The Infinity of Complete Riemannian Manifolds under Yamabe Flow

Fei Yang*and Jingfang Shen†

Abstract

We study complete locally conformally flat Riemannian manifolds M^n.We prove that the flatness of the infinity geometry can be preserved under the Yamabe flow. As an immediate application, the asymptotic volume ratio of the manifold is a constant.
Key words: Yamabe flow, locally conformally flat, asymptotic volume ratio.

1 Introduction

The geometric flow has been proved to be an effective tool in the study of the geometry and topology of Riemannian manifolds. For a complete Riemannian manifold, RS.Hamilton[1] has defined several geometric invariants such as the aperture

$$\alpha = \limsup_{s\to\infty} \frac{diam S_s}{2s}$$

where S_s is the sphere of radius s around an origin, the set of points whose distancd to the origin is s,

*1.Center of Mathematics and Science, Zhejiang University, Hangzhou, Zhejiang, 310027, 2.School of Mathematics and Physics, China University of geosciences, Wuhan, Hubei, 430000 P. R. China E-mail:yangfei712@yahoo.cn

†School of Mathematics and Physics, Huazhong Agricultural University, Wuhan, Hubei, 430000 P. R. China E-mail:sjf712@mail.hzau.edu.cn

the asymptotic volume ratio

$$v=\lim_{s\to\infty}\frac{V(B_s(P))}{s^n}$$

where $V(B_s(P))$ is the volume of a ball of radius s around an origin,

and the asymptotic scalar curvature ratio

$$A=\limsup_{s\to\infty} Rs^2.$$

Hamilton has proved that the above invariants are constants for complete solutions to the Ricci Flow under some conditions. A natural question is, how about the conclusions of the manifold under other geometric flows. Here we consider the complete solution of Yamabe Flow that $g(t)$ evolves under the flow equation

$$\frac{\partial}{\partial t}g=-Rg \tag{1.1}$$

Let M be a smooth complete locally conformally flat Riemannian n-manifold, we will represent the curvature tensors in an orthonormal frame and evolve the frame so that it remains orthonormal. Let V be a vector field over M isomorphic to the tangent bundle TM. The frame $F=\{F_1,\cdots,F_a,\cdots,F_n\}$ of V is given by $F_a=F_a^i\frac{\partial}{\partial x^i}$ locally with the isomorphism F_a^i. At initial time $t=0$, we can choose F_a^i such that $F=\{F_1,\cdots,F_a,\cdots,F_n\}$ is an orthonormal frame. Then evolve F_a^i by the equation

$$\frac{\partial}{\partial t}F_a^i=\frac{1}{2}RF_a^i$$

Then the frame $F=\{F_1,\cdots,F_a,\cdots,F_n\}$ will remain orthonormal for all times since the pull back metric on V

$$h_{ab}=g_{ij}F_a^iF_b^j$$

remains constant in time.

From now on, we will use indices $a,b,\cdots$ on a tensor to denote it's components in the evolving orthonormal frame. In the frame we have:

$$R_{abcd}=F_a^iF_b^jF_c^kF_d^lR_{ijkl}$$
$$\Gamma_{jb}^a=F_i^a\frac{\partial F_b^i}{\partial x^j}+\Gamma_{jk}^iF_a^iF_b^k$$
$$D_iV^a=\frac{\partial}{\partial x^i}V^a+\Gamma_{ib}^aV^b$$
$$D_bV^a=F_b^iD_iV^a$$

where Γ^a_{jb} is the metric connection of V with the metric h_{ab}, and $(F^a_i) = (F^i_a)^{-1}$. By direct computation, $D_i F^i_b = 0$ and $D_i h_{ab} = 0$.

Now we state our main theorem.

Theorem 1.1 *Suppose we have a solution to the Yamabe Flow on a complete locally conformally flat n-manifold with bounded curvature, If $|Rm| \to 0$ as $s \to \infty$ at initial time $t = 0$, then it will remains true for $t \geq 0$.*

Theorem 1.2 *Under the assumption of Theorem 1.1, if the manifold has nonnegative Ricci curvature, then the asymptotic volume ratio of the manifold is a constant.*

2 Preliminary

As the metric changes under the Yamabe Flow, the distance on the manifold will change. At first, e give the bound on the changing distance.

