Hyperbolic Gradient Flow: Evolution of Graphs in $\mathbb{R}^{n+1}$

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Dedicated to Professor Yi-Bing Shen on the occasion of his 70th birthday

Abstract

In this paper we introduce a new geometric flow — the hyperbolic gradient flow for graphs in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. This kind of flow is new and very natural to understand the geometry of manifolds. We particularly investigate the global existence of the evolution of convex hypersurfaces in $\mathbb{R}^{n+1}$ and the evolution of plane curves, and prove that, under the hyperbolic gradient flow, they converge to the hyperplane and the straight line, respectively, when t goes to the infinity. Our results show that the theory of shock waves of hyperbolic conservation laws can be naturally applied to do surgery on manifolds. Some fundamental but open problems are also given.

Key words and phrases: Hyperbolic gradient flow, geometry of manifold, global existence, smooth solution, shock wave.

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1 Introduction

Classical differential geometry has been the study of curved spaces, shapes and structures of manifolds in which the time does not play a role. However, in the last several decades geometers have made great strides in understanding the shapes and structures of manifolds that evolve in time. There are many processes in the evolution of a manifold, among them the Ricci flow is arguably the most successful (see Hamilton [8]), since it plays a fundamental role in the solution of the famous Poincaré conjecture (see [21]-[23]). The Ricci flow is described by a fully nonlinear system of parabolic partial differential equations of second order. Another famous geometric flow — mean curvature flow is also described by a fully nonlinear system of parabolic partial differential equations of second order. The (inverse) mean curvature flow has been used to prove the Riemannian-Penrose inequality in general relativity by Huisken and Ilmanen (see [10]) and also has been used to study many problems arising from applied fields, i.e., imaging processing (see [2]). In fact, the traditional geometric analysis has been successfully applied the theory of elliptic and parabolic partial differential equations to differential geometry and physics (see [26]). There are three typical examples: the Hamilton's Ricci flow, the (inverse) mean curvature flow and the Schoen-Yau's solution of the positive mass conjecture (see [24]-[25]). On the other hand, since the hyperbolic equation or system is one of the most natural models in the nature, a natural and important question is if we can apply the theory of hyperbolic differential equations to solve some problems arising from differential geometry and theoretical physics (in particular, general relativity). Recently, we introduced the hyperbolic geometric flow which is an attempt to answer the above question. The hyperbolic geometric flow is a very natural tool to understand the wave character of the metrics, wave phenomenon of the curvatures, the evolution of manifolds and their structures (see [12], [13], [16], [5], [6], [14], [9], [15], [17]).

In this paper we introduce a new geometric flow — the hyperbolic gradient flow for graphs in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. The flow is described by hyperbolic evolution partial differential equations of first order for a family of vector fields X_t defined on $\mathbb{R}^n$. Roughly speaking, the hyperbolic gradient flow evolves the tangent planes of the graph under consideration, this is different from the Ricci flow, the mean curvature flow or our hyperbolic geometric flow. This kind of flow is new and very natural to understand deformation phenomena of manifolds (in particular, graphs in $\mathbb{R}^{n+1}$) as well as the geometry of manifolds. It possesses many interesting properties from both mathematics and physics. In the present paper, we particularly investigate the global existence of the evolution of convex hypersurfaces in $\mathbb{R}^{n+1}$ and the evolution of plane curves, and prove that, under the hyperbolic gradient flow, they converge to the hyperplane and the straight line, respectively, when t goes to the infinity. Our results show that the theory of shock waves of hyperbolic conservation laws can be naturally applied to do surgery on manifolds. Some fundamental but open problems are also given.

2 Hyperbolic gradient flow for graphs in $\mathbb{R}^{n+1}$

Let Σ_t be a family of graphs in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with coordinates $(x_1, \cdots, x_{n+1})$. Without loss of generality, we may assume that the graphs Σ_t are given by

$$x_{n+1} = f(t, x_1, \cdots, x_n), \tag{2.1}$$

where f is a smooth function defined on $\mathbb{R} \times \mathbb{R}^n$. Let X_t be a family of tangent vector fields induced by Σ_t, or say,

$$X_t = (X_1, \cdots X_n) = (\partial_{x_1} f, \cdots, \partial_{x_n} f), \tag{2.2}$$

where $\partial_{x_i} f$ $(i = 1, \cdots, n)$ stand for $\frac{\partial f}{\partial x_i}$. The hyperbolic gradient flow under considered here is given by the following evolution equations

$$\frac{\partial X_t}{\partial t} + \nabla \left(\frac{\|X_t\|^2}{2} \right) = 0, \tag{2.3}$$

where $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$ and

$$\| \cdot \|^2 = \langle \cdot, \cdot \rangle, \tag{2.4}$$

in which $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$.

