Modulation and natural valued quiver of an algebra *

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Abstract

The concept of modulation is generalized to pseudo-modulation and its subclasses including pre-modulation, generalized modulation and regular modulation. The motivation is to define the valued analogue of natural quiver, called *natural valued quiver*, of an artinian algebra so as to correspond to its valued Ext-quiver when this algebra is not k-splitting over the field k. Moreover, we illustrate the relation between the valued Ext-quiver and the natural valued quiver.

The interesting fact we find is that the representation categories of a pseudo-modulation and of a pre-modulation are equivalent respectively to that of a tensor algebra of $\mathcal{A}$-path type and of a generalized path algebra. Their examples are given respectively from two kinds of artinian hereditary algebras. Furthermore, the isomorphism theorem is given for normal generalized path algebras with finite (acyclic) quivers and normal pre-modulations.

Four examples of pseudo-modulations are given: (i) group species in mutation theory as a semi-normal generalized modulation; (ii) viewing a path algebra with loops as a pre-modulation with valued quiver which has not loops; (iii) differential pseudo-modulation and its relation with differential tensor algebras; (iv) a pseudo-modulation is considered as a free graded category.

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1 Introduction

Throughout this paper, k denotes the ground field.

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It is well-known that for an artinian k-algebra A, one has either the Ext-quiver in the case A is k-splitting, or the valued Ext-quiver in otherwise case. This quiver Γ is used to characterize the structure of A by Gabriel theorem when A is basic, that is, $A \cong k\Gamma/I$ with admissible ideal I if A is k-splitting (e.g. if k is algebraically closed). Motivated by it, in [18], we define the notion of natural quiver for any artinian algebra in order to constitute the analogue of Gabriel theorem in the case that A is not k-splitting and even not basic. This aim has been achieved in the case when A is splitting over radical in [18]. The other important hand is the relation between natural quiver and Ext-quiver of a k-splitting artinian algebra, which is also given in [18].

However, when A is not k-splitting, the valued Ext-quiver of A can not be compared with the natural quiver of A. Hence, in general case, we have to consider the questions:

(i) How to define the valued analogue of natural quiver of A so as to correspond to the valued Ext-quiver of A?

(ii) Following (i), give the relation between the valued Ext-quiver and the valued analogue of natural quiver of A.

The first aim of this paper is to answer these questions. For this, the concept of modulation is generalized to the so-called *pseudo-modulation* and its subclasses including *pre-modulation*, *generalized modulation* and *regular modulation*, in Section 3.

For an artinian algebra A, the alteration of natural quiver corresponding to valued Ext-quiver, called as *natural valued quiver*, is introduced via the valued quiver of the corresponding pre-modulation of A. In the case A is basic, it is shown that the natural valued quiver is pair-opposite equal to the valued Ext-quiver (see Theorem 7.4). Moreover, in Theorem 7.5, for any artinian algebra A, the relation between its natural valued quiver and valued Ext-quiver is obtained, as an improvement of the relation between the natural quiver and Ext-quiver in the case A is over an algebraically closed field in [18].

The representation categories of a pseudo-modulation and of a pre-modulation are equivalent respectively to that of a tensor algebra of $\mathcal{A}$-path type and that of a generalized path algebra (Theorem 3.2 and Corollary 5.4). Their examples are given respectively from two kinds of artinian hereditary algebras (Corollary 3.3 and Proposition 7.1). Furthermore, the isomorphism theorem is given for normal generalized path algebras with finite (acyclic) quivers and normal pre-modulations in Theorem 5.5.

The notion of modulation was introduced in [6] and [7] to characterize representations of a valued quiver over a field k, which is not necessarily algebraically closed, using the method of Coxter functors in Bernstein-Gelfand-Ponomarev's theory. This aspect will be discussed for pseudo-modulations in the follow-up work.

Pseudo-modulations, as well as generalized path algebras in [17][18], can be realized as the tool to investigate some properties of structures and representations of an algebra which are not Morita invariants in the reason that (valued) natural quiver is not Morita invariant.

A kind of (semi-)normal generalized modulation is characterized, see Theorem 4.4. In Section 6, its interesting example is given from group species in mutation theory [5].

In the theory of mutations [15][5], it is known that for a finite dimensional basic hereditary algebra $A \cong k\Gamma$ for a quiver Γ, under the condition the mutation can be defined, the mutation of A is isomorphic to the path algebra of the quiver which is the mutation of Γ. Since mutations are perverse equivalent but not Morita equivalent (see [15]), it is interesting to constitute the mutation theory of finite dimensional (non-basic in general) algebras via semi-normal generalized modulations, due to Proposition 6.1.

Moreover, in Section 6, we suggest the method to transfer the study on path algebras whose quiver has loops into that on generalized path algebras and pre-modulations with valued quiver which has not loops. And, we still give the notion of differential pseudo-modulation and its relation with differential tensor algebras. Lastly, a k-pseudo-modulation $\mathcal{M}$ and also the related tensor algebra of $\mathcal{A}$-path-type $T(\mathcal{M})$ are equivalently considered as a free graded category $\mathcal{T}$.

2 Some preliminaries

2.1 A *quiver* Q can be understood as a two sets Q_0 and Q_1 together with a map $Q_1 \to Q_0 \times Q_0$ denoted by $a \mapsto (t(a), h(a))$ with $h(a)$ being called the *head* of the arrow a and $t(a)$ being called the *tail* of a. For each pair $(i,j) \in Q_0 \times Q_0$, we define

$$\Omega(i,j) = \{a \in Q_1 \mid t(a) = j, h(a) = i\}.$$

Note that Q_1 is the disjoint union of all $\Omega(i,j)$ for $i,j \in Q_0$.

Forgetting the orientation of all arrows in the quiver Q, we get the underlying graph of Q, which is denoted by $\overline{Q}$.

2.2 A *pseudo-valued graph* $(\mathcal{G}, \mathcal{D})$ consists of:

(i) A finite set $\mathcal{G} = \{i, j, \cdots\}$ whose elements are called *vertices*;

(ii) To any ordered pair $(i,j) \in \mathcal{G} \times \mathcal{G}$, there corresponds a non-negative integer d_{ij} satisfying that if $d_{ij} \neq 0$ then $d_{ji} \neq 0$ for any $(i,j) \in \mathcal{G} \times \mathcal{G}$. If $d_{ij} \neq 0$, such a pair (i,j) is called an *edge* between the vertices i and j, which is written as $i\bullet\overset{(d_{ij}, d_{ji})}{\overline{\hspace{3em}}}\bullet j$. If $d_{ij} = d_{ji} = 1$, write simply $i\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet j$.

Of course, $d_{ij} = d_{ji}$ when $i = j$.

The family $\mathcal{D} = \{(d_{ij}, d_{ji}) : (i,j) \in \mathcal{G} \times \mathcal{G}\}$ is called a *valuation* of the graph $\mathcal{G}$.

Moreover, due to [7][6], for a pseudo-valued graph $(\mathcal{G}, \mathcal{D})$, if there exist positive integers ε_i $(i \in \mathcal{G})$ such that $d_{ij}\varepsilon_j = d_{ji}\varepsilon_i$ for all $i, j \in \mathcal{G}$, then $(\mathcal{G}, \mathcal{D})$ is called a *valued graph.*

An *orientation* Ω of a (resp. pseudo-)valued graph $(\mathcal{G}, \mathcal{D})$ is given by prescribing for each edge an ordering, indicated by an oriented edge, that is,

$$\text{either } i\bullet\xrightarrow{(d_{ij}, d_{ji})}\bullet j \quad \text{or} \quad i\bullet\xleftarrow{(d_{ij}, d_{ji})}\bullet j.$$

We call a (resp. pseudo-)valued graph with orientation a *(resp. pseudo-)valued quiver*, which is denoted as $(\mathcal{G}, \mathcal{D}, \Omega)$.

A vertex $k \in \mathcal{G}$ in the valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ is called a *sink* (respectively, a *source*) if $i \neq k$ (respectively, $j \neq k$) for any oriented edge $i \bullet \xrightarrow{(d_{ij}, d_{ji})} \bullet j$.

A *path* of the pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ is a sequence $k_1, k_2, \cdots, k_t$ of vertices such that there is a valued oriented edge from k_s to k_{s+1} for $s = 1, 2, \cdots, t-1$. Its *length* is defined to be the number of the valued oriented edges in this path, that is, $t-1$.

2.3 Due to [6], a *k-modulation* $\mathcal{M} = (F_i, {}_iM_j)$ *of a valued graph* $(\mathcal{G}, \mathcal{D})$ is a set of division algebras $\{F_i\}_{i \in \mathcal{G}}$ which are finite-dimensional over a common central subfield k, together with a set $\{{}_iM_j\}_{i,j \in \mathcal{G}}$ of F_i-F_j-bimodules on which k acts centrally such that $dim({}_iM_j)_{F_j} = d_{ij}$, $dim_{F_i}({}_iM_j) = d_{ji}$ and ${}_jM_i$ is a dual of the bimodule ${}_iM_j$ in the sense that we have bimodule isomorphisms:

$$ {}_jM_i \cong Hom_{F_i}({}_iM_j, F_i) \cong Hom_{F_j}({}_iM_j, F_j). $$

Note that the final isomorphism is from [6]Lemma 0.2; there is an edge between i and j if and only if ${}_iM_j$ and ${}_jM_i$ are nonzero.

Now, let $(\mathcal{M}, \Omega)$ be a pair consisting of a k-modulation $\mathcal{M}$ of the connected valued graph $(\mathcal{G}, \mathcal{D})$, equivalently say, let $\mathcal{M} = (F_i, {}_iM_j)$ be a *k-modulation of a valued quiver* $(\mathcal{G}, \mathcal{D}, \Omega)$.

2.4 Associated with the pair $(A, {}_AM_A)$ for a k-algebra A and an A-bimodule M, we write the n-fold A-tensor product $M \otimes_A M \otimes \cdots \otimes_A M$ as M^n, then $T(A, M) = A \oplus M \oplus M^2 \oplus \cdots \oplus M^n \oplus \cdots$ as an abelian group. Writing $M^0 = A$, then $T(A, M)$ becomes a k-algebra with multiplication induced by the natural A-bilinear maps $M^i \times M^j \to M^{i+j}$ for $i \geq 0$ and $j \geq 0$. $T(A, M)$ is called the *tensor algebra* of M over A.

