Weakly Ordered Multiplicative Basis of An Algebra Related to Quiver Theory

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Abstract

In this paper, for an algebra, weakly ordered multiplicative basis with weakly admissible order is introduced as an invariant under almost Morita equivalence and it is shown that an algebra with weakly ordered multiplicative basis is a quotient of its corresponding generalized path algebra, as a generalization of the classical Gabriel structure theorem. Finally, it is shown that the skew group algebra of a cyclic group over an algebra with ordered multiplicative basis has really a weakly ordered multiplicative basis.

Keywords: ordered multiplicative basis, quiver, generalized path algebra, skew group algebra

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1. Introduction

Originally, the classical theory of Gröbner bases gives efficient methods of computation of Hilbert polynomials of graded and filtered modules over polynomial rings. Mora in [11] presented the method of Gröbner bases for free algebras over a field when such a basis is finite. Since path algebras are quotients of free algebras, Green et al [5] extended the theory to path algebras. Furthermore, Green [6] introduces algebras with ordered multiplicative basis and proves this kind of algebras are quotients of path algebras, see Theorem 1.1. From our discussion in this paper, we know indeed such algebras with ordered multiplicative bases are a generalization of artinian basic algebras as said in Remark 3.7, or we say, algebras with ordered multiplicative bases are non-artinian "basic" algebras. So, one can think that this result in [6] is a generalization of the well-known Gabriel structure theorem about artinian basic algebras in [3][2].

From [11][5][6], we know the property for an algebra with an ordered multiplicative basis is non-invariant under Morita equivalence. In this reason, one of our motivations is to find a similar notion of ordered multiplicative basis as the replacement which is "almost" invariant under Morita equivalence (we will say it *almost Morita equivalent*) such that an algebra satisfying such property possesses some similar characterizations. This notion is just the so-called *weakly ordered multiplicative basis with weakly admissible order* which we will introduce in Section 2. Its main example here is from generalized path algebras in [4][7][8][9][10].

On the other hand, in [7][10], we generalized the classical Gabriel structure theorem for artinian basic algebras to artinian non-basic algebras if they are splitting over radicals as the only condition. As mentioned above, the Green's result in [6] can be thought as a generalization of classical Gabriel theorem to algebras with ordered multiplicative bases as a kind of non-artinian "basic" algebras. As the other motivation of this paper, we want to find a generalization of algebras with ordered multiplicative bases which can be realzied as a kind of non-artinian "non-basic" algebras and but also the generalized Gabriel structure theorem can be obtained for this kind of algebras. It is interesting that they are still algebras with weakly ordered multiplicative bases. The main result in this paper is that an algebra with weakly ordered multiplicative basis are a quotient of its corresponding generalized path algebra.

This paper can be regarded as a further step of representation theory of non-artinian algebras. As we know, due to the classical Gabriel structure theorem, the representation theory of artinian basic algebras is related to the representation theory of quivers. Moreover, by the generalized Gabriel structure theorem in [7][10], the representation theory of artinian (non-basic) algebras is related to the representation theory of generalized path algebras, equivalently, by [8], to that of pre-modulations. Therefore, as a development, it would be interesting to study the representation theory of algebras with weakly ordered multiplicative bases (in particular, that with ordered multiplicative bases) related to the representation theory of generalized path algebras (in particular, that of quivers) via the main theorem in this paper. We will do it in our successor research.

Finally, it is shown that the skew group algebra of a cyclic group over an algebra with ordered multiplicative basis has a weakly ordered multiplicative basis. This gives an example of an algebra with weakly ordered multiplicative basis which is not really an ordered multiplicative basis.

Now, we give some preliminaries including notions and results for our need.

In this paper, $\mathbb{K}$ always denotes an algebraically closed field, R a $\mathbb{K}$-algebra. For two subset S, T of R, write $ST = \{st | s \in S, t \in T\} \backslash \{0\}$. If $S^2 \subset S \cup \{0\}$, we say S to be **0-closed**. If a set $\mathcal{B} = \{b_\omega \in R : \omega \in \Omega\}$ is 0-closed and also a $\mathbb{K}$-basis of R, then $\mathcal{B}$ is called a **multiplicative basis** for R.

By [6], $>$ is called an **admissible order** on $\mathcal{B}$ if the following properties hold:

A0. $>$ is well-order on $\mathcal{B}$;

A1. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_1b_3 > b_2b_3$ if both b_1b_3 and b_2b_3 are nonzero;

A2. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_3b_1 > b_3b_2$ if both b_3b_1 and b_3b_2 are nonzero;

A3. For all $b_1, b_2, b_3, b_4 \in \mathcal{B}$, if $b_1 = b_2b_3b_4$ then $b_1 \geqslant b_3$.

R is said to have **an ordered multiplicative basis** $(\mathcal{B}, >)$ if $\mathcal{B}$ is a multiplicative basis for R with an admissible order $>$ on $\mathcal{B}$.

For a multiplicative basis $\mathcal{B} = \{b_\omega \in R : \omega \in \Omega\}$ for R and any $r \in R$, let $r = \sum_{\omega\in\Omega} k_\omega b_\omega$ for $k_\omega \in \mathbb{K}$, the support of r, denoted as $\mathrm{Supp}(r)$, is the set of basis elements in $\mathcal{B}$ that occur in r, that is, $\mathrm{Supp}(r) = \{b_\omega : \text{if } k_\omega \neq 0 \text{ for } \omega \in \Omega\}$.

The tip of r, denoted as $\mathrm{Tip}(r)$, is the largest basis elements b_0 in $\mathrm{Supp}(r)$ on the order $>$. That is, $\mathrm{Tip}(r) = b_0 \in \mathrm{Supp}(r)$ such that for any $b_\omega \in \mathrm{Supp}(r)$, $b_\omega \leqslant b_0$.

In [6], we know that for a $\mathbb{K}$-algebra R with ordered multiplicative basis $(\mathcal{B}, >)$, the identity 1 of R can be decomposed as $1 = v_1 + \cdots + v_n$ such that $v_1, \cdots, v_n \in \mathcal{B}$ form a full set of primitive orthogonal idempotents for R and the set $\mathcal{B}\backslash\{v_1, \cdots, v_n\}$ has no idempotents. Denote $\Gamma_0 = \{v_1, \cdots, v_n\}$. Define

$\Gamma_1 = \{b \in \mathcal{B} : b \notin \Gamma_0$ and b cannot be written as a product b_1b_2 for $b_1, b_2 \in \mathcal{B}\backslash\Gamma_0\}$.
That is, Γ_1 is the **set of product indecomposable elements** in $\mathcal{B}\backslash\Gamma_0$. Note that Γ_1 is a unique set, as is Γ_0.

In [6], it was proved that $\Gamma_0 \cup \Gamma_1$ generates the multiplicative basis $\mathcal{B}$ of R; moreover, see [6, Proposition 3.3], for any $b \in \mathcal{B}$, there exist the unique i, j such that $v_i b = b$ and $bv_j = b$, and if $l \neq i$ then $v_l b = 0$, if $l \neq j$ then $bv_l = 0$, denote $o(b) = v_i$ and $t(b) = v_j$.

Then, the quiver Γ associated to $\mathcal{B}$ can be defined with Γ_0 as the vertex set and Γ_1 as the arrow set such that for $b \in \Gamma_1$, one views b as an arrow from $o(b)$ to $t(b)$, that is, $o(b)$ is the starting vertex of b and $t(b)$ is the end vertex of b. We call this quiver **the Green quiver of R on $\mathcal{B}$**.

The classical notations, concepts and results on quivers can be seen in [2][3]. The following theorem from [6] is fundamnental for our discussion.

Theorem 1.1. *Let R be a $\mathbb{K}$-algebra with an ordered multiplicative basis $(\mathcal{B}, >)$. Let Γ be the Green quiver of R on $\mathcal{B}$. Then there is a surjective $\mathbb{K}$-algebra homomorphism $\phi : K\Gamma \to R$ such that:*

(1) if p is a path in Γ, then $\phi(p) \in \mathcal{B} \cup \{0\}$;

(2) if $b \in \mathcal{B}$, then there is a path $p \in Q$ such that $\phi(p) = b$;

(3) the kernel of ϕ is a 2-nomial ideal; that is, it is generated by all elements of form p or $p - q$ where p and q paths in Γ.

The concept of generalized path algebra was introduced early in [4]. Here we review the different but equivalent method of definition which is given in [10].

Let $Q = (Q_0, Q_1)$ be a quiver. For each pair $(i, j) \in Q_0 \times Q_0$, define $\Omega(i, j) = \{a \in Q_1 : t(a) = j, o(a) = i\}$. Note that Q_1 is the disjoint union of all $\Omega(i, j)$.

Given a collection of $\mathbb{K}$-algebras $\mathcal{A} = \{A_i \mid i \in Q_0\}$, let e_i be the identity of A_i and $A_0 = \prod_{i\in Q_0} A_i$ the direct product of $\mathbb{K}$-algebras. Note that all e_i are orthogonal central idempotents of A_0. Let ${}_iM_j \overset{def}{=} A_i\Omega(i, j)A_j$ be the free A_i-A_j-bimodule with basis $\Omega(i, j)$. This is the free $A_i \otimes_k A_j^{op}$-module over the set $\Omega(i, j)$. Then, the rank of ${}_iM_j$ as A_i-A_j-bimodule is just the number of arrows from i to j in the quiver Q. Thus,

$$M = \oplus_{(i,j)\in Q_0\times Q_0} A_i\Omega(i, j)A_j \tag{1.1}$$

is an A_0-A_0-bimodule. The **generalized path algebra** is defined to be the tensor algebra

$$T(M, A_0) = \oplus_{n=0}^{\infty} M^{\otimes_{A_0} n}.$$

Here $M^{\otimes_{A_0} n} = M \otimes_{A_0} M \otimes_{A_0} \cdots \otimes_{A_0} M$ and $M^{\otimes_{A_0} 0} = A_0$. Usually, denote the generalized path algebra $T(M, A_0)$ by $\mathbb{K}(Q, \mathcal{A})$.

As a matter of fact, the classical path algebras are the special cases by taking $A_i = k$.

The generalized path algebra $\mathbb{K}(Q, \mathcal{A})$ is called **normal** if all A_i are simple $\mathbb{K}$-algebras. This is the most interested case of generalized path algebras, in particular when all A_i are central simple algebras. In [7][9][10], normal generalized path algebras are used to characterize the structures and representations of artinian algebras via the method of natural quivers as unlike as the classical method to depending upon the corresponding basic algebras.

Let A be an artinian algebra with radical r_A. Write $A/r_A = \bigoplus_{i=1}^{s} A_i$ the block decomposition of the algebra A/r_A. Then, r_A/r_A^2 is an A/r_A-bimodule by $\bar{a}\cdot(x+r_A^2)\cdot\bar{b} = axb+r_A^2$ for any $\bar{a} = a + r_A, \bar{b} = b + r_A \in A/r_A$ and $x \in r_A$. Let ${}_iM_j = A_i \cdot r_A/r_A^2 \cdot A_j$, then ${}_iM_j$ is finitely generated as A_i-A_j-bimodule for each pair (i, j).

Let $\Delta_0 = \{1, \cdots, s\}$ be the set of isomorphism classes of simple A-modules, which is also corresponding to the set of blocks of A/r_A. For $i, j \in \Delta_0$, set the number t_{ij} of arrows from i to j to be $rank_{A_i}({}_iM_j)_{A_j}$, that is, the least number of generators of ${}_iM_j$ as A_i-A_j-bimodule. Obviously, if ${}_iM_j = 0$, there is no arrows from i to j. Then we associate A the quiver $\Delta_A = (\Delta_0, \Delta_1)$ which is called the **natural quiver** of A.

Moreover, for A and its natural quiver Δ_A, we obtain the normal generalized path algebra $\mathbb{K}(\Delta_A, \mathcal{A})$ with $\mathcal{A} = \{A_1, \cdots, A_s\}$, which is defined as **the associated generalized path algebra** of an artinian algebra A.

In [10], it was shown that for an artinian $\mathbb{K}$-algebra A which is splitting over its radical, there is a surjective algebra homomorphism $\phi : \mathbb{K}(\Delta_A, \mathcal{A}) \to A$ with $J^s \subseteq \ker(\phi) \subseteq J$ for some positive integer s, that is, this algebra A is a class of Gabriel-type.

Moreover, we gave in [10] that if an artinian algebra A of Gabriel-type with admissible ideal is hereditary, then A is isomorphic to its related generalized path algebra $\mathbb{K}(\Delta_A, \mathcal{A})$.

2. **Weakly Ordered Multiplicative Bases**

We firstly a property of a $\mathbb{K}$-algebra A with an ordered multiplicative basis $(\mathcal{B}_A, >)$.

