1. Let \( f(z) \) be an entire function that is not a polynomial. Assume \( f \) has only finitely many zeros. Let \( m(r) = \min_{|z|=r} |f(z)| \). Show \( m(r) \to 0 \) as \( r \to \infty \).

2(a). Let \( f(z) \) be analytic in a bounded region \( \Omega \) and be continuous up to the boundary \( B \) of \( \Omega \). Let \( E = f(B) \). If \( a \) and \( b \) are in the same component of \( \mathbb{C} \setminus E \), show that \( a \) and \( b \) are taken the same number of times by \( f \).

(b). Prove that \( z^4 + z^2 + 2 \) has a zero in each quadrant.

3. Show:
\[
\int_0^\infty \frac{x^\alpha}{(x+2)^2} \, dx = \frac{\alpha \pi 2^{\alpha-1}}{\sin \pi \alpha} \quad \text{for} \quad -1 < \alpha < 1.
\]

4. Show that the annulus can be mapped conformally onto the Riemann sphere with two segments of the real axis removed.

**Hint:** You may use the Riemann mapping theorem, including boundary behavior.

5(a). Let \( f(z) \) be an entire function such that \( |f(z)| < A \exp |z|^{\alpha} \). Show that the number of zeros of \( f \) in the disk \( |z| < r \) is \( \leq Cr^\alpha \) for some constant \( C \) and for \( r > 1 \).

(b). Let \( f(z) = \sum z^n n^{-an} \). Show that \( f \) satisfies
\[
|f(z)| \leq C_1 \exp(C_2|z|^{1/\alpha}) \quad \text{for some} \ C_1, C_2 > 0.
\]

6. Let \( \Omega \) be a bounded connected region. Show that there is an analytic function \( f(z) \) defined in \( \Omega \) such that \( f \) cannot be extended to be analytic in any larger connected region.
Work all problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Evaluate

\[ \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx \]

by contour integration.

2(a). Prove that

\[ f(z) = \sum_{n=-\infty}^{\infty} 2^{-n^2} z^n \]

has infinitely many zeros.

(b). Suppose \( f \) is entire, \( |f(z)| \leq e^{A|z|} \), and \( f(z) = f(-z) \). Prove that either \( f \) is a polynomial or \( f \) has infinitely many zeros.

3(a). Give an example of a region \( D \) in \( \mathbb{C} \) whose complement is an infinite set and such that there is no conformal map of \( D \) onto a bounded region.

(b). Let \( \Omega \) be the complement of \( \{ e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \} \) with the point at infinity adjoined. Find a function \( f(z) \) that maps \( \Omega \) conformally onto the unit disk.

4(a). Let \( |f(z)| \) assume its maximum on \( \overline{D} = \{ z : |z| \leq 1 \} \) at \( z_0 \), where \( f \) is analytic in a neighborhood of \( \overline{D} \). Show that \( f'(z_0) \neq 0 \).

(b). Assume \( f \) is entire, \( f(z+1) = f(z) \), and \( |f(z)| \leq e^{c|z|} \) for some \( c < 2\pi \). Show that \( f \) is constant.

5. Assume that \( f \) is analytic in a neighborhood of \( \overline{D} = \{ z : |z| \leq 1 \} \). Assume also that \( |f(e^{i\theta})| \leq m \) if \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \) and that \( |f(e^{i\theta})| \leq M \) if \( \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \). Find the best possible bound for \( |f(0)| \).

**Hint:** First assume that \( f \) has no zeros.

6. Let \( u(z) \) be a real harmonic function on \( \{ z : 1 < |z| < 2 \} \). Show that for some \( \alpha \) and for some function \( f(z) \) analytic in \( \{ z : 1 < |z| < 2 \} \),

\[ u(z) = \alpha \log r + \text{Re } f(z). \]
Work all six problems.