Lemma 2.1 *If $(M^n, g(t))$ is a complete solution to the Yamabe Flow and $|Rm(x,t)| \leq K$ for a nonnegative constant K, then we have*

$$e^{-(n-1)K(t_1-t_0)} \leq \frac{dist_{t_1}(x_0, x_1)}{dist_{t_0}(x_0, x_1)} \leq e^{(n-1)K(t_1-t_0)}$$

for $t_1 > t_0$ and any $x_0, x_1 \in M$.

Proof: Since

$$|Rm(x,t)| \leq K$$

on $M \times [0,T]$, we get

$$|R(x,t)| \leq n(n-1)K$$

on $M \times [0,T]$. Thus

$$|\frac{\partial}{\partial t} g_{ij}(x,t)| \leq |R g_{ij}(x,t)| \leq n(n-1)K g_{ij}(x,t)$$

That is,

$$-n(n-1)K g_{ij}(x,t) \leq \frac{\partial}{\partial t} g_{ij}(x,t) \leq n(n-1)K g_{ij}(x,t)$$

Thus we get

$$e^{-(n-1)Kt}g_{ij}(x,0) \le g_{ij}(x,t) \le e^{(n-1)Kt}g_{ij}(x,0)$$

and

$$e^{-(n-1)Kt}ds_0^2 \le ds_t^2 \le e^{(n-1)Kt}ds_0^2$$

Now we have

$$e^{-(n-1)Kt}\gamma_0(x_0,x_1) \le \gamma_t(x_0,x_1) \le e^{(n-1)Kt}\gamma_0(x_0,x_1)$$

for any $x_0, x_1 \in M$, where $\gamma_t(x_0,x_1)$ represents the length of the minimal geodesic that join x_0 and x_1. □

Lemma 2.2 [4] *If (M^n, g) is locally conformally flat, then under Yamabe Flow,*

(1) $\partial_t R_{ijkl} = (n-1)\triangle R_{ijkl} - RR_{ijkl} + \frac{1}{(n-2)^2}(B_{ik}g_{jl} + B_{jl}g_{ik} - B_{il}g_{jk} - B_{jk}g_{il})$,

(2) $\partial_t R_{ij} = (n-1)\triangle R_{ij} + \frac{1}{n-2}B_{ij}$,

(3) $\partial_t R = (n-1)\triangle R + R^2$,

where $B_{ij} = (n-1)\left|Ric\right|^2 g_{ij} + nRR_{ij} - n(n-1)R_{ij}^2 - R^2 g_{ij}$.

In the proof of the main theorem, we need a global derivative estimate for the curvature.

Lemma 2.3 *If $(M^n, g(t))$ is a complete locally conformally flat solution to the Yamabe Flow and*

$$|Rm(x,t)| \le K$$

for a nonnegative constant K, then the covariant derivative of the curvature is bounded by

$$|DRm| \le CK/\sqrt{t}$$

and the k^{th} covariant derivative of the curvature is bounded by

$$|D^k Rm| \le C_k K/t^{\frac{k}{2}}.$$

Proof:From Lemma2.1.2 we have

$$\frac{\partial}{\partial t}Rm = (n-1)\triangle Rm + Rm * Rm.$$

where the symbol $A * B$ denotes any tensor product of two tensors A and B when we do not need the precise expression.

It follows that

$$\frac{\partial}{\partial t}(DRm) = (n-1)\triangle(DRm) + Rm * (DRm).$$

and

$$\frac{\partial}{\partial t}|Rm|^2 \leq (n-1)\triangle|Rm|^2 - 2(n-1)|DRm|^2 + C|Rm|^3$$

$$\frac{\partial}{\partial t}|DRm|^2 \leq (n-1)\triangle|DRm|^2 - 2(n-1)|D^2Rm|^2 + C|Rm|\cdot|DRm|^2$$

for some constant C depending only on the dimension n.

Then we can follow the proof in Shi's derivative estimates of Ricci flow, choose a test function with slight modification. □

3 Proof of the main theorem

Proof of Theorem 1.1: Suppose $|Rm| \leq K$ for some constant K. Since $|Rm| \to 0$ as $s \to \infty$, for every $\varepsilon > 0$ we can find $\sigma < \infty$ such that $|Rm| \leq \varepsilon$ for $s \geq \sigma$.

The curvature tensor evolves by a formula

$$D_t Rm = (n-1)\triangle Rm + Rm * Rm$$

which gives a formula

$$\frac{\partial}{\partial t}|Rm|^2 = (n-1)\triangle|Rm|^2 - 2|DRm|^2 + Rm * Rm * Rm$$

and an estimate

$$\frac{\partial}{\partial t}|Rm|^2 \leq (n-1)\triangle|Rm|^2 + C_1|Rm|^3$$

for some constant C_1 depending only on the dimension.