For a k-modulation $\mathcal{M} = (F_i, {}_iM_j)$ of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$, we get the tensor algebra $T(\mathcal{M}) \overset{def}{=} T(F, M)$ for $F = \oplus_{i \in \mathcal{G}} F_i$ and $M = \oplus_{(i,j) \in \mathcal{G} \times \mathcal{G}} {}_iM_j$, where M is acted by F as an F-F-bimodule through the projection maps $F \to F_i$ for $i \in \mathcal{G}$.

The interest of representations is displayed by the following:

Theorem 2.1. [6] *Let $\mathcal{M} = (F_i, {}_iM_j)$ be a k-modulation of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$. Then the category $rep(\mathcal{M})$ of all finite-dimensional representations is equivalent to the category $mod_{T(\mathcal{M})}$ of all finitely generated right $T(\mathcal{M})$-modules.*

This result is the generalization of that for the representation category of a finite-dimensional path algebra (see Theorem III.1.5 in [1]).

Modulation and its representations will be generalized in Section 3 such that they are only in the special case with linear spaces over division algebras.

2.5 For two rings A and B, the *rank* of a finitely generated left A-module (resp. right B-module, A-B-bimodule) M is defined as the minimal cardinal number of the sets generators of M as left A-module (resp. right B-module, A-B-bimodule), which is denoted by $rank_A M$ (resp. $rank M_B$, $rank_A M_B$). Clearly, if M is finitely generated, such rank always exists. As a convention, the rank of the module 0 is said to be 0.

Let $X = \{m_i\}_{i=1}^s$ be the set of generators of a finitely generated A-B-bimodule M, i.e. $M = \sum_{i=1}^s Am_iB$. If there never exist k-linearly independent sets

$$\{a_{iu} \in A : i = 1, \cdots, s; u = 1, \cdots, p\} \quad \text{and} \quad \{b_{iu} \in B : i = 1, \cdots, s; u = 1, \cdots, p\}$$

satisfying $\sum_{i=1,\cdots,s;u=1,\cdots,p} a_{iu}m_ib_{iu} = 0$, we say the set X to be A-B-*linearly independent*. In this case, we call M a *free A-B-bimodule with basis* X.

Clearly, if M is a free A-B-bimodule with basis $\{m_i\}_{i=1}^s$ and $\{b_j\}_{j=1}^t$ is a k-basis of B (resp. $\{a_j\}_{j=1}^t$ is a k-basis of A), then M is a left free A-module with basis $\{m_ib_j\}_{i=1,\cdot,s;j=1,\cdots,t}$ (resp. right free B-module with basis $\{a_jm_i\}_{i=1,\cdot,s;j=1,\cdots,t}$.).

Each A-B-bimodule M can be realized as a right $B \otimes A^{op}$-module. So, M is a free A-B-bimodule M if and only if M is a free right $B \otimes A^{op}$-module. In this case, let $M \cong \sum_i m_i(B \otimes A^{op})$ with basis $\{m_i\}$. Let $\{a_j\}$ be a k-basis of A. Then $M \cong \sum_{ij} m_ia_j \otimes B$ as B-modules where $m_ia_j \overset{def}{=} a_jm_i$. It says that $\{a_jm_i\}$ is an B-basis of $M = \sum_i m_i(B \otimes A^{op})$.

2.6 The concept of generalized path algebra was introduced early in [4]. Here we review the different but equivalent definition which is given in [18].

Let $Q = (Q_0, Q_1)$ be a quiver. Given a collection of k-algebras $\mathcal{A} = \{A_i \mid i \in Q_0\}$ with the identity $e_i \in A_i$. Let $A_0 = \prod_{i\in Q_0} A_i$ be the direct product k-algebra. Clearly, e_i are orthogonal central idempotents of A_0. Let

$$_iM_j \overset{def}{=} A_i\Omega(i,j)A_j \tag{1}$$

be the free A_i-A_j-bimodule with basis $\Omega(i,j)$. This is the free $A_i \otimes_k A_j^{op}$-module over the set $\Omega(i,j)$. Then, the rank of $_iM_j$ as A_i-A_j-bimodule is just the number of arrows from i to j in the quiver Q. Thus,

$$M = \oplus_{(i,j)\in Q_0\times Q_0} A_i\Omega(i,j)A_j \tag{2}$$

is an A_0-A_0-bimodule. The *generalized path algebra*[4][16][18] is defined to the tensor algebra

$$T(A_0, M) = \oplus_{n=0}^{\infty} M^{\otimes_{A_0} n}.$$

Here $M^{\otimes_{A_0} n} = M \otimes_{A_0} M \otimes_{A_0} \cdots \otimes_{A_0} M$ and $M^{\otimes_{A_0} 0} = A_0$. We denote the generalized path algebra by $k(Q, \mathcal{A})$. The generalized path algebra $k(Q, \mathcal{A})$ is called *(semi-)normal* if all A_i are (semi-)simple k-algebras.

3 Pseudo-modulations and representations of algebras

As we have seen, modulation essentially is determined by a tensor algebra. At this viewpoint, we will give the notion of pseudo-modulation in a more general way. According to our need, the discussion will be restricted to some special cases of pseudo-modulations.

Definition 3.1. *(i) A* k-pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a pseudo-valued graph $(\mathcal{G}, \mathcal{D})$ *is defined as a set of artinian k-algebras $\{A_i\}_{i\in\mathcal{G}}$, together with a set $\{{}_iM_j\}_{(i,j)\in\mathcal{G}\times\mathcal{G}}$ of finitely generated unital A_i-A_j-bimodules ${}_iM_j$ such that*

$$rank({}_iM_j)_{A_j} = d_{ij} \quad \text{and} \quad rank_{A_i}({}_iM_j) = d_{ji}.$$

(ii) A k-pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a pseudo-valued graph $(\mathcal{G}, \mathcal{D})$ is said to be (semi-)normal *if all A_i $(i \in \mathcal{G})$ are (semi-)simple algebras.*

(iii) For a k-pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a valued graph $(\mathcal{G}, \mathcal{D})$, if all ${}_iM_j$ are free as A_i-A_j-bimodule, then this pseudo-modulation is called a k-pre-modulation.

(iv) If a k-pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a pseudo-valued graph $(\mathcal{G}, \mathcal{D})$ satisfies

$$Hom_{A_i}({}_iM_j, A_i) \cong Hom_{A_j}({}_iM_j, A_j) \tag{3}$$

as A_j-A_i-bimodules for any $(i, j) \in \mathcal{G} \times \mathcal{G}$, then this pseudo-modulation is called a generalized k-modulation*.*

(v) For a generalized modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a valued graph $(\mathcal{G}, \mathcal{D})$, if all ${}_iM_j$ are free as A_i-A_j-bimodule for $i, j \in \mathcal{G}$, then $\mathcal{M} = (A_i, {}_iM_j)$ is called a regular k-modulation*.*

Trivially, a regular k-modulation is a generalized modulation and also a pre-modulation.

Note that each ${}_iM_j$ is required to be finite generated and ${}_iM_j \neq 0$ (meanwhile ${}_jM_i \neq 0$) if and only if there is an edge between i and j in the (pseudo-)valued graph $(\mathcal{G}, \mathcal{D})$.

Example 3.1. *(i) For a k-pseudo-modulation $\mathcal{M} = (F_i, {}_iM_j)$ of a pseudo-valued graph $(\mathcal{G}, \mathcal{D})$, if each F_i $(i \in \mathcal{G})$ is a division k-algebra, then $\mathcal{M} = (F_i, {}_iM_j)$ is just the modulation studied in [6][7].*

In fact, let $t_{ij} = rank_{F_i}{}_iM_j{}_{F_j}$, $dim_kF_i = \varepsilon_i$, $dim_kF_j = \varepsilon_j$. Then, $d_{ij} = dim({}_iM_j)_{F_j} = t_{ij}\varepsilon_i$, $d_{ji} = dim_{F_i}({}_iM_j) = t_{ij}\varepsilon_j$. Thus, $d_{ij}\varepsilon_j = d_{ji}\varepsilon_i$, which means $(\mathcal{G}, \mathcal{D})$ is a valued graph. The condition $Hom_{F_i}({}_iM_j, F_i) \cong Hom_{F_j}({}_iM_j, F_j)$ as F_j-F_i-bimodules for any $(i, j) \in \mathcal{G} \times \mathcal{G}$ is ensured by Lemma 0.2 in [6].

Hence, the classical modulation in [6][7] is a special class of regular modulations.

(ii) In particular, in (i), if F_i $(i \in \mathcal{G})$ are finite extension fields of k, then $\mathcal{M} = (F_i, {}_iM_j)$ is called a k-species of the valued graph $(\mathcal{G}, \mathcal{D})$*.*

(iii) Moreover, in (i), if the valued graph $(\mathcal{G}, \mathcal{D})$ is given an orientation Ω and $F_i = k$ $(i \in \mathcal{G})$, the bimodule ${}_iM_j$ is only a k-linear space such that $t_{ij} \overset{def}{=} d_{ij} = d_{ji}$ for any

pair $(i,j)\in\mathcal{G}\times\mathcal{G}$. Then the valued quiver $(\mathcal{G},\mathcal{D},\Omega)$ degenerates to a (non-valued) quiver $G=(G_0,G_1)$ whose arrow number from i to j is just t_{ij} if the pair (i,j) is oriented from i to j. Thus, in this case, $T(\mathcal{M})$ is just the path algebra kG.

In order to introduce representations of a pseudo-modulation, the pseudo-valued graph has to be given an orientation as below.

Given a k-pseudo-modulation $\mathcal{M}=(A_i,\ {}_iM_j)$ over a pseudo-valued quiver $(\mathcal{G},\mathcal{D},\Omega)$, we define a *representation* of $\mathcal{M}$ to be an object $\mathcal{V}=(V_i,\ {}_j\varphi_i)$, where to each vertex $i\in\mathcal{G}$ corresponds an A_i-module V_i and to each oriented edge $i\to j$ corresponds an A_j-homomorphism ${}_j\varphi_i:\ V_i\otimes_{A_i}{}_iM_j\longrightarrow V_j$. If each V_i is finitely generated as A_i-module, this representation $\mathcal{V}=(V_i,\ {}_j\varphi_i)$ is said to be *finitely generated*.