1. Show that for \( a > 0 \),
\[
\int_0^\infty \frac{\cos ax}{(1 + x^2)^2} \, dx = \frac{\pi (a + 1) e^{-a}}{4} \quad \text{if} \quad a > 0.
\]

2. Let \( D \) be the open unit disk centered at the origin and let \( f : \overline{D} \to \mathbb{C} \) be a function. Suppose \( f \) is analytic in \( D \), \( f \) is continuous in \( \overline{D} \setminus \{1\} \) and
\[
\lim_{z \to 1} \frac{|f(z)|}{\log |z - 1|} = 0.
\]
Suppose further that \( |f(w)| \leq 1 \) for \( w \in \partial D \setminus \{1\} \). Show that
\[
\max_{z \in \overline{D}} |f(z)| \leq 1.
\]

3. Let \( f(z) \) be an analytic function on the punctured disk \( 0 < |z| < 1 \) with 0 an essential singularity. Let \( \xi \) be any complex number. Show that, with the possible exception of one value of \( \xi \),
\[
\lim_{r \to 0^+} \frac{1}{2\pi \sqrt{-1}} \int_{|z|=r} \frac{f'(z)}{f(z) - \xi} \, dz = \infty,
\]
where the limit is taken for those \( r > 0 \) for which \( f(z) - \xi \) has no zeros on \( |z| = r \).

4(a). Let \( \Omega \) be the (open) ball of radius 1 centered at the origin and \( E \) be a compact subset in \( \Omega \). Use Poisson’s formula for harmonic functions to prove the following version of the Harnack inequality: There is a constant \( M \), depending only on \( E \), such that every positive harmonic function \( u(z) \) in \( \Omega \) satisfies
\[
u(z_1) \leq Mu(z_2), \quad z_1, z_2 \in E.
\]
(Do not quote Harnack’s inequality directly.)

4(b). Find the best possible \( M \) in case \( E \) is the closed disk of radius \( \frac{1}{2} \) centered at the origin.

5. Let \( D \) be the interior of the triangle whose vertices are 0, 1 and \( i \).
   (a) Prove that there is a unique conformal mapping \( w = f(z) \) of \( D \) onto the upper half plane such that
   \[
   \lim_{z \to 0} f(z) = 0, \quad \lim_{z \to 1} f(z) = 1 \quad \text{and} \quad \lim_{z \to i} f(z) = \infty.
   \]
(b) Prove that \( f \) extends by reflection to an elliptic function. Find the poles and the periods of this function.

(c) Find the singular part of this function near its poles. Find an explicit formula for \( f \) as an infinite sum.

(Hint. For (c), study the rotational symmetry of \( f \) near a pole.)

6. Let \( H \) be the upper half-plane and let \( F(z) \) be the function defined by

\[
F(z) = \int_0^z \frac{dw}{(1-w)(1+w)\sqrt{w}}, \quad z \in H
\]

where the integral is over a path from 0 to \( w \) in \( H \) and where \( \sqrt{w} \) is the single-value branch of the square root function such that \( \sqrt{1} = 1 \). Show that \( F \) is a one-one conformal map onto the region

\[
\Omega = \{(x, y) : x > 0, y > 0, \min(x, y) < \frac{\pi}{2}\}.
\]
Work all 6 problems. All problems have equal weight. Write each solution in a separate bluebook.

1. Let \( f(x):(-\frac{1}{2}, \infty) \to \mathbb{C} \) be a continuous function. Suppose \( f \) is analytic in a neighborhood of the origin and that there is a positive constant \( N \) so that
\[
\lim_{x \to \infty} f(x)e^{Nx} = 0.
\]
For the complex variable \( s \) we define
\[
F(s) = \int_0^\infty f(x)x^sdx.
\]
(a) Show that the integral converges for \( \text{Re}(s) > -1 \) and that \( F(s) \) has a meromorphic continuation to all \( s \) with possible poles only at \( s = -1, -2, \ldots \).

(b) Determine the exact location of the poles of \( F \) and its singular parts at poles.

Hint: The answer to the second part depends on the coefficients of the Taylor expansion of \( f \) at \( x = 0 \).

2. Let \( D \) be a bounded region in \( \mathbb{C} \) whose boundary consists of \( n \)-smooth disjoint Jordan arcs. Thus \( D \) is \( n \)-connected. We denote by \( \overline{D} \) the closure of \( D \).