Then choose a smooth function φ at $t = 0$ such that

$$\varphi = K^2 if s \leq \sigma$$
$$\varphi = \varepsilon^2 if s \geq 2\sigma$$
$$-\delta \leq D\varphi \leq 0$$
$$|D^2\varphi| \geq \rho > 0$$

for some constant $\delta > 0$ and constant $\rho > \delta$.

Now define φ for $t \geq 0$ by solving the scalar heat equation

$$\frac{\partial \varphi}{\partial t} = (n-1)\triangle\varphi$$

in the Laplacian of the metric evolving by the Yamabe Flow. By the maximum principle we still have $|\varphi| \leq \varepsilon^2$ everywhere for $t \geq 0$.

The derivative $D_a\varphi$ evolves by the formula

$$\begin{aligned}
D_t D_a\varphi &= D_t(F_a^i D_i\varphi) \\
&= \frac{1}{2} R F_a^i D_i\varphi + F_a^i D_t D_i\varphi \\
&= \frac{1}{2} R F_a^i D_i\varphi + (n-1)F_a^i D_i \triangle\varphi \\
&= \frac{1}{2} R F_a^i D_i\varphi + (n-1)F_a^i D_i(g^{mn} D_m D_n\varphi) \\
&= \frac{1}{2} R F_a^i D_i\varphi + (n-1)F_a^i g^{mn}(D_m D_i D_n\varphi + R_{imnl} g^{ls}\nabla_s\varphi) \\
&= (n-1)\triangle D_a\varphi + \frac{1}{2} R D_a\varphi - (n-1)F_a^i R_{il} g^{ls}\nabla_s\varphi)
\end{aligned}$$

hence

$$\frac{\partial}{\partial t}|D\varphi|^2 \leq (n-1)\triangle|D\varphi|^2 - 2(n-1)|D^2\varphi|^2 + C_1 K|D\varphi|^2$$

for some constant $C_1 > 0$ depending only on the dimension.

Let us put

$$F = t|D\varphi|^2 + |\varphi|^2$$

and compute

$$\begin{aligned}
\frac{\partial F}{\partial t} &= |D\varphi|^2 + t \cdot \frac{\partial}{\partial t}|D\varphi|^2 + \frac{\partial}{\partial t}|\varphi|^2 \\
&\leq (|D\varphi|^2 + t[(n-1)\triangle|D\varphi|^2 + C_1 K|D\varphi|^2] + (n-1)\triangle|D\varphi|^2 \\
&\quad -2(n-1)|D\varphi|^2
\end{aligned}$$

and

$$\triangle F = t\triangle|D\varphi|^2 + \triangle|\varphi|^2$$

thus

$$\frac{\partial F}{\partial t} \leq (n-1)\triangle F - [2(n-1) - 1 - tC_1K]|D\varphi|^2$$

Then if $t \leq \frac{2n-3}{C_1K}$, we have

$$\frac{\partial F}{\partial t} \leq (n-1)\triangle F$$

and the maximum of F decreases.

At initial time $t = 0$, $F = |\varphi|^2 = \varepsilon^2$, hence $F \leq \varepsilon^2$ for $t \geq 0$. Thus $|D\varphi| \leq \frac{\varepsilon}{\sqrt{t}}$ for $0 < t \leq \frac{2n-3}{C_1K}$.

Then we consider the second derivative $D_aD_b\varphi$.

$$\begin{aligned}
D_tD_aD_b\varphi &= D_t(F_a^iD_iD_b\varphi) \\
&= \frac{1}{2}RF_a^iD_iD_b\varphi + F_a^iD_iD_tD_b\varphi \\
&= \frac{1}{2}RF_a^iF_b^jD_iD_j\varphi + F_a^iD_i(\frac{1}{2}RF_b^jD_j\varphi + F_b^jD_tD_j\varphi) \\
&= \frac{1}{2}RF_a^iF_b^jD_iD_j\varphi + \frac{1}{2}F_a^iF_b^j(D_iR\cdot D_j\varphi + RD_iD_j\varphi) \\
&\quad +D_aD_b\triangle\varphi
\end{aligned}$$

