{1}{F^{-1}(x)^2} - [x^{n+1}]\frac{1}{F^{-1}}\right) - \frac{1}{2}[x^n]\frac{1}{(F^{-1}(x)^2}$$

$$= -[x^{n+1}]\frac{1}{F^{-1}}.$$

So we have proved Proposition 3.6. $\qquad\square$

We now present equivalent formulations of Proposition 3.6 that may be useful elsewhere. We use the notation introduced in (50).

Proposition A.2. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{N}^\infty$, $\mathbf{c} \in \mathbb{N}^\infty$ *and* $||\mathbf{c}|| \geq 2$. *Then the following identity holds*

$$\frac{(|\mathbf{c}| + ||\mathbf{c}|| - 3)!}{(|\mathbf{c}| - 1)!} \cdot (||\mathbf{c}|| - 1) = \frac{1}{2}\sum_{\substack{\mathbf{c}=\mathbf{a}+\mathbf{b} \\ \mathbf{a},\mathbf{b}\neq 0}}\binom{\mathbf{c}}{\mathbf{a},\mathbf{b}}\frac{(|\mathbf{a}| + ||\mathbf{a}|| - 2)! \cdot (|\mathbf{b}| + ||\mathbf{b}|| - 2)!}{(|\mathbf{a}| - 1)! \cdot (|\mathbf{b}| - 1)!},$$

where $\binom{\mathbf{c}}{\mathbf{a},\mathbf{b}}$ *is defined as* $\prod_{i\geq 1}\frac{\mathbf{c}!}{\mathbf{a}!\mathbf{b}!} = \prod_{i\geq 1}\binom{c_i}{a_i,b_i}$ *(cf. (50)).*

Proof. Take any $c = (c_1, c_2, \dots) \in \mathbb{N}^\infty$, compare the coefficient $\prod_{j\geq 2} a_j^{c_{j-1}}$ in both sides of equation (19). We have

$$\frac{(|\mathbf{c}| + ||\mathbf{c}|| - 2)!}{(|\mathbf{c}| - 1)!\mathbf{c}!} = \frac{1}{2}\sum_{\substack{\mathbf{c}=\mathbf{a}+\mathbf{b} \\ \mathbf{a},\mathbf{b}\neq 0}}\frac{(|\mathbf{a}| + ||\mathbf{a}|| - 2)!}{(|\mathbf{a}| - 1)!\mathbf{a}!}\frac{(|\mathbf{b}| + ||\mathbf{b}|| - 2)!}{(|\mathbf{b}| - 1)!\mathbf{b}!} + \frac{(|\mathbf{c}| + ||\mathbf{c}|| - 3)!}{(|\mathbf{c}| - 2)!\mathbf{c}!}.$$

By moving the last term in the right hand side to the left, we get the desired identity. $\qquad\square$

A partition is a sequence of integers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$. We write

$$|\mu| = \mu_1 + \cdots + \mu_k, \qquad \ell(\mu) = k.$$

Define $m_j(\mu)$ to be the number of j's among $\mu_1, \dots, \mu_k$, $z_\mu = \prod_j m_j(\mu)!j^{m_j(\mu)}$, and $p_\mu = \prod_j p_j^{m_j(\mu)}$.

Proposition A.3.

$$\sum_{\ell(\mu)\geq 2}\frac{(|\mu| + \ell(\mu) - 3)!(\ell(\mu) - 1)p_\mu}{(|\mu| - 1)!z_\mu} = \frac{1}{2}\left(\sum_{\mu\neq 0}\frac{(|\mu| + \ell(\mu) - 2)!p_\mu}{(|\mu| - 1)!z_\mu}\right)^2$$

Proof. Take $\mathbf{c} = (m_1(\mu), m_2(\mu), \dots) \in \mathbb{N}^\infty$. Then the identity in the proposition is just a reformulation of Proposition A.2. $\qquad\square$

APPENDIX B. THE DIFFERENTIAL POLYNOMIAL $W_r(z)$

From §3, γ_{-1}^{r+1} can be expressed as a differential polynomial in $\gamma_{-1}^1, \ldots, \gamma_{-1}^{r-1}$. If $2 \leq i \leq r$, denote by $p_i(r)$ the coefficient of $D^i \gamma_{-1}^{r+1-i}$ in the resulting differential polynomial $S(\gamma_{-1}^{r+1})$. From the proof of Proposition 3.8, it is straightforward to obtain the following recursive formula for $p_i(r)$,

$$(61) \quad \begin{aligned} p_i(r) =& \frac{1}{r+1-i} [D^i w_{-(r+1-i)}] (\gamma_{-1}^{r+1} - \frac{r+1}{r} \gamma_{-2}^r - \frac{r+1}{2r} D\gamma_{-1}^r) \\ & - \frac{1}{r+1-i} \sum_{j=2}^{i-1} \binom{r+1-j}{i+1-j} p_j(r) \\ =& \binom{r+1}{i} \frac{1}{r} \frac{i-1}{2(i+1)} - \frac{1}{r+1-i} \sum_{j=2}^{i-1} \binom{r+1-j}{i+1-j} p_j(r). \end{aligned}$$