In the above definition, in the case that $A_i=A_j=k$ and let $dim_k{}_iM_j=t_{ij}$, then $d_{ij}=d_{ji}=t_{ij}$ and we get a representation $\mathcal{V}=(V_i,\ {}_j\varphi_i)$ of the non-valued quiver G (that is, representation of kG) with ${}_j\varphi_i:\ V_i\longrightarrow V_j$.

A *morphism* α from a representation $\mathcal{V}=(V_i,\ {}_j\varphi_i)$ to another representation $\mathcal{U}=(U_i,\ {}_j\psi_i)$ consists of A_i-module homomorphisms $\alpha_i:\ V_i\to U_i$ for all $i\in\mathcal{G}$ preserving the structure of the objects, that is, such that all diagrams:

$$\begin{tikzcd} V_i\otimes_{A_i}{}_iM_j \arrow[r, "{}_j\varphi_i"] \arrow[d, "\alpha_i\otimes_{A_i}id_{{}_iM_j}"] & V_j \arrow[d, "\alpha_j"] \\ U_i\otimes_{A_i}{}_iM_j \arrow[r, "{}_j\psi_i"] & U_j \end{tikzcd}$$

commute for each oriented edge $i\to j$.

Let $Rep(\mathcal{M})$ (resp. $rep(\mathcal{M})$) be the category consisting of all (resp. finitely generated) representations of $\mathcal{M}$.

For a k-pseudo-modulation $\mathcal{M}=(A_i,\ {}_iM_j)$ of a pseudo-valued quiver $(\mathcal{G},\mathcal{D},\Omega)$, we get the tensor algebra $T(\mathcal{M})\overset{def}{=}T(A,M)$ for $A=\oplus_{i\in\mathcal{G}}A_i$ and $M=\oplus_{(i,j)\in\mathcal{G}\times\mathcal{G}}{}_iM_j$, where M is acted by A as an A-A-bimodule through the projection maps $A\to A_i$ for $i\in\mathcal{G}$.

Conversely, for a tensor algebra $T(A,M)$ with $A=\oplus_{i\in I}A_i$ and $M=\oplus_{(i,j)\in I\times I}{}_iM_j$ for subalgebras A_i and A_i-A_j-bimodules ${}_iM_j$ $(i,j\in I)$. Let $d_{ij}=rank({}_iM_j)_{A_j}$ and $d_{ji}=rank_{A_i}({}_iM_j)$. Denote $\mathcal{D}=\{d_{ij},d_{ji}:(i,j)\in I\times I\}$, $\mathcal{G}=I$. For any ${}_iM_j\neq 0$, give an oriented edge from i to j. Then we get a pseudo-valued quiver $(\mathcal{G},\mathcal{D},\Omega)$ and a k-pseudo-modulation $\mathcal{M}=(A_i,\ {}_iM_j)$. We call such a tensor algebra as above *an $\mathcal{A}$-path-type tensor algebra*[16] on the pseudo-valued quiver $(\mathcal{G},\mathcal{D},\Omega)$.

Therefore, we have

Proposition 3.1. *Pseudo-modulations and tensor algebras of $\mathcal{A}$-path-type with finitely generated bimodules can be constructed with each other in the above way.*

Clearly, representations of the classical modulations and their morphisms in [6][7] are both the special cases of that of pseudo-modulations given here.

As a generalization of Theorem 2.1, we have the following result about k-pseudo-modulation:

Theorem 3.2. *Let $\mathcal{M} = (A_i, {}_iM_j)$ be a k-pseudo-modulation of a pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$. Then the category $Rep(\mathcal{M})$ (resp. $rep(\mathcal{M})$) of all (resp. finitely generated) representations of $\mathcal{M}$ is equivalent to the category $Mod_{T(\mathcal{M})}$ (resp. $mod_{T(\mathcal{M})}$) of (resp. finitely generated) right $T(\mathcal{M})$-modules.*

Proof. Let $\mathcal{V} = (V_i, {}_j\varphi_i)$ be a representation of $\mathcal{M}$. Define the corresponding right $T(\mathcal{M})$-module V as follows.

Let $V = \oplus_{i\in\mathcal{G}} V_i$. Firstly, the right A-action on V is given via the projections $A \to A_i$ for $i \in \mathcal{G}$, and then the right M-action on V is defined by the ${}_j\varphi_i$, that is, for the oriented edge $i \to j$, $v_i m_{ij} = {}_j\varphi_i(v_i \otimes m_{ij})$ for $v_i \in V_i$ and $m_{ij} \in {}_iM_j$, and moreover, extending by distributivity; finally, the $T(\mathcal{M})$-action on V is determined inductively in a unique manner, by the M-action, that is,

$$v_i(m_{ij} \otimes \cdots \otimes m_{pq} \otimes m_{qs}) \;=\; {}_s\varphi_q((v_i(m_{ij} \otimes \cdots \otimes m_{pq})) \otimes m_{qs}).$$

Thus, V becomes a $T(\mathcal{M})$-module.

And, if α is a morphism of representations from $\mathcal{V}$ to $\mathcal{U}$, then we can define the $T(\mathcal{M})$-module morphism $\overline{\alpha}$ from V to U with $\overline{\alpha}(\oplus_{i\in\mathcal{G}} v_i) = \oplus_{i\in\mathcal{G}} \alpha_i(v_i)$. Thus, we get the functor $F : Rep(\mathcal{M}) \to mod_{T(\mathcal{M})}$ with $F(\mathcal{V}) = V$ and $F(\alpha) = \overline{\alpha}$. In fact, for $\alpha : \mathcal{V} \to \mathcal{U}$, $\beta : \mathcal{U} \to \mathcal{W}$, we have $\beta \cdot \alpha = \{\beta_i\alpha_i : i \in \mathcal{G}\}$, then $F(\beta \cdot \alpha) = F(\beta) \cdot F(\alpha)$.

Conversely, we can define the inverse functor G. Given $V \in Mod_{T(\mathcal{M})}$, let $V_i = VA_i$. Then $V = VA = \oplus_{i\in\mathcal{G}} VA_i = \oplus_{i\in\mathcal{G}} V_i$. When there is an oriented edge $i \to j$, ${}_iM_j \neq 0$. We have $V_i \cdot {}_iM_j = VA_i \cdot {}_iM_j = V \cdot {}_iM_jA_j \subset VA_j = V_j$. Then, we can induce the A_j-module morphisms ${}_j\varphi_i : V_i \otimes_{A_i} {}_iM_j \to V_j$ under this M-action. Thus, by the definition, $\mathcal{V} = (V_i, {}_j\varphi_i)$ is a representation of $\mathcal{M}$, that is, $\mathcal{V} \in Rep(\mathcal{M})$.

For V, $U \in Mod_{T(\mathcal{M})}$ and $\overline{\alpha} : V \to U$ a $T(\mathcal{M})$-homomorphism, let $\alpha_i = \overline{\alpha}|_{V_i}$, then $\alpha_i(V_i) = \alpha_i(VA_i) = \overline{\alpha}(V)A_i \subset UA_i = U_i$. From the $T(\mathcal{M})$-linearity of $\overline{\alpha}$, we get that the diagram

$$\begin{array}{ccc} V_i \otimes_{A_i} {}_iM_j & \xrightarrow{\;{}_j\varphi_i\;} & V_j \\ \Big\downarrow{\scriptstyle \alpha_i \otimes_{A_i} 1_{{}_iM_j}} & & \Big\downarrow{\scriptstyle \alpha_j} \\ U_i \otimes_{A_i} {}_iM_j & \xrightarrow{\;{}_j\psi_i\;} & U_j \end{array}$$

commutes for each oriented edge $i \to j$, where ${}_j\psi_i$ is defined as similarly as ${}_j\varphi_i$. So, $\alpha = \{\alpha_i : i \in \mathcal{G}\}$ is a morphism from $\mathcal{V}$ to $\mathcal{U}$ in $Rep(\mathcal{M})$. Define the functor G satisfying $G(V) = \mathcal{V}$ and $G(\overline{\alpha}) = \alpha$. For $\overline{\alpha} : V \to U$ and $\overline{\beta} : U \to W$, it follows that $\overline{\alpha} = \oplus_{i\in\mathcal{G}} \alpha_i$ and $\overline{\beta} = \oplus_{i\in\mathcal{G}} \beta_i$. Then, $\overline{\beta} \cdot \overline{\alpha} = \oplus_{i\in\mathcal{G}} \beta_i\alpha_i$. Hence, $G(\overline{\beta} \cdot \overline{\alpha}) = \{\beta_i\alpha_i : i \in \mathcal{G}\} = \beta \cdot \alpha = G(\overline{\beta}) \cdot G(\overline{\alpha})$.

Obviously, F and G are mutual-inverse equivalence functors between $Rep(\mathcal{M})$ and $Mod_{T(\mathcal{M})}$. $\square$

In [8], it was proved that for a finite dimensional algebra A with radical r, if the quotient algebra A/r is separable, then A is isomorphic to a quotient algebra of $T(A/r, r/r^2)$ by an admissible ideal I, that is, $J^s \subset I \subset J^2$ for some positive integer s.

Moreover, if this algebra A is hereditary, then $I = 0$ such that $A \cong T(A/r, r/r^2)$. Let $A/r = \oplus_{i=1}^{s} A_i$ where A_i are simple ideals of A/r. Then, r/r^2 is an A/r-A/r-bimodule with natural left and right module actions. Let ${}_iM_j = A_i r/r^2 A_j$ for any $i, j = 1, \cdots s$, then ${}_iM_j$ is an A_i-A_j-bimodule. By Proposition 3.1, the corresponding pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ can be constructed from this tensor algebra $T(A/r, r/r^2)$, which is called the *related pseudo-modulation* of the finite dimensional hereditary A. Therefore, by Theorem 3.2, we can state that

Corollary 3.3. *For a finite dimensional hereditary algebra A with radical r and its related pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$, if A/r is separable, then the (resp. finitely generated) representation category $Rep(\mathcal{M})$ (resp. $rep(\mathcal{M})$) is equivalent to the (resp. finitely generated) module category Mod_A (resp. mod_A).*

4 A kind of generalized k-modulations

By the Wedderburn-Artin Theorem, the center of A is just the field k. Define $\mu : A \to End_k(A)$ with $\mu(a) = \rho_a$ where ρ_a is the right translation on A by the right multiplication of a. Obviously, $\rho_a \in End_k(A)$. It is easy to check that μ is a monomorphism of algebras.

