DEFORMING SUBMANIFOLDS OF ARBITRARY CODIMENSION IN A SPHERE

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Abstract. In this paper, we prove some convergence theorems for the mean curvature flow of closed submanifolds in the unit sphere $\mathbb{S}^{n+d}$ under integral curvature conditions. As a consequence, we obtain several differentiable sphere theorems for certain submanifolds in $\mathbb{S}^{n+d}$.

1. Introduction

Let M be an n-dimensional immersed submanifold in a Riemannian manifold N^{n+d}. Throughout this paper, we always assume M is connected. Let $F_0 : M \to N^{n+d}$ denote the isometric immersion. Consider the deformation of M under mean curvature flow, i.e., consider the one-parameter family $F_t = F(\cdot, t)$ of immersions $F_t : M \to N^{n+d}$ with corresponding images $M_t = F_t(M)$ such that

$$(1.1) \qquad \begin{cases} \frac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(x,0) = F_0(x), \end{cases}$$

where H is the mean curvature vector of M_t.

The mean curvature flow was proposed by Mullins [14] to describe the formation of grain boundaries in annealing metals. In [3], Brakke introduced the motion of a submanifold in the Euclidean space by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, most works have been done on hypersurfaces. For the initial hypersurface satisfying certain convexity condition, Huisken [6, 7] proved that the solution of the mean curvature flow converges to a point as the time approaches the finite maximal time. Later, Huisken [8] investigated the mean curvature flow of a hypersurface in the unit sphere $\mathbb{S}^{n+1}$. He proved that if the initial hypersurface satisfies certain pointwise pinching condition, then either M_t converges to a round point in finite time, or M_t converges to a total geodesic sphere of $\mathbb{S}^{n+1}$ as $t \to \infty$.

For the mean curvature flow of submanifolds with higher codimension, some important results have been obtained by Wang, Smoczyk and many others, see [18, 19, 20, 21, 22, 23] etc. for example. Recently, Andrews-Baker [1, 2] and Liu-Xu-Ye-Zhao [12] proved convergence theorems for the mean curvature flows of submanifolds satisfying certain pinching conditions in space forms. This generalizes the convergence results of mean curvature flow for hypersurfaces due to Huisken [6, 7, 8] to the case of arbitrary codimensions.

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An attractive question is: can one prove the convergence theorem of the mean curvature flow of submanifolds satisfying suitable integral curvature pinching condition? The study of convergence for the mean curvature flow of hypersurfaces with small total curvature was initiated in [28]. In [11], Liu-Xu-Ye-Zhao proved two convergence theorems for the mean curvature flow of closed submanifolds of arbitrary codimension in Euclidean space under suitable integral curvature conditions.

In this paper, we investigate the convergence of mean curvature flow of submanifolds with integral curvature bound in a sphere. In particular, we obtain the following theorem.

Theorem 1.1. *Let $F : M \to \mathbb{S}^{n+d}$ be an n-dimensional $(n \geq 3)$ smooth closed submanifold in the unit sphere with codimension $d \geq 1$. Let F_t be the solution of the mean curvature flow with F as initial value. For any positive number $p \in (n, \infty)$, there exists an explicitly computable positive constant $C_{n,p}$ depending only on n and p, such that if*

$$||A||_{L^p} < C_{n,p},$$

then either M_t converges to a round point in finite time, or M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t \to \infty$.

In Theorem 1.1, A and $||\cdot||_{L^p}$ denote the second fundamental form of a submanifold and the L^p-norm of a tensor or a function, respectively.

Remark 1.2. Let M be a totally umbilical sphere $\mathbb{S}^n\left(\frac{n}{\sqrt{|H|^2+n^2}}\right)$ in $\mathbb{S}^{n+d}$. Then $||A||_{L^p} = \mathrm{Vol}(M)^{\frac{1}{p}} \cdot |A| \leq \mathrm{Vol}(\mathbb{S}^n)^{\frac{1}{p}} \cdot |A|$. If the mean curvature of M satisfies $|H| \leq \frac{\sqrt{n}C_{n,p}}{\mathrm{Vol}(\mathbb{S}^n)^{\frac{1}{p}}}$, then $||A||_{L^p} < C_{n,p}$. Obviously, if $H = 0$, M_t is unchanged along the mean curvature flow, and if $H \neq 0$, M_t shrinks to a round point in finite time. Moreover, we can construct submanifolds from $\mathbb{S}^n\left(\frac{n}{\sqrt{|H|^2+n^2}}\right)$ by small perturbations such that they satisfy $||A||_{L^p} < C_{n,p}$. If the perturbation is small enough, then we can find the submanifold M such that along the mean curvature flow M_t shrinks to a round point in finite time or converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as time tends to infinity.

Let M be an n-dimensional closed submanifold immersed in a complete simply connected $(n+d)$-dimensional space form $\mathbb{F}^{n+d}(c)$ of constant sectional curvature c. The following theorem was proved by K. Shiohama and the second author [16].

Theorem 1.3 ([16])**.** *Let M be an n-dimensional $(n \geq 2)$ smooth closed submanifold in $\mathbb{F}^{n+d}(c)$ with $c \geq 0$. There is an explicitly given positive constant B_n depending only on n such that if $||\mathring{A}||_{L^n} < B_n$, then M is homeomorphic to a sphere.*

In Theorem 1.3, $\mathring{A}$ is the traceless second fundamental form of a submanifold. Motivated by Theorem 1.3, we proposed the following conjecture in [27].

Conjecture 1.4. *Let M be an n-dimensional $(n \geq 2)$ smooth closed submanifold in $F^{n+d}(c)$ with $c \geq 0$. There is an positive constant C_n depending only on n such that if $||\mathring{A}||_{L^n} < C_n$, then M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.*

As a consequence of Theorem 1.1, we have the following differentiable sphere theorem, which can be considered as a partial solution to Conjecture 1.4.

Corollary 1.5. *Let M be an n-dimensional $(n \geq 3)$ smooth closed submanifold in the unit sphere $\mathbb{S}^{n+d}$. For any positive number $p \in (n, \infty)$, there exists an explicitly computable positive constant $C_{n,p}$ depending only on n and p, such that if*

$$||A||_{L^p} < C_{n,p},$$

then M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

At the end of this section, we would like to propose the following conjecture which we will study by developing further the techniques in this paper.

Conjecture 1.6. *Let $F : M \to \mathbb{R}^{n+d}$ be an n-dimensional $(n \geq 2)$ smooth closed submanifold in the Euclidean space with codimension $d \geq 1$. Let F_t be the solution of the mean curvature flow with F as initial value. There exists a positive constant C_n depending only on n, such that if*

$$||H||_{L^n} < n\mathrm{Vol}(\mathbb{S}^n)^{\frac{1}{n}} + C_n,$$

then M_t converges to a round point in finite time. In particular, M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

When $n = 2$ and $d = 1$, Conjecture 1.6 is closely related to the well-known Willmore conjecture.

2. Preliminaries

Let $F : M^n \to N^{n+d}$ be a smooth immersion from an n-dimensional Riemannian manifold M^n without boundary to an $(n+d)$-dimensional Riemannian manifold N^{n+d}. We shall make use of the following convention on the range of indices.

$$1 \leq i, j, k, \cdots \leq n, \quad 1 \leq A, B, C, \cdots \leq n+d, \quad \textit{and} \quad n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+d.$$

Choose a local orthonormal frame field $\{e_A\}$ in N such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$. The metric g and the volume form $d\mu$ of M are $g = \sum_i \omega_i \otimes \omega_i$ and $d\mu = \omega_1 \wedge \cdots \wedge \omega_n$.

For any $x \in M$, denoted by $N_x M$ the normal space of M in N at point x, which is the orthogonal complement of $T_x M$ in $F^* T_{F(x)} N$. Denote by $\bar{\nabla}$ the Levi-Civita connection on N. The Riemannian curvature tensor $\bar{R}$ of N is defined by

$$\bar{R}(U, V)W = -\bar{\nabla}_U \bar{\nabla}_V W + \bar{\nabla}_V \bar{\nabla}_U W + \bar{\nabla}_{[U,V]} W$$

for vector fields U, V and W tangent to N. The induced connection ∇ on M is defined by

$$\nabla_X Y = (\bar{\nabla}_X Y)^{\top}$$

for X, Y tangent to M, where $(\ )^{\top}$ denotes tangential component. Let R be the Riemannian curvature tensor of M.

Given a normal vector field ξ along M, the induced connection $\nabla^{\perp}$ on the normal bundle is defined by

$$\nabla^{\perp}_X \xi = (\bar{\nabla}_X \xi)^{\perp},$$

where $(\ )^{\perp}$ denotes the normal component. Let $R^{\perp}$ denote the normal curvature tensor.

The second fundamental form is defined to be

$$A(X, Y) = (\bar{\nabla}_X Y)^{\perp}$$

as a section of the tensor bundle $T^*M\otimes T^*M\otimes NM$, where T^*M and NM are the cotangential bundle and the normal bundle over M. The mean curvature vector H is the trace of the second fundamental form.