We have proved $p_2(r) = \frac{r+1}{12}$ in Proposition 3.8. The relation of $p_i(r)$ to the coefficients of $W_r(z)$ is given by

$$(62) \quad [z_{r-i}^{(i)}] W_r(z) = \frac{-\sqrt{-1}^i (r-i+1)}{r^{\frac{i}{2}-1}(r+1)} p_i(r).$$

For $i \geq 2$, define quantities C_i by

$$(63) \quad p_i(r) = \frac{1}{r} \binom{r+1}{i} C_i.$$

We will see shortly that C_i are in fact constants independent of r.

Substituting (63) into the recursion formula (61), we get

$$\frac{1}{r}\binom{r+1}{i} C_i = \binom{r+1}{i} \frac{1}{r} \frac{i-1}{2(i+1)} - \frac{1}{r+1-i} \sum_{j=2}^{i-1} \binom{r+1-j}{i+1-j} \frac{1}{r}\binom{r+1}{j} C_j$$

$$(64) \quad C_i = \frac{i-1}{2(i+1)} - \sum_{j=2}^{i-1} \binom{i+1}{j} \frac{C_j}{i+1}.$$

Hence by induction (starting from $C_0 = -\frac{1}{2}$, $C_1 = 0$), we see that the C_i are constants. Using the values of C_0 and C_1, we may simplify (64) as

$$(65) \quad \sum_{j=0}^{i} \binom{i+1}{j} C_j = \frac{i-2}{2}, \qquad i \geq 1.$$

Proposition B.1. Let $i \geq 2$. Then $C_i = B_i$ the Bernoulli numbers. In particular, $C_{2k+1} = 0$.

Proof. We define a new sequence C_j' by $C_0' = 1, C_1' = -\frac{1}{2}$ and $C_j' = C_j, j \geq 2$. From (65), we have

$$\sum_{j=0}^{i} \binom{i+1}{j} C_j' = 0, \qquad i \geq 1,$$

which is the usual recursion for Bernoulli numbers. Since $C_0' = B_0, C_1' = B_1$, we must have $C_j = C_j' = B_j$ for all $j \geq 2$. $\qquad\square$

From (62) and (63), we thus proved Proposition 3.10.

APPENDIX C. AN IDENTITY OF BERNOULLI NUMBERS

Let $f_n(k)$, $n \geq 1$ be given by the recursion

(66) $$f_{n+1}(k) = -(k+1)f_n(k+2) + (2k+1)f_n(k+1) - kf_n(k)$$

starting with $f_1(k) = \frac{-1}{(k+1)(k+2)}$.

Proposition C.1. *Let $n \geq 2$. We have $f_n(0) = -B_n/n$.*

The rest of the appendix is devoted to proving the above proposition. First we record the following combinatorial identities.

(67) $$\sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}\frac{1}{n+i-k} = \frac{n!(i-1)!}{(n+i)!},$$

(68) $$\sum_{i=w}^{n} \frac{(i-1)!}{(i-w)!} = \frac{n!}{(n-w)!w},$$

(69) $$\left(\frac{e^t - 1}{t}\right)^w = \sum_{j=w}^{\infty} \frac{t^{j-w}}{j!}\sum_{s=0}^{w}(-1)^s\binom{w}{s}(w-s)^j.$$

Consider the generating function $F_n(x) = \sum_{k=0}^{\infty} f_n(k)x^k$, then we have $F_1(x) = \frac{-1}{x} + \frac{x-1}{x^2}\ln(1-x)$ and (66) implies

$$F_n(x) = -\frac{\partial}{\partial x}\left(\frac{F_{n-1}(x)}{x}\right) + 2\frac{\partial}{\partial x}F_{n-1}(x) - \frac{1}{x}F_{n-1}(x) - x\frac{\partial}{\partial x}F_{n-1}(x)$$

$$= \frac{-(x-1)^2}{x}F_{n-1}(x) + \frac{1-x}{x^2}F_{n-1}(x).$$

More precisely, $F_n(x)$ from the above recursion differs from the true generating function by a finite sum of negative powers of x. Thus Proposition C.1 is equivalent to prove that the constant term of $-nF_n(x)$ equals B_n.