Define $t : A \to k$ with $t(a) = tr(\mu(a))$. Then, t is the character of the right regular representation of A satisfying that $t(ab) = t(ba)$ for any $a, b \in A$. In fact, trivially, t is k-linear and $t(ab) = tr(\mu(ab)) = tr(\mu(a)\mu(b)) = tr(\mu(b)\mu(a)) = t(ba)$.

Lemma 4.1. *Assume A is a finite-dimensional simple k-algebra with k algebraically closed whose characteristic $chark \nmid \sqrt{dim_k A}$. For any $a \neq 0$ in A, there holds that $t(aA) \neq 0$.*

Proof. Thanks to the Wedderburn-Artin theorem, $A \cong M_n(k)$ the $n \times n$ full matrix algebra over k for $n = \sqrt{dim_k A}$. For simply, we think a is a non-zero $n \times n$ matrix and as right ideal of A, $aA \neq 0$ consists of all $n \times n$ matrices over k whose all rows are 0 except for some $i_1, i_2, \cdots i_s$-rows. Choose a matrix $X = E_{i_1 i_1}$ in aA with the element 1 in position (i_1, i_1) and 0 in all other positions. Then under the k-basis $\{E_{ij}\}_{i,j=1}^{n}$ of A, $t(X) = tr\mu(X) = n \cdot 1 \neq 0$ since $chark \nmid n$. Therefore, we have $t(aA) \neq 0$. $\square$

Lemma 4.2. *Let A and B be finite-dimensional simple k-algebras with k algebraically closed whose characteristic $chark \nmid \sqrt{dim_k A}$ and $chark \nmid \sqrt{dim_k B}$. Then, for an A-B-bimodule M, $Hom_A(M, A) \cong Hom_B(M, B)$ as B-A-bimodules.*

Proof. Firstly, we prove $Hom_A(M, {}_AA_A) \cong Hom_k(M, k)$ as B-A-bimodules, where $Hom_k(M, k)$ consists of all k-homomorphism with the bimodule structure defined by $(b\psi a)(m) = \psi(amb)$ for $a \in A$, $b \in B$, $m \in M$, $\psi \in Hom_k(M, k)$.

Indeed, for b_1, $b_2 \in B$, $m \in M$,

$$((b_1b_2)\psi)(m) = \psi(mb_1b_2) = \psi((mb_1)b_2) = (b_2\psi)(mb_1) = (b_1(b_2\psi))(m),$$

then $(b_1b_2)\psi = b_1(b_2\psi)$, and similarly, for $a_1, a_2 \in A$, $\psi(a_1a_2) = (\psi a_1)a_2$.

Now, define the map $\tau : A \to Hom_k(A,k)$ by $\tau(a) = at$ for $a \in A$, where $at \in Hom_k(A,k)$ by $(at)(x) = t(ax)$ for $x \in A$. Obviously, τ is k-linear.

Moreover, τ is injective. In fact, for $a \in ker\tau$, it means that for any $x \in A$, $(at)(x) = 0$, then $t(ax) = 0$, or say, $aA \subset kert$ for the right ideal aA of A, which is equivalent to $t(aA) = tr\mu(aA) = 0$. Thus, $a = 0$ according to Lemma 4.1. Hence, $ker\tau = 0$.

Since $dim_kA = dim_kHom_k(A,k)$ are finite, we obtain that τ is a k-linear isomorphism.

Similarly, define $ta \in Hom_k(A,k)$ by $(ta)(x) = t(xa)$ for $x \in A$. Since $t(ax) = t(xa)$ for any $a, x \in A$, we get $\tau(a) = at = ta$. Naturally, it follows that ${}_AA_A \overset{\tau}{\cong} Hom_k(A,k)$ as A-A-bimodules. Consequently,

$$Hom_A(M,\ {}_AA_A) \cong Hom_A(M, Hom_k(A,k)) \cong Hom_k(A \otimes_A M, k) \cong Hom_k(M,k)$$

as required as B-A-bimodules.

Similarly, $Hom_B(M,\ {}_BB_B) \cong Hom_k(M,k)$ holds as B-A-bimodules. Therefore, we have $Hom_A(M,\ {}_AA_A) \cong Hom_B(M,\ {}_BB_B)$. □

This lemma is an improvement of Lemma 0.2 in [6].

Trivially, the condition in Lemma 4.2 is always satisfied if the field k is algebraically closed of characteristic 0.

Lemma 4.3. *Let A and B be finite-dimensional semisimple algebras over k algebraically closed of characteristic 0. Then, for an A-B-bimodule M, $Hom_A(M,A) \cong Hom_B(M,B)$ holds as B-A-bimodules.*

Proof. Let $A = \oplus_{i=1}^s A_i$, $B = \oplus_{j=1}^t B_j$ with simple ideals A_i and B_j. Then,

$$\begin{aligned} Hom_A(M,A) &\cong \oplus_{i=1}^s Hom_A(A_iM, A_i) \\ &\cong \oplus_{i=1}^s Hom_{A_i}(A_iM, A_i) \\ &\cong \oplus_{i=1}^s \oplus_{j=1}^t Hom_{A_i}(A_iMB_j, A_i) \\ &\cong \oplus_{i=1}^s \oplus_{j=1}^t Hom_{B_j}(A_iMB_j, B_j) \\ &\cong \oplus_{j=1}^t Hom_{B_j}(AMB_j, B_j) \\ &\cong Hom_B(M,B). \end{aligned}$$

□

Using Lemma 4.3 to the A_i and ${}_iM_j$ below, by Definition 3.1, we can obtain:

Theorem 4.4. *$\mathcal{M} = (A_i, {}_iM_j)$ be a pseudo-modulation of a pseudo-valued graph $(\mathcal{G}, \mathcal{D})$ over an algebraically closed filed k of characteristic* 0*. If all A_i $(i \in \mathcal{G})$ are (semi-)simple algebras, then $\mathcal{M} = (A_i, {}_iM_j)$ is a (semi-)normal generalized modulation.*

From this theorem and its proof, we see that the condition (3) in Definition 3.1, which is required by the definition of the classical modulation in Section 2, is not always true for pseudo-modulations, in the reason that for a pseudo-modulation, its pseudo-valued quiver is a *natural quiver* analogue, not Ext-quiver, of its corresponding tensor algebra of $\mathcal{A}$-path type as similar as that of a generalized path algebra, see Section 5 and Second 7.

5 Pre-modulations and generalized path algebras

In this part, we give some pre-modulations and their applications to generalized path algebras and artinian algebras.

As an generalization of path algebras, in [16][17][18][21], normal generalized path algebras are used to characterize the structures and representations of artinian algebras via the method of natural quivers. This is unlike to the classical method to depend upon the corresponding basic algebras.

In [6][7], k-representation types of valued quivers are classified through the corresponding relations between valued quivers and k-modulations. In the sequeal, we will see that the corresponding relationship still holds between (semi-)normal generalized path algebras and (semi-)normal regular k-modulations.

Lemma 5.1. *For a generalized path algebra $k(Q, \mathcal{A})$ and M and ${}_iM_j$ defined as in (1) and (2), let $\varepsilon_i = dim_k A_i$ and $d_{ij} = rank({}_iM_j)_{A_j}$, $d_{ji} = rank_{A_i}({}_iM_j)$ for all $i, j \in Q_0$. Then, $d_{ij}\varepsilon_j = d_{ji}\varepsilon_i$ for any $i, j \in Q_0$.*

Proof. Let $\{m_l\}_{l\in\Lambda}$ be an A_i-A_j-basis of ${}_iM_j$ as free A_i-A_j-bimodule. And, let $\{a_s\}_{s\in\Phi}$ and $\{b_t\}_{t\in\Psi}$ are respectively k-bases of A_i and A_j. Then, ${}_iM_j$ is right A_j-free and left A_i-free with A_j-basis $\{a_s m_l\}_{s\in\Phi, l\in\Lambda}$ and A_i-basis $\{m_l b_t\}_{l\in\Lambda, t\in\Psi}$ respectively. Thus, $|\Phi| = \varepsilon_i$, $|\Psi| = \varepsilon_j$, and $|\Phi||\Lambda| = d_{ij}$, $|\Lambda||\Psi| = d_{ji}$. So, $|\Lambda| = d_{ij}/\varepsilon_i = d_{ji}/\varepsilon_j$. It follows that $d_{ij}\varepsilon_j = d_{ji}\varepsilon_i$. □

By this lemma, we can get the valued quiver $(Q_0, \mathcal{D}, \Omega)$, which is called the *induced valued quiver* from $k(Q, \mathcal{A})$, where the valuation $\mathcal{D} = \{(d_{ij}, d_{ji}) : (i, j) \in Q_0 \times Q_0\}$ and there is just a unique oriented edge from i to j when ${}_iM_j \neq 0$.

By Definition 3.1 and Theorem 4.4, we have:

Proposition 5.2. *For a generalized path algebra $k(Q, \mathcal{A})$ over a field k and M and ${}_iM_j$ defined as in (1) and (2), then* *(i) A k-pre-modulation $\mathcal{M} = (A_i,\ {}_iM_j)$ is obtained from the induced valued quiver $(Q_0, \mathcal{D}, \Omega)$ with $d_{ij} = rank({}_iM_j)_{A_j}$, $d_{ji} = rank_{A_i}({}_iM_j)$ for the valuation $\mathcal{D} = \{(d_{ij}, d_{ji}) : (i,j) \in Q_0 \times Q_0\}$;*

(ii) Moreover, if k is an algebraically closed field of characteristic 0 and $k(Q,\mathcal{A})$ is semi-normal, then for an arrow $i \to j$, i.e. ${}_iM_j \neq 0$, it holds that $Hom({}_iM_j, A_i)_{A_i} \cong Hom({}_iM_j, A_j)_{A_j}$ by Theorem 4.4, which means that in this case, $\mathcal{M} = (A_i,\ {}_iM_j)$ is a regular modulation.