$$\begin{aligned}
\triangle D_aD_b\varphi &= g^{ij}D_iD_jD_aD_b\varphi \\
&= g^{ij}F_a^mF_b^nD_iD_jD_mD_n\varphi \\
&= g^{ij}F_a^mF_b^nD_i(D_mD_jD_n\varphi - R_{jmn}^lD_l\varphi) \\
&= g^{ij}F_a^mF_b^nD_iD_mD_nD_j\varphi - g^{ij}F_a^mF_b^nD_i(R_{jmn}^lD_l\varphi) \\
&= g^{ij}F_a^mF_b^n(D_mD_iD_nD_j\varphi - R_{imn}^pD_pD_j\varphi - R_{imj}^pD_nD_p\varphi) \\
&\quad -g^{ij}F_a^mF_b^nD_i(R_{jmn}^lD_l\varphi) \\
&= g^{ij}F_a^mF_b^nD_m(D_nD_iD_j\varphi - R_{inj}^lD_l\varphi) \\
&\quad -g^{ij}F_a^mF_b^n(R_{imn}^pD_pD_j\varphi + R_{imj}^pD_nD_p\varphi) - g^{ij}F_a^mF_b^nD_i(R_{jmn}^lD_l\varphi) \\
&= D_aD_b\triangle\varphi - g^{ij}F_a^mF_b^nD_m(R_{inj}^lD_l\varphi) \\
&\quad -g^{ij}F_a^mF_b^n(R_{imn}^pD_pD_j\varphi + R_{imj}^pD_nD_p\varphi) - g^{ij}F_a^mF_b^nD_i(R_{jmn}^lD_l\varphi)
\end{aligned}$$

Now we have

$$\begin{aligned} D_t D_a D_b \varphi &= \triangle D_a D_b \varphi + \frac{1}{2} R F_a^i F_b^j D_i D_j \varphi + \frac{1}{2} F_a^i F_b^j (D_i R \cdot D_j \varphi + R D_i D_j \varphi) \\ &\quad - g^{ij} F_a^m F_b^n D_m (R_{inj}^l D_l \varphi) - g^{ij} F_a^m F_b^n (R_{imn}^p D_p D_j \varphi + R_{imj}^p D_n D_p \varphi) \\ &\quad - g^{ij} F_a^m F_b^n D_i (R_{jmn}^l D_l \varphi) \end{aligned}$$

Since the curvature is bounded, the solution to the evolution equation of the orthonormal frame remains bounded on finite time interval $[0, t_0]$. We can denote the upper bound as an nonnegative constant M. Thus we have the inequality

$$\frac{\partial}{\partial t} |D^2 \varphi|^2 \leq (n-1) \triangle |D^2 \varphi|^2 + \frac{1}{t} C_2 M |D^2 \varphi|^2$$

where $\frac{1}{t}$ comes from the local derivative estimate in Lemma2.3.

Now we put

$$G = t^2 |D^2 \varphi|^2 + |D\varphi|^2$$

and compute

$$\begin{aligned} \frac{\partial G}{\partial t} &= 2t |D^2 \varphi|^2 + t^2 \frac{\partial}{\partial t} |D^2 \varphi|^2 + \frac{\partial}{\partial t} |D\varphi|^2 \\ &\leq 2t |D^2 \varphi|^2 + t^2 [(n-1) \triangle |D^2 \varphi|^2 + \frac{1}{t} C_2 M |D^2 \varphi|^2] \\ &\quad + (n-1) \triangle |D\varphi|^2 - 2(n-1) |D^2 \varphi|^2 + C_1 K |D\varphi|^2 \end{aligned}$$

and

$$\triangle G = t^2 \triangle |D^2 \varphi|^2 + \triangle |D\varphi|^2$$

thus

$$\frac{\partial G}{\partial t} \leq (n-1) \triangle G + (2t + t C_2 K) |D^2 \varphi|^2 - 2(n-1) |D^2 \varphi|^2 + C_1 K |D\varphi|^2.$$

Since $|D\varphi| \leq \delta < \rho \leq |D^2 \varphi|$ at $t = 0$, there is a constant $t_1 > 0$ such that $|D\varphi| \leq |D^2 \varphi|$ when $0 < t \leq t_1$. Then if $t \leq \min\{\frac{2n-3}{2+C_2 K}, t_0, t_1\}$, we have

$$\frac{\partial G}{\partial t} \leq (n-1) \triangle G$$

and the maximum of the function G decreases from the maximum principle. Denote $\min\{\frac{2n-3}{2+C_2 K}, t_0, t_1\}$ by t_2 from now on.