(a) Suppose \( f(z) \) is a non-constant continuous function on \( \overline{D} \) and is analytic in \( D \). Suppose further that
\[
|f(w)| = 1 \quad \text{for all } w \in \partial D.
\]
(*) Show that \( f \) has at least \( n \) zeros (counting multiplicities) in \( D \).

(b) For any \( n > 0 \), find an \( n \)-connected region \( D \) and an analytic function \( f:D \to \mathbb{C} \) such that \( f \) satisfies (*) and has exactly \( n \) zeros in \( D \).

Remark: You will receive partial credits if you work out the special case where \( D = \{1 < |z| < c\} \) in part (a) and/or (b).

3. Show that
\[
\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin \pi c} \quad \text{if } 0 < c < 1.
\]
Remark: You need to provide details to justify each step in your computation.

4. Let \( D \) be the open unit disk and \( f:D \to \mathbb{C} \) be a bounded analytic function.

(a) Let \( \{a_n\}_{n \geq 1} \) be the non-zero zeros of \( f \) in \( D \) counted according to multiplicity. We assume \( \{a_n\} \) is an infinite sequence. Then prove that
\[
\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.
\]
(b) Let \( f \) and \( \{a_n\} \) be as above. We define
\[
B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left( \frac{a_n - z}{1 - \bar{a}_n z} \right).
\]
Show that \( B(z) \) is a bounded analytic function on \( D \) with zeros \( \{a_n\} \). Show further that there is an integer \( m \) and a bounded non-vanishing holomorphic function \( h(z) \) so that
\[
f(z) = z^m B(z) h(z).
\]

5. Let \( D \) be the open unit disk and let \( f \) be a non-constant analytic function in \( D \).
   (a) Suppose for every \( a \in \partial D \setminus \{1\} \) we have
\[
\lim_{z \to a} |f(z)| \leq 1
\]
and for any \( \delta > 0 \) we have
\[
\lim_{z \to 1} |f(z)||z - 1|^\delta = 0.
\]
Show that \( |f(z)| < 1 \) in \( D \).
   (b) Construct an analytic function on the unit disk \( D \) that is not bounded in \( D \) and that satisfies \((*)\) but not \((***)\).

Hint: Consider \( u(z) = (z - 1)^\delta f(z) \).

6. Let \( \omega_1 \) and \( \omega_2 \) be two non-zero complex numbers with non-real ratio \( \omega_1/\omega_2 \). Let \( \Lambda \) be the lattice \( \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) and let \( a \) and \( b \) be two complex numbers not congruent to each other. We form the linear space \( V \) of all elliptic functions of period \( \Lambda \) with at most simple poles at \( a \) and \( b \).
   (a) Prove that \( \dim_{\mathbb{C}} V \) is at most 2.
   (b) Using the method of infinite series, construct explicitly a two dimensional families of elliptic functions in \( V \), thereby proving that \( \dim V = 2 \).

Remark: In (b) one needs to provide details to why the series converges, why they have period \( \Lambda \) and why they provide a two dimensional family.
Work all problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let
   \[ f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \]
   be a polynomial with complex coefficients \( a_1, \ldots, a_n \). Let \( \alpha_k \) be the real part of \( a_k \). Suppose \( f(z) \) has \( n \) zeros in the upper-half plane \( \text{Im } z > 0 \). Prove that the polynomial
   \[ \alpha(x) = x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n \]
   has \( n \) distinct real roots.

2. Let \( D \) be the open unit disk and let \( f : D \to \mathbb{C} \) be an odd univalent function. (That is, \( f(-z) = -f(z) \) and \( f \) is one-one.) Show that there is a univalent analytic function \( g : D \to \mathbb{C} \) such that
   \[ f(z) = \sqrt{g(z^2)}. \]

3. Let \( \Omega \) be the region \( -1 < \text{Re}(z) < 1 \) and let \( \mathcal{F} \) be the collection of all analytic functions \( f(z) \) defined on \( \Omega \) such that \( f(0) = 0 \) and \( |f(z)| < 1 \) for all \( z \in \Omega \). Find
   \[ \sup_{f \in \mathcal{F}} \left\{ \left| f \left( \frac{1}{2} \right) \right| \right\}. \]

4. Define
   \[ F(z) = \int_0^\infty x^{z-1} e^{-x^2} \, dx \]
   for \( \text{Re}(z) > 0 \).
   (a) Prove that \( F \) is an analytic function on the region \( \text{Re}(z) > 0 \).
   (b) Prove that \( F \) extends to a meromorphic function on the whole complex plane.
   (c) Find all poles of \( F \) and find the singular parts of \( F \) at these poles.