By definition, such k-pre-modulation $\mathcal{M} = (A_i,\ {}_iM_j)$ built from the $\mathcal{A}$-path algebra $k(Q,\mathcal{A})$ is unique, which is called the *corresponding k-pre-modulation of* $k(Q,\mathcal{A})$, denoted as $\mathcal{M}_{k(Q,\mathcal{A})}$, whose valued quiver is just the induced valued quiver from $k(Q,\mathcal{A})$.

Conversely, given a k-pre-modulation $\mathcal{M} = (A_{i,i}M_j)$ of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ with semisimple algebras A_i $(i \in \mathcal{G})$, we illustrate how to build its generalized path algebra. In fact, we only need to set up the quiver Q for a generalzied path algebra. Let the vertex set $Q_0 = \mathcal{G}$. For any oriented pair $(i,j) \in \mathcal{G} \times \mathcal{G}$, let t_{ij} be the number of generators in the A_i-A_j-basis of ${}_iM_j$ as free A_i-A_j-bimodule and set t_{ij} arrows from i to j. Then, the arrow set Q_1 is given when the oriented pair (i,j) runs over the whole $\mathcal{G} \times \mathcal{G}$. Thus, the quiver Q is constructed and then the normal path algebra $k(Q,\mathcal{A}) = T(A_0, M)$ is obtained where $M = \oplus_{i,j} A_i\Omega(i,j)A_j$ and $A_0 = \prod_{i \in Q_0} A_i$.

Since ${}_iM_j$ and $A_i\Omega(i,j)A_j$ have the same numbers of generators in their bases as free A_i-A_j-bomodules, we get ${}_iM_j \cong A_i\Omega(i,j)A_j$ for any $(i,j) \in \mathcal{G} \times \mathcal{G}$ following the invariant basis property of all A_i as semi-simple algebras. Hence, the pre-modulation constructed from $k(Q,\mathcal{A})$ in the way of Proposition 5.2 is just $\mathcal{M} = (A_{i,i}M_j)$.

Thus, we have obtained the following:

Theorem 5.3. *Pre-modulations and generalized path algebras can be constructed with each other in the above way. When the field k is algebraically closed of characteristic 0, (semi-)normal pre-modulations are (semi-)normal regular modulations.*

By this, as a corollary of Theorem 3.2, we have:

Corollary 5.4. *For a generalized path algebra $k(Q,\mathcal{A})$ and the corresponding k-pre-modulation $\mathcal{M} = (A_i,\ {}_iM_j)$, the category $Rep(\mathcal{M})$ (resp. $rep(\mathcal{M})$) is equivalent to the category $Mod_{k(Q,\mathcal{A})}$ (resp. $mod_{k(Q,\mathcal{A})}$).*

Concretely, using of the functors in the proof of Theorem 3.2, we can give the mutual constructions between representations of a generalized path algebra $k(Q,\mathcal{A})$ and that of its corresponding k-pre-modulation $\mathcal{M} = (A_{i,i}M_j)$.

Two k-pseudo-modulations $\mathcal{M} = (A_i,\ {}_iM_j)$ of the pseudo-valued quiver $(Q_0, \mathcal{D}, \Omega)$ and $\mathcal{N} = (B_i,\ {}_iN_j)$ of $(P_0, \mathcal{C}, \Psi)$ are said to be *isomorphic* if there exists a permutation θ such that $(Q_0, \mathcal{D}, \Omega) \overset{\theta}{\cong} (P_0, \mathcal{C}, \Psi)$ as pseudo-valued quivers and $A_i \cong B_{\theta(i)}$ as k-algebras,

${}_iM_j \cong {}_{\theta(i)}N_{\theta(j)}$ as bimodules for any $(i,j) \in Q_0 \times Q_0$. Here, $(Q_0, \mathcal{D}, \Omega) \overset{\theta}{\cong} (P_0, \mathcal{C}, \Psi)$ as pseudo-valued quivers means that they are isomorphic via a permutation θ as directed graphs and $d_{ij} = c_{\theta(i)\theta(j)}$, $d_{ji} = c_{\theta(j)\theta(i)}$ for any $(i,j) \in Q_0 \times Q_0$.

Although pseudo-modulation and tensor algebra of $\mathcal{A}$-path type can be constructed with each other as stated in Proposition 3.1, isomorphism condition can not be shifted between them. In fact, if two pseudo-modulations are isomorphic, then their related tensor algebras are isomorphic, too. But, the converse is not true.

For example, let Δ be a quiver consisting of a unique vertex without loops and Δ' be a quiver consisting of two vertices without loops and arrows, then clearly $\Delta \ncong \Delta'$. For any two artinian algebras S_1 and S_2, we have $k(\Delta, \{S_1 \oplus S_2\}) \cong S_1 \oplus S_2 \cong k(\Delta', \{S_1, S_2\})$. However, trivially, their related pre-modulations $\mathcal{M}_{k(\Delta,\{S_1 \oplus S_2\})} \ncong \mathcal{M}_{k(\Delta',\{S_1,S_2\})}$.

This example means that the isomorphism theorem does not hold for generalized path algebras, in general. Now, we give some cases of generalized path algebras in which the isomorphism theorem holds.

(i)[24] The path algebras $kQ \cong kP$ if and only if $Q \cong P$ as quivers.

(ii)[3] If finite quivers Δ and Δ' are acyclic, then normal generalized path algebras $k(\Delta, \mathcal{A}) \cong k(\Delta', \mathcal{A}')$ if and only if there is $\Delta \overset{\theta}{\cong} \Delta'$ as quivers such that $A_i \cong A'_{\theta(i)}$ as algebras for $i \in Q_0$.

(iii) When Δ and Δ' have oriented cycles, the isomorphism theorem for $k(\Delta, \mathcal{A})$ and $k(\Delta', \mathcal{A}')$ as in (ii) can also be proved in the similar method of (i) given in [24] or the dual method for generalized path coalgebras given in [19].

As a summary, we have:

Theorem 5.5. (Isomorphism Theorem) *Two normal generalized path algebras with finite (acyclic) quivers are isomorphic if and only if their corresponding normal k-pre-modulations are isomorphic.*

Another example of k-pre-modulation for which isomorphism theorem holds is the classical k-modulation (see Example 3.1(i)). For k-modulations $\mathcal{M} = (F_i, {}_iM_j)$ of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ and $\mathcal{M}' = (F'_s, {}_sM'_t)$ of $(\mathcal{G}', \mathcal{D}', \Omega')$ with division k-algebras F_i, F'_s, denote by $T(\mathcal{M})$ and $T(\mathcal{M}')$ the corresponding tensor algebras as given in the proof of Proposition 3.1, then in [22] it was shown that $\mathcal{M} \cong \mathcal{M}'$ if and only if $\mathcal{T}(M) \cong \mathcal{T}(M')$.

6 Some examples from related topics

(1) Group species

A **group species**[5] is a triple $G = (I, (\Gamma_i)_{i \in I}, (M_{ij})_{(i,j) \in I^2})$ where I is a finite set and for each $i \in I$, Γ_i is a finite group and for each $(i,j) \in I^2$, M_{ij} is a finite dimensional $(k\Gamma_i, k\Gamma_j)$-bimodule.

A group species can be seen as a k-pseudo-modulation of a pseudo-valued quiver as follows. Consider $Q_0 = I$ as the vertex set. For an ordered pair $(i,j) \in I \times I$, if $M_{ij} \neq 0$, set an arrow ρ_{ij} from i to j with valuation (d_{ij}, d_{ji}) for $d_{ij} = rank({}_{k\Gamma_i}M_{ij})$ and $d_{ji} = rank({M_{ij}}_{k\Gamma_j})$. Let such arrows compose the arrow set Q_1. Let $\mathcal{D} = \{(d_{ij}, d_{ji}) : \forall \rho_{ij} \in Q_1\}$. Thus, $(\mathcal{Q}, \mathcal{D})$ is a pseudo-valued quiver with $\mathcal{Q} = (Q_0, Q_1)$ and the group species G can be thought as $((k\Gamma_i)_{i\in Q_0}, (M_{ij})_{(i,j)\in Q_1})$ a pseudo-modulation of $(\mathcal{Q}, \mathcal{D})$.

In [5], let a group species $G = (I, (\Gamma_i)_{i\in I}, (M_{ij})_{(i,j)\in I^2})$ be over a field k with char$k \not| \mid \Gamma_i \mid$ for $i \in I$. In this case, all $k\Gamma_i$ are semisimple algebras. By Theorem 4.4, we have

Proposition 6.1. *Suppose k is an algebraically closed field of characteristic 0. Then the pseudo-modulation $((k\Gamma_i)_{i\in Q_0}, (M_{ij})_{(i,j)\in Q_1})$ from a group species $G = (I, (\Gamma_i)_{i\in I}, (M_{ij})_{(i,j)\in I^2})$ is a semi-normal generalized modulation, where $Q_0 = I$, $Q_1 = \{(i,j) \in I^2 : \ M_{ij} \neq 0\}$.*

As mentioned in [5], the category of representations of a group species is equivalent to the category of finite generated representations over its "path algebra" (i.e. its tensor algebra). According to Proposition 6.1, this statement is a special case of Theorem 3.2.

The notion of group species is introduced in [5] with potentials and dcorated representations. In some *good* cases, said to be *non-degenerate*, their mutations are defined in such a way that these mutations mimic the mutations of seeds defined by Fomin and Zelevinsky [9] for a skew-symmetrizable exchange matrix defined from group species. When an exchange matrix can be associated to a non-degenerate group species with potential, an interpretation of the F-polynomials and the g-vectors in [10] is given in the term of the mutation of group species with potentials and their decorated representations.

Due to Proposition 6.1, we will be motivated to plan to generalize the conclusions in [5] as said above to semi-normal generalized modulation. In the theory of mutations, it is known that for a finite dimensional basic hereditary algebra $A = k\Gamma$, under the condition the mutation can be defined, the mutation of A is just isomorphic to the path algebra of the quiver which is the mutation of Γ. However, since mutations are perverse equivalent but not Morita equivalent (see [15]), it is interesting to constitute the mutation theory of finite dimensional (possibly, non-basic) algebras via semi-normal generalized modulations.

(2) Path algebras with loops

As well-known, many subjects will be difficult if the underlying quiver has loops. For examples, Kac conjectures were discussed for quivers without loops, see [11][12][29]; the mutation theory of basic algebras was given in the case for quivers without loops, see [15].

