Light-likeness of Bordlines of Extremal Surfaces of Mixed Type in Physical Space-times

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Abstract

In this paper, we prove the light-likeness of bordlines of smooth extremal surfaces of mixed type in general physical space-times, and for the case of the $(1+2)$-dimensional Minkowski space $\mathbb{R}^{1+2}$, we improve a Gu's theorem on the light-likeness of bordlines of extremal surfaces in $\mathbb{R}^{1+2}$. As a consequence, we show that a curve moving in a physical space-time keeps its like-property and the bordline only exists when its world sheet at the initial time has light-like points. This implies that any extremal surface of mixed type is generated by an "initial curve of mixed type".

Key words and phrases: Space-time, Extremal surface of mixed type, Bordline, Light-likeness.

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1 Introduction

The theory of minimal surfaces in the n-dimensional Euclidean space and Riemannian manifolds has its roots in the calculus of variations developed by Euler and Lagrange in the 18-th century which was promoted by the famous Plateau problem. There are plenty of deep and beautiful results obtained by many mathematicians. This theory plays an important role in general relativity, the theory of black hole, particle physics, fluid mechanics and so on. In a space-time, since the metric is not positive definite, a surface can be space-like, time-like, light-like or of mixed type. A surface is called an extremal surface of mixed type (ESMT) if it is a connected C^2 surface with vanishing mean curvature and containing a space-like part and a time-like part simultaneously. To the authors' knowledge, only a few results on ESMT has been known. By the classical Legendre transformation, Gu constructs many complete extremal surfaces of mixed type with explicit expressions by using representation formulas of the space-like extremal surfaces and time-like extremal surfaces and gives a method to construct the ESMT globally (see [2, 3, 6]). In [5], Gu shows that the space-like parts and the time-like parts of the ESMT are separated by a null curve and that the solution is analytic in a neighborhood of this curve and not only in the part where the equations are elliptic. Moreover, Gu [6] proves that there exist complete extremal surfaces of mixed type which have a given number of time-like spans and a given number of annular ends. In [4], Gu considers a special case of boundary problems for extremal surfaces of mixed type in $\mathbb{R}^{1+2}$ such that these problems could be considered as the counterparts of the famous Plateau problem for minimal surfaces in the Euclidean space $\mathbb{R}^3$ and shows that there exists a mixed-type extremal surface (with some modification) which passes through a given null curve. To do so, the author reduces the boundary problem to a special case of the Riemann-Hilbert problems for holomorphic functions, and then by the corresponding classical result he proves the existence of the solution of the boundary problem. Here we particularly point out that, Gu [6] shows that an ESMT $\mathbb{R}^{1+2}$ without flat point on the bordline is analytic around the bordline and the bordline is an analytic null curve.

In this paper, we investigate the geometric properties of bordlines of smooth ESMT in general space-times. The paper is organized as follows. In Section 2, we extend Gu's theorem in [6] for ESMT in $\mathbb{R}^{1+2}$ to general flat space-time, i.e., the Minkowski space-time $\mathbb{R}^{1+(1+n)}$. However, in our theorem, we do not require Gu's assumption that there is no flat point on the bordline. In Section 3, we introduce a new system which governs the ESMT in the general curved space-time $\mathscr{L}$ and extend our above result to the case $\mathscr{L}$. In order to illustrate our result, Section 4 is devoted to an intuitive example. In Appendix, we improve Gu's result in $\mathbb{R}^{1+2}$ on the light-likeness of bordlines without Gu's assumption.

2 Flat space-time —Minkowski space $\mathbb{R}^{1+(1+n)}$

Let $(t, x, y) = (t, x, y_1, ..., y_n)$ be points in the $(1 + (1 + n))$-dimensional Minkowski space $\mathbb{R}^{1+(1+n)}$ with the Lorentz metric

$$ds^2 = -dt^2 + dx^2 + dy^2. \tag{2.1}$$

For simplicity, we consider the equation for the extremal surface which takes the following form

$$y = \phi(t, x). \tag{2.2}$$

Let

$$u = \phi_x, \quad v = \phi_t. \tag{2.3}$$

Define

$$\Delta(u, v) = 1 + |u|^2 - |v|^2 - |v|^2|u|^2 + \langle v, u\rangle^2. \tag{2.4}$$