5. Calculate the following integral:
   \[ \int_0^\infty \frac{\cos x - 1}{x^2} \, dx. \]

6. Let \( \Omega \) be a connected open subset of \( \mathbb{C} \).
   (a) Let \( h(z) \) be a non-trivial analytic function defined on \( \Omega \). Let \( \{a_n\}_{n \geq 1} \) be all the (distinct) zeros of \( h(z) \) and let \( \{c_n\}_{n \geq 1} \) be a sequence of complex numbers. Show
that there is an analytic function $H(z)$ defined on $\Omega$ such that $H(a_n) = c_n$ for all $n$.

(b) Let $f(z)$ and $g(z)$ be two analytic functions defined on $\Omega$ with no common zeros in $\Omega$. Assume that both $f(z)$ and $g(z)$ have only simple zeros. Prove that there are analytic functions $F(z)$ and $G(z)$ defined on $\Omega$ such that over $\Omega$,

$$F(z)f(z) + G(z)g(z) = 1.$$ 

Hint: One possible approach to (a) is to apply the Mittag-Leffler Theorem for the domain $\Omega$. See below for the exact statement of the theorem. For (b), consider

$$F(z) = \frac{1 - G(z)g(z)}{f(z)}.$$

Mittag-Leffler Theorem: Let $\{b_k\}$ be a sequence of distinct points in $\Omega$ without limit points in $\Omega$, and let $\{P_k(z)\}$ be a sequence of polynomials without constant terms. Then there are meromorphic functions $\phi$ defined on $\Omega$ such that the poles of $\phi$ are the points $\{b_k\}$ and such that (for each $k$) the singular part of $\phi$ at $z = b_k$ is $P_k(\frac{1}{z - b_k})$. 

2
Work all 6 problems. All problems have equal weight. Write each problem in a separate bluebook.

1. \textbf{a.} Let $\omega_1, \omega_2$ be two complex numbers such that $\omega_1/\omega_2$ is not real and let $\Lambda$ be the lattice \{ $n_1\omega_1 + n_2\omega_2$ $| n_1, n_2 \in \mathbb{Z}$ \}. Prove directly that
\[ \zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda - \{0\}} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \]
is a doubly periodic meromorphic function up to constants.
\textbf{b.} Prove that the necessary and sufficient condition for the existence of a doubly periodic meromorphic function with (finitely many) prescribed poles and singular parts in a fundamental parallelogram is that the sum of the residues of the poles is zero.

2. Let $a_i$ and $b_i$ be real numbers for $1 \leq i \leq 3$ such that $0 < b_i < 1$. Let $u = u(z)$ be a holomorphic function in the upper half plane that solves the second order differential equation
\[ u'' + u' \left( \frac{b_1}{z - a_1} + \frac{b_2}{z - a_2} + \frac{b_3}{z - a_3} \right) = 0. \]

\textbf{a.} Solve $u$ and express the solution in integral form.
\textbf{b.} Show directly, without quoting the Schwarz-Christoffel theorem, that $u$ maps the real axis to the boundary of a polygon.
\textbf{c.} Determine under what condition on the $b_i$’s does $u$ map the upper half plane conformally onto a triangle? Prove, again without quoting Schwarz-Christoffel theorem, your assertion and determine the angles of this triangle.