We have $1_A = \Sigma_{i=1}^{n} v_i$ where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\} \subseteq \mathcal{B}_A$ is the full set of primitive orthogonal idempotents for A by [6, Lemma 3.6], mentioned above. By [6, Proposition 3.3] (see Section 1) and A3, it is easy to see that

Lemma 2.1. *If $b_1 = b_2 b_3$ with each $b_i \in \mathcal{B}_A$, then $b_1 \geqslant b_2$ and $b_1 \geqslant b_3$.*

Proposition 2.2. *Using of the notation as above, $v_i A \ncong v_j A$ for any $i \neq j$.*

Proof. Assume that $v_i A \cong v_j A$, then there is $x \in v_i A v_j$ and $x' \in v_j A v_i$ such that $v_i = xx'$ and $v_j = x'x$. Note that $x = \Sigma_{i=1}^{n} k_i b_i$ and $x' = \Sigma_{j=1}^{m} k'_j b'_j$ with $k'_i, k'_j \in \mathbb{K}^*$, $v_i b_i v_j = b_i \in \mathcal{B}$ and $v_j b'_j v_i = b'_j \in \mathcal{B}$. It follows that $v_i = \Sigma_{i=1}^{n} k_i b_i \Sigma_{j=1}^{m} k'_j b'_j = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} k_i k'_j b_i b'_j$, and so there is some i and j such that $v_i = b_i b'_j$. By Lemma 2.1, we get $v_i \geqslant b_i$ which contradicts to [6, Lemma 3.4]. Thus $v_i A \ncong v_j A$ if $i \neq j$. $\square$

Of course, $\mathbb{K}$ is a trivial $\mathbb{K}$-algebra with ordered multiplicative basis. However, by Proposition 2.2, for any positive integer n, the matrix algebra $\mathrm{M}_n(K)$ has not an ordered multiplicative basis. This means that the property for an algebra with ordered multiplicative basis is not Morita invariant, even not invariant under the expansion by a full matrix algebra.

Now we will define a similar notion of ordered multiplicative basis as the replacement which is better as invariant under a certain sense.

Let R be a $\mathbb{K}$-algebra with a multiplicative basis $\mathcal{B}$. For convenience, we denote two subsets by $\mathcal{V} = \{b \in \mathcal{B} : b^2 = b\}$ and $\widetilde{\mathcal{V}} = \{b \in \mathcal{B} : b \text{ have a local inverse}\}$ (i.e., there is a $b^{-1} \in \mathcal{B}$ such that bb^{-1} and $b^{-1}b$ are two different idempotents). Define $\widetilde{\mathcal{B}} = \mathcal{B} \backslash (\mathcal{V} \cup \widetilde{\mathcal{V}})$.

We say that $>$ is a **weakly admissible order** on $\mathcal{B}$ if the following properties hold:

W0. $>$ is well-order on $\mathcal{B}$;

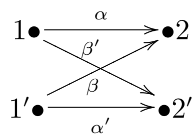
W1. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_1 b_3 > b_2 b_3$ if both $b_1 b_3$ and $b_2 b_3$ are nonzero;

W2. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_3 b_1 > b_3 b_2$ if both $b_3 b_1$ and $b_3 b_2$ are nonzero;

W3. For all $b_1, b_3 \in \mathcal{B}$, if there are $b_2, b_4 \in \mathcal{B} \backslash \widetilde{\mathcal{V}}$ such that $b_1 = b_2 b_3 b_4$ then $b_1 \geqslant b_3$.

With this condition, $(\mathcal{B}, >)$ is called a **weakly ordered multiplicative basis** for R.

Example 2.3. *Let $\mathbb{K}$ be a field of char$K \neq 2$ and Γ be the quiver*



Let $\Lambda = K\Gamma$ and let $G = \langle \sigma \rangle$ be the automorphism group of Γ with order 2, where σ is defined satisfying that

$$\sigma e_1 = e_{1'},\ \sigma e_2 = e_{2'},\ \sigma e_{1'} = e_1,\ \sigma e_{2'} = e_2,\ \sigma\alpha = \alpha',\ \sigma\beta = \beta',\ \sigma\alpha' = \alpha,\ \sigma\beta' = \beta.$$

Then,

$$\mathcal{B} = \{e_1, e_2, e_{1'}, e_{2'}, \alpha, \beta, \alpha', \beta', e_1\sigma, e_2\sigma, e_{1'}\sigma, e_{2'}\sigma, \alpha\sigma, \beta\sigma, \alpha'\sigma, \beta'\sigma\}$$

is a $\mathbb{K}$-basis of ΛG with $\mathcal{V} = \{e_1, e_2, e_{1'}, e_{2'}\}$, $\widetilde{\mathcal{V}} = \{e_1\sigma, e_2\sigma, e_{1'}\sigma, e_{2'}\sigma\}$. Define an order $<$ such that

$$e_1 < e_1\sigma < e_2 < e_2\sigma < e_{1'} < e_{1'}\sigma < e_{2'} < e_{2'}\sigma < \alpha < \alpha\sigma < \beta < \beta\sigma < \alpha' < \alpha'\sigma < \beta' < \beta'\sigma.$$

Then, it is trivial to check that the skew group algebra ΛG has a weakly ordered multiplicative basis $(\mathcal{B}, <)$.

Example 2.4. *Every normal generalized path algebra $\Lambda = \mathbb{K}(Q, \mathcal{A})$ has a weakly ordered multiplicative basis $(\mathcal{B}, <)$, where $\mathcal{A} = \{A_i | i \in Q_0\}$ with $A_i = M_{n_i}(K)$.*

Proof. Let $\widetilde{\mathcal{V}} = \{E^k_{ij}\}$ with $E^k_{ij} = (1)_{ij} \in A_k$ and $k \in Q_0$, then it is easy to see that

$\mathcal{B} = \widetilde{\mathcal{V}} \cup \{a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1} : \ n \geqslant 1, a_i \in \widetilde{\mathcal{V}}, \beta_1\beta_2\cdots\beta_n \text{ a path in } Q\}$ is a multiplicative basis for Λ. Define a well order $<$ in the multiplicative basis $\mathcal{B}$ as follows.

Let $<_Q$ be the left length-lexicographic order on Q. Then $E^k_{ij} <_{\widetilde{\mathcal{V}}} E^l_{st}$ provided $k <_Q l$. And, for E^k_{ij}, E^k_{st}, if $i < s$ then let $E^k_{ij} <_{\widetilde{\mathcal{V}}} E^k_{st}$. If $i = s$ and $j < t$, then $E^k_{ij} <_{\widetilde{\mathcal{V}}} E^k_{st}$. Such $<_{\widetilde{\mathcal{V}}}$ is obvious a well order on $\widetilde{\mathcal{V}}$. Furthermore, for any $b', b'' \in \mathcal{B}$ and $b' \neq b''$, if $b' \in \widetilde{\mathcal{V}}$ but $b'' \in \mathcal{B}\backslash\widetilde{\mathcal{V}}$, then we define $b' < b''$. Suppose $b', b'' \in \mathcal{B}\backslash\widetilde{\mathcal{V}}$ then there are two paths $p' = \beta'_1\beta'_2\cdots\beta'_n$ and $p'' = \beta''_1\beta''_2\cdots\beta''_m$ in Q such that $b' = a'_1\beta'_1a'_2\beta'_2\cdots a'_n\beta'_na'_{n+1}$ and $b'' = a''_1\beta''_1a''_2\beta''_2\cdots a''_m\beta''_ma''_{m+1}$. Define $b' < b''$ if $p' < p''$. Next, assume $p' = p''$, then there is a least positive integer $\mathbb{K}$ such that $a'_k \neq a''_k$. No loss of generality, if $a'_k <_{\widetilde{\mathcal{V}}} a''_k$, then define $b' < b''$.

It is easy to check that this order $<$ on $\mathcal{B}$ is weakly admissible, as required. □

For the remainder of this section, let R be always a $\mathbb{K}$-algebra with weakly ordered multiplicative basis $(\mathcal{B}, >)$.

Lemma 2.5. *For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1b_3 = b_2b_3 \neq 0$ or $b_3b_1 = b_3b_2 \neq 0$ then $b_1 = b_2$.*

Proof. Suppose that $b_1 \neq b_2$. No loss of generality, say $b_1 > b_2$. Then $b_1b_3 > b_2b_3$ and $b_3b_1 > b_3b_2$ by W1 and W2. This contradicts to the known condition. □

Denote by 1 the identity of R on multiplication. We have the following:

Lemma 2.6. *(i) The set $\mathcal{V}$ of all idempotents in $\mathcal{B}$ is a finite set;*

(ii) Write $\mathcal{V} = \{v_1 \cdots, v_n\}$, then $1 = \sum_{i=1}^n v_i$ with $v_iv_j = \begin{cases} v_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$;

(iii) For any $b \in \mathcal{B}$, there exist unique $v_i, v_j \in \mathcal{V}$ such that $v_ib = b = bv_j$.

Proof. Let

$$1 = \sum_{i=1}^n \alpha_i v_i \tag{2.1}$$

where each $0 \neq \alpha_i \in \mathbb{K}$ and $v_i \in \mathcal{B}$, $v_i \neq v_j$ for $i \neq j$. For any v_l here, we get $v_l = \sum_{i=1}^n \alpha_i v_i v_l$, then there is t such that $v_l = v_tv_l$ and by Lemma 2.5, $v_iv_l = 0$ for all $i \neq t$ and $\alpha_t = 1$. For v_t, similarly, there is p such that $v_t = v_tv_p$ and $v_tv_i = 0$ for all $i \neq p$. But, we have $v_tv_l = v_l \neq 0$ as said above. Hence $p = l$. Thus $v_t = v_tv_p = v_tv_l = v_l$. It follows that $v_l = v_l^2$ and $\alpha_l = 1$ for any $l = 1, \cdots, n$. Meantime, $1 = \sum_{i=1}^n v_i$.

For any j, $v_j = \sum_{i=1}^n v_iv_j$. We have $v_j = v_j^2$. Then, by Lemma 2.5, we can get $v_iv_j = 0$ for any $i \neq j$.

Let $\mathcal{V}_0 = \{v_1, \cdots, v_n\}$. Then $\mathcal{V}_0 \subseteq \mathcal{V}$.

Now, we show that for any $b \in \mathcal{B}$, there exist unique $v_{i_0}, v_{j_0} \in \mathcal{V}_0$ such that

$$v_{i_0}b = b = bv_{j_0}. \tag{2.2}$$

In fact, for any $b \in \mathcal{B}$, we have $b = \sum_{i=1}^{n} v_i b$. So, there is unique i_0 such that $b = v_{i_0} b$ and $v_i b = 0$ for all $i \neq i_0$. Similarly, there is unique j_0 such that $b = b v_{j_0}$ and $b v_j = 0$ for all $j \neq j_0$.

Next, we can obtain $\mathcal{V} = \mathcal{V}_0$. It is enough to prove $\mathcal{V} \subseteq \mathcal{V}_0$. For any $v \in \mathcal{V}$, of course $v = v^2$. According to the above discussion, there is $v_{j_0} \in \mathcal{V}_0$ such that $v v_{j_0} = v$. Thus, $v v_{j_0} = v^2$. By Lemma 2.5, we get $v = v_{j_0} \in \mathcal{V}_0$. Hence, $\mathcal{V} \subseteq \mathcal{V}_0$.

From $\mathcal{V} = \mathcal{V}_0 = \{v_1, \cdots, v_n\}$ and by (2.1), (2.2), the statements (i), (ii), (iii) follow. □

Due to (iii) of this lemma, for any $b \in \mathcal{B}$, let v_i, v_j be the unique elements $\mathcal{V}$ such that $v_i b = b = b v_j$ and denote $o(b) = v_i$ and $t(b) = v_j$.

Corollary 2.7. (i) *For $b \in \mathcal{B}$, if there is either $a \in \mathcal{B}$ or $b \in \mathcal{B}$ such that $b = ab$ or $b = bc$, then $a = o(b)$ or $c = t(b)$;*

(ii) *For $b \in \mathcal{B}$, if there is either $a, c \in \mathcal{B} \backslash \widetilde{\mathcal{V}}$ such that $b = abc$, then $a, c \in \mathcal{V}$ with $a = o(b)$ and $c = t(b)$.*

Proof. (i) If $b = ab$, then since $b = o(b)b$ by Lemma 2.6, we get $ab = o(b)b = b \neq 0$. Thus, by Lemma 2.5, it follows that $a = o(b)$. It is similar for the case that $b = bc$.

(ii) Since $b = abc$, then by Lemma 2.6, we have $o(a) = o(b)$ and $t(c) = t(b)$. Moreover, $bc = o(b)bc$ and $abc = abct(c)$, then by W3, we get $bc \geqslant b$ and $abc \geqslant bc$ respectively. Then,

$$b = abc \geqslant bc \geqslant b$$

Hence, $bc = b$ and $abc = bc$. Thus from (i), $c = t(b)$ and $a = o(bc) = o(b)$, which means $a, c \in \mathcal{V}$. □

Lemma 2.8. *Let $b \in \widetilde{\mathcal{V}}$. Then,*

(i) *The local inverse of b is unique;*

(ii) *$o(b) \neq t(b)$;*

(iii) *Denote by b^{-1} the local inverse of b, then $b^{-1} \in \widetilde{\mathcal{V}}$ and $bb^{-1} = o(b)$, $b^{-1}b = t(b)$.*

Proof. (i) Let b' is a local inverse of b, then $bb', b'b \in \mathcal{V}$. Then by Lemma 2.6, $bb' = v_i, b'b = v_j$ for some i and j. Thus $o(b) = o(bb') = o(v_i) = v_i$ and $t(b) = t(b'b) = t(v_j) = v_j$. Similarly, $t(b') = v_i$ and $o(b') = v_j$. Then, $o(b) = t(b')$ and $o(b') = t(b)$.