By the definition, the hyperbolic gradient flow introduced in this note is a geometric flow for the evolution of a family of tangent vector fields induced by a family of graphs, it is quite different from the Ricci flow and the mean curvature flow: the Ricci flow is described by evolution equations for a family of Riemannian metrics $g_{ij}(t)$ defined on the manifold under consideration, while the mean curvature flow is on the evolution of the manifold itself.

3 The evolution of convex hypersurfaces in $\mathbb{R}^{n+1}$

In this section, we shall investigate the evolution of convex hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

As before, let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space with coordinates $(x_1, \cdots, x_{n+1})$, and $x_{n+1} = \mathscr{S}(t, x_1, \cdots, x_n)$ be a family of hypersurfaces in $\mathbb{R}^{n+1}$. Introduce the vector field

$$\vec{v} = \{\mathscr{S}_1, \cdots, \mathscr{S}_n\}, \tag{3.1}$$

where $\mathscr{S}_i$ $(i = 1, \cdots, n)$ stand for $\frac{\partial \mathscr{S}}{\partial x_i}$. In the present situation, the hyperbolic gradient flow (2.3) reads

$$\vec{v}_t + \vec{v} \cdot \nabla \vec{v} = 0. \tag{3.2}$$

In this case, (3.2) is nothing but the transport equation for $\vec{v}$.

Example 3.1. *Consider the evolution of the hypersurface $x_n = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)$ under the hyperbolic gradient flow. In the present situation, we need to consider the Cauchy problem for the equation (3.2) with the following initial data*

$$t = 0: \quad \vec{v} = \vec{v}^0 \triangleq (x_1, \cdots, x_n). \tag{3.3}$$

It is easy to see that the solution of the Cauchy problem (3.2), (3.3) reads

$$\vec{v} = \left(\frac{x_1}{t+1}, \cdots, \frac{x_n}{t+1}\right), \tag{3.4}$$

moreover, the solution is unique. Obviously, the vector field defined by (3.4) gives a potential function $x_n = \frac{1}{2(t+1)}\left(x_1^2 + \cdots + x_n^2\right) + C$, where C is a constant independent of x. Noting that the initial hypersurface is $x_n = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)$ leads to that the constant C must be zero. Thus, the evolution of the hypersurface $x_n = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)$ under the hyperbolic gradient flow is described by the family of hypersurfaces $x_n = \frac{1}{2(t+1)}\left(x_1^2 + \cdots + x_n^2\right)$. Clearly, for any fixed x, the hypersurfaces tend to flat under the hyperbolic gradient flow when t goes to the infinity. $\square$

Consider the Cauchy problem for the equation (3.2) with the following initial data

$$t = 0: \quad \vec{v} = \vec{v}^0(x_1, \cdots, x_n), \tag{3.5}$$

where $\vec{v}^0$ is a smooth vector field defined on $\mathbb{R}^n$. We now consider the global existence and decay property of smooth solutions of the the Cauchy problem (3.2) and (3.5).

In fact, we can obtain a sufficient and necessary condition on the global existence of smooth solutions of the following Cauchy problem for more general quasilinear systems of first order

$$\frac{\partial u}{\partial t} + \sum_{j}^{n} \lambda_j(u) \frac{\partial u}{\partial x_j} = 0, \quad \forall\, (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \tag{3.6}$$

with the initial data

$$u(0, x) = \phi(x), \quad \forall\, x \in \mathbb{R}^n, \tag{3.7}$$

where $x = (x_1, \cdots, x_n)$ stands for the special variable, $u = (u_1(x,t), \cdots u_m(x,t))^T$ is the unknown vector-valued function of $(t, x) = (t, x_1, \cdots, x_n) \in \mathbb{R}^+ \times \mathbb{R}$, $\lambda_i(u)$ $(i = 1, \cdots, n)$ are given C^1 functions and $\phi(x) = (\phi_1(x), \cdots, \phi_m(x))^T$ is a given C^1 vector-valued function with bounded C^1 norm. The following lemma comes from Conway [3], Li [20], Dafermos [4] or Kong [11].