We hope to consider quivers with loops under the viewpoint of pseudo-modulations so as to give a possible approach to study such quivers for some related theories.

For a quiver $\Gamma = (\Gamma_0, \Gamma_1)$, divide the vertex set Γ_0 into two parts: $\Gamma_0 = \Gamma_0^0 \cup \Gamma_0^1$ where Γ_0^0 consists of all vertices without loops, Γ_0^1 consists of all vertices with loops. For a vertex $i \in \Gamma_0^1$, let Φ_i be the subquiver consisting of all loops at i. Then the whole set of loops in

Γ is just $\Phi = \bigcup_{i\in\Gamma_0^1}(\Phi_i)_1$. Define a new quiver $\breve{\Gamma}$ related to Γ with the vertex set $\breve{\Gamma}_0 = \Gamma_0$ and the arrow set $\breve{\Gamma}_1 = \Gamma_1\backslash\Phi$. Clearly, this quiver $\breve{\Gamma}$ is one without loops.

The important fact is that $k\Gamma$ can be considered as a k-pre-modulation over the quiver $\breve{\Gamma}$ without loops.

In fact, let a collection of k-algebras be $\mathcal{A} = \{A_i \mid i \in \Gamma_0 = \breve{\Gamma}_0\}$ with $A_i = k$ for $i \in \Gamma_0^0$ and $A_i = k\Phi_i$ for $i \in \Gamma_0^1$; let $\breve{\Omega}(i,j) = \{a \in \breve{\Gamma}_1 : t(a) = j, h(a) = i\}$. Then, for any $i,j \in \breve{\Gamma}_0, i \neq j$, ${}_iM_j \overset{def}{=} A_i\breve{\Omega}(i,j)A_j$ is the free A_i-A_j-bimodule with basis $\breve{\Omega}(i,j)$; for any $i \in \breve{\Gamma}_0$, ${}_iM_i \overset{def}{=} 0$ and $\breve{\Omega}(i,i) = \emptyset$. Thus, the path algebra $k\Gamma$ is just the generalized path algebra $k(\breve{\Gamma},\mathcal{A})$ over the quiver $\breve{\Gamma}$ without loops.

By Proposition 5.2, $k\Gamma = k(\breve{\Gamma},\mathcal{A})$ is considered as the pre-modulation $\mathcal{M} = (A_i, {}_iM_j)$ over the valued quiver $(\Gamma_0,\mathcal{D},\Omega)$ for the valuation $\mathcal{D} = \{(d_{ij}, d_{ji}) : (i,j) \in \breve{\Gamma}_0 \times \breve{\Gamma}_0\}$ with $d_{ij} = |\breve{\Omega}(i,j)||(\Phi_i)_1|$, $d_{ji} = |\breve{\Omega}(i,j)||(\Phi_j)_1|$ and the orientation Ω is given from j to i for any $i \neq j$ if $|\breve{\Omega}(i,j)| \neq 0$. Note that the valued quiver $(\Gamma_0,\mathcal{D},\Omega)$ has not loops.

This discussion means one can transfer the study on path algebras with loops into that on generalized path algebras and pre-modulations of valued quivers without loops. This viewpoint gives us a new approach to those subjects whose underlying quiver has loops.

(3) Differential tensor algebras

In [2], the theory of differential tensor algebras is introduced as a natural generalization of the theory of algebras and their module categories, which is a useful tool in establishing some deep results in the representation theory of algebras. It has some common features with the original theory in terms of differential graded categories as well as with formulation given in terms of bocses.

A tensor algebra $T = T(A,M)$ is graded standardly by $T_l = M^{\otimes l}$ for all $l \geq 0$ with $T_0 = A$.

For a graded k-algebra T, a linear transformation δ on T is said to be a *differential* if it satisfies $\delta([T]_i) \subseteq [T]_{i+1}$ for all i and the Leibniz rule $\delta(ab) = \delta(a)b + (-1)^{deg(a)}a\delta(b)$ for all homogeneous elements $a, b \in T$.

A *differential tensor algebra* or *ditalgebra*[2] $\mathcal{A}$ is by definition a pair $\mathcal{A} = (T,\delta)$ where T is a tensor algebra and δ is a differential on T satisfying $\delta^2 = 0$.

Now we define differential pseudo-modulation and give its relation with ditalgebra.

Definition 6.1. *(1) Given a k-pseudo-modulation $\mathcal{M} = (A_i, {}_iM_j)_{i\in\mathcal{G}}$ of a pseudo-valued quiver $(\mathcal{G},\mathcal{D})$ and its related tensor algebra of $\mathcal{A}$-path type $T(A,M)$ as in Proposition 3.1, we say that δ is a* differential *on $\mathcal{M}$ if $\delta : T(A,M) \to T(A,M)$ is a linear transformation such that*

(i) $\delta(A_i) \subseteq {}_iM_i$;

(ii) $\delta({}_iM_{i_1} \otimes_{A_{i_1}} \cdots \otimes_{A_{i_{s-1}}} {}_{i_{s-1}}M_j) \subseteq \sum_{l\in\mathcal{G}} {}_iM_l \otimes_{A_l} {}_lM_{i_1} \otimes_{A_{i_1}} \cdots \otimes_{A_{i_{s-1}}} {}_{i_{s-1}}M_j +$ $+ \sum_{l\in\mathcal{G}} {}_iM_{i_1} \otimes_{A_{i_1}} {}_{i_1}M_l \otimes_{A_l} {}_lM_{i_2} \otimes_{A_{i_2}} \cdots \otimes_{A_{i_{s-1}}} {}_{i_{s-1}}M_j + \cdots + \sum_{l\in\mathcal{G}} {}_iM_{i_1} \otimes_{A_{i_1}} \cdots \otimes_{A_{i_{s-1}}} {}_{i_{s-1}}M_l \otimes_{A_l} {}_lM_j$

and the Leibniz rule $\delta(ab) = \delta(a)b + (-1)^{deg(a)}a\delta(b)$ *for all* $a \in {}_iM_{i_1} \otimes \cdots \otimes {}_{i_{s-1}}M_j,\ b \in {}_uM_{u_1} \otimes \cdots \otimes {}_{u_{t-1}}M_v$.

(2) A differential pseudo-modulation $\mathcal{M}$ *is by definition a pair* $(\mathcal{M}, \delta)$ *with a differential* δ *on* $\mathcal{M}$ *satisfying* $\delta^2 = 0$.

It is easy to check that the Leibniz rule is satisfied by all homogeneous elements about the standard grading of $T(A, M)$. Therefore, by Proposition 3.1, the related tensor algebra $T(A, M)$ of a differential pseudo-modulation $\mathcal{M}$ is a differential tensor algebra with differential δ.

(4) Differential graded category

A category $\mathcal{T}$ is called a *graded category (GC)*[13][27] if for any objects a, b in $\mathcal{T}$, the set $Hom_{\mathcal{T}}(a, b)$ of morphisms is a set-theoretical union of the sets $T_i(a, b),\ 0 \le i < +\infty$, and for any $\alpha \in T_i(a, b), \beta \in T_j(b, c)$, then $\beta\alpha \in T_{i+j}(a, c)$, where α is said to be of *degree* i. If each set $T_i(a, b)$ is a vector space over k and the multiplication by a fixed morphism is a homomorphism of these spaces, then $\mathcal{T}$ is said to be a *GC over the field k*.

For a positive integer n, a graded category $\mathcal{T}$ over a field k is said to be a *differential n-graded category* (briefly, *n-DGC*) if there is a k-linear map $D : T \to T$ for $T = \oplus_{a,b\in\mathcal{T}} Hom_{\mathcal{T}}(a, b)$ such that $D^2 = 0$ and $D(T_i(a, b)) \subseteq T_{i+n}(a, b)$ for each $a, b \in \mathcal{T},\ i \ge 0$, and the Leibnitz formula holds:

$$D(\beta\alpha) = D(\beta)\alpha + (-1)^{ndeg\beta}\beta D(\alpha)$$

for all homogeneous elements $\alpha, \beta \in T$. This D is called an *n-differential* of $\mathcal{T}$.

From [25], we know that for any bimodule $\mathcal{M}$ over a category $\mathcal{K}$, one can construct a tensor category $T(\mathcal{M})$ of $\mathcal{M}$, i.e. a graded category $T(\mathcal{M})$ such that $T_0 = \mathcal{M}$, $T_1 = \mathcal{M}$ and for $n > 1$, $T_n = \mathcal{M} \otimes_{\mathcal{K}} \mathcal{M} \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} \mathcal{M}$ with n factors. A graded category which is a tensor algebra of a bimodule is called a *semifree GC* in [27][28][14].

For an k-pseudo-modulation $\mathcal{M} = (A_a,\ {}_aM_b)$ of a pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ and its related tensor algebra of $\mathcal{A}$-path type $T(\mathcal{M}) \overset{def}{=} T(A, M)$ for $A = \oplus_{a\in\mathcal{G}} A_a$ and $M = \oplus_{(a,b)\in\mathcal{G}\times\mathcal{G}}\, {}_aM_b$, we can define the GC $\mathcal{T}$ whose objects are the vertices in $\mathcal{G}$ and for $a, b \in \mathcal{G}$ whose morphism set $Hom_{\mathcal{T}}(a, b) = \bigcup_{i\ge 0} T_i(a, b)$ with

$T_i(a, b) = \sum_{(a\alpha_1 a_1\alpha_2 a_2\cdots a_{i-1}\alpha_i b)} {}_aM_{a_1} \otimes_{A_{a_1}} {}_{a_1}M_{a_2} \otimes_{A_{a_2}} \cdots \otimes_{A_{a_{i-1}}} {}_{a_{i-1}}M_b$

where the sum runs over all paths $(a\alpha_1 a_1\alpha_2 a_2 \cdots a_{i-1}\alpha_i b)$ from a to b in the pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$. Trivially, $T_i(a, b)T_j(b, c) \subseteq T_{i+j}(a, c)$.

In this case, we call it a *free* graded category generated by the pseudo-valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ due to [27].

Hence, a k-pseudo-modulation $\mathcal{M}$ and also the related tensor algebra of $\mathcal{A}$-path-type $T(\mathcal{M})$ can equivalently be considered as this free graded category $\mathcal{T}$.