Due to the vanishing mean curvature, the equation of ESMT in $\mathbb{R}^{1+(1+n)}$ reads

$$(1 + |u|^2)O - 2\langle u, v\rangle P - (1 - |v|^2)Q = 0, \tag{2.5}$$

where

$$\begin{cases} O = \Delta(u, v)v_t + (1 + |u|^2)\langle v, v_t\rangle v - \langle u, v\rangle\langle u, v_t\rangle v - \langle u, v\rangle\langle v, v_t\rangle u - (1 - |v|^2)\langle u, v_t\rangle u, \\ P = \Delta(u, v)v_x + (1 + |u|^2)\langle v, v_x\rangle v - \langle u, v\rangle\langle u, v_x\rangle v - \langle u, v\rangle\langle v, v_x\rangle u - (1 - |v|^2)\langle u, v_x\rangle u, \\ Q = \Delta(u, v)u_x + (1 + |u|^2)\langle v, u_x\rangle v - \langle u, v\rangle\langle u, u_x\rangle v - \langle u, v\rangle\langle v, u_x\rangle u - (1 - |v|^2)\langle u, u_x\rangle u. \end{cases} \tag{2.6}$$

Assume that Σ is an extremal surface of mixed type in $\mathbb{R}^{1+(1+n)}$. The bordline ℓ of Σ is a curve on Σ on which it holds that

$$\Delta(u, v) = 1 + |u|^2 - |v|^2 - |v|^2|u|^2 + \langle v, u\rangle^2 = 0. \tag{2.7}$$

The main result in this section is the following theorem.

Theorem 2.1 *Let Σ be a connected C^2 extremal surface of mixed type in $\mathbb{R}^{1+(1+n)}$. If ℓ is a bordline of Σ, then ℓ is light-like.*

Proof. Suppose that ℓ can be expressed by $(t, x(t), y(t, x(t)))$. Since ℓ is a bordline of Σ, then on ℓ, the conditions (2.5) and (2.7) are satisfied. Thus, it suffices to show that the length of tangent vector of ℓ is zero everywhere. Equivalently, it suffices to prove that

$$1 = \dot{x}^2 + \dot{y}^2, \tag{2.8}$$

where $\dot{y} = \frac{dy(t,x(t))}{dt}$.

In fact, noting (2.3), we obtain from (2.8) that

$$\dot{x}^2 + |v|^2 + |u|^2\dot{x}^2 + 2\langle u, v\rangle\dot{x} - 1 = 0. \tag{2.9}$$

On the other hand, differentiating (2.7) with respect to t leads to

$$\begin{aligned}&\langle u, \phi_{tx}\rangle + \langle u, \phi_{xx}\rangle\dot{x} - \langle v, \phi_{tt}\rangle - \langle v, \phi_{tx}\rangle\dot{x} - |v|^2\langle u, \phi_{tx}\rangle - |v|^2\langle u, \phi_{xx}\rangle\dot{x} - |u|^2\langle v, \phi_{tt}\rangle - \\ &\quad |u|^2\langle v, \phi_{tx}\rangle\dot{x} + \langle u.v\rangle(\langle v, \phi_{tx}\rangle + \langle v, \phi_{xx}\rangle\dot{x} + \langle u, \phi_{tt}\rangle + \langle u, \phi_{tx}\rangle\dot{x}) = 0.\end{aligned} \tag{2.10}$$

For the sake of simplicity, we introduce the following notations

$$\begin{cases} k = 1 + |u|^2, \quad l = \langle u, v\rangle, \quad m = 1 - |v|^2, \quad a = \langle v, \phi_{tt}\rangle, \quad b = \langle u, \phi_{tt}\rangle, \\ c = \langle v, \phi_{tx}\rangle, \quad d = \langle u, \phi_{tx}\rangle, \quad e = \langle v, \phi_{xx}\rangle, \quad f = \langle u, \phi_{xx}\rangle. \end{cases} \tag{2.11}$$

Then (2.5) becomes

$$\begin{aligned}F \triangleq & k(kav - lbv - lau - mbu) - 2l(kcv - ldv - lcu - mdu) - \\ & m(kev - lfv - leu - mfu) = 0.\end{aligned} \tag{2.12}$$