3. Let $f$ be a univalent analytic function defined on the unit disk $D$ centered at the origin. Suppose $f(0) = 0$. Show that the function
\[ g(z) = \sqrt{f(z^2)} \]
has a single-valued branch, and is also univalent.
\textit{(Hint: Univalent means one-one. Use $\sqrt{f(z^2)} = z \sqrt{f(z^2)}$.)}

4. Let $f$ be a nonconstant harmonic function in the unit disk $D$ which restricts smoothly to the boundary. Assume $f(0) = 0$ and $\nabla f(0) = 0$. Show that $f(x) = 0$ for at least four distinct points $x$ on the boundary.

5. Let $\Delta$ be the half-strip
\[ \text{Re } z > 0, \quad -\pi < \text{Im } z < \pi \]
and let $L$ be the boundary of $\Delta$ consisting of three straight pieces. We orient $L$ so that $\Delta$ is on the right hand side of $L$. 


a. Show that the integral
\[ \frac{1}{2\pi i} \int_L \frac{e^{\zeta}}{\zeta - z} d\zeta = E(z) \]
defines an analytic function on the complement of $\Delta$.

b. Show that $E(z)$ extends to an entire function on $\mathbb{C}$.

c. Show that $E(z)$ assumes real values for real $z$.

6. Let $f(z)$ be an entire function that satisfies the relation
\[ p_n(z)(f(z))^n + p_{n-1}(z)(f(z))^{n-1} + \cdots + p_0(z) = 0, \]
where $p_0(z), \ldots, p_n(z)$, not all trivial, are rational functions on $\mathbb{C}$. Show that $f$ must be a polynomial.
Do all six problems, each in a separate blue book. All problems have equal weight.

1. Let $\Omega$ be the domain $|z| < 2$ and let $[0, 1] \subset \Omega$ be the line segment between 0 and 1.
   (a) Suppose $f : \Omega \to \mathbb{C}$ is a continuous function that is analytic over $\Omega \setminus [0, 1]$. Show that $f$ is analytic over $\Omega$.
   (b) Show that there are bounded analytic functions $f : \Omega \setminus [0, 1] \to \mathbb{C}$ that cannot be extended to analytic functions over $\Omega$.

2. Let $f(z)$ be an analytic function defined over $|z| < 1$ that maps $|z| < 1$ one-one and onto $\mathbb{C} \setminus (-\infty, -1/4]$. Find the most general form of $f(z)$.

3. Establish the following identity
   \[
   \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{1}{t} = 2 \sum_{n=1}^{\infty} \frac{1}{t^2 + 4n^2\pi^2}.
   \]
   (Hint: Consider the difference of the terms on the left and the right hand side.)

4. Let $f$ and $g \neq 0, g_2, \ldots, g_n$ be analytic functions over the punctured disk $D^* = \{0 < |z| < 1\}$. Suppose $g_0, \ldots, g_n$ all have at most poles at $z = 0$. Suppose further that $f$ satisfies the identity
   \[
   g_0 f^n + g_1 f^{n-1} + \cdots + g_n = 0.
   \]
   Show that $f$ has at most a pole at $z = 0$.

5. Let $B$ be the square $\{|x| \leq 1, |y| \leq 1\}$ and let $T$ be the boundary of $B$, in the $xy$-plane. Let $f : T \to \mathbb{R}$ be a piecewise smooth function defined over $T$. Show that there is a sequence of real polynomials $p_n(x, y)$ so that $p_n$ converges uniformly to $f$ on $T$.
   (Hint: The Runge’s approximation theorem might be helpful.)

6. Let $f(z)$ be an analytic function defined over the disk $|z| < 2$. Show that
   \[
   \int_0^1 f(x)dx = \frac{1}{2\pi i} \oint_{|z|=1} f(z) \log z dz.
   \]
   Here the path integral is taken counterclockwise.
Work all six problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let $P(z)$ be a polynomial in $z$.
   (a) Assume that $P(z) \neq 0$ for $\text{Re}(z) > 0$. Show that $P'(z) \neq 0$ for $\text{Re}(z) > 0$. (Hint: Try taking a logarithmic derivative.)
   (b) Show that for any polynomial $P(z)$, the zeroes of $P'$ are contained in the convex hull of the zeroes of $P$.