Assume that b'' is another local inverse of b. Similarly, $v_j = b''b$ and $v_i = t(b'')$. Then, $b' = o(b')b' = t(b)b' = v_j b' = b''bb' = b''v_i = b''t(b'') = b''$.

(ii) By (i), $o(b) = v_i = bb'$, $t(b) = v_j = b'b$. Then, from definition of $\widetilde{\mathcal{V}}$, $o(b) \neq t(b)$.

(iii) By (i) and its proof, $bb^{-1} = o(b)$, $b^{-1}b = t(b)$. And by (ii), $b^{-1} \in \widetilde{\mathcal{V}}$. □

From Lemma 2.6 and 2.8, we have firstly:

Corollary 2.9. *In any weakly ordered multiplicative basis $(\mathcal{B}, >)$, $\mathcal{V} \bigcap \widetilde{\mathcal{V}} = \varnothing$.*

Corollary 2.10. *A weakly ordered multiplicative basis* $(\mathcal{B}, >)$ is an ordered multiplicative basis of R if and only if its $\widetilde{\mathcal{V}} = \varnothing$.

Proof. "If": It is obvious by definitions of (weakly) ordered multiplicative basis.

"Only if": If there exists $v \in \widetilde{\mathcal{V}}$, then there is $v^{-1} \in \widetilde{\mathcal{V}}$ such that $vv^{-1}, v^{-1}v \in \mathcal{V}$. Thus, $vv^{-1}v = v$ and $v^{-1}vv^{-1} = v^{-1}$ by Lemma 2.8 (ii). Since $>$ is admissible, $v \geqslant v^{-1}$ and $v^{-1} \geqslant v$ by A3. So, $v = v^{-1}$. It follows that $o(v) = vv^{-1} = v^{-1}v = t(v)$, which contradicts to Lemma 2.8 (ii). □

Theorem 2.11. *Let R be a $\mathbb{K}$-algebra with weakly ordered multiplicative basis $(\mathcal{B}, >)$ and I be a monomial ideal (i.e., I is generated by some basis-elements in $\mathcal{B}$), then the quotient algebra R/I has a weakly ordered multiplicative basis.*

Proof. Let $\pi : R \to R/I$ be a canonical surjection, then since I is monomial, $\pi(\mathcal{B})$ is a multiplicative basis of R/I by [6, Theorem 2.3]. Also, because I is monomial, for any $0 \neq \bar{b} \in \pi(\mathcal{B})$, there is a unique element $b \in \mathcal{B} \setminus I$ such that $\pi(b) = b + I = \bar{b}$. We call this fact as the unique lifting property (briefly, ULP) of $\pi(\mathcal{B})$ in $\mathcal{B}$.

For any $\bar{b} \in \mathcal{V}_{R/I}$ with $b \in \mathcal{B}$, we have $b - b^2 \in I$. Assume that $b - b^2 \neq 0$, then $\mathrm{Supp}(b - b^2) = \{b, b^2\} \subseteq \mathcal{B} \cap I$. So, $b \in I$, then $\bar{b} = 0$. It contradicts to $\bar{b} \neq 0$. Thus $b^2 = b \in \mathcal{V}$, and so $\mathcal{V}_{R/I} \subseteq \{v + I | v \in \mathcal{V}\} \setminus \{0\}$. The converse inclusion is clear, and hence

$$\mathcal{V}_{R/I} = \{v + I | v \in \mathcal{V}\} \setminus \{0\}. \tag{2.3}$$

For any $\tilde{v} + I \in \widetilde{\mathcal{V}}_{R/I}$ with $\tilde{v} \in \mathcal{B}$, there is $v' + I \in \widetilde{\mathcal{V}}_{R/I}$ with $v' \in \mathcal{B}$ such that $v'\tilde{v} + I, \tilde{v}v' + I \in \mathcal{V}_{R/I}$ and $v'\tilde{v} + I \neq \tilde{v}v' + I$ with $v'\tilde{v}, \tilde{v}v' \in \mathcal{B}$. By (2.3), we have $v'\tilde{v}, \tilde{v}v' \in \mathcal{V}$ with $v'\tilde{v} \neq \tilde{v}v'$. It follows that $\tilde{v} \in \widetilde{\mathcal{V}}$. So, $\widetilde{\mathcal{V}}_{R/I} \subseteq \{\tilde{v} + I | \tilde{v} \in \widetilde{\mathcal{V}}\} \setminus \{0\}$.

Conversely, let $\tilde{v} \in \widetilde{\mathcal{V}} \setminus I$, then $\tilde{v}^{-1} \in \widetilde{\mathcal{V}}$ and $\tilde{v}^{-1}\tilde{v}, \tilde{v}\tilde{v}^{-1} \in \mathcal{V}$ with $\tilde{v}^{-1}\tilde{v} \neq \tilde{v}\tilde{v}^{-1}$. We can say that $\tilde{v}^{-1}, \tilde{v}^{-1}\tilde{v}, \tilde{v}\tilde{v}^{-1} \notin I$. Otherwise, $\tilde{v} = \tilde{v}\tilde{v}^{-1}\tilde{v} \in I$ by Lemma 2.8, which contradicts to $\tilde{v} \notin I$. Moreover, due to the above ULP, $(\tilde{v} + I)(\tilde{v}^{-1} + I) = \tilde{v}\tilde{v}^{-1} + I$ and $(\tilde{v}^{-1} + I)(\tilde{v} + I) = \tilde{v}^{-1}\tilde{v} + I$ are not equal each other in $\mathcal{V}_{R/I}$. Hence, $\tilde{v} + I \in \widetilde{\mathcal{V}}_{R/I}$ and thus, $\widetilde{\mathcal{V}}_{R/I} \supseteq \{\tilde{v} + I | \tilde{v} \in \tilde{\mathcal{V}}\} \setminus \{0\}$. Therefore,

$$\widetilde{\mathcal{V}}_{R/I} = \{\tilde{v} + I | \tilde{v} \in \tilde{\mathcal{V}}\} \setminus \{0\}. \tag{2.4}$$

Now, define the order $\succ$ in $\pi(\mathcal{B})$ satisfying that for any $\bar{b}_1 = b_1 + I, \bar{b}_2 = b_2 + I \in \pi(\mathcal{B})$ with $b_1, b_2 \in \mathcal{B}$, let $\bar{b}_1 \succ \bar{b}_2$ if $b_1 > b_2$. Then by the above ULP and (2.4), it is trivial to see that $\succ$ is a weakly admissible order on $\pi(\mathcal{B})$. □

One can think Theorem 2.11 as a method for constructing a new algebra with weakly ordered multiplicative basis from a known one. On the other hand, when R is a generalized path algebra, this theorem supplies an example of the converse problem of the latter Theorem 3.16, that is, under what condition, has the quotient of a generalized path algebra a weakly ordered multiplicative basis?

Lemma 2.12. *For any $b \in \mathcal{B} \backslash \widetilde{\mathcal{V}}$, $o(b) \leqslant b$ and $t(b) \leqslant b$.*

Proof. Note that $b = bt(b)t(b) = o(b)o(b)b$, then this relation follows from W3. □

Proposition 2.13. *In a weakly ordered multiplicative basis $\mathcal{B}$ of a $\mathbb{K}$-algebra R, all idempotents are primitive.*

Proof. By Lemma 2.6, it is just to prove all elements in $\mathcal{V} = \{v_1, \cdots, v_n\}$ are primitive.

For any $v_i \in \mathcal{V}$, suppose there are two nonzero orthogonal idempotents x and y such that $v_i = x + y$.

Since $0 = xy = x(v_i - x) = xv_i - x^2 = xv_i - x$, we have $xv_i = x$. Similarly, $v_i x = x$. So, $v_i x v_i = x$ and $o(x) = t(x) = v_i$. Similarly, $v_i y v_i = y$ and $o(y) = t(y) = v_i$.

Thus, we can write $x = k_i v_i + \sum_l k_l b_l$ with $k_i, k_l \in \mathbb{K}$ and $b_l \in \mathcal{B} \backslash \mathcal{V}$ satisfying $o(b_l) = t(b_l) = v_i$, which implies $b_l \in \widetilde{\mathcal{B}}$. Then $y = (1 - k_i)v_i - \sum_l k_l b_l$. It follows that

$$0 = xy = k_i(1 - k_i)v_i + \sum_l k_l b_l + \sum_l \sum_j k_l k_j b_l b_j \tag{2.5}$$

If there are l, j such that $b_l b_j = v_i$, then $b_l b_j v_i = v_i$. By W3, $v_i \geqslant b_j$. But, $b_j \neq v_i$. So, $t(b_j) = v_i \gneqq b_j$, which contradicts to Lemma 2.12.

Therefore, $b_l b_j \neq v_i$ for any l, j and thus by (2.5),

$$\sum_l k_l b_l + \sum_l \sum_j k_l k_j b_l b_j = 0. \tag{2.6}$$

We conclude that $k_i(1 - k_i) = 0$. Then, $x = k_i v_i + \sum_l k_l b_l$ with $k_i = 1$ or $k_i = 0$.

Assume that $\sum_l k_l b_l \neq 0$. Since $>$ is well-order, there is b as the minimal b_l occurring in $\sum_l k_l b_l$. By (2.6), $b_{l_0} b_{j_0} = b$ for some l_0 and j_0, then $b \geq b_{l_0}$ and $b \geq b_{j_0}$. Due to $b \notin \mathcal{V}$, it is impossible that $b = b_{l_0}$ and $b = b_{j_0}$ Hence, $b \gneqq b_{l_0}$ or $b \gneqq b_{j_0}$, which contradict to the minimality of b.

Thus, $\sum_l k_l b_l = 0$ and then $x = v_i$ or $x = 0$. It follows that v_i is primitive. □

Theorem 2.14. *Let n be a positive integer. A $\mathbb{K}$-algebra R has a weakly ordered multiplicative basis if and only if $M_n(R)$ has a weakly ordered multiplicative basis.*

Proof. Assume R has a weakly ordered multiplicative basis $(\mathcal{B}, >)$, then the set

$$M_n(\mathcal{B}) = \{E_{ij}(b) : b \in \mathcal{B}\}$$

is a $\mathbb{K}$-basis of $\mathrm{M}_n(R)$, where $E_{ij}(b)$ means the matrix whose all elements are zero except the (i,j)-element is b. For any $E_{ij}(b),\ E_{kl}(b') \in M_n(\mathcal{B})$, we have

$$E_{ij}(b)E_{kl}(b') = \begin{cases} 0, & \text{if } k \neq j \text{ or } bb' = 0; \\ E_{il}(bb'), & \text{otherwise }. \end{cases}$$

Since $bb' \in \mathcal{B}$, it follows $E_{il}(bb') \in M_n(\mathcal{B})$. Therefore, $\mathrm{M}_n(\mathcal{B})$ is a multiplicative basis.

Denote that

$$\mathcal{V}_{M_n(R)} = \{E_{ii}(v) | v \in \mathcal{V}, 1 \leqslant i \leqslant n\}$$

$$\widetilde{\mathcal{V}}_{M_n(R)} = \{E_{ij}(v) | v \in \widetilde{\mathcal{V}}, 1 \leqslant i, j \leqslant n\} \cup \{E_{ij}(v) | v \in \mathcal{V}, 1 \leqslant i \neq j \leqslant n\}$$

Define an order $\succ$ on $\mathrm{M}_n(\mathcal{B})$ satisfying that if $b > b'$, then $E_{ij}(b) \succ E_{kl}(b')$; if $b = b'$ and $i > k$, then $E_{ij}(b) \succ E_{kl}(b')$; if $b = b', i = k$ and $j > l$, then $E_{ij}(b) \succ E_{kl}(b')$. The only we have to show is that $\succ$ is a weakly admissible ordered on $\mathrm{M}_n(\mathcal{B})$.

W0. Let S be any subset of $\mathrm{M}_n(\mathcal{B})$. Denote $T = \{b \in \mathcal{B} : \exists i, j \text{ such that } E_{ij}(b) \in S\}$. Since $>$ is well-order on $\mathcal{B}$, we have a least element b_0 in T. Then for any $E_{kl}(b') \in S$, we have $b' \geq b$. Thus, for any i, j, it holds $E_{kl}(b') \succ E_{ij}(b)$. Let $K_0 = \{E_{ij}(b_0) : 1 \leq i, j \leq n\}$.

Then, there are i_0, j_0 such that $E_{i_0 j_0}(b_0)$ is the least element in K_0 on $\succ$. It is easy to see that $E_{i_0 j_0}(b_0)$ is also the least element in S on $\succ$. Therefore, $\succ$ is a well order on $\mathrm{M}_n\mathcal{B}$.

W1. No loss of generality, we only need to consider $E_{ij}(b_1),\ E_{kj}(b_2),\ E_{jl}(b_3) \in M_n(\mathcal{B})$ and $b_1b_3,\ b_2b_3$ are nonzero. If $E_{ij}(b_1) \succ E_{kj}(b_2)$, then $b_1 > b_2$ or $b_1 = b_2$ and $i \geqslant k$. Note that $E_{ij}(b_1)E_{jl}(b_3) = E_{il}(b_1b_3)$ and $E_{kj}(b_2)E_{jl}(b_3) = E_{kl}(b_2b_3)$. Since $b_1 > b_2$ implies $b_1b_3 > b_2b_3$ by W1 on $>$, it follows that $i \geqslant k$ implies $E_{ij}(b_1)E_{jl}(b_3) \succ E_{kj}(b_2)E_{jl}(b_3)$ by definition of $\succ$.