It is not difficult to see that $F_n(x)$ decomposes as

(70) $$F_n(x) = G_n(x) + R_n(x)\ln(1-x), \quad n \geq 1,$$

where $G_n(x)$ and $R_n(x)$ are rational functions in x satisfying the recursions

(71) $$G_n(x) = \frac{-(x-1)^2}{x}\frac{\partial}{\partial x}G_{n-1}(x) + \frac{1-x}{x^2}G_{n-1}(x) + \frac{1-x}{x}R_{n-1}(x),$$

(72) $$R_n(x) = \frac{-(x-1)^2}{x}\frac{\partial}{\partial x}R_{n-1}(x) + \frac{1-x}{x^2}R_{n-1}(x).$$

We may solve (72) to get

(73) $$R_n(x) = (x-1)^n\sum_{i=1}^{n}(-1)^{i+1}a(n,i)x^{-n-i},$$

where $a(n,i)$ is given by

$$a(n,i) = \sum_{w=0}^{i}(-1)^{i-w}\binom{n+i}{i-w}\sum_{s=0}^{w}(-1)^s\frac{(w-s)^{n+w}}{s!(w-s)!}.$$

From (71), we may prove that $[x^k]G_n(x) = 0$, $\forall k \geq 0$. By using (67), (68), (69), the constant term of $-nF_n(x)$ equals

$$(74) \quad [x^0](-n)R_n(x)\ln(1-x) = (-n)\sum_{i=1}^{n}(-1)^i a(n,i)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{1}{n+i-k}$$

$$= (-n)\sum_{i=1}^{n}(-1)^i a(n,i)\frac{n!(i-1)!}{(n+i)!}$$

$$= (-n)\sum_{w=1}^{n}\sum_{s=0}^{w}(-1)^s\binom{w}{s}\frac{(w-s)^{w+n}}{w!}(-1)^w\frac{n!}{(w+n)!}\sum_{i=w}^{n}\frac{(i-1)!}{(i-w)!}$$

$$= (-n)\sum_{w=1}^{n}\sum_{s=0}^{w}(-1)^s\binom{w}{s}\frac{(w-s)^{w+n}}{w!}(-1)^w\frac{n!}{(w+n)!}\frac{n!}{(n-w)!w}$$

$$= (-n)\sum_{w=1}^{n}(-1)^w\binom{n}{w}\frac{n!}{w}[t^n]\left(\frac{e^t-1}{t}\right)^w$$

$$= -\sum_{w=1}^{n}(-1)^w\binom{n}{w}n![t^{n-1}]\left(\left(\frac{e^t-1}{t}\right)^{w-1}\frac{d}{dt}\left(\frac{e^t-1}{t}\right)\right)$$

$$= -n![t^{n-1}]\left(\sum_{w=0}^{n}(-1)^w\binom{n}{w}\left(\frac{e^t-1}{t}\right)^w\frac{te^t-e^t+1}{t(e^t-1)}\right) + n![t^{n-1}]\frac{te^t-e^t+1}{t(e^t-1)}$$

$$= -n![t^{n-1}]\left(\left(1-\frac{e^t-1}{t}\right)^n\frac{te^t-e^t+1}{t(e^t-1)}\right) + n![t^n]\frac{te^t-e^t+1}{e^t-1}$$

$$= n![t^n]\frac{te^t-e^t+1}{e^t-1},$$

where the last equation follows by noting

$$1-\frac{e^t-1}{t} = \frac{-1}{2}t - \frac{1}{6}t^2 + \cdots,$$

$$\frac{te^t-e^t+1}{t(e^t-1)} = \frac{1}{2} + \frac{1}{12}t + \cdots.$$

Finally, Proposition C.1 follows from

$$(75) \qquad\qquad 1+\frac{te^t-e^t+1}{e^t-1} = \frac{t}{1-e^{-t}} = 1+\frac{t}{2}+\sum_{n=2}^{\infty}\frac{B_n t^n}{n!}.$$

Remark C.2. Another way of proving Proposition C.1 is by studying the function $h_j(k) = \prod_{i=1}^{2j}(k+i)f_j(k)$. Then (66) becomes

$$(76) \quad h_{j+1}(k) = -(k+1)^2(k+2)h_j(k+2)$$
$$+ (k+1)(2k+1)(k+2j+2)h_j(k+1) - k(k+2j+1)(k+2j+2)h_j(k)$$

starting with $h_1(k) = -1$, $h_2(k) = 4k-2$, $h_3(k) = -36k^2+84k$.

We may prove from the recursion (76) (although more difficult) that $h_j(k)$ is a degree $j-1$ polynomial whose leading term equals $(-1)^j(j!)^2 k^{j-1}$ and the constant term equals $-(2j)!B_j/j$ when $j \geq 2$, as claimed in Proposition C.1.

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