2(a). Let $f(z)$ be an entire function with $\text{Re}(f(z)) < c(1+|z|)^p$ for some positive constants $c$ and $p$. Show that $f$ is a polynomial.
   (b). Construct an entire function $f(z)$ that is not a polynomial such that for every $\epsilon > 0$,
   $$\lim_{r \to \infty} e^{-\epsilon r} M(r) = 0,$$
   where $M(r) = \max_{|z|=r} |f(z)|$. Justify your answer.

3. Let $u(z)$ be a continuous real-valued function on $\mathbb{C}$ that satisfies
   $$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \, d\theta$$
   for all $z$ and all $r > 0$.
   (a) Prove that $\sup_{\Omega} u = \sup_{\partial \Omega} u$ for every bounded smooth domain $\Omega$.
   (b) Show that $u$ is a smooth harmonic function, i.e., that $\Delta u = 0$. (Hint: You may quote the solution for the Dirichlet problem.)

4(a). Find the radius of convergence of the power series $\sum_{n=1}^{\infty} nz^n$. Find the largest domain to which the corresponding analytic function can be analytically extended.
   (b). Suppose that the power series $\sum_{n=1}^{\infty} a_n z^n$ converges for $|z| < 1$ and that the coefficients $a_n$ are real and nonnegative. Show that if the corresponding analytic function has an analytic continuation to a neighborhood of $z = 1$, then the radius of convergence of the series is greater than 1.

5. Let $u$ be a subharmonic function defined on $\mathbb{C}$ and let $M(r) = \max_{|z|=r} u(z)$.
   (a) Prove that
   $$u(z) \leq \frac{\log r_2 - \log |z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log |z| - \log r_1}{\log r_2 - \log r_1} M(r_2)$$
   for $0 < r_1 \leq |z| \leq r_2$. (Hint: use the maximum principle.)
   (b) Show that $\lim_{r \to \infty} M(r)/\log r$ exists (possibly infinite).
   (c) Show that if the the limit $\lambda(u)$ in part (b) is 0, then $u$ is a constant function.
6(a). Let $\Gamma$ be a smooth simple closed curve in $\mathbb{C}$. Describe the set of points $z$ such that $|P(z)| \leq \max_{w \in \Gamma} |P(w)|$ for all polynomials $P(z)$. Justify your answer.

(b). Let $L$ be a line in $\mathbb{C}$ and let $z_0$ be a point not on $L$. Show that there is a sequence of polynomials $P_j(z)$ with $P_j(z_0) = 1$ such that $P_j \to 0$ uniformly on compact subsets of $L$. 


Work all six problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let $P(w, z) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z)$ be a polynomial in two variables. Suppose $z_0 \in \mathbb{C}$ is a point such that $a_0(z_0) \neq 0$ and $P(w, z_0)$ has $n$-distinct zeros, say $w_1, \ldots, w_n$. Show that there exists an open disc $\Delta \subset \mathbb{C}$ containing $z_0$ and $n$-holomorphic functions $f_i(z) : \Delta \to \mathbb{C}$, $i = 1, \ldots, n$, such that: (1) $P(f_i(z), z) = 0$ on $\Delta$; (2) $f_i(z_0) = w_i$ and (3) whenever $P(w, z) = 0$ and $z \in \Delta$, then $w = f_i(z)$ for some $i$.

2. Find the Green’s function for the region consisting of the complement in the $\mathbb{C}$-plane of the intervals $(-\infty, -1]$ and $[1, \infty)$ on the real axis.

3. Let $H$ be the upper half plane and $\bar{H}$ be its closure in $\mathbb{C}$. Let $f : \bar{H} \to \mathbb{C}$ be a continuous function that is analytic in $H$. Suppose $f$ is bounded on $H$ and
$$\lim_{t \to \pm \infty} f(t) = 0, \quad t \in \mathbb{R}.$$ 
Show that
$$\lim_{|z| \to \infty} f(z) = 0, \quad t \in H.$$ 
(Hint 1: You can use the following result if you like: Let $D$ be the open unit disk and $g : D - \{1\} \to \mathbb{C}$ be a continuous map that is analytic in $D$. Suppose $|g| \leq M$ on $D$ and $|g(e^{i\theta})| \leq 1$ for $0 < \theta < 2\pi$. Then $|g(z)| \leq 1$ on $D$.)