Lemma 3.1 *Under the assumptions mentioned above, the Cauchy problem (3.6)-(3.7) has a unique global C^1 smooth solution on the domain $\mathbb{R}^+ \times \mathbb{R}^n$ if only if, for any given $x \in \mathbb{R}^n$, it holds that*

$$d(S_p V_0(x), \mathbb{R}^-) \geq 0, \tag{3.8}$$

i.e., all eigenvalues of the $n \times n$ matrix

$$V_0(x) = \left(\sum_{k=1}^{m} \frac{\partial \lambda_i}{\partial u_k}(\phi(x)) \frac{\partial \phi_k}{\partial x_j} \right)_{i,j=1}^{n} \tag{3.9}$$

are non-negative, where $S_p V_0(x)$ stands for the spectrum of the matrix $V_0(x)$.

Lemma 3.2 *Under the assumptions of Lemma 3.2, suppose that ϕ is a C^2 vector-valued function with bounded C^2 norm and suppose furthermore that there exists a positive constant $\delta > 0$ such that*

$$d(S_p V_0(x), \mathbb{R}^-) \geq \delta, \quad \forall\, x \in \mathbb{R}^n. \tag{3.10}$$

Then the Cauchy problem (3.6)-(3.7) admits a unique global C^2 smooth solution $u = u(t,x)$ on the domain $\mathbb{R}^+ \times \mathbb{R}^n$, moreover it holds that

$$\| Du(t,x) \|_{L^\infty(\mathbb{R}^n)} = C_1 (1+t)^{-1} \tag{3.11}$$

and

$$\| D^2 u(t,x) \|_{L^\infty(\mathbb{R}^n)} \leq C_2 (1+t)^{-2}, \tag{3.12}$$

where C_1 is a positive constant independent of t but depending on δ and the C^1 norm of ϕ, while C_2 is a positive constant independent of t but depending on δ and the C^2 norm of ϕ.

The proof of Lemma 3.3 can be found in Grassin [7] for the case of scalar equation and in Kong [11] for general case.

If $m = n$ and $\lambda_i(u) = u_i$ $(i = 1, \cdots, n)$ (in this case, the system (3.6) goes back to the system (3.2)), then in the present situation, $V_0(x)$ defined by (3.9) reads

$$V_0(x) = \left(\frac{\partial \phi_i}{\partial x_j} \right)_{i,j=1}^{n}. \tag{3.13}$$

In particular, if there exists a potential function $\Phi(x)$ such that

$$\frac{\partial \Phi}{\partial x_i} = \phi_i(x) \quad (i = 1, \cdots, n), \tag{3.14}$$

then

$$V_0(x) = \text{Hess}\,(\Phi(x)). \tag{3.15}$$

We now turn to consider the Cauchy problem for this special case, i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{j=1}^{n} u_j \dfrac{\partial u}{\partial x_j} = 0, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ t = 0 : u = \phi(x) = \left(\dfrac{\partial \Phi}{\partial x_1}, \cdots, \dfrac{\partial \Phi}{\partial x_n} \right)^T, \quad \forall\, x \in \mathbb{R}^n. \end{cases} \tag{3.16}$$

By Lemma 3.1-3.2, we have

Lemma 3.3 *Suppose that the potential function $\Phi = \Phi(x)$ is C^2 smooth and its derivates Φ_{k_i} $(i = 1, ..., n)$ has a bounded C^1 norm. Then the Cauchy problem (3.16) has a unique global C^1 smooth solution on the domain $\mathbb{R}^+ \times \mathbb{R}^n$ if only if* $\mathrm{Hess}\,(\Phi(x))$ *is non-negative for all $x \in \mathbb{R}^n$.*

Moreover, if the following assumptions are satisfied: (i) Φ is a C^3 smooth function, (ii) the derivative $D\Phi = (\Phi_{x_1}, \cdots, \Phi_{x_n})^T$ is a vector-valued function with bounded C^2 norm, (iii) there exists a positive constant δ independent of x such that

$$d(\mathrm{Hess}\,(\Phi(x)), \mathbb{R}^-) \geq \delta, \quad \forall\, x \in \mathbb{R}^n, \tag{3.17}$$

then the global smooth solution $u = u(t,x)$ to the Cauchy problem (3.16) satisfies the following properties:

(I) there exists a C^3 potential function $U = U(t,x)$ such that

$$u_i(t,x) = \frac{\partial U}{\partial x_i}(t,x) \quad (i = 1, ..., n), \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{3.18}$$

(II) there exist two positive constants C_3 and C_4 independent of t but depending on δ and the C^1 norm (for C_3), the C^2 norm (for C_4) of $D\Phi(x)$, respectively, such that

$$\|D^2 U(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_3 (1+t)^{-1}, \tag{3.19}$$

and

$$\|D^3 U(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_4 (1+t)^{-2}, \tag{3.20}$$

where $D = (\partial_{x_1}, ..., \partial_{x_n})$.