The proof of W2 is similar to that of W1.

W3. If $E_{il}(b_1) = E_{ij}(b_2)E_{jk}(b_3)E_{kl}(b_4)$ with $E_{ij}(b_2), E_{kl}(b_4) \notin \widetilde{\mathcal{V}}_{M_n(R)}$, then $b_1 = b_2b_3b_4$ and $b_2, b_4 \notin \widetilde{\mathcal{V}}$. It follows that $b_1 \geqslant b_3$. If $b_1 > b_3$, then $E_{il}(b_1) \geqslant E_{jk}(b_3)$ holds clearly. If $b_1 = b_3$, then $b_2, b_4 \in \mathcal{V}$ by Corollary 2.7. Note that $E_{ij}(b_2), E_{kl}(b_4) \notin \widetilde{\mathcal{V}}_{M_n(R)}$, then $i = j$ and $k = l$, and hence $E_{il}(b_1) \geqslant E_{jk}(b_3)$. □

The idempotents in the $\mathbb{K}$-subalgebra $\mathbb{K}\mathcal{V}$ of R generated by $\mathcal{V}$ can be presented as $e = u_1 + \cdots + u_k$ with $u_1, \cdots, u_k \in \mathcal{V}$ and $u_i \neq u_j$ for $i \neq j$.

Lemma 2.15. *Let an idempotent $e = u_1 + \cdots + u_k$ with $u_1, \cdots, u_k \in \mathcal{V}$ and $u_i \neq u_j$ for $i \neq j$. Then for any $b \in \mathcal{B}$, (i) either $eb = b$ or $eb = 0$ and (ii) either $be = b$ or $be = 0$.*

Proof. (i) For any $b \in \mathcal{B}$, let $o(b) = v_i$ then $v_ib = b$. Denote $Supp(e) = \{u_1, \cdots, u_k\}$. If $v_i \in Supp(e)$, then $eb = b$. Otherwise, $eb = 0$. (ii) can be proved similarly. □

Proposition 2.16. *Assume R is a $\mathbb{K}$-algebra with weakly ordered multiplicative basis $\mathcal{B}$. Let an idempotent $e = u_1 + \cdots + u_k$ with $u_1, \cdots, u_k \in \mathcal{V}$ and $u_i \neq u_j$ for $i \neq j$. Then the $\mathbb{K}$-algebra eRe has also a weakly ordered multiplicative basis.*

Proof. It is easy to see $e\mathcal{B}e = \{ebe, b \in \mathcal{B}\}\backslash\{0\}$ is a $\mathbb{K}$-basis of eRe. For any $ebe, eb'e \in e\mathcal{B}e$, $ebeeb'e = \begin{cases} 0, & \text{if } be = 0 \text{ or } eb' = 0 \\ ebb'e, & \text{otherwise} \end{cases}$ by Lemma 2.15. From this and $bb' \in \mathcal{B} \cup \{0\}$, then $ebeeb'e \in e\mathcal{B}e \cup \{0\}$. That is, $e\mathcal{B}e$ is a multiplicative basis.

Clearly, $\mathcal{V}_{eRe} = \{u_1, \cdots, u_k\}$ and $\widetilde{\mathcal{V}}_{eRe} = \{eve : v \in \widetilde{\mathcal{V}}\}\backslash\{0\}$.

Note that $e\mathcal{B}e \subseteq \mathcal{B}$. The restriction of $>$ to $e\mathcal{B}e$ is trivially a weakly admissible order. □

It is well-known that two rings R and S are Morita equivalent if and only if there is a positive integer n and an idempotent matrix $e \in M_n(R)$ such that $S \cong eM_n(R)e$. Theorem 2.14 means that possessing weakly ordered multiplicative basis is invariant only for the special Mroita equivalence of R in the case e is the identity matrix; Proposition 2.16 means this property is invariant in the other case of Mroita equivalence of R, that is, $n = 1$ and e is only in $\mathbb{K}\mathcal{V}$, not any idempotent in R. Hence, the property for a $\mathbb{K}$-algebra to have weakly ordered multiplicative basis is not Morite invariant in general, but it has analogously invariant in the above cases. In this reason, we call the property for a $\mathbb{K}$-algebra to have weakly ordered multiplicative basis is **almost Morita invariant**.

However, from some examples in [11][5][6], the property for an algebra to have an ordered multiplicative basis is not such almost Morita invariant. This is just one of the motivations for us to introduce the notion of weakly ordered multiplicative basis.

3. Generalized Gabriel structure theorem

In this section, we assume that R is a $\mathbb{K}$-algebra with weakly ordered multiplicative basis $(\mathcal{B}, >)$ and maintain the notation of the above section. The aim in this part is to give the generalized Gabriel structure theorem for a $\mathbb{K}$-algebra with weakly ordered multiplicative basis as a reform of the Gabriel structure theorem for an artinian algebra in [7][10] and the classical Gabriel structure theorem for a finite dimensional basic algebra over an algebraically closed field in [2][3].

Lemma 3.1. *$\mathcal{V} \cup \widetilde{\mathcal{V}}$ is 0-closed under multiplication.*

Proof. Assume that $b_1, b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$ and $b_1 b_2 \neq 0$.

If $b_1 \in \mathcal{V}$ or $b_2 \in \mathcal{V}$, then $b_1 = o(b_2)$ or $b_2 = t(b_1)$. Thus $b_1 b_2 = b_2$ or $b_1 b_2 = b_1$, which means $b_1 b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$.

Otherwise, $b_1, b_2 \in \widetilde{\mathcal{V}}$. Then there are some $b_1^{-1}, b_2^{-1} \in \widetilde{\mathcal{V}}$ such that $b_1 b_1^{-1} = o(b_1)$, $b_2 b_2^{-1} = o(b_2) = t(b_1)$. Thus $b_1 b_2 (b_2^{-1} b_1^{-1}) = o(b_1)$ and $(b_2^{-1} b_1^{-1}) b_1 b_2 = o(b_2)$. Then by definition, we get $b_1 b_2 \in \widetilde{\mathcal{V}}$. Hence $b_1 b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$. □

We define the relation $\sim$ on $\mathcal{B}$: for any b, $b' \in \mathcal{B}$, $b \sim b'$ if $b = b_1 b' b_2$ for some $b_1, b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$.

Lemma 3.2. *$\sim$ is an equivalence relation on $\mathcal{B}$.*

Proof. 1. Reflexivity: for any $b \in \mathcal{B}$, we have $b = o(b) b t(b)$.

2. Symmetry: if $b \sim b'$, then there are $b_1, b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$ such that $b = b_1 b' b_2$. It follows that $b' = \widetilde{b}_1 b \widetilde{b}_2$ where $\widetilde{b}_1$ is ${b_1}^{-1}$ or $o(b')$, $\widetilde{b}_2$ is ${b_2}^{-1}$ or $t(b')$.

3. Transitivity: if $b \sim b'$ and $b' \sim b''$, then there are $b_i \in \mathcal{V} \cup \widetilde{\mathcal{V}}$ $(i = 1, 2, 3, 4)$ such that $b = b_1 b' b_2$ and $b' = b_3 b'' b_4$. Thus $b = b_1 b_3 b'' b_4 b_2$. By Lemma 3.1, $b_1 b_3, b_4 b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$. Hence $b \sim b''$. □

And, as the restriction on $\mathcal{V}$ of the relation $\sim$, we get the decomposition $\mathcal{V} = \bar{v}^1 \cup \cdots \cup \bar{v}^m$ with $\bar{v}^i$ all equivalences classes for $i = 1, \cdots, m$. In the sequel, we always set $|\bar{v}^i| = k_i$, then $|\mathcal{V}| = k_1 + \cdots + k_m$.

Lemma 3.3. *For $v', v'' \in \mathcal{V}$ and $v' \neq v''$, $v' \sim v''$ if and only if there is a unique $b \in \widetilde{\mathcal{V}}$ such that $v' b v'' = b$, that is, $o(b) = v'$, $t(b) = v''$.*

Proof. "if": In this case, $o(b) = v'$, $t(b) = v''$. Then, $v' b v'' b^{-1} = v'$ where $v' b, b^{-1} \in \widetilde{\mathcal{V}}$. It means $v' \sim v''$.

"only if": By definition, $v' = b_1 v'' b_2$ for $b_1, b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$. If $b_1 \in \mathcal{V}$ or $b_2 \in \mathcal{V}$, then $v' = v''$ which contradicts to $v' \neq v''$. So, $b_1, b_2 \in \widetilde{\mathcal{V}}$. It follows that $v' b_2^{-1} = b_1 v''$. But, $b_2^{-1} = v' b_2^{-1}$ and $b_1 = b_1 v''$. Thus, $b_1 = b_2^{-1}$. Then, $b_1 = v' b_1 = v' b_2^{-1} = b_1 v'' = v' b_1 v''$. So, $b = b_1$ is just that we hope.

Assume there another $b_3 \in \widetilde{\mathcal{V}}$ such that $b_3 = v' b_3 v''$. Then $v' = b_3 b_3^{-1}$. Since $b_2 = b_2 v' = b_2 b_3 b_3^{-1}$, it follows $b_3 b_2 = b_2 b_3$. Then $b_3 b_2 \neq 0$ (otherwise, $b_2 = 0$). Hence, $b_3 b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$ by Lemma 3.1. Note that $o(b_3 b_2) = o(b_3) = v' = t(b_2) = t(b_3 b_2)$, and so $b_3 b_2 \in \mathcal{V}$ by

Lemma 2.8 (ii), that is, $b_3b_2 = v'$. Thus $b_1 = v'b_1 = b_3b_2b_1 = b_3v'' = b_3$, which means the uniqueness of b. $\square$

In the sequel, if $v', v'' \in \bar{v}^i$ with $v' \neq v''$, we denote v' and v'' respectively by v^i_{jj} and v^i_{ll} for some $j, l = 1, \cdots, k_i$ with $j \neq l$, and denote the unique b in Lemma 3.3 by v^i_{jl}. Then, $v^i_{jj}v^i_{jl}v^i_{ll} = v^i_{jl}$ and $o(v^i_{jl}) = v^i_{jj}$, $t(v^i_{jl}) = v^i_{ll}$.

Furthermore, for any $t = 1, \cdots, k_i$, $v^i_{jj}(v^i_{jt}v^i_{tl})v^i_{ll} = (v^i_{jj}v^i_{jt})(v^i_{tl}v^i_{ll}) = v^i_{jt}v^i_{tl}$. Then $v^i_{jt}v^i_{tl}$ can only be in $\widetilde{\mathcal{V}}$ and, due to the uniqueness of b in Lemma 3.3, we have for any $j, t, l = 1, \cdots, k_i$ and $i = 1, \cdots, m$,

$$v^i_{jt}v^i_{tl} = v^i_{jl}. \tag{3.1}$$

And, for any $i_1 \neq i_2$ and $p, q = 1, \cdots, k_{i_1}$ and $u, w = 1, \cdots, k_{i_2}$, we have

$$v^{i_1}_{pq}v^{i_2}_{uw} = 0 \tag{3.2}$$

since $t(v^{i_1}_{pq}) = v^{i_1}_{qq}$ and $o(v^{i_2}_{uw}) = v^{i_2}_{ww}$ are different in $\mathcal{V}$.

For any $b \in \widetilde{\mathcal{V}}$, by Lemma 3.3, it holds $o(b) \sim t(b)$, so we can write $o(b) = v^i_{jj}, t(b) = v^i_{ll}$ for some $i = 1, \cdots, m$ and $j, l = 1, \cdots, k_i$ with $j \neq l$. Then, as seen above, we have $b = v^i_{jl}$. Therefore, the set $\{v^i_{jl} : i = 1, \cdots, m, j, l = 1, \cdots, k_i \text{ with } j \neq l\}$ consists of all elements of $\widetilde{\mathcal{V}}$, that is, we obtain:

Proposition 3.4. *(i)* $\widetilde{\mathcal{V}} = \{v^i_{jl} : i = 1, \cdots, m,\ j, l = 1, \cdots, k_i$ *with* $j \neq l\}$*;*

(ii) $\widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}_1 \cup \cdots \cup \widetilde{\mathcal{V}}_m$ *where* $\widetilde{\mathcal{V}}_i = \{v^i_{jl} : j, l = 1, \cdots, k_i$ *with* $j \neq l\}$ *for* $i = 1, \cdots, m$ *as the equivalence classes of restriction on* $\widetilde{\mathcal{V}}$ *of the relation* $\sim$*;*

(iii) $|\widetilde{\mathcal{V}}| = k_1(k_1 - 1) + \cdots + k_m(k_m - 1)$ *with* $|\widetilde{\mathcal{V}}_i| = k_i(k_i - 1)$ *for* $i = 1, \cdots, m$*.*

Proof. (i) has been proved in above.