(Hint 2: Consider functions of the form $\frac{\log z}{A + B \log z} f(z)$.)

4. Use Argument Principle to show that the function $f(z) = e^{\pi z} - e^{-\pi z}$ assumes any value $w$ with positive real part once and only once in the half strip $\text{Re} z > 0$, $-\frac{1}{2} < \text{Im} z < \frac{1}{2}$.

5. Show that
$$\frac{\pi^2 \cos \pi z}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z - n)^2}.$$ 
(Hint: Study the principal parts at the poles and use periodicity.)

6(a). Describe the Riemann surface of $w = \sqrt{z(z-1)(z-\lambda)}$, $\lambda \neq 0, 1$.

(b). Show that $dz/w$ is a holomorphic differential on the Riemann surface and describe the mapping defined by
$$f(\zeta) = \int_{\zeta_0}^{\zeta} \frac{dz}{w},$$
where $\zeta$ is a general point on the Riemann surface and $\zeta_0$ is a chosen basepoint.
Work all six problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

Convention: We denote by $\Delta$ the open unit disk and by $\bar{A}$ the closure of $A$ in $\mathbb{C}$.

1. It is known that for any smooth complex valued function $g$ on $\bar{\Delta}$, there is a smooth function in $\Delta$ so that $\frac{\partial u}{\partial \bar{z}} = g$. Now let $\Omega \subset \mathbb{C}$ be $\Delta$ with the origin and the line segment $[\frac{1}{2}, 1]$ deleted:
\[
\Omega = \Delta \setminus \{z \in \mathbb{R} : z = 0 \text{ or } \frac{1}{2} \leq z \leq 1\}.
\]
Prove that given any function $f \in C^\infty(\Omega)$ there is a smooth function $u \in C^\infty(\Omega)$ so that $\frac{\partial u}{\partial \bar{z}} = f$.

Hint: Use the Runge approximation theorem.

2. (1). Write down an entire function $f : \mathbb{C} \to \mathbb{C}$ in the form of an infinite product so that its set of zeros equals $\{\log n : n = 2, 3, 4, \ldots \}$.
(2) State the definition of an entire function being finite order.
(3) Is there an entire function of finite order whose zero set is $\{\log n : n = 2, 3, 4, \ldots \}$? Prove the existence or the non-existence of such functions.

3. Prove that
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{\pi}{2} e^{i\left(\frac{1}{2} - t\right)z}}{z} \, dz = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } t < 0 \text{ or } t > 1 \end{cases}
\]

4. Let $\Delta^*$ be the punctured disc. We let $\mathcal{P}$ be the set of all subharmonic functions $v$ on $\Delta^*$ such that $\limsup_{z \to z_0} v(z) \leq 0$ for all $z_0 \in \partial \Delta$ and $\limsup_{z \to 0} v(z) \leq 2002$. Prove that for any $v \in \mathcal{P}$, $v(z) \leq 0$ in $\Delta^*$.

5. Show that the function
\[
w = \log \left(\frac{1 + z}{1 - z}\right) + \frac{2z}{1 + z^2},
\]
maps $\Delta$ one-one and onto the full $w$-plane with four half-lines deleted. Find the locations of the four end points of the four half-lines.

6. (1). We let $A$ be the interior of an equilateral triangle. Prove that any one-one, onto holomorphic map $f : A \to \Delta$ can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$ defined on an open neighborhood $U \supset A$.
(2). We let $B$ be the interior of a triangle whose three interior angles are $\frac{\pi}{5}$, $\frac{2\pi}{5}$ and $\frac{3\pi}{5}$. Prove that no one-one, onto holomorphic map $f : B \to \Delta$ can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$ defined on an open neighborhood $U \supset B$. 

Work all 6 problems. All problems have equal weight. Write each problem in a separate bluebook.

1. Let $D$ be a bounded region in $\mathbb{C}$ whose boundary consists of $n$-smooth disjoint Jordan curves. Thus $D$ is $n$-connected. We denote by $\overline{D}$ the closure of $D$.