Proof. By Lemmas 3.1-3.2, we only need to prove (I) in Lemma 3.3.

In order to prove (I), it suffices to show

$$\frac{\partial u_i(t,x)}{\partial x_j} = \frac{\partial u_j(t,x)}{\partial x_i}, \quad \forall\, i \neq j, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{3.21}$$

In fact, introduce

$$\omega^i_j = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \quad (i,j = 1, \cdots, n; \quad i \neq j). \tag{3.22}$$

Obviously, when $t=0$,

$$\omega^i_j(t,0)=\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\Phi\right)-\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\Phi\right)=0 \quad (i,j=1,\cdots,n). \tag{3.23}$$

On the one hand, differentiating the i-th equation in (3.16) with respect to x_j gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right)+\sum_{k=1}^{n}u_k\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}\right)=-\sum_{k=1}^{n}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}. \tag{3.24}$$

On the other hand, differentiating the j-th equation in (3.16) with respective to x_i yields

$$\frac{\partial}{\partial t}\left(\frac{\partial u_j}{\partial x_i}\right)+\sum_{k=1}^{n}u_k\frac{\partial}{\partial x_k}\left(\frac{\partial u_j}{\partial x_i}\right)=-\sum_{k=1}^{n}\frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_i}. \tag{3.25}$$

Combing (3.24)-(3.25) leads to

$$\frac{\partial \omega^i_j}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial \omega^i_j}{\partial x_k}=\sum_{p\neq q}\Gamma^{ij}_{pq}\omega^q_p, \quad \forall\, i\neq j, \tag{3.26}$$

where Γ^{ij}_{pq} stands for the coefficients of ω^q_p which are smooth functions of $\frac{\partial u_l}{\partial x_h}$ $(l,h=1,\cdots,n)$. Clearly, $\omega^i_j=0$ $(i,j=1,\cdots,n;\ i\neq j)$ is a solution of the Cauchy problem (3.26), (3.23). By the uniqueness of the smooth solution of the Cauchy problem for hyperbolic partial differential equation, we have

$$\omega^i_j\equiv 0 \quad (i\neq j), \quad \forall\,(t,x)\in\mathbb{R}^+\times\mathbb{R}^n. \tag{3.27}$$

This proves (3.21). Thus the proof of Lemma 3.3 is completed. ∎

Remark 3.1 *In Lemma 3.2, we need the C^1 norm of ϕ and the C^2 norm of ϕ is bounded for the estimates (3.11) and (3.12), respectively. For Lemma 3.3, the situation is similar.*

However, in many cases (i.g., Example 3.1), the assumption that the C^1 norm or C^2 norm of the initial data is bounded is not satisfied. The following discussion is devoted to the case of unbounded initial data. For simplicity, we only consider the Cauchy problem (3.16).

Lemma 3.4 *Suppose that $\Phi=\Phi(x)$ is a C^3 convex function, i.e., $\Phi(x)\in C^3(\mathbb{R}^n)$ and*

$$\mathrm{Hess}\,(\Phi)\geq 0. \tag{3.28}$$

Then the Cauchy problem (3.16) admits a unique C^2 solution $u=u(t,x)$ on the domain $\mathbb{R}^+\times\mathbb{R}^n$. Moreover, there exists a potential function $U=U(t,x)\in C^3(\mathbb{R}^+\times\mathbb{R}^n)$ such that (3.18) is satisfied. In particular, if there exists a positive constant δ independent of x such that (3.17) holds, then for any fixed $\alpha\in\mathbb{R}^n$ along the characteristic curve $x=x(t,\alpha)$ it holds that

$$|D^2U(t,x(t,\alpha))|\leq \tilde{C}_1(1+t)^{-1} \tag{3.29}$$

and

$$|D^3U(t,x(t,\alpha))| \le \tilde{C}_2(1+t)^{-2}, \tag{3.30}$$

where $\tilde{C}_1$ *and* $\tilde{C}_2$ *are tow constants independent of* t *but depending on* δ *and* α.

The following corollary comes from Lemma 3.4 directly.