The first covariant derivative of A is defined as

$$(\widetilde{\nabla}_X A)(Y,Z)=\nabla^{\perp}_X A(Y,Z)-A(\nabla_X Y,Z)-A(Y,\nabla_X Z),$$

where $\widetilde{\nabla}$ is the connection on $T^*M\otimes T^*M\otimes NM$. Similarly, we can define the second covariant derivative of A. Under the local orthonormal frame field, the components of A and its first and second covariant derivatives are

$$\begin{aligned}h^{\alpha}_{ij}=&\langle A(e_i,e_j),e_\alpha\rangle,\\ h^{\alpha}_{ijk}=&\langle(\widetilde{\nabla}_{e_k}A)(e_i,e_j),e_\alpha\rangle,\\ h^{\alpha}_{ijkl}=&\langle(\widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k}A)(e_i,e_j),e_\alpha\rangle.\end{aligned}$$

Then A and H can be written as

$$A=\sum_{i,j,\alpha}h^{\alpha}_{ij}\omega_i\otimes\omega_j\otimes e_\alpha,\quad H=\sum_{i,\alpha}h^{\alpha}_{ii}e_\alpha=\sum_{\alpha}H^{\alpha}e_\alpha.$$

The Laplacian of A is defined by $\Delta h^{\alpha}_{ij}=\sum_k h^{\alpha}_{ijkk}$.

We define the traceless second fundamental form $\mathring{A}$ by $\mathring{A}=A-\frac{1}{n}g\otimes H$, whose components are $\mathring{h}^{\alpha}_{ij}=h^{\alpha}_{ij}-\frac{1}{n}H^{\alpha}\delta_{ij}$. Obviously, we have $\sum_i\mathring{h}^{\alpha}_{ii}=0$.

Now we recall evolution equations for some geometric quantities associated with the evolving submanifold in the unit sphere $\mathbb{S}^{n+d}$.

Lemma 2.1 ([1, 2, 21])**.** *Along the mean curvature flow (1.1) where the ambient space is the unit sphere* $\mathbb{S}^{n+d}$*, we have*

$$\begin{aligned}\frac{\partial}{\partial t}d\mu_t=&-H^2d\mu_t;\\ \frac{\partial}{\partial t}|A|^2=&\triangle|A|^2-2|\nabla A|^2+2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^{\alpha}_{ij}h^{\beta}_{ij}\Big)^2\\ &+2\sum_{i,j,\alpha,\beta}\Big[\sum_p\Big(h^{\alpha}_{ip}h^{\beta}_{jp}-h^{\alpha}_{jp}h^{\beta}_{ip}\Big)\Big]^2+4|H|^2-2n|A|^2;\\ \frac{\partial}{\partial t}|H|^2=&\triangle|H|^2-2|\nabla H|^2+2\sum_{i,j}\Big(\sum_\alpha H^{\alpha}h^{\alpha}_{ij}\Big)^2+2n|H|^2.\end{aligned}$$

From an inequality in [10], we have

$$2\sum_{\alpha,\beta}\Big(\sum_{i,j}h^{\alpha}_{ij}h^{\beta}_{ij}\Big)^2+2\sum_{i,j,\alpha,\beta}\Big[\sum_p\Big(h^{\alpha}_{ip}h^{\beta}_{jp}-h^{\alpha}_{jp}h^{\beta}_{ip}\Big)\Big]^2\le 3|A|^4.$$

Then we have the following inequality.

$$\frac{\partial}{\partial t}|A|^2\le\triangle|A|^2-2|\nabla A|^2+3|A|^4+4|H|^2-2n|A|^2. \tag{2.1}$$

By the Schwarz inequality, we have

$$\begin{aligned}\sum_{i,j}\Big(\sum_\alpha H^{\alpha}h^{\alpha}_{ij}\Big)^2\le&\sum_{i,j}\Big(\sum_\alpha(H^{\alpha})^2\Big)\Big(\sum_\alpha(h^{\alpha}_{ij})^2\Big)\\ =&\sum_\alpha(H^{\alpha})^2\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2=|A|^2|H|^2.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 \leq & \triangle |H|^2 - 2|\nabla H|^2 + 2|A|^2|H|^2 + 2n|H|^2 \\ = & \triangle |H|^2 - 2|\nabla H|^2 + 2|\mathring{A}|^2|H|^2 + \frac{2}{n}|H|^4 + 2n|H|^2.\end{aligned} \tag{2.2}$$

From Lemma 2.1, we have the evolution equation of $|\mathring{A}|^2$.

$$\begin{aligned}\frac{\partial}{\partial t}|\mathring{A}|^2 = & \triangle|\mathring{A}|^2 - 2|\nabla \mathring{A}|^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\Big)^2 \\ & + 2\sum_{i,j,\alpha,\beta}\Big[\sum_p \Big(h^\alpha_{ip}h^\beta_{jp} - h^\alpha_{jp}h^\beta_{ip}\Big)\Big]^2 - \frac{2}{n}\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2 \\ & - 2n|\mathring{A}|^2.\end{aligned} \tag{2.3}$$

At the point where $H \neq 0$, we choose $\{\nu_\alpha\}$ such that $e_{n+1} = \frac{H}{|H|}$. Let $A_H = \sum_{i,j} h^{n+1}_{ij}\omega_i \otimes \omega_j$. Set $\mathring{A}_H = A_H - \frac{|H|}{n}\mathrm{Id}$ and $|\mathring{A}_I|^2 = |\mathring{A}|^2 - |\mathring{A}_H|^2$. As in [1], we have the following estimate.

(2.4)

$$\begin{aligned}& 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big[\sum_p \Big(h^\alpha_{ip}h^\beta_{jp} - h^\alpha_{jp}h^\beta_{ip}\Big)\Big]^2 - \frac{2}{n}\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2 \\ & \leq 2|\mathring{A}_H|^4 + \frac{2}{n}|\mathring{A}_H|^2|H|^2 + 8|\mathring{A}_H|^2|\mathring{A}_I|^2 + 3|\mathring{A}_I|^4 \\ & \leq 2|A|^2|\mathring{A}|^2 + 11|\mathring{A}|^4.\end{aligned}$$

Combining (2.3) and (2.4) we obtain

$$\frac{\partial}{\partial t}|\mathring{A}|^2 \leq \triangle|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + 13|A|^2|\mathring{A}|^2. \tag{2.5}$$

Note that this inequality also holds at the point where $H = 0$.

3. A Sobolev inequality for submanifolds in a sphere

Firstly we recall the well-know Michael-Simon inequality.

Lemma 3.1 ([13]). *Let M^n $(n \geq 2)$ be a compact submanifold with or without boundary in the Euclidean space $\mathbb{R}^{n+d}$ with $d \geq 1$. For a nonnegative function $h \in C^1(M)$ such that $h|_{\partial M} = 0$ if $\partial M \neq \emptyset$, we have*

$$\left(\int_M h^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq c_n \int_M (|\nabla h| + |H|h) d\mu, \tag{3.1}$$

where $c_n = 4^{n+1}\sigma_n^{-1/n}$ and σ_n is the volume of the unit ball in $\mathbb{R}^n$.

An improvement of the constant c_n in Lemma 3.1 was given in [15]. We derive a Sobolev type inequality in a proper form, which will be used in the proof of our theorems.

Lemma 3.2. *Let M be an n-dimensional $(n \geq 3)$ closed submanifold in $\mathbb{S}^{n+d}$. Then for all Lipschitz functions v on M and all $\alpha \geq \alpha_0 > n$, we have*

$$||v||^2_{L^{\frac{2n}{n-2}}(M)} \leq C_{n,\alpha_0}\left(||\nabla v||^2_{L^2(M)} + \Big(1 + ||H||^{\frac{2\alpha}{\alpha-n}}_{L^\alpha(M)}\Big)||v||^2_{L^2(M)}\right),$$

where C_{n,α_0} is a positive constant depending only on n and α_0.