It follows from (2.7) that

$$\Delta = km + l^2 = 0. \tag{2.13}$$

By (2.10), we obtain

$$\dot{x} = \frac{md - ka + lc + lb}{-mf + kc - le - ld}. \tag{2.14}$$

Substituting (2.14) into (2.9) leads to

$$\begin{aligned}\dot{x}^2 + &|v|^2 + |u|^2\dot{x}^2 + 2\langle u, v\rangle\dot{x} - 1 \\ =& k\left(\frac{md - ka + lc + lb}{-mf + kc - le - ld}\right)^2 + 2l\left(\frac{md - ka + lc + lb}{-mf + kc - le - ld}\right) - m \\ =& \frac{1}{(-mf + kc - le - ld)^2}[k(md - ka + lc + lb)^2 + 2l(md - ka + lc + lb) \\ & (-mf + kc - le - ld) - m(-mf + kc - le - ld)^2].\end{aligned} \tag{2.15}$$

A direct calculation gives

$$\begin{aligned}&k(md - ka + lc + lb)^2 + 2l(md - ka + lc + lb)(-mf + kc - le - ld) - m(-mf + kc - le - ld)^2 \\ &= k^3a^2 + kl^2b^2 + (3kl^2 - mk^2)c^2 + (km^2 - 3ml^2)d^2 - ml^2e^2 - m^3f^2 - 2k^2lab - 4k^2lac + \\ &\quad (2kl^2 - 2mk^2)ad + 2kl^2ae + 2mklaf + 4kl^2bc + (2mkl - 2l^3)bd - 2l^3be - 2ml^2bf + \\ &\quad (6mkl - 2l^3)cd + (2kml - 2l^3)ce + (2km^2 - 2ml^2)cf - 4ml^2de - 4m^2ldf - 2m^2lef.\end{aligned} \tag{2.16}$$

On the other hand, noting (2.12) we have

$$
\begin{aligned}
0 = F^2 =& k^2(kv-lu)^2a^2 + k^2(lv+mu)^2b^2 + 4l^2(-kv+lu)^2c^2 + 4l^2(lv+mu)^2d^2 + \\
& m^2(kv-lu)^2e^2 + m^2(lv+mu)^2f^2 - 2k^2(kv-lu)(lv+mu)ab + \\
& 4kl(kv-lu)(-kv+lu)ac + 4kl(kv-lu)(lv+mu)ad - 2mk(kv-lu)^2ae + \\
& 2mk(kv-lu)(lv+mu)af - 4kl(lv+mu)(-kv+lu)bc - 4kl(lv+mu)^2bd + \\
& 2mk(lv+mu)(kv-lu)be - 2km(lv+mu)^2bf + 8l^2(-kv+lu)(lv+mu)cd + \\
& 4ml(-kv+lu)^2ce + 4lm(-kv+lu)(lv+mu)cf + 4ml(lv+mu)(-kv+lu)de + \\
& 4ml(lv+mu)^2df + 2m^2(-kv+lu)(lv+mu)ef.
\end{aligned}
\tag{2.17}
$$

Thus, using the condition (2.13), we can easily check that the coefficients of the right-side of (2.16) is proportional to the coefficients of the right-side of (2.17). Noting (2.12), i.e., $F = 0$, we observe that the left-side of (2.16) equals to zero. This implies that the bordline is light-like. Thus, Theorem 2.1 is proved. □

For a light-like curve C in $\mathbb{R}^{1+(1+n)}$, without loss of generality, we assume that it can be parameterized as $C(x) = (t(x), x, y(x))$, where $y = (y_1, ..., y_n)$. According to the definition of light-likeness, we have

$$
-\left(\frac{dt}{dx}\right)^2 + 1 + \left(\frac{dy}{dx}\right)^2 = 0, \tag{2.18}
$$

that is,

$$
\left(\frac{dt}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \geq 1. \tag{2.19}
$$

This implies that

$$
\frac{dt}{dx} \geq 1 \quad \text{or} \frac{dt}{dx} \leq -1,
$$

namely,

$$
-1 \leq \frac{dx}{dt} \leq 1. \tag{2.20}
$$

Hence we can extend C toward the plane $\{t = 0\}$ and $C \bigcap \{t = 0\} \neq \emptyset$.

From the above discussion, it is obvious that the bordline of the ESMT, which is generated by a curve moving in the Minkowski space-time $\mathbb{R}^{1+(1+n)}$ goes across the plane $\{t = 0\}$.