(ii) By (3.2) and the definition of the relation $\sim$, it is easy to see that for any $i_1 \neq i_2$, $v^{i_1}_{pq} \not\sim v^{i_2}_{uw}$ always holds; for any $i = i_1 = i_2$, $v^i_{pq} \sim v^i_{uw}$ always holds, since $v^i_{pq} = v^i_{pu}v^i_{uw}v^i_{wq}$. Thus, $\widetilde{\mathcal{V}}_i = \{v^i_{jl} : j, l = 1, \cdots, k_i \text{ with } j \neq l\}$.

(iii) is from (ii). $\square$

Note $\widetilde{\mathcal{B}} = \mathcal{B} \backslash (\mathcal{V} \cup \widetilde{\mathcal{V}})$. Now, define $\mathcal{E} = \{b \in \widetilde{\mathcal{B}} :\ b$ cannot be written as a product b_1b_2 with $b_1, b_2 \in \widetilde{\mathcal{B}}\}$. Then, $\mathcal{E} = \bigcup_{\lambda \in \Lambda} \mathcal{E}_\lambda$ where $\mathcal{E}_\lambda$ ($\lambda \in \Lambda$) are all nonempty equivalence classes of $\mathcal{E}$ on the equivalence relation $\sim$.

Then, for any $v^i_{kk} \in \bar{v}^i$ and $v^j_{ll} \in \bar{v}^j$, we get $v^i_{kk}\mathcal{E}v^j_{ll} = \bigcup_{\lambda \in \Lambda} v^i_{kk}\mathcal{E}_\lambda v^j_{ll}$.

Lemma 3.5. *For any* $b \in \mathcal{V} \cup \widetilde{\mathcal{V}}$*,* $b\mathcal{E} \cup \mathcal{E}b \subseteq \mathcal{E} \cup \{0\}$*.*

Proof. The case of $b \in \mathcal{V}$ is obvious. Now, let $b \in \widetilde{\mathcal{V}}$.

For any $b_1 \in \mathcal{E}$. First, $bb_1 \in \widetilde{\mathcal{B}}$ holds (otherwise, $b_1 = b^{-1}(bb_1) \in \widetilde{\mathcal{V}}(\mathcal{V} \cup \widetilde{\mathcal{V}}) \subset \mathcal{V} \cup \widetilde{\mathcal{V}}$ by Lemma 3.1). Assume that $bb_1 = b_2b_3 \neq 0$ with $b_2, b_3 \in \widetilde{\mathcal{B}}$. It follows that $b_1 = b^{-1}b_2b_3$. Note that $b^{-1}b_2, b_3 \in \widetilde{\mathcal{B}}$ by Lemma 3.1 and hypothesis. It contradicts to $b_1 \in \mathcal{E}$. Hence, $b\mathcal{E} \subset \mathcal{E} \cup \{0\}$. Similarly, $\mathcal{E}b \subset \mathcal{E} \cup \{0\}$. $\square$

Proposition 3.6. *Assume R is a $\mathbb{K}$-algebra with weakly ordered multiplicative basis $\mathcal{B}$. Let an idempotent $e = u_1 + \cdots + u_k$ with $u_1, \cdots, u_k \in \mathcal{V}$ and $u_i \neq u_j$ for $i \neq j$. If $|\mathrm{Supp}(e) \cap \bar{v}^i| \leqslant 1$ for any $i = 1, \cdots, m$, then eRe has an ordered multiplicative basis.*

Proof. By Corollary 2.10 and Proposition 2.16, it is sufficient to show that

$$\widetilde{\mathcal{V}}_{eRe} = \{eve | v \in \widetilde{\mathcal{V}}\} \setminus \{0\} = \emptyset.$$

In fact, for any $v \in \widetilde{\mathcal{V}}$, since $v^{-1}v = t(v)$ by Lemma 2.8, $o(v) \sim t(v)$ by the definition of the relation $\sim$. Hence, there is some integer i such that $o(v), t(v) \in \bar{v}^i$. Because at most one of $o(v)$ and $t(v)$ belongs to $\mathrm{Supp}(e)$, and so $eve = 0$, as desired. □

Now, let R be an artinian $\mathbb{K}$-algebra with weakly ordered multiplicative basis $\mathcal{B}$ and an idempotent $e = u_1 + \cdots + u_k$ with $u_1, \cdots, u_k \in \mathcal{V}$ and $u_i \neq u_j$ for $i \neq j$ satisfying $|\mathrm{Supp}(e) \cap \bar{v}^i| = 1$ for any $i = 1, \cdots, m$. Then the algebra eRe is just the basic algebra of R.

Therefore, in this sense, for a general R satisfying Proposition 3.6 which may be not artinian, we say eRe to be the *basic algebra* of R, as a generalization of the notion when R is artinian.

Example 3.7. For ΛG in Example 2.3, it is easy to check that for $i = 1, 2$, the projective modules of e_i and e_i' are isomorphic with each other. So, due to the above discussion, the basic algebra $B_{\Lambda G}$ of ΛG is $e\Lambda Ge$ with $e = e_1 + e_2$. Then $B_{\Lambda G}$ has an ordered multiplicative basis $\{e_1, e_2, \alpha, \beta'\sigma\}$ by Proposition 3.6 and $B_{\Lambda G} \cong K\widetilde{\Gamma}/I$ by [6, Theorem 3.8] where

$$\widetilde{\Gamma}: \ 1\bullet \rightrightarrows \bullet 2 \ .$$

Note that $dim B_{\Lambda G} = 4 = K\widetilde{\Gamma}$, and hence $I = 0$ and $B_{\Lambda G} \cong K\widetilde{\Gamma}$. Thus $\widetilde{\Gamma}$ is the Ext-quiver of $B_{\Lambda G}$ and $\Lambda G \cong M_2(B_{\Lambda G}) \cong M_2(K\widetilde{\Gamma})$, this fact will be mentioned again in Example 3.17.

Proposition 3.8. *For any i, j, k, l, s, $|v^i_{kk}\mathcal{E}_s v^j_{ll}| \leq 1$ with $v^i_{kk} \in \bar{v}^i$ and $v^j_{ll} \in \bar{v}^j$.*

Proof. Note that $v^i_{kk}\mathcal{E}_s v^j_{ll} = \{v^i_{kk} b v^j_{ll}, b \in \mathcal{E}_s\} \setminus \{0\}$ by definition. If $v^i_{kk} b v^j_{ll} = 0$ for any $b \in \mathcal{E}_s$, then $v^i_{kk}\mathcal{E}_s v^j_{ll} = \emptyset$. So $|v^i_{kk}\mathcal{E}_s v^j_{ll}| = 0$. Otherwise, $v^i_{kk}\mathcal{E}_s v^j_{ll} \neq \emptyset$. Let $b,\ b' \in v^i_{kk}\mathcal{E} v^j_{ll}$, then $\exists\ b_1,\ b_2 \in \mathcal{V} \cup \widetilde{\mathcal{V}}$ such that $b' = b_1 b b_2$. It follows that

$$o(b_1) = o(b') = v^i_{kk} = o(b) = t(b_1) \text{ and } t(b_2) = t(b') = v^j_{ll} = t(b) = o(b_2),$$

and hence $b_1,\ b_2 \in \mathcal{V}$ by Lemma 2.8. Hence, $b' = b$. □

Corollary 3.9. *For any $v^i_{kk}, v^i_{k'k'} \in \bar{v}^i$ and $v^j_{ll}, v^j_{l'l'} \in \bar{v}^j$, then $|\ v^i_{kk}\mathcal{E}v^j_{ll}\ | = |\ v^i_{k'k'}\mathcal{E}v^j_{l'l'}\ |$.*

Proof. It is easy to see by Proposition 3.8 and $v^i_{kk}\mathcal{E}v^j_{ll} = \bigcup_{\lambda \in \Lambda} v^i_{kk}\mathcal{E}_\lambda v^j_{ll}$. □

Proposition 3.10. *Every $b \in \widetilde{\mathcal{B}}$ can be written as a product $b_1 \cdots b_r$ with $b_i \in \mathcal{E}$. In particular, $\mathcal{V} \cup \widetilde{\mathcal{V}} \cup \mathcal{E}$ generate the multiplicative basis $\mathcal{B}$.*

Proof. The proof is similar to [6, Proposition 3.7]. Without loss of generality, let $b \in \widetilde{B}$ and $b = b_1 b_2$ with $b_1, b_2 \in \widetilde{B}$. By W3, Lemma 2.5 and assumption, we get $b > b_1$ and $b > b_2$. If both b_1 and b_2 are in $\mathcal{E}$, then we are done. Continuing in this fashion, we get $b = b_{i1} b_{i2} \dots b_{ir}$ with $b_{ij} \in \widetilde{B}$. Since $>$ is a total order, we have a proper descending chain $b > b_{i_{s_1}} > \cdots > b_{i_{s_u}}$ where the chain is $r+1$ elements long. But $>$ is a well-order and this process must stop. That is, each b_{ij} must be product indecomposable and we are done. $\square$

For any rational number q, denote $\lceil q \rceil = ceil(q) = \min\{n \in \mathbb{Z} | q \leqslant n\}$. Now we define the **Green quiver Q of R on $\mathcal{B}$** as follows:

(i) The vertex set Q_0 is the upper index set $\{1, \cdots, n\}$ of $\{\bar{v}^1, \cdots, \bar{v}^n\}$;

(ii) $t_{ij} = \lceil \frac{m_{ij}}{k_i k_j} \rceil$ is the number of arrows from i to j where $m_{ij} = |v^i_{kk} \mathcal{E} v^j_{ll}|$ for some fixed k and l.

Note that (1) t_{ij} is unique determined by the pair (i,j) due to Corollary 3.9; (2) if $m_{ij} = +\infty$ for some i, j, then $t_{ij} = +\infty$, that is, in Q there are infinite arrows from i to j.

Let A_i be the $\mathbb{K}$-subalgebra of R generated by the bases set $\{v^i_{st}\}$ with $1 \leqslant s, t \leqslant k_i$, and $A = \bigoplus_{i=1}^{n} A_i$, $M = \bigoplus_{i,j=1}^{n} {}_iM_j$ with ${}_iM_j$ the free A_i-A_j-bimodule with basis consisting of the m_{ij} arrows from i to j, then the generalized path algebras $\mathbb{K}(Q, \mathcal{A}) \cong T(A, M)$ with $\mathcal{A} = \{A_i\}_{i=1}^n$. It is easy to check that each $A_i \cong M_{k_i}(K)$.

Proposition 3.11. *If R is an Artinian $\mathbb{K}$-algebra with a multiplicative basis $\mathcal{B}$, then $J = R\langle \mathcal{E} \rangle = K\langle \widetilde{\mathcal{B}} \rangle$ is the radical of R.*

Here $R\langle \mathcal{E} \rangle$ denotes the ideal of R generated by $\mathcal{E}$, $K\langle \widetilde{\mathcal{B}} \rangle$ the $\mathbb{K}$-linear space generated by $\widetilde{\mathcal{B}}$.

Proof. Consider the chain $\cdots \subset J^i \subset \cdots \subset J^2 \subset J$, then there is an integer n such that $J^{n+1} = J^n$. First, we show that $J^n = 0$.

Assume $J^{n+1} = J^n \neq 0$. Then there is a minimal element b of $\underbrace{\widetilde{\mathcal{B}} \cdots \widetilde{\mathcal{B}}}_{n}$ by W0, and so there are $r_1, \cdots, r_n, r_{n+1} \in J$ such that $b = r_1 r_2 \cdots r_n r_{n+1}$. Set $r_i = \sum_{j_i} k_{j_i} b_{j_i}$ with $b_{j_i} \in \widetilde{\mathcal{B}}$, then $b = \sum_{j_1 \cdots j_{n+1}} k_{j_1} \cdots k_{j_{n+1}} b_{j_1} \cdots b_{j_n} b_{j_{n+1}}$. So $b = b_{j_1} b_{j_2} \cdots b_{j_n} b_{j_{n+1}}$ for some $j_1 \cdots j_{n+1}$. By W3 and Lemma 2.5, we get $b > b_{j_1} b_{j_2} \cdots b_{j_n} \in \underbrace{\widetilde{\mathcal{B}} \cdots \widetilde{\mathcal{B}}}_{n}$ which contradicts to the minimal property.

Thus $J^n = 0$, that is, J is nilpotent, so $J \subset rad(R)$.

Now, it is sufficient to show that $rad(R) \subset J$. Let $0 \neq r \in rad(R)$.

If there is a $v \in \mathcal{V}$ such that $v \in \mathrm{Supp}(r)$, then $vrv \in rad(R)$ and $\mathrm{Supp}(vrv) \setminus \{v\} \subset \widetilde{\mathcal{B}}$. Set $vrv = kv + \sum_i k_i b_i$ with $b_i \in \widetilde{\mathcal{B}}$ and $k, k_i \in \mathbb{K}$. Note that $\sum_i k_i b_i \in J \subset rad(R)$, then $v \in rad(R)$. It follows that v is nilpotent, a contradiction. Thus $\mathcal{V} \cap \mathrm{Supp}(r) = \emptyset$.