Proof. Since a Lipschitz function is differentiable almost everywhere, we only have to prove the lemma for $v \in C^1(M)$ and $v \geq 0$. We consider the composition of isometric immersions $M^n \to \mathbb{S}^{n+d} \subset \mathbb{R}^{n+d+1}$. Denote by $\bar{H}$ the mean curvature vector of M as a submanifold in $\mathbb{R}^{n+d+1}$. Then we have $|\bar{H}|^2 = |H|^2 + n^2$. By Lemma 3.1, we have for any nonnegative function $h \in C^1(M)$,

$$\begin{aligned}\left(\int_M h^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq & c_n \int_M (|\nabla h| + |\bar{H}|h) d\mu \\ \leq & c_n \int_M (|\nabla h| + (n+|H|)h) d\mu.\end{aligned} \tag{3.2}$$

Let $h = v^{\frac{2(n-1)}{n-2}}$ in (3.2), we have

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-1}{n}} \leq c_n \left(\int_M |\nabla v| v^{\frac{n}{n-2}} d\mu + \int_M (n+|H|) v^{\frac{2(n-1)}{n-2}} d\mu\right).$$

Denote by V the volume of M and let $Q = \frac{n}{n-2}$. By the Hölder inequality, we have for $\alpha, \alpha' \geq \alpha_0 > n$

$$\begin{aligned}\left(\int_M v^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leq & c_n^{\frac{n-2}{n-1}} \left(\int_M |\nabla v| v^{\frac{n}{n-2}} d\mu + \int_M (n+|H|) v^{\frac{2(n-1)}{n-2}} d\mu\right)^{\frac{n-2}{n-1}} \\ \leq & \tilde{c}_n \Bigg(||\nabla v||_{L^2(M)} ||v||^{\frac{n}{n-2}}_{L^{2Q}(M)} + ||H||_{L^\alpha(M)} ||v||^{\frac{2(n-1)}{n-2}}_{L^{2m}(M)} \\ & + V^{\frac{1}{\alpha'}} ||v||^{\frac{2(n-1)}{n-2}}_{L^{2m'}(M)} \Bigg)^{\frac{n-2}{n-1}} \\ \leq & \tilde{c}_n \Bigg(||\nabla v||^{\frac{n-2}{n-1}}_{L^2(M)} ||v||^{\frac{n}{n-1}}_{L^{2Q}(M)} + ||H||^{\frac{n-2}{n-1}}_{L^\alpha(M)} ||v||^2_{L^{2m}(M)} \\ & + V^{\frac{1}{\alpha'} \cdot \frac{n-2}{n-1}} ||v||^2_{L^{2m'}(M)} \Bigg).\end{aligned}$$

Here $\tilde{c}_n = n^{\frac{n-2}{n-1}} c_n^{\frac{n-2}{n-1}}$, $m = \frac{(n-1)\alpha}{(n-2)(\alpha-1)} < \frac{n}{n-2} = Q$ and $m' = \frac{(n-1)\alpha'}{(n-2)(\alpha'-1)} < \frac{n}{n-2} = Q$. Hence

$$\begin{aligned}||v||^2_{L^{2Q}(M)} \leq & \tilde{c}_n \Bigg(||\nabla v||^{\frac{n-2}{n-1}}_{L^2(M)} ||v||^{\frac{n}{n-1}}_{L^{2Q}(M)} \\ & + ||H||^{\frac{n-2}{n-1}}_{L^\alpha(M)} ||v||^2_{L^{2m}(M)} + V^{\frac{1}{\alpha'} \cdot \frac{n-2}{n-1}} ||v||^2_{L^{2m'}(M)} \Bigg).\end{aligned} \tag{3.3}$$

By using Young's inequality

$$ab \leq \varepsilon a^p + \varepsilon^{-\frac{q}{p}} b^q,$$

for $a,\ b,\ \varepsilon > 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, with

$$a = ||v||^{\frac{n}{n-1}}_{L^{2Q}(M)},\ b = ||\nabla v||^{\frac{n-2}{n-1}}_{L^2(M)},\ \varepsilon = \frac{1}{2\tilde{c}_n},\ p = \frac{2(n-1)}{n},\ q = \frac{2(n-1)}{n-2},$$

we obtain from (3.3)

$$\begin{aligned}||v||^2_{L^{2Q}(M)} \leq &\tilde{c}_n\left(\frac{1}{2\tilde{c}_n}||v||^2_{L^{2Q}(M)} + \left(\frac{1}{2\tilde{c}_n}\right)^{-\frac{n}{n-2}}||\nabla v||^2_{L^2(M)}\right.\\ &\left.+||H||^{\frac{n-2}{n-1}}_{L^{\alpha}(M)}||v||^2_{L^{2m}(M)} + V^{\frac{1}{\alpha'}\cdot\frac{n-2}{n-1}}||v||^2_{L^{2m'}(M)}\right).\end{aligned}$$

This implies

$$\begin{aligned}||v||^2_{L^{2Q}(M)} \leq &\hat{c}_n\left(||\nabla v||^2_{L^2(M)}\right.\\ &\left.+||H||^{\frac{n-2}{n-1}}_{L^{\alpha}(M)}||v||^2_{L^{2m}(M)} + V^{\frac{1}{\alpha'}\cdot\frac{n-2}{n-1}}||v||^2_{L^{2m'}(M)}\right).\end{aligned}\tag{3.4}$$

Here $\hat{c}_n = (2\tilde{c}_n)^{\frac{2(n-1)}{n-2}}$. Recall the interpolation inequality

$$||u||_{L^r(M)} \leq \varepsilon||u||_{L^s(M)} + \varepsilon^{-\mu}||u||_{L^t(M)},$$

where $t < r < s$, $\mu = \frac{\frac{1}{t}-\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}}$. Since $1 < m < Q$, $1 < m' < Q$, we have

$$\begin{aligned}||v||_{L^{2m}(M)} &\leq \varepsilon||v||_{L^{2Q}(M)} + \varepsilon^{-\gamma}||v||_{L^2(M)},\\ ||v||_{L^{2m'}(M)} &\leq \varepsilon'||v||_{L^{2Q}(M)} + \varepsilon'^{-\gamma'}||v||_{L^2(M)}.\end{aligned}$$

Here $\varepsilon, \varepsilon' > 0$ to be determined, $\gamma = \frac{Q(m-1)}{Q-m}$ and $\gamma' = \frac{Q(m'-1)}{Q-m'}$. So we obtain from (3.4)

$$\begin{aligned}||v||^2_{L^{2Q}(M)} \leq &\hat{c}_n||\nabla v||^2_{L^2(M)} + \hat{c}_n||H||^{\frac{n-2}{n-1}}_{L^{\alpha}(M)}\left(\varepsilon||v||_{L^{2Q}(M)} + \varepsilon^{-\gamma}||v||_{L^2(M)}\right)^2\\ &+\hat{c}_nV^{\frac{1}{\alpha'}\cdot\frac{n-2}{n-1}}\left(\varepsilon'||v||_{L^{2Q}(M)} + \varepsilon^{-\gamma'}||v||_{L^2(M)}\right)^2\\ \leq &\hat{c}_n||\nabla v||^2_{L^2(M)} + \hat{c}_n||H||^{\frac{n-2}{n-1}}_{L^{\alpha}(M)}\left(\varepsilon^2||v||^2_{L^{2Q}(M)} + \varepsilon^{-2\gamma}||v||^2_{L^2(M)}\right)\\ &+\hat{c}_nV^{\frac{1}{\alpha'}\cdot\frac{n-2}{n-1}}\left(\varepsilon'^2||v||^2_{L^{2Q}(M)} + \varepsilon'^{-2\gamma'}||v||^2_{L^2(M)}\right).\end{aligned}\tag{3.5}$$

Now set $\varepsilon^2 = \frac{1}{4\hat{c}_n}||H||^{-\frac{n-2}{n-1}}_{L^{\alpha}(M)}$ and $\varepsilon'^2 = \frac{1}{4\hat{c}_n}V^{-\frac{1}{\alpha'}\cdot\frac{n-2}{n-1}}$. Then from (3.5) we have

$$\begin{aligned}||v||^2_{L^{2Q}(M)} \leq &C_n||\nabla v||^2_{L^2(M)} + C_n^{\gamma+1}||H||^{\frac{(n-2)(1+\gamma)}{n-1}}_{L^{\alpha}(M)}||v||^2_{L^2(M)}\\ &+C_n^{\gamma'+1}V^{\frac{(n-2)(1+\gamma')}{(n-1)\alpha'}}||v||^2_{L^2(M)}.\end{aligned}\tag{3.6}$$

Here $C_n = 4\hat{c}_n \geq 1$. Notice that $C_n^{\gamma+1}$ and $C_n^{\gamma'+1}$ are decreasing functions with respect to α and α', respectively. Then we have $C_n^{\gamma+1} \leq C_n^{\gamma_0+1}$ and $C_n^{\gamma'+1} \leq C_n^{\gamma_0+1}$, where $\gamma_0 = \frac{n(\alpha_0+n-2)}{(n-2)(\alpha_0-n)}$. Set $C_{n,\alpha_0} = C_n^{\gamma_0+1}$. Letting $\alpha' \to +\infty$, we obtain from (3.6)

$$||v||^2_{L^{2Q}(M)} \leq C_{n,\alpha_0}||\nabla v||^2_{L^2(M)} + C_{n,\alpha_0}\left(1 + ||H||^{\frac{(n-2)(1+\gamma)}{n-1}}_{L^{\alpha}(M)}\right)||v||^2_{L^2(M)}.\tag{3.7}$$

Substituting $\gamma = \frac{Q(m-1)}{Q-m}$ and $m = \frac{(n-1)\alpha}{(n-2)(\alpha-1)}$ into (3.7), we obtain

$$||v||^2_{L^{2Q}(M)} \leq C_{n,\alpha_0}\left(||\nabla v||^2_{L^2(M)} + \left(1 + ||H||^{\frac{2\alpha}{\alpha-n}}_{L^\alpha(M)}\right)||v||^2_{L^2(M)}\right).$$

This completes the proof of Lemma 3.2. □

4. The extension of the mean curvature flow

In this section we investigate the extension of the mean curvature flow under finite integral curvature condition. Huisken [6, 7] and Wang [21] showed that if the second fundamental form is uniformly bounded in $[0,T)$, then the solution can be extended to $[0,T+\omega)$ for some $\omega > 0$. In [9, 11, 25, 26], the integral condition conditions that assure the extension of the mean curvature flow were investigated, respectively.