Remark 2.1 *Theorem 2.1 implies that the bordline of ESMT in the* $1+(1+n)$*-dimensional Minkowski space-time passes through the plane* $\{t = 0\}$*. It means that a bordline exists only in the case when the initial curve is "of mixed type".*

Theorem 2.2 *A relativistic string in the Minkowski space-time* $\mathbb{R}^{1+(1+n)}$ *will not meet a light-like point when it moves physically in* $\mathbb{R}^{1+(1+n)}$*, provided that its world sheet is regular.*

Remark 2.2 *By Theorem 2.1, we observe that arbitrary two bordlines will not meet together.*

3 General curved space-time

This section is devoted to the discussion on bordlines of extremal surfaces of mixed type in a enveloping space-time $(\mathscr{N}, \tilde{g})$, which is described by a given general Lorentzian manifold. We shall extend Theorem 2.1 to the case of arbitrary Lorentz manifold. Denoting a two-dimensional extremal sub-manifold by $\mathscr{M}$, we may choose the local coordinates (ζ^0, ζ^1) in $\mathscr{M}$. For simplicity, we also denote $\zeta^0 = t, \zeta^1 = \theta$. Let a position vector in the space-time $(\mathscr{N}, \tilde{g})$ be

$$X(t,\theta) = (x^0(t,\theta), x^1(t,\theta), \cdots, x^n(t,\theta)). \tag{3.1}$$

Let

$$x^A_\mu = \frac{\partial x^A}{\partial \zeta^\mu} \quad \text{and} \quad x^A_{\mu\nu} = \frac{\partial^2 x^A}{\partial \zeta^\mu \partial \zeta^\nu} \quad (A = 0, 1, \cdots, n;\ \mu,\, \nu = 0, 1), \tag{3.2}$$

then the induced metric of the sub-manifold $\mathscr{M}$ can be written as $g = (g_{\mu\nu})$, where

$$g_{\mu\nu} = \tilde{g}_{AB} x^A_\mu x^B_\nu \quad (A, B = 0, 1, \cdots, n;\ \mu, \nu = 0, 1). \tag{3.3}$$

By the vanishing mean curvature, we may define the equations for extremal surface of mixed type in the general curved space-times in the following way (see He and Kong [8])

$$g_{11} x^C_{tt} - 2g_{01} x^C_{t\theta} + g_{00} x^C_{\theta\theta} + g_{11} \tilde{\Gamma}^C_{AB} x^A_t x^B_t - 2g_{01} \tilde{\Gamma}^C_{AB} x^A_t x^B_\theta + g_{00} \tilde{\Gamma}^C_{AB} x^A_\theta x^B_\theta = 0, \tag{3.4}$$

where $\tilde{\Gamma}^C_{AB}$ stands for the connections of the metric $\tilde{g}$.

Remark 3.1 *This definition is given based on the following two reasons:*

(I) In the flat Minkowski space-time, the equation (3.4) is reduced to the equation (2.5).

(II) We recall that the basic equations governing the motion of a p-dimensional extended object in $(\mathscr{N}, \tilde{g})$ *read*

$$g^{\mu\nu} \left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_\mu x^B_\nu - \Gamma^\rho_{\mu\nu} x^C_\rho \right) = 0 \quad (C = 0, 1, \cdots, n), \tag{3.5}$$

where $g^{-1} \triangleq (g^{\mu\nu})$ *is the inverse of the metric* g, $\tilde{\Gamma}^C_{AB}$ *and* $\Gamma^\rho_{\mu\nu}$ *stand for the connections of the metric* $\tilde{g}$ *and the induced metric* g*, respectively.*

Based on Aurilia and Christodoulou's work [1], He and Huang [7] proved that any solution to (3.4) which describe the motion of a relativistic string and membrane also satisfies (3.5), and if the sub-manifold $\mathscr{M}$ *is time-like or space-like, (3.5) can be reduced to (3.4) by wave coordinate or harmonic coordinate, respectively. Moreover, the solutions to (3.4) and (3.5) with the same Cauchy data are diffeomorphic.*

The bordline ℓ is a curve on $\mathscr{M}$ along which

$$\Delta \triangleq det g = g_{00} g_{11} - g_{01}^2 = 0. \tag{3.6}$$

Theorem 3.1 *Let $\mathscr{M}$ be a connected C^2 extremal surface of mixed type in $\mathscr{N}$. If ℓ is a bordline of $\mathscr{M}$, then it is light-like.*

Proof. We parameterize the bordline as

$$\mathcal{C}(t) = (x^0(t,\theta(t)), ..., x^n(t,\theta(t))). \tag{3.7}$$

Differentiating (3.7) gives

$$\dot{\mathcal{C}}(t) = (x^0_t + x^0_\theta \dot{\theta}, ..., x^n_t + x^n_\theta \dot{\theta}). \tag{3.8}$$