If there is a $\tilde{v} \in \widetilde{\mathcal{V}}$ such that $\tilde{v} \in \mathrm{Supp}(r)$, then

$$o(\tilde{v}) r t(\tilde{v}) \in rad(R) \text{ and } \widetilde{\mathcal{V}} \cap \mathrm{Supp}(o(\tilde{v}) r t(\tilde{v})) = \{\tilde{v}\}$$

by Lemma 3.3. Similar to the above proof, we get $\tilde{v} \in rad(R)$. And so $o(\tilde{v}) = \tilde{v}\tilde{v}^{-1} \in rad(R)$, which is contradict to the above proof. Thus $\widetilde{\mathcal{V}} \cap \mathrm{Supp}(r) = \emptyset$.

Therefore $\mathrm{Supp}(r) \subset \widetilde{\mathcal{B}}$, that is, $r \in J$. So $J = rad(R)$. $\square$

Corollary 3.12. *If R is Artinian with weakly ordered multiplicative basis $\mathcal{B}$, then the Green quiver Q of R on $\mathcal{B}$ is just the natural quiver of R.*

Proof. By Proposition 3.11, $R/J = \bigoplus_{i=1}^{n} R_i$ with $R_i = \langle v_{st}^i + J : 1 \leqslant s,t \leqslant k_i \rangle$. Put ${}_iM_j = R_i \cdot J/J^2 \cdot R_j$,

$$ {}_iM_j = K\langle v_{st}^i b v_{pq}^j + J^2 : b \in \mathcal{E}, 1 \leqslant s,t \leqslant k_i, 1 \leqslant p,q \leqslant k_j \rangle. \tag{1} $$

$$ = R_i \langle v_{1t}^i b v_{p1}^j + J^2 : b \in \mathcal{E}, 1 \leqslant t \leqslant k_i, 1 \leqslant p \leqslant k_j \rangle R_j. \tag{2} $$

Let $T = \{v_{st}^i b v_{pq}^j + J^2 : b \in \mathcal{E}, 1 \leqslant s,t \leqslant k_i, 1 \leqslant p,q \leqslant k_j\}$, then we construct a map $\chi : T \to v_{kk}^i \mathcal{E} v_{ll}^j$ via $\chi(v_{st}^i b v_{pq}^j + J^2) = v_{kt}^i b v_{pl}^j$. Note that $v_{kt}^i b v_{pl}^j = v_{ks} v_{st}^i b v_{pq}^j v_{ql}$ and $v_{st}^i b v_{pq}^j = v_{sk} v_{kt}^i b v_{pl}^j v_{lq}$, then χ is well defined and bijective by Lemma 3.5. By (1), we get ${}_iM_j = \langle T \rangle$. It follows that $\mathrm{rk}_{R_i}({}_iM_j)_{R_j} \overset{(2)}{=} \lceil \frac{\dim {}_iM_j}{k_i k_j} \rceil = \lceil \frac{|v_{kk}^i \mathcal{E} v_{ll}^j|}{k_i k_j} \rceil$, as desired. $\square$

From Corollary 3.12, we know that the Green quiver of an algebra R on weakly ordered multiplicative basis $\mathcal{B}$ can be thought as a generalization of the natural quiver of an artinian algebra defined in [7][9][10].

Let ${}_i\Omega_j = \{(s, \alpha, t)\}$ be a triple set with $1 \leqslant s \leqslant k_i$, $1 \leqslant t \leqslant k_j$ and α an arrow from i to j. Set $S_{ij} = \{v_{kk}^i \mathcal{E}_\lambda v_{ll}^j | \lambda \in \Lambda, v_{kk}^i \mathcal{E}_\lambda v_{ll}^j \neq 0\}$ for some fixed k and l, then $|S_{ij}| = |v_{kk}^i \mathcal{E} v_{ll}^j| = m_{ij}$.

Note that $|{}_i\Omega_j| = m_{ij} k_i k_j$ and $m_{ij} \geqslant \frac{|v_{kk}^i \mathcal{E} v_{ll}^j|}{k_i \times k_j}$, then $|{}_i\Omega_j| \geqslant |S_{ij}| = m_{ij}$. Thus we get

Lemma 3.13. *There is a surjective map $\sigma_{ij} : {}_i\Omega_j \to S_{ij}$.*

Lemma 3.14. *The set ${}_i\mathfrak{M}_j = \{v_{kl}^i \alpha v_{st}^j | \alpha$ is an arrow from i to $j\}$ is the $\mathbb{K}$-basis of ${}_iM_j$.*

Proof. By definition, ${}_i\mathfrak{M}_j$ is linearly independent. For any $x \in {}_iM_j$, there are $\{a_p^i\}_{p=1,\cdots,m_i} \subseteq A_i$ and $\{a_q^j\}_{q=1,\cdots,m_j} \subseteq A_j$ such that $x = \sum_{p,q} a_p^i \alpha a_q^j$. Note that $a_p^i = \sum_{s,t} k_{st} v_{st}^i$ and $a_q^j = \sum_{s',t'} k_{s't'} v_{s't'}^j$ for $k_{st}, k_{s't'} \in \mathbb{K}$, then

$$ x = \sum_{p,q} (\sum_{s,t} k_{st} v_{st}^i) \alpha \sum_{s',t'} k_{s't'} v_{s't'}^j = \sum_{p,q} \sum_{k,l} \sum_{s,t} k_{st} k_{s't'} v_{st}^i \alpha v_{s't'}^j. $$

Thus ${}_i\mathfrak{M}_j$ is a $\mathbb{K}$-basis of ${}_iM_j$. $\square$

Corollary 3.15. $v_i\mathfrak{M}_j \subseteq {}_i\mathfrak{M}_j \cup \{0\}$ *and* ${}_i\mathfrak{M}_j v \subseteq {}_i\mathfrak{M}_j \cup \{0\}$ *for any* $v \in \mathcal{V} \cup \widetilde{\mathcal{V}}$.

Theorem 3.16. *Let R be a $\mathbb{K}$-algebra with a weakly ordered multiplicative basis $(\mathcal{B}, >)$ and Q be the quiver associated to R. Then there is a surjective $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}(Q, \mathcal{A}) \to R$.*

Proof. As above, we have known the unique set of algebras $\mathcal{A}$ and the quiver Q associated to R. Now, we firstly define a map $\widetilde{\varphi} : A \oplus M \to R$.

Denote $v_{kl}^{[i_0]} = (0, \cdots, 0, \underset{i_0-th}{v_{kl}^{i_0}}, 0, \cdots, 0) \in A$ and $v_{kl}^{i_0}[\alpha]v_{st}^{j_0} = \bigoplus_{i,j=1}^{n} \delta_{ij}$ where only $\delta_{i_0 j_0} = v_{kl}^{i_0} \alpha v_{st}^{j_0}$ and all other $\delta_{ij} = 0$. Then, the sets $\overline{A} = \{v_{kl}^{[i]} : 1 \le i \le n, 1 \le k, l \le k_i\}$ and $\overline{M} = \{v_{kl}^{i}[\alpha]v_{st}^{j} : 1 \le i \le n, 1 \le k, l \le k_i; 1 \le j \le n, 1 \le s, t \le k_j\}$ are respectively the $\mathbb{K}$-basis of A and M.

For any k, l, i, α, define $\widetilde{\varphi}(v_{kl}^{[i]}) := v_{kl}^{i}$, $\widetilde{\varphi}(v_{kk}^{i}[\alpha]v_{ll}^{j}) := b$ the unique element in $\sigma_{ij}((k, \alpha, l))$ by Proposition 3.8, moreover define $\widetilde{\varphi}(v_{sk}^{i}[\alpha]v_{lt}^{j}) := v_{sk}^{i}\widetilde{\varphi}(v_{kk}^{i}[\alpha]v_{ll}^{j})v_{lt}^{j}$ $(\forall 1 \leqslant s \leqslant k_i, 1 \leqslant t \leqslant k_j)$. Then $\widetilde{\varphi}$ can be extended linearly to a $\mathbb{K}$-linear map.

And,

$$1_A = \Sigma_{i=1}^{n} 1_{A_i} = \Sigma_{i=1}^{n} \Sigma_{k=1}^{k_i} v_{kk}^{[i]}$$

which implies that $\widetilde{\varphi}(1_A) = 1_R$. Since $\widetilde{\varphi}(v_{kl}^{[i]} v_{st}^{[j]}) = 0 = v_{kl}^{i} v_{st}^{j} = \widetilde{\varphi}(v_{kl}^{[i]})\widetilde{\varphi}(v_{st}^{[j]})$ if $i \neq j$ or $l \neq s$ and $\widetilde{\varphi}(v_{kl}^{[i]} v_{lt}^{[i]}) = \widetilde{\varphi}(v_{kt}^{[i]}) = v_{kt}^{i} = v_{kl}^{i} v_{lt}^{i} = \widetilde{\varphi}(v_{kl}^{[i]})\widetilde{\varphi}(v_{lt}^{[i]})$, we have that $\widetilde{\varphi}|_A$ is a $\mathbb{K}$-algebra homomorphism.

Furthermore,

$$v_{kp}^{[i]}\widetilde{\varphi}(v_{pl}^{i}[\alpha]v_{st}^{j}) = v_{kp}^{i} v_{pl}^{i} \widetilde{\varphi}(v_{ll}^{i}[\alpha]v_{ss}^{j})v_{st}^{j} = v_{kl}^{i}\widetilde{\varphi}(v_{ll}^{i}[\alpha]v_{ss}^{j})v_{st}^{j},$$

and $\widetilde{\varphi}(v_{kp}^{[i]} v_{pl}^{i}[\alpha]v_{st}^{j}) = \widetilde{\varphi}(v_{kl}^{i}[\alpha]v_{st}^{j}) = v_{kl}^{i}\widetilde{\varphi}(v_{ll}^{i}[\alpha]v_{ss}^{j})v_{st}^{j}$, thus we get that

$$v_{kp}^{[i]}\widetilde{\varphi}(v_{pl}^{i}[\alpha]v_{st}^{j}) = \widetilde{\varphi}(v_{kp}^{[i]} v_{pl}^{i}[\alpha]v_{st}^{j})$$

which means that $\widetilde{\varphi}|_M$ is a left A-module homomorphism. Similarly, $\widetilde{\varphi}|_M$ is a right A-module homomorphism.

Conversely, for any $b \in \mathcal{E}$, there are $v_{kk}^{i}, v_{ll}^{j} \in \mathcal{V}$ such that $v_{kk}^{i} b v_{ll}^{j} = b$ by Lemma 2.6, then $b \in v_{kk}^{i}\mathcal{E}v_{ll}^{j}$. Since $\mathcal{E} = \bigcup_{\lambda \in \Lambda} \mathcal{E}_\lambda$, there is an equivalent class $\mathcal{E}_\lambda$ such that b is the unique element $v_{kk}^{i}\mathcal{E}_\lambda v_{ll}^{j}$, and hence $v_{kk}^{i}\mathcal{E}_\lambda v_{ll}^{j} \neq 0$ such that $v_{kk}^{i}\mathcal{E}_\lambda v_{ll}^{j} \in S_{ij}$. Choose a triple $(k, \alpha, l) \in \sigma_{ij}^{-1}(v_{kk}^{i}\mathcal{E}_\lambda v_{ll}^{j})$, then $\widetilde{\varphi}(v_{kk}^{i}[\alpha]v_{ll}^{j}) = b$.

On the other hand, by definition of $\widetilde{\varphi}$, $\mathcal{V} \cup \widetilde{\mathcal{V}} \subseteq Im\widetilde{\varphi}$. And, by Proposition 3.10, R is generated as an algebra by $\mathcal{V} \cup \widetilde{\mathcal{V}} \cup \mathcal{E}$. Therefore, it means that $\widetilde{\varphi}$ is surjective.

By the universal property of $\mathbb{K}(Q, \mathcal{A})$ as a tensor algebra, φ can be constructed and is also surjective. $\square$

Example 3.17. (i) Let ΛG is a skew group algebra in Example 2.3. Since $e_1 \sim e_2, e_{1'} \sim e_{2'}$, $\mathcal{E} = \underbrace{\{\alpha, \alpha', \alpha\sigma, \alpha'\sigma\}}_{\mathcal{E}_1} \bigcup \underbrace{\{\beta, \beta', \beta\sigma, \beta'\sigma\}}_{\mathcal{E}_2}$. Thus $Q: 1\bullet \xrightarrow{a} \bullet 2$ is the Green quiver of ΛG and $\mathcal{A} = \{A_1 = M_2(K), A_2 = M_2(K)\}$. Moreover, let

$$\sigma_{12} : \{(1, a, 1), (1, a, 2), (2, a, 1), (2, a, 2)\} \to \{\alpha, \beta'\sigma\}$$

via $\sigma_{12}((1, a, 1)) = \alpha = \sigma_{12}((1, a, 2))$ and $\sigma_{12}((2, a, 1)) = \beta'\sigma = \sigma_{12}((2, a, 2))$.

Thus, by Theorem 3.16 and its proof, there is an isomorphism: $\Lambda G \cong \mathbb{K}(Q, \mathcal{A})/I$ where $I = \langle E_{11}aF_{11} - E_{11}aF_{22}, E_{22}aF_{11} - E_{22}aF_{22}\rangle$.

(ii) We can prove that ΛG is not a normal generalized path algebra.