At time $t = 0$, $G = |D\varphi|^2 \leq \delta^2$, hence for $t \geq 0$ also. Thus $|D^2\varphi| \leq \frac{\delta}{t}$ for $0 < t \leq t_2$.

Since $|\triangle\varphi|^2 \leq n|D^2\varphi|^2$ and φ solves the equation $\frac{\partial\varphi}{\partial t} = (n-1)\triangle\varphi$,

$$|\frac{\partial\varphi}{\partial t}| \leq \frac{C_3\delta}{t}$$

for $0 < t \leq t_2$ where $C_3 = \sqrt{n(n-1)}$.

Thus for every point $P \in M - \overline{B_{t=0}(O, 2\sigma)}$,

$$|\varphi(P,t) - \varphi(P,0)| \leq C_3\delta \ln t$$

for $0 < t \leq t_2$, we can take

$$\delta \leq \frac{\varepsilon^2}{C_3 \ln t_2}$$

so that $C_3\delta \ln t \leq \varepsilon^2$ for $0 < t \leq t_2$. Then $\varphi \leq 2\varepsilon^2$ at times $0 < t \leq t_2$ on $M - \overline{B_{t=0}(O, 2\sigma)}$.

Notice that distances may expand, but from Lemma 2.1.1 we have $s(0)e^{-(n-1)Kt} \leq s(t) \leq s(0)e^{(n-1)Kt}$. This gives us a constant C_4 such that if $s \geq C_4 \cdot 2\sigma$ at P at $t \leq t_2$, then $s \geq 2\sigma$ and $\varphi \leq 2\varepsilon^2$ at P at time t.

Now at initial time $t = 0$, $|Rm|^2 \leq M^2 = \varphi$ if $s \leq \sigma$, and $|Rm|^2 \leq \varepsilon^2 \leq \varphi$ if $s \geq \sigma$. Thus $|Rm|^2 \leq \varphi$ everywhere at $t = 0$.

Since

$$\frac{\partial}{\partial t}|Rm|^2 \leq (n-1)\triangle|Rm|^2 + C|Rm|^3$$

we have

$$\frac{\partial}{\partial t}|Rm|^2 \leq (n-1)\triangle|Rm|^2 + CK|Rm|^2$$

while

$$\frac{\partial}{\partial t}(e^{CKt}\varphi) = (n-1)\triangle(e^{CKt}\varphi) + CK(e^{CKt}\varphi)$$

so $|Rm|^2 \leq e^{CKt}\varphi$ on $M - \overline{B_{t=0}(O, 2\sigma)}$ from the maximum principle.

For $t \leq t_2$, this gives $|Rm|^2 \leq C_5\varphi$ for some constant C_5 depending only on the dimension. Hence at time t we have

$$|Rm|^2 \leq C_6\varepsilon^2$$

for $s \geq C_6\rho$ where the constant C depends only on the dimension and is independent of ε. Thus $|Rm| \to 0$ for $t \leq t_2$ as $s \to \infty$.

Since the time interval can always be advanced by t_2, we get the result. $\square$

Now we can follow get Theorem 1.2 immediately.

Proof of Theorem 1.2: Let M^n be a complete manifold under the assumption of Theorem 1.1 and has nonnegative Ricci curvature. Choose an origin $O \in M$ and by the Bishop-Gromove volume comparison theorem,

$$s \mapsto \frac{Vol(B(O,s))}{s^n}$$

is nonincreasing. Denote ω_n the volume of the unit ball in the n-dimensional Euclidean space. We have

$$AVR(g) \leq \frac{Vol(B(O,s))}{s^n} \leq \omega^n$$

frome the Bishop-Gromove volume comparison theorem. For any $\delta > 0$ and $s > 0$, consider the annulus

$$\begin{aligned} A_{\delta s,s} &\doteq \{x \in M^n | \delta s \leq d_g(x,O) \leq s\} \\ &= B_g(O,s) - B_g(O,\delta s) \end{aligned}$$

Using Lemma 3.5 in [3](also Lemma B.40 in [2]), the distance of any two points changes as

$$\begin{aligned} \frac{\partial}{\partial t}[dist_{g(t)}(x,y)] &= -\frac{1}{2}\sup_{\gamma}\int_{\gamma} Rds \\ &\geq -4n(n-1)K^{\frac{1}{2}} \end{aligned}$$

where the inequality above comes from the upper bound of $|Rm|$ and Prop 1.94 in [2].

Now consider the volume of the annulus and follow the proof of Prop 8.37 in [2], we can get the result immediately. $\square$

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