In order to prove the theorem, it suffices to show that $\mathcal{C}$ is a null curve, i.e.,

$$\tilde{g}_{ij}(x^i_t + x^i_\theta \dot{\theta})(x^j_t + x^j_\theta \dot{\theta}) = 0. \tag{3.9}$$

By previous notations, we rewrite (3.9) as

$$H = \tilde{g}_{ij} x^i_0 x^j_0 + 2\tilde{g}_{ij} x^i_0 x^j_1 \dot{\theta} + \tilde{g}_{ij} x^i_1 x^j_1 \dot{\theta}^2 = 0, \tag{3.10}$$

that is,

$$H = g_{00} + 2g_{01}\dot{\theta} + g_{11}\dot{\theta}^2 = 0. \tag{3.11}$$

On the one hand, differentiating (3.6) gives the expression of $\dot{\theta}$ as follows

$$\begin{aligned}
\frac{\partial}{\partial t}\det g =& \frac{\partial}{\partial t}[\tilde{g}_{AB}\tilde{g}_{CD}(x^A_0 x^B_0 x^C_1 x^D_1 - x^A_0 x^B_1 x^C_0 x^D_1)] \\
=& \frac{\partial \tilde{g}_{AB}}{\partial E}(x^E_0 + x^E_1\dot{\theta})\tilde{g}_{CD} x^A_0 x^B_0 x^C_1 x^D_1 + \tilde{g}_{AB}\frac{\partial \tilde{g}_{CD}}{\partial E}(x^E_0 + x^E_1\dot{\theta}) x^A_0 x^B_0 x^C_1 x^D_1 + \\
& \tilde{g}_{AB}\tilde{g}_{CD}(x^A_{00} + x^A_{01}\dot{\theta}) x^B_0 x^C_1 x^D_1 + \tilde{g}_{AB}\tilde{g}_{CD} x^A_0 (x^B_{00} + x^B_{01}\dot{\theta}) x^C_1 x^D_1 + \\
& \tilde{g}_{AB}\tilde{g}_{CD} x^A_0 x^B_0 (x^C_{01} + x^C_{11}\dot{\theta}) x^D_1 + \tilde{g}_{AB}\tilde{g}_{CD} x^A_0 x^B_0 x^C_1 (x^D_{01} + x^D_{11}\dot{\theta}) - \\
& \frac{\partial \tilde{g}_{AB}}{\partial E}(x^E_0 + x^E_1\dot{\theta})\tilde{g}_{CD} x^A_0 x^B_1 x^C_0 x^D_1 - \tilde{g}_{AB}\frac{\partial \tilde{g}_{CD}}{\partial E}(x^E_0 + x^E_1\dot{\theta}) x^A_0 x^B_1 x^C_0 x^D_1 - \\
& \tilde{g}_{AB}\tilde{g}_{CD}(x^A_{00} + x^A_{01}\dot{\theta}) x^B_1 x^C_0 x^D_1 - \tilde{g}_{AB}\tilde{g}_{CD} x^A_0 (x^B_{01} + x^B_{11}\dot{\theta}) x^C_0 x^D_1 - \\
& \tilde{g}_{AB}\tilde{g}_{CD} x^A_0 x^B_1 (x^C_{00} + x^C_{01}\dot{\theta}) x^D_1 - \tilde{g}_{AB}\tilde{g}_{CD} x^A_0 x^B_1 x^C_0 (x^D_{01} + x^D_{11}\dot{\theta}) \\
=& 0.
\end{aligned} \tag{3.12}$$

On the other hand, after a series of rotations and symmetry of index, we get

$$I_1 + \dot{\theta} I_2 = 0, \tag{3.13}$$

where

$$I_1 = \frac{\partial \tilde{g}_{AB}}{\partial E}\tilde{g}_{CD} R_{AC} R_{BD} x^E_0 + 2\tilde{g}_{AB}\tilde{g}_{CD}(x^A_{01} x^C_0 - x^A_{00} x^C_1) R_{BD} \tag{3.14}$$

and

$$I_2 = \frac{\partial \tilde{g}_{AB}}{\partial E}\tilde{g}_{CD} R_{AC} R_{BD} x^E_1 + 2\tilde{g}_{AB}\tilde{g}_{CD}(x^A_{11} x^C_0 - x^A_{01} x^C_1) R_{BD}, \tag{3.15}$$