Corollary 3.1 *Under the assumptions of Lemma 3.4, for any compact set* $\Omega \subseteq \mathbb{R}^n$ *it holds that*

$$\|D^2U(t,\cdot)\|_{L^\infty(\Omega(t))} \le \tilde{C}_3(1+t)^{-1}, \tag{3.31}$$

and

$$\|D^3U(t,\cdot)\|_{L^\infty(\Omega(t))} \le \tilde{C}_4(1+t)^{-2}, \tag{3.32}$$

where

$$\Omega(t) = \{(t,x) | x = x(t,\alpha), \quad \alpha \in \Omega\}, \tag{3.33}$$

$\tilde{C}_3$ *and* $\tilde{C}_4$ *are two constants independent of* t *but depending on* δ *and the set* Ω. □

Proof of Lemma 3.4. Noting (3.28), we have

$$\Phi_{x_ix_i}(x) \ge 0 \quad (i = 1,\cdots,n), \quad \forall\, x \in \mathbb{R}^n. \tag{3.34}$$

In the present situation, the characteristic curve pasting through any fixed point $(0,\alpha)$ in the initial hyperplane $t = 0$ reads

$$x_i = \alpha_i + \frac{\partial \Phi}{\partial \alpha_i}(\alpha)t \quad (i = 1,\cdots,n). \tag{3.35}$$

By (3.34), it is easy to check that the mapping $\Pi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by (3.35) is *proper*. On the other hand,

$$J(\Pi_t) = I + t\mathrm{Hess}\,(\Phi). \tag{3.36}$$

Using (3.28) again, we have

$$\det J(\Pi_t) \ge 1, \quad \forall\, (t,\alpha) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{3.37}$$

This implies that for any fixed $x \in \mathbb{R}^+$, the mapping Π_t is a global C^1 deffeomorphism. Solving α from (3.35) gives

$$\alpha = \alpha(t,x) \in C^2(\mathbb{R}^+ \times \mathbb{R}^n). \tag{3.38}$$

The rest of the proof is standard (See [20] or [11]), here we omit the details. The proof of Lemma 3.4 is completed. ■

Lemma 3.3 guarantees that, if the initial vector field is induced by a graph, then so does the solution vector-field. That is, if there exists a fuction $\varphi_0(x_1, \cdots, x_n)$ such that $v_i^0 = \frac{\partial \varphi_0}{\partial x_i}$ $(i = 1, \cdots, n)$, then there is a family of functions $\varphi(t, x_1, \cdots, x_n)$ such that

$$v_i(t, x_1 \cdots, x_n) = \frac{\partial \varphi}{\partial x_i}(t, x_1 \cdots, x_n) \quad (i = 1, \cdots, n) \tag{3.39}$$

and

$$\varphi(0, x_1, \cdots, x_n) = \varphi_0(x_1, \cdots, x_n). \tag{3.40}$$

From the point of view of geometry, the hyperbolic gradient flow evolves a graph as a family of graphs in the Euclidean space $\mathbb{R}^{n+1}$.

Summarizing the above argument leads to

Theorem 3.1 *For any given initial vector field induced by a convex graph $x_{n+1} = \varphi_0(x_1, \cdots, x_n)$, the solution $\vec{v} = \vec{v}(t, x_1, \cdots, x_n)$ to the hyperbolic gradient flow (3.2) exists for all time, and there exists a unique family of graphs $x_{n+1} = \varphi(t, x_1, \cdots, x_n)$ such that the solution vector-field $\vec{v} = \vec{v}(t, x_1, \cdots, x_n)$ is induced by the family of graphs $x_{n+1} = \varphi(t, x_1, \cdots, x_n)$. Moreover, if the initial graph is strictly convex, then for any fixed point $(x_1, \cdots, x_n) \in \mathbb{R}^n$ the graphs $x_{n+1} = \varphi(t, x_1, \cdots, x_n)$ tends to be flat at an algebraic rate $(t+1)^{-1}$, when t goes to the infinity.*

4 The evolution of plane curves

In this section, we particularly investigate the evolution of plane curves under the hyperbolic gradient flow, here we still consider the graph case, however we do not assume that the graph is convex.

Let $y = f(x)$ be a smooth curve in the (x, y)-plane, and

$$v_0(x) = f'(x) \tag{4.1}$$

be the slope function of the curve. In the present situation, the hyperbolic gradient flow equation (3.2) becomes one-dimensional case, i.e.,

$$v_t + v v_x = 0. \tag{4.2}$$

This equation can be rewritten as a conservative form

$$v_t + (v^2/2)_x = 0. \tag{4.3}$$

We next consider the Cauchy problem for the conservation law (4.3) with the initial data

$$t = 0: \quad v = v_0(x). \tag{4.4}$$

As in Lax [19], we introduce

Definition 4.1 *A function ψ has mean value M, if*

$$\lim_{L\to\infty} \frac{1}{L} \int_a^{a+L} \psi(x)dx = M \tag{4.5}$$

uniformly in a.