Suppose $f(z)$ is a non-constant continuous function on $\overline{D}$ and is analytic in $D$. Suppose further that $|f(w)| = 1$ for all $w \in \partial D$.

Show that $f$ has at least $n$ zeros (counting multiplicities) in $D$.

2. Show that
\[
\int_0^\infty \frac{(\log x)^2}{1 + x^2} \, dx = \frac{\pi^3}{8}.
\]

Hint: You may need to compute $\int_0^\infty \frac{1}{1 + x^2} \, dx$ along the way.

Remark: You need to provide details to justify each step in your computation.

3. Let $D$ be the open unit disk and let $f : D \to \mathbb{C}$ be an odd univalent (i.e. one-one) function. Show that there is a univalent analytic function $g : D \to \mathbb{C}$ such that $f(z) = \sqrt{g(z^2)}$. 
4. Prove that the function
\[ w = \log(z) + \frac{z^2 - 1}{z^2 + 1} \]
is a 1-1 mapping from the half-plane defined by \( Re(z) > 0 \) onto a region \( \Omega \) in the \( w \) plane. Describe the region \( \Omega \) as explicitly as you can.

5. Define
\[ F(z) = \int_0^\infty x^{z-1} e^{-x^2} \, dx. \]
(a) Prove that \( F \) is an analytic function on the region \( Re(z) > 0 \).
(b) Prove that \( F \) extends to a meromorphic function on the whole complex plane.
(c) Find all the poles of \( F \) and find the singular parts of \( F \) at these poles.

6. Let \( \omega_1 \) and \( \omega_2 \) be two non-zero complex numbers with non-real ratio \( \omega_1/\omega_2 \). Let \( \Lambda \) be the lattice \( \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) and let \( a \) and \( b \) be two complex numbers not congruent to each other. We form the linear space \( V \) of all elliptic functions of period \( \Lambda \) with at most simple poles at \( a \) and \( b \).
(a) Prove that \( \dim_{\mathbb{C}} V \) is at most 2.
(b) Using the method of infinite series, construct explicitly a two dimensional family of elliptic functions in \( V \), thereby proving that \( \dim V = 2 \).

Remark: In (b) one needs to provide details to why the series converge, why they have period \( \Lambda \), and why they provide a two dimensional family.
1. Let \( \mathcal{H} \) be the class of analytic functions \( f(z) \) on \( |z| \leq 1 \) satisfying \( f(0) = 0, f'(0) = 1 \) and \( |f(z)| \leq 100 \) for all \( |z| < 1 \). Prove that there is a constant \( c > 0 \) so that for any \( f \in \mathcal{H} \) the image of the unit disk under \( f \) contains the disk \( |z| < c \).

2. Let \( H \) be the upper half plane and let \( F : H \to \mathbb{C} \) be defined by
\[
F(z) = \int_0^z \frac{dw}{(4 - w^2)\sqrt{w - 1}}.
\]
Prove, using the argument principle but not quoting the Christoffel-Schwarz Lemma directly, that \( F \) maps \( H \) one-one and onto a domain \( \Omega \) in \( \mathbb{C} \). Identify this domain \( \Omega \).

3. Factor the function
\[
\cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right)
\]
into an infinite product.

4. Prove that
\[
\int_0^\pi \ln \sin \theta d\theta = -\pi \ln 2.
\]

5. Let \( f(z) \) be an analytic function defined on the unit disk \( |z| < 1 \) so that \( f(0) = 0 \) and \(-1 < \text{Re} f(z) < 1 \) for all \( |z| < 1 \). Prove that
\[
|\text{Im} f(z)| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}, \quad \text{for} \ 0 < |z| < 1.
\]
Find an explicit form of \( f(z) \) when equality holds for some \( 0 < |z| < 1 \).

6. Let \( \wp(z) \) be the Weierstrass \( \wp \) function of periods 1 and \( \tau \). Prove that there is a single value branch of the meromorphic function
\[
F(z) = \sqrt{\wp(z) - \wp\left(\frac{1}{2}\right)}
\]
with \( F(\frac{1}{2}) = 0 \). What are the periods of this function? Verify your assertion.