in which

$$R_{ab} = x_1^a x_0^b - x_0^a x_1^b. \tag{3.16}$$

Substituting (3.13), (3.14) and (3.15) into (3.11) leads to

$$\hat{H} = g_{11} I_1^2 - 2g_{01} I_1 I_2 + g_{00} I_2^2 = 0. \tag{3.17}$$

A direct calculation gives

$$\begin{aligned}
\hat{H} =& g_{11} I_1^2 - 2g_{01} I_1 I_2 + g_{00} I_2^2 \\
=& \frac{\partial \tilde{g}_{AB}}{x^E} \tilde{g}_{CD} \frac{\partial \tilde{g}_{FG}}{\partial x^M} \tilde{g}_{KL} R_{AC} R_{BD} R_{FK} R_{GL} (g_{11} x_0^E x_0^M - 2g_{01} x_0^E x_1^M + g_{00} x_1^E x_1^M) \\
& + 2\tilde{g}_{AB} \tilde{g}_{CD} \frac{\partial \tilde{g}_{FG}}{\partial x^M} \tilde{g}_{KL} R_{BD} R_{FK} R_{GL} [(g_{11} x_0^C x_0^M - 2g_{01} x_0^C x_1^M - g_{00} x_1^C x_1^M) x_{01}^A + \\
& (2g_{01} x_1^C x_1^M - g_{11} x_1^C x_0^M) x_{00}^A + g_{00} x_0^C x_1^M x_{11}^A] \\
& + 2\frac{\partial \tilde{g}_{AB}}{\partial x^E} \tilde{g}_{CD} \tilde{g}_{FG} \tilde{g}_{KL} R_{AC} R_{BD} R_{GL} [(g_{11} x_0^E x_0^K + 2g_{01} x_0^E x_1^K - g_{00} x_1^E x_1^K) x_{01}^F + \\
& (g_{00} x_1^E x_0^K - 2g_{01} x_0^E x_0^K) x_{11}^F - g_{11} x_0^E x_1^K x_{00}^F] \\
& + 4\tilde{g}_{AB} \tilde{g}_{CD} \tilde{g}_{FG} \tilde{g}_{KL} R_{BD} R_{GL} [x_{01}^A x_{01}^F (g_{11} x_0^C x_0^K + 2g_{01} x_0^C x_1^K + g_{00} x_1^C x_1^K) + \\
& x_{00}^A x_{01}^F (-g_{11} x_1^C x_0^K - 2g_{01} x_1^C x_1^K) + x_{01}^A x_{00}^F (-g_{11} x_0^C x_1^K) + \\
& x_{00}^A x_{00}^F (g_{11} x_1^C x_1^K) + x_{01}^A x_{11}^F (-2g_{01} x_0^C x_0^K - g_{00} x_1^C x_0^K) + \\
& x_{00}^A x_{11}^F (2g_{01} x_1^C x_0^K) + x_{11}^A x_{11}^F (g_{00} x_0^C x_0^K) + x_{11}^A x_{01}^F (-g_{00} x_0^C x_1^K)].
\end{aligned} \tag{3.18}$$

Let

$$\begin{aligned}
f^i &= g_{11} x_{00}^i - 2g_{01} x_{01}^i + g_{00} x_{11}^i + g_{11} \tilde{\Gamma}_{AB}^i x_0^A x_0^B - 2g_{01} \tilde{\Gamma}_{AB}^i x_0^A x_1^B + g_{00} \tilde{\Gamma}_{AB}^i x_1^A x_1^B \\
&= (g_{11} x_{00}^i - 2g_{01} x_{01}^i + g_{00} x_{11}^i) + \tilde{\Gamma}_{AB}^i (g_{11} x_0^A x_0^B - 2g_{01} x_0^A x_1^B + g_{00} x_1^A x_1^B).
\end{aligned} \tag{3.19}$$