Corollary 4.1 *If a function ψ is periodic with p period, then it has mean value M, and M is given by*

$$M = \frac{1}{p} \int_0^p \psi(x)dx. \tag{4.6}$$

If $\psi \in L^1(\mathbb{R})$, then it has mean value 0.

The following lemma comes from Lax [19].

Lemma 4.1 *Let $v(t,x)$ be a bounded weak solution of the Cauchy problem (4.3), (4.4). Suppose that the initial data $v_0(x)$ has a mean value, then $v(t,x)$ has the same mean value for all t.*

The following important lemma comes from Kruzkov [18].

Lemma 4.2 *Suppose that the initial data v_0 is bounded measurable, then the Cauchy problem (4.3), (4.4) has a unique entropy solution $v = v(t,x)$ on the half plane $t \geq 0$.*

Lemma 4.3 *Under the assumption of Lemma 4.2, if the initial data v_0 is periodic with p period, then the entropy solution $v = v(t,x)$ of the Cauchy problem (4.3), (4.4) tends to M uniformly in x at an algebraic rate $(t+1)^{-1}$, when t tends to infinity, where M is given by*

$$M = \frac{1}{p} \int_0^p v_0(x)dx.$$

Lemma 4.4 *Suppose that the initial data v_0 is in the class of $L^1(\mathbb{R})$, then the Cauchy problem (4.3), (4.4) has a unique entropy solution $v = v(t,x)$ on the half plane $t \geq 0$. Moreover, $v(t,x)$ tends to 0 uniformly in x at an algebraic rate $(t+1)^{-1}$, when t tends to infinity.*

Lemmas 4.3 and 4.4 can be found in Serre [27] and Bressan [1], respectively.

Remark 4.1 *The entropy solution $v = v(t,x)$ mentioned in Lemmas 4.2, 4.3 and 4.4 means that (i) $v = v(t,x)$ is a weak solution of the Cauchy problem (4.3), (4.4); (ii) it satisfies the entropy condition. In fact, the entropy solution may includes shock waves, rarefaction waves, and other physical discontinuities.*

We now consider the evolution of the initial curve $y = f(x)$ under the hyperbolic gradient flow. Without loss of generality, we may assume that

$$f(0) = 0. \tag{4.7}$$

By Lemmas 4.2-4.4, we have

Theorem 4.1 *Suppose that $f'(x)$ is bounded measurable, and suppose furthermore that $f'(x)$ is periodic or is in the class of $L^1(\mathbb{R})$, then the family of curves $y = F(t, x)$ tends to the straight line $y = Mx$ uniformly in x at an algebraic rate $(t+1)^{-1}$, when t tends to infinity, where M is the mean value of $f'(x)$, and $y = F(t, x)$ is generated by the hyperbolic gradient flow, i.e., $F(t, x)$ satisfies*

$$\frac{\partial F}{\partial x}(t, x) = v(t, x), \tag{4.8}$$

in which $v = v(t, x)$ is the entropy solution of the Cauchy problem (4.3), (4.4) (in the present situation, $v_0(x) = f'(x)$).

Remark 4.2 *In geometry, one is, in general, interested in the case that the initial data $f(x)$, or say v_0, is smooth and bounded. However, in the evolution process under the hyperbolic gradient flow, discontinuities may appear. See Example 4.1 below for the details.*

Example 4.1. *Consider the evolution of the curve $y = -\cos x$ in the (x, y)-plane under the hyperbolic gradient flow. In the present situation, the initial data reads*

$$t = 0: \quad v = v_0(x) = \sin x. \tag{4.9}$$

By the method of characteristics, the solution of the Cauchy problem (4.3), (4.9) can be constructed and is given by Figure 4.1.

Notice that Figure 4.1 only describes the solution on one space-periodic domain, i.e., $\mathbb{R}^+ \times [0, 2\pi]$. Corresponding to the solution shown in Figure 4.1, the evolution of the curve $y = -\cos x$ under the hyperbolic gradient flow can be described by Figure 4.2.

We observe from Figure 4.2 that the singularity have appeared in the evolutionary process (see Figure 4.3 for the details).