In fact, assume that $\Lambda G \cong \mathbb{K}(\widetilde{Q}, \widetilde{\mathcal{A}})$ for a quiver $\widetilde{Q}$ and a collection $\widetilde{\mathcal{A}}$ of simple $\mathbb{K}$-algebras. Then the number $|\widetilde{Q}_0|$ of the vertex set of $\widetilde{Q}$ is two by [9, The diagram]. It follows that $\widetilde{\mathcal{A}} = \mathcal{A}$ or $\widetilde{\mathcal{A}} = \{M_3(K), K\}$ since the identity of ΛG is the sum of four primitive orthogonal idempotent elements. It is easy to check $\dim\mathbb{K}(\widetilde{Q}, \mathcal{A}) \geqslant 20$ and $\dim\mathbb{K}(\widetilde{Q}, \{M_3(K), K\}) \geqslant 19$ in any cases of the numbers of arrows of $\widetilde{Q}$ which are contradict to $\dim\Lambda G = 16$.

Remark 3.18. (1) This example says that the matrix algebra of a generalized path algebra is not necessarily a generalized path algebra, and so the property that an algebra possesses the structure of generalized path algebra is not morita invariant.

(2) By Example 3.7, the basic algebra $B_{\Lambda G}$ of ΛG is a path algebra and then is hereditary. It follows that ΛG is hereditary. It means that NOT any hereditary artinian algebras is a generalized path algebra, even if it is splitting over radical, e.g. when the ground-field $\mathbb{K}$ is of characteristic 0. In [10], the authors gave that for a hereditary Artinian algebra A splitting over radical such that the surjective homomorphism $\pi : \mathbb{K}(\Delta_A, \mathcal{A}) \to A$ given in [10] possesses the kernel $ker(\pi) \subseteq J^2$, then π is an isomorphism and thus A is a normal generalized path algebra. The example given here implies the condition $ker(\pi) \subseteq J^2$ in this result of [10] can not be omitted.

4. Skew Group Algebras

Throughout this section, Λ is a $\mathbb{K}$-algebra with a ordered multiplicative basis $(\mathcal{B}, >)$. Then $1_\Lambda = v_1 + \cdots + v_h$, and denote by

$\Gamma_0 = \{v_1 \cdots v_h\}$;

$\Gamma_1 = \{b \in \mathcal{B} | b \notin \Gamma_0$ and b cannot be written as a product $b_1 b_2$ with $b_1, b_2 \in \mathcal{B} \setminus \Gamma_0\}$.

Suppose that $\sigma : \Gamma_0 \cup \Gamma_1 \to \Gamma_0 \cup \Gamma_1$ is a bijection which can be extended to a $\mathbb{K}$-linear automorphism of Λ satisfying the following:

(i) $\sigma(\Gamma_0) = \Gamma_0$ and $\sigma(\Gamma_1) = \Gamma_1$.

(ii) $\sigma^i(\lambda) = \sigma(\sigma^{i-1}(\lambda))$ for all $\lambda \in \Lambda$ and $i \geqslant 1$.

(iii) $\sigma^n(\lambda) = \lambda$ for all λ in Λ.

Let $G = \langle\sigma\rangle$ be a finite cyclic group with $|G| = n$ and $\mathrm{char}K \nmid n$. Then the skew group algebra of G over Λ, denote by ΛG, is given by the following data:

(i) ΛG is the free left Λ-module with the elements of G as a basis.

(ii) The multiplication in ΛG is defined by the rule $(\lambda_i\sigma^i)(\lambda_j\sigma^j) = \lambda_i\sigma^i(\lambda_j)\sigma^{i+j}$ for any λ_i and λ_j in Λ and $1 \leqslant i, j \leqslant n$.

In the following, let ε be the n-th primitive root of 1 and $\bar{b}$ a G-orbit of $b \in \mathcal{B}$. Since $(\mathcal{B}, >)$ is well-order, any orbit $\bar{b}$ (or $\bar{v}$) has a unique minimal element, denoted as $\mathfrak{b}$ (or $\mathfrak{v}$).

For any $v_i, v_j \in \Gamma_0$, let $d_i = |\bar{v}_i|$, $d_j = |\bar{v}_j|$, and t_{ij} is the least common multiple of d_i and d_j. Trivially, $\sigma^{t_{ij}}$ is an automorphism of $v_i\Gamma_1 v_j$, then $t_{ij} \big| |\bar{b}|$.

Lemma 4.1.

$$\frac{d}{n}\sum_{i=0}^{n/d-1} \varepsilon^{d\lambda i} = \begin{cases} 1, & \lambda = \frac{n}{d} \mathrm{mod} n \\ 0, & \text{otherwise} \end{cases}$$

Proposition 4.2. *Denote* $E_{s,t}^{\mathfrak{v},\mu} = \frac{d}{n}\sum_{m=0}^{n/d-1}\varepsilon^{m\mu d}\sigma^s\mathfrak{v}\sigma^{md-t}$ *for any* $0 \leqslant \mu < \frac{n}{d}, 0 \leqslant s,t < d, \mathfrak{v} \in \bar{v} \subset \Gamma_0\}$*, then the set*

$$\{E_{s,t}^{\mathfrak{v},\mu} | 0 \leqslant \mu < \frac{n}{d}, 0 \leqslant s,t < d, \bar{v} \in \Gamma_0\}$$

is a 0-closed set under multiplicative where $d = |\bar{v}|$.

Proof. In the sequel, we show that

$$E_{s,t}^{\mathfrak{v},\mu}E_{s',t'}^{\mathfrak{v}',\mu'} = \begin{cases} E_{s,t'}^{\mathfrak{v},\mu}, & \mathfrak{v} = \mathfrak{v}', t = s' \text{ and } \mu = \mu' \\ 0, & \text{otherwise} \end{cases}$$

In fact,

$$\begin{aligned} E_{s,t}^{\mathfrak{v},\mu}E_{s',t'}^{\mathfrak{v}',\mu'} &= \frac{dd'}{n^2}\sum_{m=0}^{n/d-1}\sum_{m'=0}^{n/d'-1}\varepsilon^{m\mu d+m'\mu' d'}\sigma^s\mathfrak{v}\sigma^{md-t}\sigma^{s'}\mathfrak{v}'\sigma^{m'd'-t'} \\ &= \frac{dd'}{n^2}\sum_{m=0}^{n/d-1}\sum_{m'=0}^{n/d'-1}\varepsilon^{m\mu d+m'\mu' d'}\sigma^s(\mathfrak{v})\sigma^{md-t+s+s'}(\mathfrak{v}')\sigma^{m'd'-t'+md-t+s+s'} \end{aligned}$$

If $\mathfrak{v} \neq \mathfrak{v}'$, that is, $\mathfrak{v}$ and $\mathfrak{v}'$ do not lie in the same orbit, then $\sigma^s(\mathfrak{v}) \neq \sigma^{md-t+s+s'}(\mathfrak{v}')$ which implies that $\sigma^s(\mathfrak{v})\sigma^{md-t+s+s'}(\mathfrak{v}') = 0$. It follows that $E_{s,t}^{\mathfrak{v},\mu}E_{s',t'}^{\mathfrak{v}',\mu'} = 0$. So we assume that $E_{s,t}^{\mathfrak{v},\mu}E_{s',t'}^{\mathfrak{v}',\mu'} \neq 0$, then $\mathfrak{v} = \mathfrak{v}'$. Thus

$$\begin{aligned} E_{s,t}^{\mathfrak{v},\mu}E_{s',t'}^{\mathfrak{v},\mu'} &= \frac{d^2}{n^2}\sum_{m=0}^{n/d-1}\sum_{m'=0}^{n/d-1}\varepsilon^{m\mu d+m'\mu' d}\sigma^s(\mathfrak{v})\sigma^{md-t+s+s'}(\mathfrak{v})\sigma^{m'd-t'+md-t+s+s'} \\ &= \frac{d^2}{n^2}\sum_{m=0}^{n/d-1}\sum_{m'=0}^{n/d-1}\varepsilon^{m\mu d+m'\mu' d}\sigma^s(\mathfrak{v}\sigma^{-t+s'}(\mathfrak{v}))\sigma^{(m'+m)d-t'-t+s+s'} \\ &= \left(\frac{d}{n}\sum_{m=0}^{n/d-1}\varepsilon^{m(\mu-\mu')d}\right)\left(\frac{d}{n}\sum_{k=0}^{n/d-1}\varepsilon^{k\mu d}\sigma^s(\mathfrak{v}\sigma^{-t+s'}(\mathfrak{v}))\sigma^{kd-t'-t+s+s'}\right) \end{aligned}$$

where $(m+m')d = kd(\text{mod}n)$ and $0 \leqslant k < \frac{n}{d}$. By Lemma 4.1 and $0 \leqslant \mu, \mu' < \frac{n}{d}$, we get $\mu = \mu'$. And $\mathfrak{v}\sigma^{-t+s'}(\mathfrak{v}) \neq 0$ if and only if $t = s'(\text{mod}n)$ if and only if $t = s'$. Therefore $E_{s,t}^{\mathfrak{v},\mu}E_{s',t'}^{\mathfrak{v}',\mu'} \neq 0$ implies that $\mathfrak{v} = \mathfrak{v}', \mu = \mu'$ and $t = s'$. On other hand, if $\mathfrak{v} = \mathfrak{v}', \mu = \mu'$ and $t = s'$, then by Lemma 4.1

$$E_{s,t}^{\mathfrak{v},\mu}E_{t,t'}^{\mathfrak{v},\mu} = \frac{d}{n}\sum_{k=0}^{n/d-1}\varepsilon^{k\mu d}\sigma^s(\mathfrak{v})\sigma^{kd-t'+s} = \frac{d}{n}\sum_{k=0}^{n/d-1}\varepsilon^{k\mu d}\sigma^s\mathfrak{v}\sigma^{kd-t'} = E_{s,t'}^{\mathfrak{v},\mu}$$

□

Corollary 4.3. *The set* $\{E_{s,s}^{\mathfrak{v},\mu} | 0 \leqslant \mu < \frac{n}{d}, 0 \leqslant s < d, \bar{v} \subset \Gamma_0\}$ *is a family of orthogonal idempotents in* ΛG.

Lemma 4.4. *For any* $\mathfrak{b} \in \bar{b}(\subset \Gamma_1)$*, let* $v_k, v_l \in \Gamma_0$ *such that* $\sigma^p(\mathfrak{v}_k)\mathfrak{b}\sigma^q(\mathfrak{v}_l) \neq 0$*. If we set*

$$(\Gamma G)_1 = \{E_{s,p}^{\mathfrak{v}_k,\mu_k}\mathfrak{b}E_{q,t}^{\mathfrak{v}_l,\mu_l} | \bar{b} \subset \Gamma_1, 0 \leqslant \mu_k < \frac{n}{d_k}, 0 \leqslant \mu_l < \frac{n}{d_l}, 0 \leqslant s < d_k, 0 \leqslant t < d_k\} \setminus \{0\},$$

then $|(\Gamma G)_1| = n|\Gamma_1|$.

Proof. For some fixed vertices v_i and v_j, let

$$(\Gamma G)_1^{ij} = \{E_{s,p}^{v_i,\mu_i}\mathfrak{b}E_{q,t}^{v_j,\mu_j} | 0 \leqslant \mu_i < \frac{n}{d_i}, 0 \leqslant \mu_j < \frac{n}{d_j}, 0 \leqslant s < d_i, 0 \leqslant t < d_j\} \setminus \{0\},$$

then it is sufficient to show that $|(\Gamma G)_1^{ij}| = n|\bar{b}|$.

In fact,

$$\begin{aligned} E_{s,p}^{\mathfrak{v}_i,\mu_i} b E_{q,t}^{\mathfrak{v}_j,\mu_j} &= \frac{d_i d_j}{n^2}\left(\sum_{m=0}^{n/d_i-1} \varepsilon^{m\mu_i d_i}\sigma^s \mathfrak{v}_i \sigma^{md_i-p}\right)\mathfrak{b}\left(\sum_{k=0}^{n/d_j-1} \varepsilon^{k\mu_j d_j}\sigma^q \mathfrak{v}_j \sigma^{md_j-t}\right) \\ &= \frac{d_i d_j}{n^2}\sum_{m=0}^{n/d_i-1}\sum_{k=0}^{n/d_j-1} \varepsilon^{m\mu_i d_i + k\mu_j d_j}\sigma^{md_i+s-p}(\mathfrak{b})\sigma^{md_i+kd_j-t+s-p+q} \end{aligned}$$

Let $|\bar{b}| = D$, and write $md_i = MD + m'd_i$ and $kd_j = KD + k'd_j$ with $0 \leqslant M, K < \frac{n}{D}$, $0 \leqslant m' < \frac{D}{d_i}$ and $0 \leqslant k' < \frac{D}{d_j}$. This yields

$$E_{s,p}^{\mathfrak{v}_i,\mu_i}\mathfrak{b}E_{q,t}^{\mathfrak{v}_j,\mu_j} = \frac{d_i d_j}{n^2}\sum_{M,K,m',k'} \varepsilon^{(M+K)D\mu_i + m'd_i\mu_i + KD(\mu_i-\mu_j) + k'd_j\mu_j}\sigma^{m'd_i+s-p}(\mathfrak{b})\sigma^{(M+K)D+m'd_i+k'd_j-t+s-p+q}$$

Finally, if we write $d_i r \equiv (M+K)D + d_i m' (\mathrm{mod} n)$ with $0 \leqslant r < n/d_i$, then we obtain the further factorization

$$E_{s,p}^{\mathfrak{v}_i,\mu_i}\mathfrak{b}E_{q,t}^{\mathfrak{v}_j,\mu_j} = \left(\frac{D}{n}\sum_K \varepsilon^{KD(\mu_i-\mu_j)}\right)\left(\frac{d_i d_j}{nD}\sum_{r,k',m'} \varepsilon^{d_i r\mu_i + k'd_j\mu_j}\sigma^{m'd_i+s-p}(\mathfrak{b})\sigma^{d_i r+k'd_j-t+s-p+q}\right)$$

Thus $E_{s,p}^{\mathfrak{v}_i,\mu_i} b E_{q,t}^{\mathfrak{v}_j,\mu_j} \neq 0$ if and only if $\mu_j \equiv \mu_i \left(\mathrm{mod}\frac{n}{D}\right)$, and hence $|(\Gamma G)_1^{ij}| = \frac{D}{d_j}\cdot\frac{D}{d_i}\cdot\frac{n}{D}\cdot d_i d_j = nD = n|\bar{b}|$. □

Let $(\Gamma G)_1^n = \underbrace{(\Gamma G)_1(\Gamma G)_1\cdots(\Gamma G)_1}_{n}\setminus\{0\}$ for any integer $n > 0$.