Then we have

$$\begin{aligned}
& 4\tilde{g}_{iE} \tilde{g}_{jF} (g_{00} x_1^E x_1^F - 2g_{01} x_0^E x_1^F + g_{11} x_0^E x_0^F) f^i f^j \\
=& 4\tilde{g}_{iE} \tilde{g}_{jF} \tilde{\Gamma}_{AB}^i \tilde{\Gamma}_{CD}^j (g_{11} x_0^E x_0^F - 2g_{01} x_0^E x_1^F + g_{00} x_1^E x_1^F) \\
& (g_{11} x_0^A x_0^B - 2g_{01} x_0^A x_1^B + g_{00} x_1^A x_1^B)(g_{11} x_0^C x_0^D - 2g_{01} x_0^C x_1^D + g_{00} x_1^C x_1^D) \\
& + 8\tilde{g}_{iE} \tilde{g}_{jF} \tilde{\Gamma}_{AB}^i (g_{00} x_1^E x_1^F - 2g_{01} x_0^E x_1^F + g_{11} x_0^E x_0^F) \\
& (g_{11} x_0^A x_0^B - 2g_{01} x_0^A x_1^B + g_{00} x_1^A x_1^B)(g_{11} x_{00}^j - 2g_{01} x_{01}^j + g_{00} x_{11}^j) \\
& + 4\tilde{g}_{iE} \tilde{g}_{jF} (g_{00} x_1^E x_1^F - 2g_{01} x_0^E x_1^F + g_{11} x_0^E x_0^F) \\
& (g_{11}^2 x_{00}^i x_{00}^j - 4g_{01} g_{11} x_{01}^i x_{00}^j + 2g_{00} g_{11} x_{11}^i x_{00}^j + 4g_{01}^2 x_{01}^i x_{01}^j - 4g_{00} g_{01} x_{11}^i x_{01}^j + g_{00}^2 x_{11}^i x_{11}^j).
\end{aligned} \tag{3.20}$$

By the condition (3.6), we have

$$\hat{H} = 4\tilde{g}_{iE} \tilde{g}_{jF} (g_{00} x_1^E x_1^F - 2g_{01} x_0^E x_1^F + g_{11} x_0^E x_0^F) f^i f^j. \tag{3.21}$$

Noting (3.4), we observe $f^i \equiv 0, \forall i = 0, 1, 2, 3, 4$. Thus, we get $\hat{H} = 0$. This proves the theorem. $\square$

4 An example

In this section, we use an example constructed by Lü [9], which is essentially due to Gu [3], to illustrate a visual sight of how a bordline act on the ESMT.

The example by [9] is as follows:

Part I: Time-like part

The time-like part is given by

$$\begin{cases} z = \frac{1}{2}(\lambda^2 + \mu^2), \\ x = \lambda sin\lambda + cos\lambda + \mu sin\mu + cos\mu, \\ y = -\lambda cos\lambda + sin\lambda - \mu cos\mu + sin\mu. \end{cases} \tag{4.1}$$

See Figure 1.

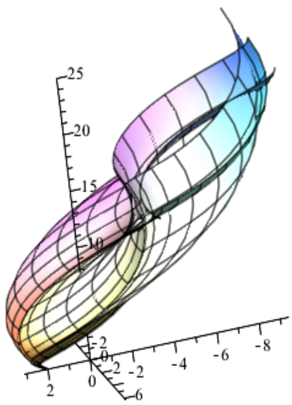

Figure 1: The time-like surface in (x, y, z)-plane

Part II: Light-like part, i.e., bordline

The bordline is given by

$$\begin{cases} z = \theta^2, \\ x = 2\theta sin\theta + 2cos\theta, \\ y = -2\theta cos\theta + 2sin\theta. \end{cases} \tag{4.2}$$

See Figure 2.

Part III: Space-like part

The space-like part is given by

$$\begin{cases} z = \theta^2 - \sigma^2, \\ x = 2\theta sin\theta cosh^2\sigma + 2cos\theta cosh\sigma - 2\sigma cos\theta sinh\sigma, \\ y = -2\theta cos\theta cosh\sigma - 2\sigma sin\theta sinh\sigma + 2sin\theta cosh\sigma. \end{cases} \tag{4.3}$$

See Figure 3.

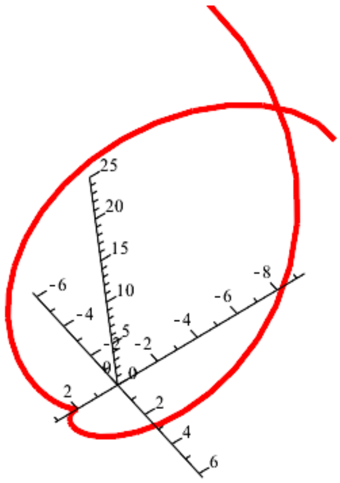

Figure 2: The bordline in (x, y, z)-plane

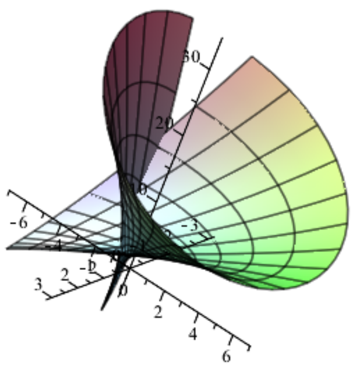

Figure 3: The space-like surface in (x, y, z)-plane

5 Appendix

In this section, we give a simple proof of Gu's theorem in the Minkowski space-time $\mathbb{R}^{1+2}$, in our proof we do not require Gu's assumption that there is no flat point on the borderline.