Now we prove another integral condition sufficiently strong to extend the mean curvature flow. Recall that a Riemannian manifold is said to have bounded geometry if (i) the sectional curvature is bounded; (ii) the injective radius is bounded from below by a positive constant.

Lemma 4.1. *Let $F : M \times [0,T) \to N$ be a mean curvature flow solution with compact initial value on a finite time interval $[0,T)$, where N has bounded geomtry. If $\int_{M_t} |A|^p d\mu_t$ is bounded in $[0,T)$ for some $n < p < +\infty$, then the solution can be extended to $[0,T+\omega)$ for some $\omega > 0$.*

Proof. We argue by contradiction.

Suppose that $T(< +\infty)$ is the maximal existence time. Firstly we choose a sequence of time $t^{(i)}$ such that $\lim_{i\to\infty} t^{(i)} = T$. Then we take a sequence of points $x^{(i)} \in M$ satisfying

$$|A|^2(x^{(i)}, t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)})} |A|^2(x,t),$$

where $\lim_{i\to\infty} |A|^2(x^{(i)}, t^{(i)}) = +\infty$.

Putting $Q^{(i)} = |A|^2(x^{(i)}, t^{(i)})$, we consider the rescaling mean curvature flow

$$F^{(i)}(t) = F\left(\frac{t}{Q^{(i)}} + t^{(i)}\right) : (M, g^{(i)}(t)) \to (N, Q^{(i)}h),$$

where h is the metric on N. Then the induced metric on M by the immersion $F^{(i)}(t)$ is $g^{(i)}(t) = Q^{(i)}g\left(\frac{t}{Q^{(i)}} + t^{(i)}\right)$, $t \in (-Q^{(i)}t^{(i)}, 0]$. For $(M, g^{(i)}(t))$, the second fundamental form $|A^{(i)}|(x,t) \leq 1$ for any i.

From [4], there exists a subsequence of $(M, g^{(i)}(t), x^{(i)})$ that converges to a Riemannian manifold $(\overline{M}, \overline{g}(t), \overline{x})$, $t \in (-\infty, 0]$, and the corresponding subsequence of immersions $F^{(i)}(t)$ converges to an immersion $\overline{F}(t) : \overline{M} \to \mathbb{R}^{n+d}$. Then we have

$$\begin{aligned}
\int_{B_{\overline{g}(0)}(\overline{x},1)} |\overline{A}|^p_{\overline{g}(0)} d\overline{\mu}_{\overline{g}(0)} \leq &\lim_{i\to\infty} \int_{B_{g^{(i)}(0)}(x^{(i)},1)} |A|^p_{g^{(i)}(0)} d\mu_{g^{(i)}(0)} \\
=&\lim_{i\to\infty} \frac{1}{(Q^i)^{\frac{p-n}{2}}} \cdot \int_{B_{g(t^{(i)})}(x^{(i)},(Q^{(i)})^{-\frac{1}{2}})} |A|^p_{g(t^{(i)})} d\mu_{g(t^{(i)})} \\
=&0.
\end{aligned} \tag{4.1}$$

The last equality in (4.1) holds since $\lim_{t\to T}\int_M |A|^p d\mu < +\infty$ and $Q^i \to \infty$ as $i \to \infty$. The equality (4.1) implies that $|\overline{A}| \equiv 0$ on the ball $B_{\overline{g}(0)}(\overline{x},1)$. In particular, $|\overline{A}|(\overline{x},0)=0$. On the other hand, the points selecting process implies that

$$|\overline{A}|(\overline{x},0)=\lim_{i\to\infty}|A|_{g^{(i)}}(x^{(i)},0)=1.$$

The contradiction completes the proof. □

Remark 4.2. Consider the totally umbilical spheres in a complete simply connected space form $\mathbb{F}^{n+d}(c)$ with constant curvature c. Suppose the totally umbilical sphere satisfies $|H|^2+n^2c>0$. Then along the mean curvature flow, these totally umbilical spheres remain totally umbilical, and converge to a round point in finite time. On the other hand, it is easy to check that $\int_{M_t}|A|^n d\mu_t$ is uniformly bounded along the mean curvature flow. From these examples we see that the condition $p>n$ in Lemma 4.1 is optimal.

5. The convergence of the mean curvature flow

In this section, we always assume that F_t is the solution of the mean curvature flow of a submanifold in the unit sphere $\mathbb{S}^{n+d}$. We first prove the following theorem.

Theorem 5.1. *Let $F_0 : M^n \to \mathbb{S}^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold. For given positive numbers $p\in(n,\infty)$ and $q\in(1,\infty)$, there is a positive constant C_1 depending on n, p, q, the upper bound Λ on the L^p-norm of the second fundamental form of the submanifold, such that if*

$$||\mathring{A}||_{L^q(M_0)} < C_1,$$

then the mean curvature flow with F_0 as initial value has a unique solution $F : M\times[0,T)\to\mathbb{S}^{n+d}$, and either

(1) *$T<\infty$ and M_t converges to a round point as $t\to T$; or*

(2) *$T=\infty$ and M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t\to\infty$.*

To prove Theorem 5.1, we need some lemmas.

Lemma 5.2. *If $||A||_{L^p(M_0)} \leq \Lambda$ for some $p>n$ at $t=0$, then there is $T_1>0$ depending only on n,p,Λ such that there holds $||A||_{L^p(M_t)}\leq 2\Lambda$ for $t\in[0,T_1]$.*

Proof. Putting $u=|A|^2$, we obtain from (2.1)

$$\frac{\partial}{\partial t}u \leq \Delta u + 3u^2 + 2nu. \tag{5.1}$$

From (5.1), we have

(5.2)
$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} u^{\frac{p}{2}}d\mu_t =& \int_{M_t}\frac{p}{2}u^{\frac{p}{2}-1}\frac{\partial}{\partial t}u d\mu_t + \int_{M_t}u^{\frac{p}{2}}\frac{\partial}{\partial t}d\mu_t\\ =&\frac{p}{2}\int_{M_t}u^{\frac{p}{2}-1}(\Delta u + c_1u^2)d\mu_t - \int_{M_t}H^2u^{\frac{p}{2}}d\mu_t\\ \leq& -\frac{4(p-2)}{p}\int_{M_t}|\nabla u^{\frac{p}{4}}|^2d\mu_t + \frac{3p}{2}\int_{M_t}u^{\frac{p}{2}+1}d\mu_t + np\int_{M_t}u^{\frac{p}{2}}d\mu_t.\end{aligned}$$

For the second term of the right hand side of (5.2), we have by Hölder's inequality and Sobolev type inequality in Lemma 3.2,

(5.3)
$$\begin{aligned}
\int_{M_t} u^{\frac{p}{2}+1} d\mu_t \leq & \Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{2}{p}} \cdot \Big(\int_{M_t} (u^{\frac{p}{2}})^{\frac{p}{p-2}} d\mu_t\Big)^{\frac{p-2}{p}} \\
\leq & \Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{2}{p}} \cdot \Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n}{p}} \cdot \Big(\int_{M_t} (u^{\frac{p}{4}})^{\frac{2n}{n-2}} d\mu_t\Big)^{\frac{n-2}{p}} \\
\leq & \Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{2}{p}} \cdot \Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n}{p}} \\
& \times \Big\{C_{n,p}\Big(\int_{M_t} |\nabla u^{\frac{p}{4}}|^2 d\mu_t + \Big[1 + \Big(\int_{M_t} |H|^p d\mu_t\Big)^{\frac{2}{p-n}}\Big] \int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)\Big\}^{\frac{n}{p}} \\
\leq & \Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p}} \cdot \Big[C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} |\nabla u^{\frac{p}{4}}|^2 d\mu_t\Big)^{\frac{n}{p}} \\
& + C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{n}{p}} + n^{\frac{n}{p-n}} C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{(1+\frac{2}{p-n})\frac{n}{p}}\Big] \\
= & C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p+2}{p}} + n^{\frac{n}{p-n}} C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p}+(1+\frac{2}{p-n})\frac{n}{p}} \\
& + C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p}} \cdot \Big(\int_{M_t} |\nabla u^{\frac{p}{4}}|^2 d\mu_t\Big)^{\frac{n}{p}} \\
\leq & C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p+2}{p}} + n^{\frac{n}{p-n}} C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p-n}} \\
& + C_{n,p}^{\frac{n}{p}} \frac{p-n}{p} \epsilon^{\frac{p}{p-n}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p-n}} + C_{n,p}^{\frac{n}{p}} \frac{n}{p} \epsilon^{-\frac{p}{n}}\Big(\int_{M_t} |\nabla u^{\frac{p}{4}}|^2 d\mu_t\Big),
\end{aligned}$$

for any $\epsilon > 0$.