Lemma 4.5. *$\mathcal{B}G = \bigcup_{n=0}^{\infty}(\Gamma G)_1^n$ is a multiplicative basis of ΛG where $(\Gamma G)_1^0$ stands for the set $\{E_{s,t}^{\mathfrak{v},\mu} | 0 \leqslant \mu < \frac{n}{d}, 0 \leqslant s, t < d, v \in \Gamma_0\}$ with $d = |\bar{v}|$.*

Proof. First, we show that $\mathcal{B}G$ is 0-closed. Assume $\alpha \in (\Gamma G)_1^n, \beta \in (\Gamma G)_1^m$ and $\alpha\beta \neq 0$, if $n, m > 0$, then it is obvious that $\alpha\beta \in (\Gamma G)_1^{n+m}$. If $n = m = 0$, then $\alpha\beta \in \Gamma_1^0$ by Proposition 4.2. Without loss of generality, we assume that $\alpha = E_{s,0}^{\mathfrak{v},\mu}$ and $\beta = E_{0,0}^{\mathfrak{v},\mu}\mathfrak{b}E_{0,t}^{\mathfrak{v}',\mu'}$, then $\alpha\beta = E_{s,0}^{\mathfrak{v},\mu}E_{0,0}^{\mathfrak{v},\mu}\mathfrak{b}E_{0,t}^{\mathfrak{v}',\mu'} = E_{s,0}^{\mathfrak{v},\mu}\mathfrak{b}E_{0,t}^{\mathfrak{v}',\mu'} \in \Gamma G_1^1$, as required.

Secondly, we show that $\mathcal{B}G$ is a $\mathbb{K}$-basis of ΛG.

We note that $\widetilde{\mathfrak{B}} \triangleq \{b\sigma^i | b \in \mathcal{B}, 0 \leqslant i < n\}$ is a $\mathbb{K}$-basis of ΛG. It is sufficient to show that $\mathcal{B}G$ is a linearly independent set by which $\mathcal{B}G$ can be linearly expressed.

Assume that there is an equation

$$\sum_{i=1}^{m} k_i E_{s_i,t_i}^{\mathfrak{v}_i,\mu_i} = 0 \qquad (1)$$

satisfying $E^{\mathfrak{v}_i,\mu_i}_{s_i,t_i} \neq E^{\mathfrak{v}_j,\mu_j}_{s_j,t_j}$ if $i \neq j$, then for any fixed $p \in \{1, \cdots, m\}$, we get

$$0 = E^{\mathfrak{v}_p,\mu_p}_{s_p,s_p}\left(\sum_{i=1}^{m} k_i E^{\mathfrak{v}_i,\mu_i}_{s_i,t_i}\right) E^{\mathfrak{v}_p,\mu_p}_{t_p,t_p} = \sum_{i=1}^{m} k_i E^{\mathfrak{v}_p,\mu_p}_{s_p,s_p} E^{\mathfrak{v}_i,\mu_i}_{s_i,t_i} E^{\mathfrak{v}_1,\mu_1}_{t_1,t_1} = k_p E^{\mathfrak{v}_p,\mu_p}_{s_p,t_p}. \qquad (*)$$

So each $k_p = 0$ in Eq.(1), that is, $(\Gamma G)^0_1$ is linearly independent. Moreover, $|(\Gamma G)^0_1| = |\{E^{\mathfrak{v},\mu}_{s,t}\}| = \dfrac{n}{d} \cdot d^2 \cdot \dfrac{|\Gamma_0|}{d} = n|\Gamma_0| = \big|\{b\sigma^i | b \in \Gamma_0, 0 \leqslant i < n\}\big|$. By definition of $E^{\mathfrak{v},\mu}_{s,t}$, $(\Gamma G)^0_1$ and $\{b\sigma^i | b \in \Gamma_0, 0 \leqslant i < n\}$ are linearly expressed each other.

In the similar way as in $(*)$, without loss of generality, we can assume that there is an equation

$$\sum_{i=1}^{m} \alpha_i E^{\mathfrak{v},\mu}_{0,p_i} \mathfrak{b} E^{\mathfrak{v}',\mu'}_{q_i,0} = 0 \qquad (2)$$

with $\alpha_i \in \mathbb{K}$, $0 \leqslant p_i < d$, $0 \leqslant q_i < d'$, either $p_i \neq p_j$ or $q_i \neq q_j$ if $i \neq j$. Note that

$$E^{\mathfrak{v},\mu}_{0,p_i} \mathfrak{b} E^{\mathfrak{v}',\mu'}_{q_i,0} = \frac{dd'}{n^2} \sum_{m=0}^{n/d-1} \sum_{k=0}^{n/d'-1} \varepsilon^{m\mu d + k\nu d'} \sigma^{md-p_i}(\mathfrak{b}) \sigma^{md+kd'-p_i+q_i},$$

then $\sigma^{md-p_i}(\mathfrak{b}) = \sigma^{m'd-p_j}(\mathfrak{b})$ if and only if $md - p_i = m'd - p_j (\mathrm{mod} |\bar{b}|)$ if and only if $p_i = p_j$ and $|\bar{b}|\Big|(m-m')d$ since $d\Big||\bar{b}|$ and $0 \leqslant p_i, p_j < d$.

If $\sigma^{md-p_i}(\mathfrak{b})\sigma^{md+kd'-p_i+q_i} = \sigma^{m'd-p_j}(\mathfrak{b})\sigma^{m'd+k'd'-p_j+q_j}$, then

$$p_i = p_j, \qquad |\bar{b}|\Big|(m-m')d$$

$$(m-m')d + (k-k')d' = q_j - q_i (\mathrm{mod} |\bar{b}|).$$

It follows that $q_i = q_j$ since $d'\Big||\bar{b}|$ and $0 \leqslant q_i, q_j < d'$.

So if either $p_i \neq p_j$ or $q_i \neq q_j$, then $\sigma^{md-p_i}(\mathfrak{b})\sigma^{md+kd'-p_i+q_i} = \sigma^{m'd-p_j}(\mathfrak{b})\sigma^{m'd+k'd'-p_j+q_j}$. Since $\{b\sigma^i | b \in \Gamma_1, 0 \leqslant i < n\}$ is a linearly independent set, each $\alpha_i = 0$ in Eq. (2). It follows that $(\Gamma G)_1$ is linearly independent. And since $|(\Gamma G)_1| = \big|\{b\sigma^i | b \in \Gamma_1, 0 \leqslant i < n\}\big|$ by Lemma 4.4, $(\Gamma G)_1$ and $\{b\sigma^i | b \in \Gamma_1, 0 \leqslant i < n\}$ are linearly expressed each other.

Note that $\mathcal{B}G$ is generated by $(\Gamma G)^0_1 \cup (\Gamma G)_1$, and so it is linearly independent. Meanwhile, for any $b\sigma^i$ with $b \in \mathcal{B}$ and $0 \leqslant i < n$, there are $\beta_1, \cdots, \beta_b \in \Gamma_0$ such that $b\sigma^i = \beta_1 \cdots \beta_b \sigma^i$. By the above proof, we know that each β_k $(k = 1, \cdots, b)$ is linearly expressed by $(\Gamma G)_1$. So there are $k_{i_1} \in \mathbb{K}^*$ and $\rho_{i_1} \in (\Gamma G)_1$ such that

$$b\sigma^i = \left(\sum_{i_1=1}^{m_1} k_{i_1}\rho_{i_1}\right) \cdots \left(\sum_{i_b=1}^{m_b} k_{i_b}\rho_{i_b}\right) \in \mathbb{K}\langle(\Gamma G)^b_1\rangle,$$

as desired. □

Theorem 4.6. *ΛG is a $\mathbb{K}$-algebra with a weakly ordered multiplicative basis.*

Proof. By Lemma 4.5, (ΓG) is a multiplicative basis of ΛG. Now we define an order $\succ$ on (ΓG) as follows.

For any $E^{\mathfrak{v},\mu}_{s,t}, E^{\mathfrak{v},\mu'}_{s',t'} \in (\Gamma G)^0_1$, define

$$E^{\mathfrak{v},\mu}_{s,t} \succ E^{\mathfrak{v}',\mu'}_{s',t'} \quad \text{if} \quad \begin{cases} \mathfrak{v} > \mathfrak{v}' \\ \mathfrak{v} = \mathfrak{v}', \mu > \mu' \\ \mathfrak{v} = \mathfrak{v}', \mu = \mu', s > s' \\ \mathfrak{v} = \mathfrak{v}', \mu = \mu', s = s', t > t' \end{cases}$$

Let $\alpha, \alpha' \in (\Gamma G)^1_1$, for convenience, we set $\alpha = e\mathfrak{b}f$ and $\alpha' = e'\mathfrak{b}'f'$ with $e, f, e', f' \in (\Gamma G)^0_1$ and $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$. Then

$$\alpha \succ \alpha' \quad \text{if} \quad \begin{cases} \mathfrak{b} > \mathfrak{b}' \\ \mathfrak{b} = \mathfrak{b}', e > e' \\ \mathfrak{b} = \mathfrak{b}', e = e', f > f' \end{cases}$$

For any $\alpha \in (\Gamma G)^n_1, \beta \in (\Gamma G)^m_1$, $n > m$ implies that $\alpha \succ \beta$. If $n = m > 1$, then we write $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and $\beta = \beta_1\beta_2\cdots\beta_n$ with $\alpha_i, \beta_i \in \Gamma G_1$. If there is some k such that $\alpha_i = \beta_i$ for any $0 \leqslant i < k$ and $\alpha_k \succ \beta_k$, then we define $\alpha \succ \beta$.

In the sequel, we check that $\succ$ is a weakly admissible order on (ΓG).

$\mathcal{V} = \{E^{\mathfrak{v},\mu}_{s,s}\}$ and $\widetilde{\mathcal{V}} = (\Gamma G)^0_1/\mathcal{V}$.

A0. $\succ$ is obvious a total order and is a well order on ΓG^0_1 (for $(\Gamma G)^0_1$ is finite). For any subset T of $(\Gamma G)_1$. Let $\widetilde{T} = \{b \in \Gamma_1 |$ there are $e, f \in (\Gamma G)^0_1$ such that $ebf \in T\}$, then there is a minimal element b_1 on $\widetilde{T}$ since $>$ is a well order on $\mathcal{B}$. Set $T_1 = \{eb_1f | e, f \in (\Gamma G)^0_1\} \cap T$, and hence there is a minimal element t_1 in T_1. Clearly, t_1 is also a minimal element in T, that is, $\succ$ is a well order on $(\Gamma G)_1$. Moreover, for any $S \subset (\Gamma G)$, there is a least n such that $S_0 = (\Gamma G)^n_1 \bigcap S \neq \emptyset$. For any $\alpha = \alpha_1\cdots\alpha_n \in (\Gamma G)^n_1$ with $\alpha_i \in (\Gamma G)_1$, then we denote by $t_i(\alpha) = \alpha_i$. Let $S_1 = \{\alpha \in S_0 | \forall\ \beta \in S_0, t_1(\beta) \succeq t_1(\alpha)\}$. By induction, $S_i = \{\alpha \in S_{i-1} | \forall\ \beta \in S_{i-1}, t_i(\beta) \succeq t_i(\alpha)\}$. Then by n steps, we have $|S_n| = 1$. So the unique element $s_n \in S_n$ is a minimal element of S.

A1 and A2 are trivial.

A3. For any $\alpha_1, \alpha_3 \in (\Gamma G)$, if there are $\alpha_2, \alpha_4 \in (\Gamma G)/\widetilde{\mathcal{V}}$ such that $\alpha_1 = \alpha_2\alpha_3\alpha_4$. Assume that α_2 or α_4 in $\bigcup\limits_{n=1}^{\infty}(\Gamma G)^n_1$, then $\alpha_1 \succ \alpha_3$. But if $\alpha_2, \alpha_4 \in \Gamma G_0/\widetilde{\mathcal{V}}$, then it is easy to check that $\alpha_2\alpha_3\alpha_4 = \alpha_3 = \alpha_1$.

Now we complete the proof. □

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