Let (x, y, z) be points in $\mathbb{R}^{1+2}$, the corresponding Lorentz metric reads

$$ds^2 = -dz^2 + dx^2 + dy^2. \tag{5.1}$$

For simplicity, we consider the graph case: $z = f(x, y)$, denote it by Σ. Due to the vanishing mean curvature of extremal surface, the equation for ESMT reads

$$(1 - p^2)t + 2pqs + (1 - q^2)r = 0, \tag{5.2}$$

where

$$p = z_x, \quad q = z_y, \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy}. \tag{5.3}$$

The bordline ℓ of Σ is a curve on Σ on which it satisfies

$$p^2 + q^2 = 1. \tag{5.4}$$

The space-like regions and time-like regions are separated by bordlines.

The main result in this section is the following theorem.

Theorem 5.1 *Let Σ be a connected C^2 extremal surface of mixed type in $\mathbb{R}^{1+2}$ which is governed by (5.2), with $t \in [0, \infty)$. If $\ell \in \Sigma$ is a bordline, then it is light-like.*

Proof. We suppose ℓ is given by $(x, y(x), z(x, y(x)))$. Because ℓ is a bordline, then the conditions (5.2) and (5.4) holds on $\ell \in \Sigma$. Thus, it suffices to show that the length of the tangent vector of ℓ is zero everywhere. That is, it suffices to prove

$$-\left(\frac{dz(x, y(x))}{dx}\right)^2 + 1 + (\dot{y})^2 = 0, \tag{5.5}$$

by (5.3),

$$-(p + q\dot{y})^2 + 1 + \dot{y}^2 = 0. \tag{5.6}$$

Differentiating (5.4) along ℓ gives

$$p(r + s\dot{y}) + q(s + t\dot{y}) = 0, \tag{5.7}$$

equivalently,

$$\dot{y} = \frac{-qs - pr}{ps + qt}. \tag{5.8}$$

Noting (5.2) and (5.4) lead to

$$s = \frac{-q^2 t - p^2 r}{2pq}. \tag{5.9}$$

Substituting (5.7) and (5.9) into (5.6) yields

$$
\begin{aligned}
&-(p+q\dot{y})^2+1+\dot{y}^2\\
&\quad=\frac{1}{(ps+qt)^2}\left[-\left(-\frac{p}{2q}r+\frac{q}{2p}t\right)^2+\left(-\frac{p^2}{2q}r+\frac{q}{2}t\right)^2+\left(\frac{p}{2}r-\frac{q^2}{2p}t\right)^2\right]\\
&\quad=\frac{1}{(ps+qt)^2}\left[\left(\frac{p^2}{4q^2}-\frac{p^2}{4}-\frac{p^4}{4q^2}\right)r^2+\left(-\frac{1}{2}+\frac{q^2}{2}+\frac{p^2}{2}\right)rt+\left(\frac{q^2}{4p^2}-\frac{q^4}{4p^2}-\frac{q^2}{4}\right)t^2\right]\\
&\quad=0.
\end{aligned}
\tag{5.10}
$$

The proof is completed.

Similar argument to the extremal surface of mixed type taking the form $y=\varphi(t,x)$, given by the trajectory of a moving curve in the Minkowski space-time $\mathbb{R}^{1+2}$, we obtain

$$\ell\bigcap\{t=0\}\neq\emptyset.$$

Remark 5.1 *Theorem 5.1 implies that the bordline of ESMT in the (1+2)-dimensional Minkowski space-time passes through the plane $\{t=0\}$. It means that a bordline exists only in the case when "the initial curve is of mixed type".*

Theorem 5.2 *A relativistic string in the Minkowski space-time $\mathbb{R}^{1+2}$ will not meet a light-like point when it moves physically in $\mathbb{R}^{1+2}$, provided that its world sheet is regular.*

Remark 5.2 *The conclusion of Remark 2.2 still keeps true in the present situation.*

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