However, a differential of degree n on $T(\mathcal{M})$ does not need to be a differential of some degree on its graded category $\mathcal{T}$. For example, particularly, in [20], for a path algebra

$k\Gamma$, we give the method to construct all differentials D on $k\Gamma$, not on its related graded category $\mathcal{T}_{k\Gamma}$ in general. It needs to find out such differential of degree n on $k\Gamma$ that with this D, the graded category $\mathcal{T}_{k\Gamma}$ of $k\Gamma$ becomes to a DGC.

The motivated question is how to construct differentials on a k-pseudo-modulation $\mathcal{M}$ and moreover to choose such ones of them that its corresponding graded category becomes to a DGC. In general, it is interesting to characterize differentials of some degrees on an arbitrary graded category and discuss the Lie algebra composed by all such differentials.

7 Natural valued quiver and valued Ext-quiver of an algebra

The natural quiver Δ_A associated to an artinian algebra A is important for some researches in [16][17][18], etc.

Denote by r the radical of A. Write $A/r = \bigoplus_{i=1}^{s} A_i$ where A_i are two-sided simple ideals of A/r for all i. Then, r/r^2 is an A/r-bimodule by $\bar{a}\cdot(x+r^2)\cdot\bar{b} = axb + r^2$ for any $\bar{a} = a + r, \bar{b} = b + r \in A/r$ and $x \in r$. Thus, ${}_iM_j = A_i \cdot r/r^2 \cdot A_j$ is a finitely generated A_i-A_j-bimodule for each pair (i,j).

Let the vertex set $\Delta_0 = \{1, \cdots, s\}$. For $i, j \in \Delta_0$, set the number t_{ij} of arrows from i to j in Δ to be $rank({}_{A_i}({}_iM_j)_{A_j})$. Then, $\Delta_A = (\Delta_0, \Delta_1)$ is called the *natural quiver*[17] of A. Moreover, construct the normal generalized path algebra $k(\Delta_A, \mathcal{A})$ with $\mathcal{A} = \{A_1, \cdots, A_s\}$, which is defined as the *associated normal generalized path algebra* of A. By Proposition 5.2, from $k(\Delta_A, \mathcal{A})$, we can get the corresponding normal pre-modulation $\mathcal{M}_A$, which is called the *corresponding normal pre-modulation* of A.

For an artinian algebra A and its related normal generalized algebra $k(\Delta_A, \mathcal{A})$, by [16], there always exists a surjective homomorphism of algebras $\pi : k(\Delta_A, \mathcal{A}) \to T(A/r, r/r^2)$, and from the result in [8], it follows that any such algebra A with separable quotient A/r is isomorphic to a quotient algebra of $k(\Delta_A, \mathcal{A})$ by an admissible ideal.

An artinian algebra A is said to be of *Gabriel-type*[18] if it is a quotient of a normal generalized path algebra. As an improvement, in [18], we show that for an artinian k-algebra A splitting over its radical, there is a surjective algebra homomorphism $\phi : k(\Delta_A, \mathcal{A}) \to A$ with $J^s \subseteq \ker(\phi) \subseteq J$ for some positive integer s, that is, A is of Gabriel-type.

Moreover, we give in [18] that if an artinian algebra A of Gabriel-type with admissible ideal is hereditary, then A is isomorphic to its related generalized path algebra $k(\Delta_A, \mathcal{A})$. Hence, according to Corollary 5.4, we have

Proposition 7.1. *For a hereditary artinian algebra A of Gabriel-type with admissible ideal and its corresponding k-pre-modulation $\mathcal{M} = (A_i, {}_iM_j)$, the category $Rep(\mathcal{M})$ (resp. $rep(\mathcal{M})$) is equivalent to the category $ModA$ (resp. $modA$).*

From the above discussion, it is better for us if the ideal J of $k(\Delta_A, \mathcal{A})$ is admissible. In general, this condition is not satisfied for arbitrary non-basic algebras. Now, we specially restrict to the case of basic algebras over an arbitrary field k.

Proposition 7.2. *Suppose that B is an artinian basic algebra with radical $r = r(B)$ over an arbitrary field k and $B/r \cong F_1 \oplus \cdots \oplus F_s$ for central division k-algebras F_i satisfying that $dim_k F_i = n_i^2$ with $(n_i, n_j) = 1$ for any $i \neq j$. Then, for the associated generalized path algebra $k(\Delta_B, \mathcal{F})$ of B with $\mathcal{F} = \{F_1, \cdots, F_s\}$ and the natural quiver Δ_B, there exists an admissible ideal I of $k(\Delta_B, \mathcal{F})$ such that $B \cong k(\Delta_B, \mathcal{F})/I$.*

Proof. By the conclusion in pp.191 of [26], B/r is separable since $dim_k F_i < +\infty$ and the center $Z(F_i) = k$ for any i. Due to this and Theorem 8.5.4 of [8], there is an admissible ideal I of $T(B/r, r/r^2)$ such that $B \cong T(B/r, r/r^2)/I$.

Furthermore, since each F_i is a central division algebra with $dim_k F_i = n_i^2 < +\infty$ and $(n_i, n_j) = 1$ for any $i \neq j$, it is known from pp.78 of [23] that $F_j \otimes F_i^{op}$ is a central division algebra. Hence, r/r^2 is free as F_i-F_j-bimodule for any i, j. Then, according to the definition of generalized path algebra, we have $k(\Delta_B, \mathcal{F}) \cong T(B/r, r/r^2)$. □

This proposition generalizes the Gabriel theorem for a k-splitting basic algebra to that which is not necessarily k-splitting.

Definition 7.1. *The* natural valued quiver *of an artinian algebra A is defined to be the induced valued quiver of $k(\Delta_A, \mathcal{A})$, or say, the valued quiver of the normal pre-modulation of A.*

Meantimes, from an artinian algebra A, one can define another valued quiver $(\mathcal{Q}_A, \mathcal{E}, \Upsilon)$ (cf. [1]) as follows.

Definition 7.2. *For a k-artinian algebra A, let $A/r = \oplus_{i=1}^s A_i$ with $A_i \cong M_{n_i}(D_i)$ where D_i are division k-algebras for $i = 1, \cdots, s$. Denote $\{T_i\}_{i=1}^s$ the complete set of non-isomorphic simple modules of A. Define the* valued Ext-quiver $(\mathcal{Q}_A, \mathcal{E}, \Upsilon)$ *of A as follows:*

(i) $\mathcal{Q}_A = \{1, \cdots, s\}$;

(ii) For $i, j \in \mathcal{Q}_A$, write an oriented edge from i to j if $Ext_A^1(T_j, T_i) \neq 0$. This gives the orientation Υ;

(iii) For $i, j \in \mathcal{Q}_A$, if $Ext_A^1(T_j, T_i) \neq 0$, i.e. there is an oriented edge from i to j, let $e_{ij} = dim_{D_i} Ext_A^1(T_j, T_i)$ and $e_{ji} = dim_{D_j^{op}} Ext_A^1(T_j, T_i)$. Then, define the valuation $\mathcal{E} = \{(e_{ij}, e_{ji}) : \forall (i, j) \in \mathcal{Q}_A \times \mathcal{Q}_A\}$.

The valued Ext-quiver is Morita invariant, but the natural valued quiver is not so.

Using the notations in Definition 7.2, note that $D_i \cong End_A(T_i)$ for $i \in \mathcal{Q}_A$. An artinian algebra A is called *k-splitting*, or say, *splitting* over the ground field k if $D_i \cong k$ for each i.

For example, A is always k-splitting if the ground field k is algebraically closed.

When A is k-splitting, the valued Ext-quiver of A degenerates to a non-valued quiver, which is just the Ext-quiver of A. In this case, we have the following results from [18]:

(i) The vertex set of the Ext-quiver of A is equal to that of the natural quiver of A.

(ii) $t_{ij} = \lceil \frac{m_{ij}}{n_i n_j} \rceil$ where t_{ij} and m_{ij} are respectively the arrow numbers of the natural quiver and the Ext-quiver of A from i to j and $n_i = dim_k T_i$ for the irreducible module T_i of A at the vertex i.

(iii) If A is a basic algebra, then the Ext-quiver is just the natural quiver.

Now, their analogues will be given in the case that A is non-k-splitting in general.

Lemma 7.3. *Let A be an artinian algebra with radical r such that $A/r = \oplus_{i=1}^s A_i$ where $A_i \cong M_{n_i}(F_i)$ for division k-algebras F_i $(i = 1, \cdots, s)$. Let $\{u_i\}_{i=1}^s$ be the complete set of primitive orthogonal idempotents of A and $\{T_i\}_{i=1}^s$ be the corresponding complete set of non-isomorphic A-simple modules. Then, for $i, j \in \{1, \cdots, s\}$,*

$$dim_k(u_i r/r^2 u_j) = dim_k Ext_A^1(T_j, T_i).$$

Proof. For $i, j = 1, \cdots, s$, let $P_j \to T_j$ be a projective cover, then there is the exact sequence $0 \to rP_j \to P_j \to T_j \to 0$. Applying the functor $Hom_A(\ ,T_i)$, we obtain the exact sequence of k-linear spaces

$$0 \to Hom_A(T_j, T_i) \to Hom_A(P_j, T_i) \xrightarrow{h} Hom_A(rP_j, T_i) \to Ext_A^1(T_j, T_i) \to 0.$$

By Schur Lemma, $Hom_A(T_j, T_i) = \begin{cases} F_j, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$ Since $T_j \cong P_j/rP_j$ and $rT_i = 0$ for any i, j, it follows that h must be zero map for $i \neq j$. Hence, we have

$$Hom_A(rP_j, T_i) \cong Ext_A^1(T_j, T_i). \tag{4}$$

On the other hand,

$$Hom_A(rP_j, T_i) = Hom_A(rP_j/r^2P_j, T_i) \cong Hom_{A/r}(rP_j/r^2P_j, T_i). \tag{5}$$

Since A/r is semisimple, rP_j/r^2P_j is a direct sum of some T_p as A/r-module. Thus,

$$Hom_{A/r}(rP_j/r^2P_j, T_i) \cong Hom_{A/r}(T_i, rP_j/r^2P_j) \cong Hom_A(P_i, rP_j/r^2P_j). \tag{6}$$

Using $P_j = Au_j$ for any j and by Proposition I.4.9 of [1], we have $Hom_A(P_i, r^m P_j) \cong u_i r^m u_j$ for any positive integer m. Via these isomorphisms for $m = 1, 2$, we can get $Hom_A(P_i, rP_j/r^2P_j) \cong u_i r/r^2 u_j$. Using this and (4), (5), (6), we get $u_i r/r^2 u_j \cong Ext_A^1(T_j, T_i)$ as k-linear spaces. Then the required result follows. $\square$

In this lemma, the ground field k is arbitrary and A is not assumed to be basic, which are different with ones in Proposition III.1.14 of [1].