Figure 4.3 shows that the singularity of cusp type of $v = v(t, x)$ appears at $x = \pi$ when $t = 1$. It is easy to see that the entropy solution $v = v(t, x)$ to the Cauchy problem (4.3), (4.9) includes space-periodic shock waves. $\square$

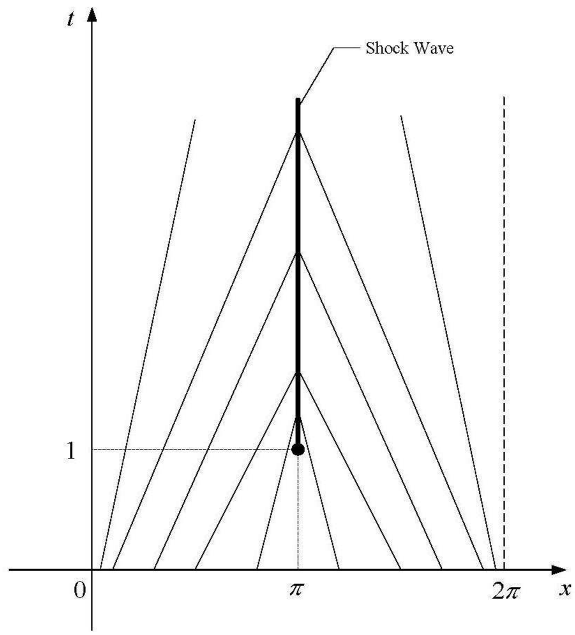

Figure 4.1: The shock solution $v = v(t, x)$ of the Cauchy problem (4.3), (4.9)

5 Conclusions and open problems

It is well known that there have been many successes of elliptic and parabolic equations applied to mathematics and physics. On the other hand, hyperbolic partial differential equation is a very important kind of PDEs, it can be used to describe the wave phenomena in the nature and engineering. Recently, we introduced the hyperbolic geometric flow, showed that the hyperbolic geometric flow possesses very interesting geometric properties and dynamical behavior, and obtain some interesting results. However, the hyperbolic geometric flow is described by a fully nonlinear system of hyperbolic partial differential equations of second order, which is very difficult to solve. In this paper we introduce a new geometric flow — the hyperbolic gradient flow, which is described by a quasilinear system of hyperbolic partial differential equations of first order. Comparing the hyperbolic geometric flow, the hyperbolic gradient flow is easier to solve. The key point of the hyperbolic gradient flow is to evolve the tangent planes of the graphs under consideration, this is different with the famous Ricci flow, the mean curvature flow or our hyperbolic geometric flow. In this paper, we investigate the evolution of convex hypersurfaces in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ and the evolution of plane curves, and prove that, under the hyperbolic gradient flow, they converge to the hyperplane and straight line, respectively, when t goes to the infinity. Our results obtained in this paper show that the theory of shock waves of hyperbolic

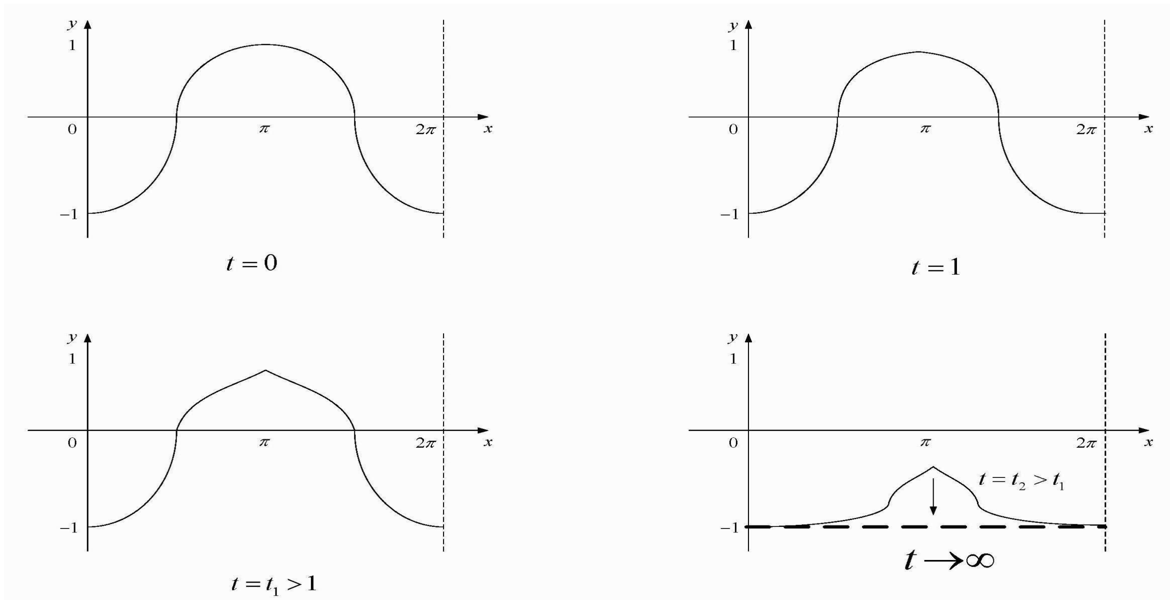

Figure 4.2: The evolution of the curve $y = -\cos x$ under the hyperbolic gradient flow