Combining (5.2) and (5.3), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} u^{\frac{p}{2}} d\mu_t \leq & \Big(\frac{3n}{2} C_{n,p}^{\frac{n}{p}} \epsilon^{-\frac{p}{n}} - \frac{4(p-2)}{p}\Big)\int_{M_t} |\nabla u^{\frac{p}{4}}|^2 d\mu_t \\
& + np\int_{M_t} u^{\frac{p}{2}} d\mu_t + \frac{3p}{2} C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p+2}{p}} \\
& + \frac{3p}{2}\Big(n^{\frac{n}{p-n}} C_{n,p}^{\frac{n}{p}} + C_{n,p}^{\frac{n}{p}} \frac{p-n}{p} \epsilon^{\frac{p}{p-n}}\Big)\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p-n}}.
\end{aligned} \tag{5.4}$$

Pick $\epsilon = \Big(\frac{3npC_{n,p}^{\frac{n}{p}}}{8(p-2)}\Big)^{\frac{n}{p}}$. Then (5.4) reduces to

(5.5)
$$\frac{\partial}{\partial t}\int_{M_t} u^{\frac{p}{2}} d\mu_t \leq np\int_{M_t} u^{\frac{p}{2}} d\mu_t + \frac{3p}{2} C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p+2}{p}} + c_1\Big(\int_{M_t} u^{\frac{p}{2}} d\mu_t\Big)^{\frac{p-n+2}{p-n}},$$

where $c_1 = \frac{3p}{2}\Big(n^{\frac{n}{p-n}}C_{n,p}^{\frac{n}{p}} + C_{n,p}^{\frac{n}{p}}\frac{p-n}{p}\Big(\frac{3npC_{n,p}^{\frac{n}{p}}}{8(p-2)}\Big)^{\frac{n}{p-n}}\Big)$. Then from the maximum principle and Lemma 4.1, there exists a positive constant T_1 depending only on n, p, Λ such that the mean curvature is smooth on $[0, T_1]$ and $||A||_{L^p(M_t)} \le 2\Lambda$ for $t \in [0, T_1]$. This completes the proof of the lemma. $\square$

Lemma 5.3. *There exists a constant $T_2 \in (0, T_1]$ depending only on n, p, q, Λ such that if $||\mathring{A}||_{L^q(M_0)} < \varepsilon$ at $t = 0$, then there holds $||\mathring{A}||_{L^q(M_t)} \le 2\varepsilon$ for $t \in [0, T_2]$.*

Proof. From Lemmas 3.2 and 5.2 we have for a Lipschitz function v and $t \in [0, T_1]$,

$$||v||^2_{L^{\frac{2n}{n-2}}(M_t)} \le C_{n,p}\Big(||\nabla v||^2_{L^2(M_t)} + \Big(1 + n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)||v||^2_{L^2(M_t)}\Big), \tag{5.6}$$

where $C_{n,p}$ is a positive constant depending only on n and p.

Define a tensor $\tilde{\mathring{A}}$ by $\tilde{\mathring{h}}^\alpha_{ij} = \mathring{h}^\alpha_{ij} + \sigma\eta^\alpha\delta_{ij}$, where $\eta^\alpha = 1$. Set $h_\sigma = |\tilde{\mathring{A}}| = (|\mathring{A}|^2 + nd\sigma^2)^{\frac{1}{2}}$. Then from (2.5), we have

$$\frac{\partial}{\partial t}h_\sigma \le \triangle h_\sigma + 13|A|^2h_\sigma. \tag{5.7}$$

For any $r \ge q > 1$, we have

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t}h^r_\sigma d\mu_t &= \int_{M_t}h^{r-1}_\sigma\frac{\partial}{\partial t}h_\sigma d\mu_t + \frac{1}{r}\int_{M_t}h^p_\sigma\frac{\partial}{\partial t}d\mu_t\\ &\le -\frac{4(r-1)}{r^2}\int_{M_t}|\nabla h^{\frac{r}{2}}_\sigma|^2d\mu_t + 13\int_{M_t}|A|^2h^r_\sigma d\mu_t.\end{aligned} \tag{5.8}$$

For the second term of the right hand side of (5.8), we have the following estimate.

$$\begin{aligned}\int_{M_t}|A|^2h^r_\sigma d\mu_t \le& \Big(\int_{M_t}|A|^pd\mu_t\Big)^{\frac{2}{p}}\cdot\Big(\int_{M_t}h_\sigma^{r\cdot\frac{p}{p-2}}d\mu_t\Big)^{\frac{p-2}{p}}\\ \le&(2\Lambda)^2\Big(\int_{M_t}h^r_\sigma d\mu_t\Big)^{\frac{p-n}{p}}\cdot\Big(\int_{M_t}(h^r_\sigma)^{\frac{n}{n-2}}d\mu_t\Big)^{\frac{n-2}{n}\cdot\frac{n}{p}}\\ \le&(2\Lambda)^2\Big(\int_{M_t}h^r_\sigma d\mu_t\Big)^{\frac{p-n}{p}}\cdot\Big[C_{n,p}\Big(\int_{M_t}|\nabla h^{\frac{r}{2}}_\sigma|^2d\mu_t\\ &+\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)\int_{M_t}h^r_\sigma d\mu_t\Big)\Big]^{\frac{n}{p}}\\ \le&(2\Lambda)^2\Big(\int_{M_t}h^r_\sigma d\mu_t\Big)^{\frac{p-n}{p}}\cdot\Big[C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t}|\nabla h^{\frac{r}{2}}_\sigma|^2d\mu_t\Big)^{\frac{n}{p}}\\ &+C_{n,p}^{\frac{n}{p}}\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\Big(\int_{M_t}h^r_\sigma d\mu_t\Big)^{\frac{n}{p}}\Big]\end{aligned} \tag{5.9}$$

$$
\begin{aligned}
=&(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\int_{M_t} h_\sigma^r d\mu_t\\
&+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\bigg(\int_{M_t} h_\sigma^r d\mu_t\bigg)^{\frac{p-n}{p}}\cdot\bigg(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\bigg)^{\frac{n}{p}}\\
\le&(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\int_{M_t} h_\sigma^r d\mu_t\\
&+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{p-n}{p}\mu^{\frac{p}{p-n}}\int_{M_t} h_\sigma^r d\mu_t\\
&+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}\mu^{-\frac{p}{n}}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t,
\end{aligned}
$$

for any $\mu > 0$.

Then from (5.8) and (5.9) we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t \le&\bigg(13r\cdot(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}\mu^{-\frac{p}{n}}-\frac{4(r-1)}{r}\bigg)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\\
&+13r\cdot\bigg((2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\\
&+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{p-n}{p}\mu^{\frac{p}{p-n}}\bigg)\int_{M_t} h_\sigma^r d\mu_t.
\end{aligned}
\tag{5.10}
$$

Pick $\mu = \Big(\frac{13r^2\cdot(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}}{3(r-1)}\Big)^{\frac{n}{p}}$. Then from (5.10) we have

$$
\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t + (1-\frac{1}{q})\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \le c_2 r^{1+\frac{n}{p-n}}\int_{M_t} h_\sigma^r d\mu_t, \tag{5.11}
$$

where $c_2 = 13\bigg((2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\cdot\frac{1}{q^{1+\frac{n}{p-n}}}+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{p-n}{p}\Big(\frac{13q\cdot(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}}{3(q-1)}\Big)^{\frac{n}{p-n}}\bigg)$.

Let $r = q$, then we have from (5.11)

$$
\frac{\partial}{\partial t}\int_{M_t} h_\sigma^q d\mu_t \le c_2 q^{1+\frac{n}{p-n}}\int_{M_t} h_\sigma^q d\mu_t,
$$

which implies that

$$
\int_{M_t} h_\sigma^q d\mu_t \le (2\varepsilon)^q
$$

for $t \le \min\{T_1, \frac{q\ln 2}{c_2 q^{\frac{p}{p-n}}}\}$. Setting $T_2 = \min\{T_1, \frac{q\ln 2}{c_2 q^{\frac{p}{p-n}}}\}$ and letting $\sigma \to 0$, we complete the proof of the lemma. ☐

Lemma 5.4. *For any $t \in (0, T_2]$, we have*

$$
|\mathring{A}|^2 \le \Big(1+\frac{2}{n}\Big)^{\frac{np(n+2)}{4q(p-n)}} c_3^{\frac{n}{q}}\Big(c_2 q^{\frac{2n}{p-n}+1}+\frac{(n+2)^2}{2nt}\Big)^{\frac{n+2}{q}}\Big(\int_0^t\int_{M_t}|\mathring{A}|^q d\mu_t dt\Big)^{\frac{2}{q}}, \tag{5.12}
$$

for some positive constants c_2 and c_3 depending only n n, p, q and Λ.