Now, we give firstly the relationship between the natural valued quiver and the Ext-valued quiver for a basic algebra B.

Two valued quivers $(\mathcal{G}, \mathcal{D}, \Omega)$ and $(\mathcal{Q}, \mathcal{E}, \Upsilon)$ are called *pair-opposite equal* if $\mathcal{G} = \mathcal{Q}$ and $\Omega = \Upsilon$ and $d_{ij} = e_{ji}$, $d_{ji} = e_{ij}$ for any $(d_{ij}, d_{ji}) \in \mathcal{D}$, $(e_{ij}, e_{ji}) \in \mathcal{E}$.

From this definition, we can think two pair-opposite equal valued quivers are indeed equal under the meaning of re-writing the order of pairs of valution.

For the radical r of B, we have $B/r \cong F_1 \oplus \cdots \oplus F_s$ for division k-algebras F_i.

The normal regular modulation $\mathcal{M} = (F_i,\ {}_iM_j)$ is constructed from $k(\Delta_B, \mathcal{F})$ with ${}_iM_j = F_i(r/r^2)F_j$ as F_i-F_j-bimodules for any $i, j \in \Delta_0$.

The natural valued quiver of B, that is, the induced valued quiver from $k(\Delta_B, \mathcal{F})$, is $(\Delta_0, \mathcal{D}, \Omega)$ with a unique oriented edge from i to j when ${}_iM_j \neq 0$ and $\mathcal{D} = \{(d_{ij}, d_{ji}) : (i,j) \in \Delta_0 \times \Delta_0\}$ for $d_{ij} = dim({}_iM_j)_{F_j} = t_{ij}\varepsilon_i$, $d_{ji} = dim_{F_i}({}_iM_j) = t_{ij}\varepsilon_j$ and $t_{ij} = dim_{(F_j^{op} \otimes F_i)}{}_iM_j$ the arrow number from i to j in Δ_B and $\varepsilon_i = dim_k F_i$, $\varepsilon_j = dim_k F_j$ satisfying $d_{ij}\varepsilon_j = d_{ji}\varepsilon_i$.

Theorem 7.4. *The natural valued quiver $(\Delta_0, \mathcal{D}, \Omega)$ and the valued Ext-quiver $(\mathcal{Q}, \mathcal{E}, \Upsilon)$ of an artinian basic k-algebra B are pair-opposite equal.*

Proof. Firstly, $\Delta_0 = \mathcal{Q} = \{1, \cdots, s\}$.

By Lemma 7.3, there is an oriented edge from i to j in $(\Delta_0, \mathcal{D}, \Omega)$ if and only if there is an oriented edge from i to j in $(\mathcal{Q}, \mathcal{E}, \Upsilon)$, that is, $\Omega = \Upsilon$.

In the natural quiver Δ_B of B, for any $i, j \in \Delta_0$, the arrow number $t_{ij} = dim_{F_j^{op} \otimes F_i}{}_iM_j$.

Denote $m_{ij} = dim_k(u_i r/r^2 u_j)$. We have $u_i r/r^2 u_j = F_i(r/r^2)F_j = {}_iM_j$. Then,

$$m_{ij} = dim_{F_j^{op} \otimes F_i}({}_iM_j) dim_k(F_j^{op} \otimes F_i) = t_{ij}\varepsilon_i\varepsilon_j \tag{7}$$

Thus,

$$dim_k Ext_B^1(T_j, T_i) = dim_{End_B(T_i)} Ext_B^1(T_j, T_i) dim_k End_B(T_i) = e_{ij} dim_k F_i = e_{ij}\varepsilon_i \tag{8}$$

By Lemma 7.3 and (8), $m_{ij} = e_{ij}\varepsilon_i$. Similarly, it can be given $m_{ij} = e_{ji}\varepsilon_j$.

By (7) and $d_{ij} = t_{ij}\varepsilon_i$, $d_{ji} = t_{ij}\varepsilon_j$, we get that $d_{ji}\varepsilon_i = t_{ij}\varepsilon_j\varepsilon_i = m_{ij} = e_{ij}\varepsilon_i$. Thus, $d_{ji} = e_{ij}$. Similarly, $d_{ij} = e_{ji}$.

In summary, $(\Delta_0, \mathcal{D}, \Omega)$ and $(\mathcal{Q}, \mathcal{E}, \Upsilon)$ are pair-opposite equal. □

We think this consequence is an evidence of the rationality of the notion of natural valued quiver of an artinian algebra A as given above.

By definitions, natural quiver and natural valued quiver can be constructed with each other. Hence, natural valued quiver is not Morita invariant since so is the frontal notion.

Now, we discuss the relation between the natural valued quiver $((\Delta_0)_A, \mathcal{D}, \Omega)$ and the valued Ext-quiver $(\mathcal{Q}_A, \mathcal{E}, \Upsilon)$ for an arbitrary artinian algebra A.

Theorem 7.5. *For an artinian algebra A, the natural valued quiver $((\Delta_0)_A, \mathcal{D}, \Omega)$ and the valued Ext-quiver $(\mathcal{Q}_A, \mathcal{E}, \Upsilon)$ are satisfied through the following relations:*

(i) The vertex sets are equal. i.e. $(\Delta_0)_A = \mathcal{Q}_A$;

(ii) The orientations are the same, i.e. $\Omega = \Upsilon$;

(iii) The valuations $\mathcal{D} = \{(d_{ij}, d_{ji}) : (i,j) \in (\Delta_0)_A \times (\Delta_0)_A\}$ and $\mathcal{E} = \{(e_{ij}, e_{ji}) : (i,j) \in \mathcal{Q}_A \times \mathcal{Q}_A\}$ hold the formulae:

$$d_{ji} = e_{ij} n_j^2 \frac{t_{ij}}{m_{ij}} \qquad\qquad d_{ij} = e_{ji} n_i^2 \frac{t_{ij}}{m_{ij}} \tag{9}$$

for any vertices i,j. Here, t_{ij} is the arrow number in the natural quiver Δ_A of A from i to j, m_{ij} is the arrow number in the natural quiver Δ_B of the associated basic algebra B of A from i to j and $n_i = \frac{dim_k S_i}{dim_k End S_i}$ for the simple module S_i of A at the vertex i.

Proof. (i) This is easy due to their definitions.

(ii) By Lemma 7.3, ${}_iM_j = A_i(r/r^2)A_j \neq 0$ iff $u_i r/r^2 u_j \neq 0$ iff $Ext_A^1(T_j, T_i) \neq 0$. Then, the claim follows from the definitions of the orientations Ω and Υ.

(iii) By the proof of Lemma 5.1, $d_{ij} = t_{ij}\varepsilon_i$, $d_{ji} = t_{ij}\varepsilon_j$, where t_{ij} is the arrow number in Δ_A from i to j, $\varepsilon_i = dim_k A_i$ for $A/r_A = A_1 \oplus \cdots \oplus A_s$.

Firstly, the valued Ext-quiver $(\mathcal{Q}_A, \mathcal{E}, \Upsilon)$ of A is equal to that of its associated basic algebra B. And, by Theorem 7.4, the latter is pair-opposite equal to the natural valued quiver $((\Delta_0)_B, \mathcal{D}_B, \Omega_B)$ of B. Hence, $(\mathcal{Q}_A, \mathcal{E}, \Upsilon)$ is pair-opposite equal to $((\Delta_0)_B, \mathcal{D}_B, \Omega_B)$. Therefore, for $\mathcal{E} = \{(e_{ij}, e_{ji}) : (i,j) \in \mathcal{Q}_A \times \mathcal{Q}_A\}$ and $\mathcal{D}_B = \{(d_{ij}^B, d_{ji}^B) : (i,j) \in (\Delta_0)_A \times (\Delta_0)_A\}$, it follows that

$$e_{ij} = d_{ji}^B = m_{ij}\varepsilon_j^B \quad \text{and} \quad e_{ji} = d_{ij}^B = m_{ij}\varepsilon_i^B \tag{10}$$

where m_{ij} is the arrow number in the natural quiver Δ_B of B from i to j and $\varepsilon_i^B = dim_k F_i$ for $B/r_B = F_1 \oplus \cdots \oplus F_s$ with division k-algebras $F_i \cong End S_i$ for the simple module S_i of A at the vertex i $(i = 1, \cdots, s)$.

Due to Wedderburn-Artin Theorem, for any i, $A_i \cong M_{n_i}(k) \otimes F_i$ for a positive integer n_i, where $n_i = \frac{dim_k S_i}{dim_k End S_i}$. Then , we get

$$d_{ji} = t_{ij} n_j^2 \varepsilon_j^B \quad \text{and} \quad d_{ij} = t_{ij} n_i^2 \varepsilon_i^B \tag{11}$$

By (10) and (11), it follows that $d_{ji} = e_{ij} n_j^2 \frac{t_{ij}}{m_{ij}}$ and $d_{ij} = e_{ji} n_i^2 \frac{t_{ij}}{m_{ij}}$. □

Obviously, Theorem 7.4 is just in the special case of Theorem 7.5 when A is basic.

By the formula given in [18], when A is k-splitting, it holds that $t_{ij} = \lceil \frac{m_{ij}}{n_i n_j} \rceil$. Then, in this case, from the formula (9) we get

Corollary 7.6. *For a k-splitting artinian algebra A, using the notations in Theorem 7.5, it holds that for any vertices i,j, $d_{ji} = e_{ij} n_j^2 \frac{1}{m_{ij}} \lceil \frac{m_{ij}}{n_i n_j} \rceil$ and $d_{ij} = e_{ji} n_i^2 \frac{1}{m_{ij}} \lceil \frac{m_{ij}}{n_i n_j} \rceil$.*

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