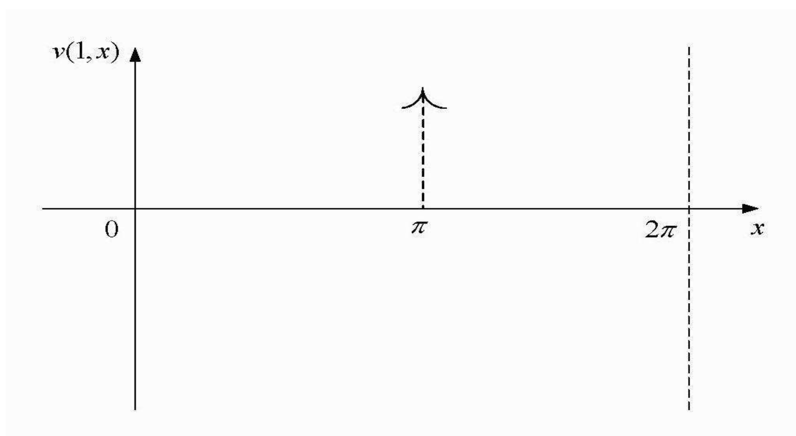

Figure 4.3: The formation of singularity of cusp type of $v = v(t, x)$

conservation laws can be naturally applied to differential geometry. We believe that the hyperbolic gradient flow is a new and powerful tool to study some problems arising from geometry and physics. However, there are many fundamental but still open problems. In particular, the following open problems seem to us more interesting and important:

1. The evolution of plane curves. *In Theorem 4.1 if we do not assume that $f'(x)$ has a mean value, what is the limit of the family of curves $F(t, x)$ as t goes to infinity? Moreover, what happens if the initial curve is not a graph, e.g., a closed curve?*

2. The evolution of surfaces in $\mathbb{R}^3$. *In Theorem 3.1 if the initial surface is a graph but is not convex, what about the limit of the family of surfaces $\varphi(t, x_1, x_2)$ as t goes to infinity? A more difficult but more natural and important question is: how to define the hyperbolic gradient flow for a family of close surfaces? If so, what is the asymptotic behaviour of a close surface under "the hyperbolic gradient flow"? This problem is related to the theory of multi-dimensional hyperbolic systems of partial differential equations of first order.*

3. The evolution of hypersurfaces in $\mathbb{R}^n$ $(n \geq 4)$. *Investigate the hyperbolic gradient flow in multi-dimensional Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 4)$. In particular, how can we define a suitable " hyperbolic gradient flow" to evolve a closed sub-maifold? if we can, what is the large time behaviour of a close hypersurface in $\mathbb{R}^n$ $(n \geq 4)$ under this kind of hyperbolic gradient flow. The convex case maybe is easier to study.*

We may also consider variations of the above hyperbolic gradient flow which can be defined intrinsically on any manifold. For example we let $(\mathscr{M}, g)$ be a Riemannian manifold, and $X_t \in \Gamma(\mathscr{M}, TM)$ be a family of tangent vector fileds, the hyperbolic gradient flow under considered here is given by the following evolution equation

$$\frac{\partial X_t}{\partial t} + \frac{1}{2}\nabla(\|X_t\|^2) = 0, \tag{5.1}$$

where, if in local coordinates $X_t = \sum_{i=1}^{n} X_t^i \frac{\partial}{\partial x_i}$, then $\|X_t\|^2$ is defined by

$$\|X_t\|^2 = g_{ij} X_t^i X_t^j \tag{5.2}$$

and ∇h stands for the gradient vector field of a function h on the manifold, and $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. By definition, for any given $h \in C^\infty(\mathscr{M}, \mathbb{R})$ and $X \in TM$, we have

$$g(X, \nabla h) = X(h). \tag{5.3}$$

The study of this flow will be very useful to understand the topological and geometrical structure of the manifold.

Finally, we would like to point out that, perhaps the method in the present paper is more important than the results obtained here. Our method may provide a new approach to some conjectures in differential geometry (see Yau [26]).

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