Proof. Fix $t_0 \in (0, T_2]$. For any τ, τ' such that $0 < \tau < \tau' < t_0$, define a function ψ on $[0, t_0]$ by

$$
\psi(t) = \begin{cases} 0 & 0 \le t \le \tau,\\ \frac{t-\tau}{\tau'-\tau} & \tau \le t \le \tau',\\ 1 & \tau' \le t \le t_0. \end{cases}
$$

Then from (5.11), we have

$$\frac{\partial}{\partial t}\Big(\psi\int_{M_t} f^q d\mu_t\Big)d\mu_t+\Big(1-\frac{1}{p}\Big)\psi\int_{M_t}|\nabla(f^{\frac{q}{2}})|^2 d\mu_t\le(c_2 r^{\frac{p}{p-n}}\psi+\psi')\int_{M_t} f^q d\mu_t. \tag{5.13}$$

For any $t\in[\tau',t_0]$, integrating both side of (5.13) on $[\tau,t]$ implies

$$\int_{M_t} h_\sigma^r d\mu_t+\Big(1-\frac{1}{p}\Big)\int_{\tau'}^{t}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t dt\le\Big(c_2 r^{\frac{p}{p-n}}+\frac{1}{\tau'-\tau}\Big)\int_{\tau}^{t_0}\int_{M_t} h_\sigma^r d\mu_t dt. \tag{5.14}$$

On the other hand, by the Sobolev inequality we have

$$\begin{aligned}
&\int_{\tau'}^{t_0}\int_{M_t} h_\sigma^{r(1+\frac{2}{n})} d\mu_t dt\\
\le&\int_{\tau'}^{t_0}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n}}\cdot\Big(\int_{M_t} h_\sigma^{\frac{nr}{n-2}} d\mu_t\Big)^{\frac{n-2}{n}} dt\\
\le&\max_{t\in[\tau',t_0]}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n}}\cdot\int_{\tau'}^{t_0}\Big(\int_{M_t} h_\sigma^{\frac{nr}{n-2}} d\mu_t\Big)^{\frac{n-2}{n}} dt\\
\le& C_{n,p}\cdot\max_{t\in[\tau',t_0]}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n}}\int_{\tau'}^{t_0}\Big(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\\
&+\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)\int_{M_t} h_\sigma^r d\mu_t\Big)dt.
\end{aligned} \tag{5.15}$$

From (5.14) and (5.15), we have

$$\begin{aligned}
\int_{\tau'}^{t_0}\int_{M_t} h_\sigma^{r(1+\frac{2}{n})} d\mu_t dt\le& c_3\Big(c_2 r^{\frac{2n}{p-n}+1}+\frac{1}{\tau'-\tau}\Big)^{1+\frac{2}{n}}\\
&\times\Big(\int_{\tau}^{t_0}\int_{M_t} h_\sigma^r d\mu_t dt\Big)^{1+\frac{2}{n}},
\end{aligned} \tag{5.16}$$

where $c_3=C_{n,p}\cdot\max\{\frac{q}{q-1},\Big(1+n^{\frac{p}{p-n}}(2\Lambda)^{\frac{2p}{p-n}}\Big)T_2\}$.

We put

$$J(r,t)=\int_t^{t_0}\int_{M_t} h_\sigma^r d\mu_t dt.$$

Then from (5.16) we have

$$J\Big(r\Big(1+\frac{2}{n}\Big),\tau'\Big)\le c_3\Big(c_2 r^{\frac{p}{p-n}}+\frac{1}{\tau'-\tau}\Big)^{1+\frac{2}{n}} J(r,\tau)^{1+\frac{2}{n}}. \tag{5.17}$$

We let

$$\mu=1+\frac{2}{n},\quad r_k=q\mu^k,\quad \tau_k=\Big(1-\frac{1}{\mu^{k+1}}\Big)t.$$

Notice that $\mu>1$. From (5.17) we have

$$J(r_{k+1},\tau_{k+1})^{\frac{1}{r_{k+1}}}\le c_3^{\frac{1}{r_{k+1}}}\Big(c_2 q^{\frac{p}{p-n}}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}\Big)^{\frac{1}{r_k}}\mu^{\frac{k}{r_k}\cdot\frac{p}{p-n}} J(r_k,\tau_k)^{\frac{1}{r_k}}.$$

Hence

$$J(r_{m+1},\tau_{m+1})^{\frac{1}{r_{m+1}}} \leq c_3^{\sum_{k=0}^{m}\frac{1}{r_{k+1}}}\left(c_2 q^{\frac{p}{p-n}}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}\right)^{\sum_{k=0}^{m}\frac{1}{r_k}} \cdot \mu^{\frac{p}{p-n}\cdot\sum_{k=0}^{m}\frac{k}{r_k}} J(p,t)^{\frac{1}{p}}.$$

As $m\to+\infty$, we conclude that
(5.18)

$$h_\sigma(x,t)\leq\left(1+\frac{2}{n}\right)^{\frac{np(n+2)}{4q(p-n)}} c_3^{\frac{n}{2q}}\left(c_2q^{\frac{p}{p-n}}+\frac{(n+2)^2}{2nt}\right)^{\frac{n+2}{2q}}\left(\int_0^{t_0}\int_{M_t}h_\sigma^q d\mu_t dt\right)^{\frac{1}{q}}.$$

Now let $\sigma\to0$. Then (5.18) implies

$$|\mathring{A}|^2\leq\left(1+\frac{2}{n}\right)^{\frac{np(n+2)}{2q(p-n)}} c_3^{\frac{n}{q}}\left(c_2q^{\frac{2n}{p-n}+1}+\frac{(n+2)^2}{2nt}\right)^{\frac{n+2}{q}}\left(\int_0^{t_0}\int_{M_t}h_\sigma^q d\mu_t dt\right)^{\frac{2}{q}}.$$

Since $t_0\in(0,T_2]$ is arbitrary, we complete the proof of the Lemma. □

Now we give the proof of Theorem 5.1.

Proof of Theorem 5.1. We consider the submanifold M_{T_2}. From Lemmas 5.3 and 5.4, we have

$$|\mathring{A}|^2\leq\left(1+\frac{2}{n}\right)^{\frac{np(n+2)}{2q(p-n)}} c_3^{\frac{n}{q}}\left(c_2q^{\frac{2n}{p-n}+1}+\frac{(n+2)^2}{2nT_2}\right)^{\frac{n+2}{q}} T_2^{\frac{2}{q}}(2\varepsilon)^2:=c_4\varepsilon^2.$$

Set $\varepsilon_0=\left(\frac{2}{c_4}\right)^{\frac{1}{2}}$ for $n\geq4$ and $\varepsilon_0=\left(\frac{4}{3c_4}\right)^{\frac{1}{2}}$ for $n=3$. If $\varepsilon\leq\varepsilon_0$, then on M_{T_2}, we have $|A|^2\leq\frac{|H|^2}{n-1}+2$ for $n\geq4$ and $|A|^2\leq\frac{4|H|^2}{9}+\frac{4}{3}$ for $n=3$. Then by the convergence theorem proved by Baker [2] and the uniqueness of the mean curvature flow, we see that the mean curvature flow with F_0 as initial value either has a solution on a finite time interval $[0,T)$ and M_t converges to a round point as $t\to T$, or has a solution on $[0,\infty)$ and M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t\to\infty$. This completes the proof of Theorem 5.1. □

Corollary 5.5. *Let $F:M^n\to\mathbb{S}^{n+d}$ $(n\geq3)$ be a smooth closed submanifold. Let C_1 be as in Theorem 5.1. If $||A||_{L^p(M)}<C_1$, then M is diffeomorphic to a unit n-sphere.*

Write the constant obtained in Theorem 5.1 as $C_1=C_1(n,p,q,\Lambda)$. Since $||\mathring{A}||_{L^p(M)}\leq||A||_{L^p(M)}$, if we put $C_{n,p}=\min\{100,C_1(n,p,p,100)\}$, then Theorem 1.1 follows.

Theorem 5.6. *Let $F_0:M^n\to\mathbb{S}^{n+d}$ $(n\geq3)$ be a smooth closed submanifold. For given positive numbers $p\in(n,\infty)$ and $q\in(n,\infty)$, there is a positive constant C_2 depending on n, p, q, the upper bound Λ on the L^p-norm of the mean curvature of the submanifold, such that if*

$$||\mathring{A}||_{L^q(M_0)}<C_2,$$

then the mean curvature flow with F_0 as initial value has a unique solution $F: M\times[0,T)\to\mathbb{S}^{n+d}$, and either

(1) *$T<\infty$ and M_t converges to a round point as $t\to T$; or*

(2) *$T=\infty$ and M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t\to\infty$.*

Proof. Suppose $||H||_{L^p(M_0)} \le \Lambda$ and $||\mathring{A}||_{L^q(M_0)} < \varepsilon$ for some fixed $p, q > n$ and assume $\varepsilon \in (0, 100]$. Set $T = \sup\{t \in [0, T_{\max}) : ||H||_{L^p(M_t)} < 2\Lambda, ||\mathring{A}||_{L^q(M_t)} < 2\varepsilon\}$. We consider the mean curvature flow on the time interval [0, T).

From (2.2) we have for $w = |H|^2$

$$\frac{\partial}{\partial t} w \le \Delta w + 2|\mathring{A}|^2 w + \frac{2}{n} w^2 + 2nw. \tag{5.19}$$

For $r \ge \frac{p}{2} \ge \frac{3}{2}$, we have from (5.19)

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} w^r d\mu_t \le & -\frac{4(r-1)}{r^2}\int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t \\ & + 2\int_{M_t} |\mathring{A}|^2 w^r d\mu_t + \frac{2}{n}\int_{M_t} w^{r+1} d\mu_t + 2n\int_{M_t} w^r d\mu_t. \end{aligned} \tag{5.20}$$

By the definition of T, we know that for any Lipschitz function v and $t \in [0, T)$, there holds

$$\Big(\int_{M_t} v^{\frac{2n}{n-2}} d\mu_t\Big)^{\frac{n-2}{n}} \le C_{n,p}\Big(\int_{M_t} |\nabla v|^2 d\mu_t + \Big(1 + (2\Lambda)^{\frac{2p}{p-n}}\Big)\int_{M_t} v^2 d\mu_t\Big). \tag{5.21}$$

For the second term of the right hand side of (5.20), we have for any $\mu > 0$

(5.22)

$$\begin{aligned} \int_{M_t} |\mathring{A}|^2 w^r d\mu_t \le & \Big(\int_{M_t} |\mathring{A}|^q d\mu_t\Big)^{\frac{2}{q}} \cdot \Big(\int_{M_t} w^{r\cdot\frac{q}{q-2}} d\mu_t\Big)^{\frac{q-2}{q}} \\ \le & 200^2 \Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{q-n}{q}} \cdot \Big(\int_{M_t} (w^r)^{\frac{n}{n-2}} d\mu_t\Big)^{\frac{n-2}{n}\cdot\frac{n}{q}} \\ \le & 200^2 \Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{q-n}{q}} \cdot \Big[C_{n,p}\Big(\int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t \\ & + \Big(1 + (2\Lambda)^{\frac{2p}{p-n}}\Big)\int_{M_t} w^r d\mu_t\Big)\Big]^{\frac{n}{q}} \\ \le & 200^2 \Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{q-n}{q}} \cdot \Big[C_{n,p}^{\frac{n}{q}}\Big(\int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t\Big)^{\frac{n}{q}} \\ & + \Big(1 + (2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}} C_{n,p}^{\frac{n}{q}}\Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{n}{q}}\Big] \\ = & 200^2\Big(1 + (2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}} C_{n,p}^{\frac{n}{q}} \int_{M_t} w^r d\mu_t \\ & + 200^2 C_{n,p}^{\frac{n}{q}} \Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{q-n}{q}} \cdot \Big(\int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t\Big)^{\frac{n}{q}} \\ \le & 200^2\Big(1 + (2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}} C_{n,p}^{\frac{n}{q}} \int_{M_t} w^r d\mu_t \\ & + 200^2 C_{n,p}^{\frac{n}{q}} \cdot \frac{q-n}{q}\mu^{\frac{q}{q-n}} \int_{M_t} w^r d\mu_t + 200^2 C_{n,p}^{\frac{n}{q}} \cdot \frac{n}{q}\mu^{-\frac{q}{n}} \int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t. \end{aligned}$$

For the third term of the right hand side of (5.20), we have for any $\epsilon > 0$

(5.23)
$$\begin{aligned}\int_{M_t} w^{r+1}d\mu_t \leq&\Big(\int_{M_t} w^{\frac{p}{2}}d\mu_t\Big)^{\frac{2}{p}}\cdot\Big(\int_{M_t}(w^r)^{\frac{p}{p-2}}d\mu_t\Big)^{\frac{p-2}{p}}\\ \leq&(2\Lambda)^2\cdot\Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{p-n}{p}}\cdot\Big(\int_{M_t}(w^{\frac{r}{2}})^{\frac{2n}{n-2}}d\mu_t\Big)^{\frac{n-2}{p}}\\ \leq&(2\Lambda)^2\cdot\Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{p-n}{p}}\\ &\times\Big[C_{n,p}\Big(\int_{M_t}|\nabla w^{\frac{r}{2}}|^2 d\mu_t+\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)\cdot\int_{M_t} w^r d\mu_t\Big)\Big]^{\frac{n}{p}}\\ \leq&(2\Lambda)^2\cdot\Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{p-n}{p}}\cdot\Big[C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t}|\nabla w^{\frac{p}{4}}|^2 d\mu_t\Big)^{\frac{n}{p}}\\ &+C_{n,p}^{\frac{n}{p}}\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{n}{p}}\Big]\end{aligned}$$

$$\begin{aligned}=&(2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\cdot C_{n,p}^{\frac{n}{p}}\cdot\int_{M_t} w^r d\mu_t\\ &+(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\Big(\int_{M_t} w^r d\mu_t\Big)^{\frac{p-n}{p}}\cdot\Big(\int_{M_t}|\nabla w^{\frac{p}{4}}|^2 d\mu_t\Big)^{\frac{n}{p}}\\ \leq&(2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\cdot C_{n,p}^{\frac{n}{p}}\cdot\int_{M_t} w^r d\mu_t\\ &+(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\frac{p-n}{p}\epsilon^{\frac{p}{p-n}}\cdot\int_{M_t} w^r d\mu_t+(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\frac{n}{p}\epsilon^{-\frac{p}{n}}\cdot\int_{M_t}|\nabla w^{\frac{p}{4}}|^2 d\mu_t.\end{aligned}$$

Combining (5.20), (5.22) and (5.23) we have

(5.24)
$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} w^r d\mu_t \leq&\Big(2r\cdot 200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{n}{q}\mu^{-\frac{q}{n}}+\frac{2}{n}r(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\frac{n}{p}\epsilon^{-\frac{p}{n}}-\frac{4(r-1)}{r}\Big)\int_{M_t}|\nabla w^{\frac{r}{2}}|^2 d\mu_t\\ &+\Big(2r\cdot 200^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}}C_{n,p}^{\frac{n}{q}}+2r\cdot 200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{q-n}{q}\mu^{\frac{q}{q-n}}\\ &+\frac{2}{n}r\cdot(2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\cdot C_{n,p}^{\frac{n}{p}}+\frac{2}{n}r\cdot(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\frac{p-n}{p}\epsilon^{\frac{p}{p-n}}\\ &+2nr\Big)\int_{M_t} w^r d\mu_t.\end{aligned}$$

Set $c_5 = 2\cdot 200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{n}{q}+\frac{2}{n}(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\frac{n}{p}$ and $c_6 = 2\cdot 200^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}}C_{n,p}^{\frac{n}{q}}+2\cdot 200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{q-n}{q}\Big(\frac{c_5 p}{4(p-2)}\Big)^{\frac{n}{q-n}}+\frac{2}{n}\cdot(2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}\cdot C_{n,p}^{\frac{n}{p}}+\frac{2}{n}\cdot(2\Lambda)^2\cdot C_{n,p}^{\frac{n}{p}}\frac{p-n}{p}\Big(\frac{c_5 p}{4(p-2)}\Big)^{\frac{n}{p-n}}+2n$. Let $\mu = \Big(\frac{c_5 r^2}{4(r-1)}\Big)^{\frac{n}{q}}$ and $\epsilon = \Big(\frac{c_5 r^2}{4(r-1)}\Big)^{\frac{n}{p}}$. Then from

(5.24) we get

$$\frac{\partial}{\partial t}\int_{M_t} w^r d\mu_t \le c_6 r^{\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1}\int_{M_t} w^r d\mu_t. \tag{5.25}$$

Take $r=\frac{p}{2}$. Then for $t\in[0,\min\{T,T_1\})$, where $T_1=\frac{p\ln\frac{3}{2}}{c_6(\frac{p}{2})^{\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1}}$, there holds $||H||_{L^p(M_t)}<\frac{3}{2}\Lambda$.

From (2.5) we have the following inequality for h_σ.

$$\frac{\partial}{\partial t}h_\sigma \le \triangle h_\sigma + 13|\mathring{A}|^2 h_\sigma + \frac{2}{n}|H|^2 h_\sigma. \tag{5.26}$$

For any $r\ge q>1$, we have

(5.27)
$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t =&\int_{M_t} h_\sigma^{r-1}\frac{\partial}{\partial t}h_\sigma d\mu_t + \frac{1}{r}\int_{M_t} h_\sigma^p \frac{\partial}{\partial t}d\mu_t\\ \le& -\frac{4(r-1)}{r^2}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t + 13\int_{M_t}|\mathring{A}|^2 h_\sigma^r d\mu_t + \frac{2}{n}\int_{M_t}|H|^2 h_\sigma^r d\mu_t.\end{aligned}$$

For the second term of the right hand side of (5.27), as (5.22) we have for any $\nu>0$

$$\begin{aligned}\int_{M_t}|\mathring{A}|^2 h_\sigma^r d\mu_t \le& 200^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}} C_{n,p}^{\frac{n}{q}}\int_{M_t} h_\sigma^r d\mu_t\\ &+200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{q-n}{q}\nu^{\frac{q}{q-n}}\int_{M_t} h_\sigma^r d\mu_t\\ &+200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{n}{q}\nu^{-\frac{q}{n}}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t.\end{aligned}\tag{5.28}$$

Similarly, for the last term of the right hand side of (5.27), we have for any $\vartheta>0$

$$\begin{aligned}\int_{M_t}|H|^2 h_\sigma^r d\mu_t \le& (2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}} C_{n,p}^{\frac{n}{p}}\int_{M_t} h_\sigma^r d\mu_t\\ &+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{p-n}{p}\vartheta^{\frac{p}{p-n}}\int_{M_t} h_\sigma^r d\mu_t\\ &+(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}\vartheta^{-\frac{p}{n}}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t.\end{aligned}\tag{5.29}$$

Combining (5.27), (5.28) and (5.29), we obtain

(5.30)
$$\begin{aligned}&\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t\\ \le&\Big(13r\cdot 200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{n}{q}\nu^{-\frac{q}{n}}+\frac{2}{n}r\cdot(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}\vartheta^{-\frac{p}{n}}-\frac{4(r-1)}{r}\Big)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\\ &+\Big(13r\cdot 200^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}} C_{n,p}^{\frac{n}{q}}+13r\cdot 200^2 C_{n,p}^{\frac{n}{q}}\cdot\frac{q-n}{q}\nu^{\frac{q}{q-n}}\\ &+\frac{2}{n}r\cdot(2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}} C_{n,p}^{\frac{n}{p}}+\frac{2}{n}r\cdot(2\Lambda)^2 C_{n,p}^{\frac{n}{p}}\cdot\frac{p-n}{p}\vartheta^{\frac{p}{p-n}}\Big)\int_{M_t} h_\sigma^r d\mu_t.\end{aligned}$$

Set $c_7=13\cdot 200^2C_{n,p}^{\frac{n}{q}}\cdot\frac{n}{q}+\frac{2}{n}\cdot(2\Lambda)^2C_{n,p}^{\frac{n}{p}}\cdot\frac{n}{p}$ and $c_8=13\cdot 200^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{q}}C_{n,p}^{\frac{n}{q}}+13\cdot 200^2C_{n,p}^{\frac{n}{q}}\cdot\frac{q-n}{q}\Big(\frac{c_7q}{3(q-1)}\Big)^{\frac{n}{q-n}}+\frac{2}{n}\cdot(2\Lambda)^2\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)^{\frac{n}{p}}C_{n,p}^{\frac{n}{p}}+\frac{2}{n}\cdot(2\Lambda)^2C_{n,p}^{\frac{n}{p}}\cdot\frac{p-n}{p}\Big(\frac{c_7q}{3(q-1)}\Big)^{\frac{n}{p-n}}$. Let $\nu=\Big(\frac{c_7r^2}{3(r-1)}\Big)^{\frac{n}{q}}$ and $\vartheta=\Big(\frac{c_7r^2}{3(r-1)}\Big)^{\frac{n}{p}}$. Then (5.30) implies

$$\frac{\partial}{\partial t}\int_{M_t}h_\sigma^r d\mu_t+\Big(1-\frac{1}{q}\Big)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t\le c_8r^{\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1}\int_{M_t}h_\sigma^r d\mu_t. \tag{5.31}$$

Take $r=p$. Then for $t\in[0,\min\{T,T_2\})$, where $T_2=\frac{p\ln\frac{3}{2}}{c_8q^{\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1}}$, we have $||\mathring{A}||_{L^q(M_t)}<\frac{3}{2}\varepsilon$.

We claim that $T>\min\{T_1,T_2\}$. Suppose not, i.e., $T\le\min\{T_1,T_2\}$. If $T<T_{\max}$, then by the smooth of the mean curvature flow and the definition of T, we get a contradiction. If $T=T_{\max}$, then $T_{\max}$ must be ∞. If not, by the definition of T, for $t\in[0,T_{\max})$ we have $||H||_{L^p(M_t)}<2\Lambda$ and $||\mathring{A}||_{L^q(M_t)}<2\varepsilon\le 200$. This implies $||A||_{L^{\min\{p,q\}}(M_t)}<\infty$ for any $t\in[0,T_{\max})$. Then by Lemma 4.1, the mean curvature flow can be extended over time $T_{\max}$, which is a contradiction. Hence we obtain that $T>\min\{T_1,T_2\}$.

Set $T_0=\min\{T_1,T_2\}$. We consider the mean curvature flow on $[0,\frac{T_0}{2}]$. Then we know that (5.31) holds for any $t\in[0,\frac{T_0}{2}]$. By a standard Moser iteration as before, we have for any $t\in(0,\frac{T_0}{2}]$, there holds

$$h_\sigma(x,t)\le\Big(1+\frac{2}{n}\Big)^{\hat{q}}c_9^{\frac{n}{2q}}\Big(c_8q^{\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1}+\frac{(n+2)^2}{2nt}\Big)^{\frac{n+2}{2q}}\Big(\int_0^{\frac{T_0}{2}}\int_{M_t}h_\sigma^q d\mu_t dt\Big)^{\frac{1}{q}},$$

where $c_9=C_{n,p}\cdot\max\Big\{\frac{q}{q-1},\Big(1+(2\Lambda)^{\frac{2p}{p-n}}\Big)\cdot T_0\Big\}$ and $\hat{q}=\frac{n(n+2)}{4q}\cdot\Big(\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1\Big)$. Letting $\sigma\to 0$, we get at time $\frac{T_0}{2}$

$$|\mathring{A}|^2\le\Big(1+\frac{2}{n}\Big)^{2\hat{q}}c_9^{\frac{n}{q}}\Big(c_{26}q^{\max\{\frac{n}{p-n},\frac{n}{q-n}\}+1}+\frac{(n+2)^2}{nT_0}\Big)^{\frac{n+2}{q}}\cdot\Big(\frac{T_0}{2}\Big)^{\frac{2}{q}}(2\varepsilon)^2:=c_{10}\varepsilon^2.$$

Set $\varepsilon_0=\Big(\frac{2}{c_{10}}\Big)^{\frac{1}{2}}$ for $n\ge 4$ and $\varepsilon_0=\Big(\frac{4}{3c_{10}}\Big)^{\frac{1}{2}}$ for $n=3$. Then if $\varepsilon\le\varepsilon_0$, we have $|A|^2\le\frac{|H|^2}{n-1}+2$ for $n\ge 4$ and $|A|^2\le\frac{4|H|^2}{9}+\frac{4}{3}$ for $n=3$ on $M_{\frac{T_0}{2}}$. By the convergence theorem proved by Baker [2] and the uniqueness of the mean curvature flow, we see that the mean curvature flow with F_0 as initial value either has a solution on a finite time interval $[0,T)$ and M_t converges to a round point as $t\to T$, or has a solution on $[0,\infty)$ and M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t\to\infty$. This completes the proof of Theorem 5.6. □

Corollary 5.7. *Let $F:M^n\to\mathbb{S}^{n+d}$ $(n\ge 3)$ be a smooth closed submanifold. Let C_2 be as in Theorem 5.6. If $||A||_{L^p(M)}<C_2$, then M is diffeomorphic to a unit n-sphere.*

Write the constant obtained in Theorem 5.6 as $C_2=C_2(n,p,q,\Lambda)$. Since $||\mathring{A}||_{L^p(M)}\le||A||_{L^p(M)}$, if we put $C_{n,p}=\min\{100,C_2(n,p,p,100\cdot n^{\frac{1}{2}})\}$, then Theorem 1.1 also follows. Hence, the pinching constant $C_{n,p}$ in Theorem 1.1 can be chosen as $C_{n,p}=\min\Big\{100,\max\{C_1(n,p,p,100),C_2(n,p,p,100\cdot n^{\frac{1}{2}})\}\Big\}$.

Remark 5.8. From the proofs of Lemma 3.2, Theorem 5.1 and Theorem 5.6, the constant $C_{n,p}$ in Theorem 1.1 can be computed explicitly.

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