Algebra Qualifying Exam, Fall 1998: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$GL(\ )$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Classify groups of order $171 = 9 \cdot 19$.

2. Find all groups which can occur as the Galois group of the splitting field over $\mathbb{F}_5$ of a polynomial of degree 9. (The polynomial is not assumed irreducible.)

3. (a) Let p be an odd prime. Explain why $-1 \in \mathbb{Z}/(p)$ is a square if and only if $p \equiv 1 \mod 4$.

(b) You may assume the fact that the ring $\mathbb{Z}[i]$ of Gaussian integers is a principal ideal domain. Show that an odd prime $p \in \mathbb{Z}$ is irreducible in $\mathbb{Z}[i]$ if and only if $p \equiv 3 \mod 4$. [**Hint**: Use (a).]

4. Suppose that V is a finite dimensional complex vector space and suppose that $S_1, \cdots, S_n$ are endomorphisms of V such that each S_i is diagonalizable and $S_i S_j = S_j S_i$ for all i, j. Show that there is a basis of V consisting of vectors each of which is an eigenvector for all S_i.

5. Let $F \subset K$ be subfields of the complex numbers such that K is a finite algebraic extension of F. Let $\zeta \in \mathbb{C}$.

(a) If ζ is *transcendental* over K, prove that $[K(\zeta) : F(\zeta)] = [K : F]$.

(b) Give an example of $F \subset K$ and ζ *algebraic* over K such that $[K(\zeta) : F(\zeta)]$ does not divide $[K : F]$.

Algebra Qualifying Exam, Fall 1998: Part II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
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$GL(\ )$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Let A be an abelian group with generators x, y and z subject to the relations

$$\begin{aligned} 2x + 2y - 16z &= 0, \\ 8x + 4y + 2z &= 0, \\ 2x + y - 22z &= 0. \end{aligned}$$

What is the structure of A as a direct sum of cyclic groups?

2. Use linear algebra to prove that if $F \subset E$ is a cyclic Galois field extension then there is an F-vector space basis of E of the form $\{\sigma(x) | \sigma \in \mathrm{Gal}(E/F)\}$, for some $x \in E$.

3. (a) Assume that A is a commutative Noetherian integral domain. Show that every nonzero noninvertible element of A can be written as a finite product of irreducible elements. [**Definition**: a noninvertible element $p \neq 0$ of A is *irreducible* if whenever $p = bc$ with $b, c \in A$ either b or c is invertible in A.]

(b) Give an example of a Noetherian integral domain which is not a unique factorization domain.

4. Let G be the group of order 20 with generators σ and τ and relations $\sigma^4 = \tau^5 = 1$, $\sigma\tau\sigma^{-1} = \tau^2$. Determine the conjugacy classes of G and compute the character table of the irreducible complex representations of G.

5. (a) Find the Galois group of $x^5 + 3x^2 + 1$ over the prime fields $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$.

Hint: The only irreducible quadratic over $\mathbb{F}_2$ is $x^2 + x + 1$.

(b) Find the Galois group of $x^5 + 3x^2 + 1$ over $\mathbb{Q}$.

Hint: Use part (a).

Algebra Qualifying Exam, Spring 1998: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$GL(,)$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Let G be a group of order p^n where p is prime and $n > 1$. Show that G has an automorphism of order p.

2. Let M be a 9×9 matrix over $\mathbb{C}$ with characteristic polynomial $(x^2+1)^3(x+1)^3$ and with minimal polynomial $(x^2+1)^2(x+1)$.

(a) Find trace(M) and det(M).
(b) How many distinct conjugacy classes of such matrices are there in $GL(9, \mathbb{C})$? Write down the Jordan form over $\mathbb{C}$ for one such matrix M.
(c) Write down a 9×9 matrix with rational coefficients with the above characteristic and minimal polynomials.

3. Let $F = \mathbb{C}(z)$ be the field of rational functions in one variable over $\mathbb{C}$, i.e. the field of fractions of the polynomial ring $A = \mathbb{C}[z]$.

(a) Show that $A = \mathbb{C}[z]$ is integrally closed in $F = \mathbb{C}(z)$.
(b) Show that $f(y) = y^5 - (z+1)(z+2) \in F[y]$ is irreducible.
(c) Let $E = F[y]/(f(y))$ and let $B = \mathbb{C}[z, y] \subset E$ be the integral closure of $A = \mathbb{C}[z]$ in E. Consider the prime ideals $\mathfrak{p}_0 = (z)$ and $\mathfrak{p}_1 = (z+1)$ in A. How many prime ideals in B lie above $\mathfrak{p}_0$ and $\mathfrak{p}_1$, respectively.

4. Suppose G is a finite group, $N \subset G$ is a subgroup, and $\rho : G \to \mathrm{End}(V)$ is an irreducible complex representation of G. Suppose there is a nonzero vector $v_0 \in V$ such that $\rho(x)v_0 = v_0$ for all $x \in N$.

(a) If N is normal in G, prove that N is contained in the kernel of ρ.
(b) Give an example to show that the conclusion to (a) need not be true if N is not normal in G.

5(a). Suppose that F is a field of characteristic $p > 0$. If α is algebraic over F, show that α is separable over F if and only if $F(\alpha) = F(\alpha^{p^n})$ for all $n \geq 1$.

(b). Suppose that k is a field of characteristic $p > 0$ and let $F = k(x, y)$ by the field of rational functions in two independent variables over k. Let $E = F(x^{1/p}, y^{1/p})$. Prove that E is not primitively generated over F. In other words, prove for all $\theta \in E$ that $F(\theta) \neq E$.

Algebra Qualifying Exam, Fall 1998: Part II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
$\mathbb{Z}$: Integers
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$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$GL(\ )$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Classify groups of order 306 that have a cyclic 3-Sylow subgroup.

2(a). Find the order of $GL(5, \mathbb{F}_2)$, the group of invertible 5×5 matrices over the field $\mathbb{F}_2$.
(b). Show that the polynomial $f(x) = x^5 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$ is irreducible. (Hint: how many irreducible quadratics are there over $\mathbb{F}_2$?)
(c). Exhibit a matrix A of order 31 in $GL(5, \mathbb{F}_2)$. (Hint: use (b) and some finite field theory.)

3. Let R be a commutative ring and let E be an R-module spanned over R by elements $e_1, \ldots, e_n$. Suppose that $b : E \times E \to R$ is an R-bilinear map such that $\det(B) \in R$ is *not* a zero divisor, where B is the $n \times n$ matrix $(b(e_i, e_j))$. Prove that E is a free R-module.

4. Let G be the group of order $136 = 8 \cdot 17$ with presentation

$$\langle x, y : y^8 = x^{17} = 1, \quad yxy^{-1} = x^4 \rangle.$$

(a) Find the center of G.
(b) Describe the number and dimensions of the irreducible complex representations of G.
(c) Find the simple summands of the group ring $\mathbb{Q}[G]$.

5(a). Let ζ be a primitive 7th root of unity in $\mathbb{C}$ and let $\beta = \zeta + \zeta^2 + \zeta^4$. Show that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$ and that $\sqrt{-7} \in \mathbb{Q}(\beta)$. (Hint: find a linear relation between 1, β, and β^2.)
(b). Let E be the splitting field of the polynomial $x^{14} + 7 = f(x)$ over $\mathbb{Q}$ and let α be a root of $f(x)$ in $\mathbb{C}$. Show that $E = \mathbb{Q}[\zeta, \alpha]$ and find the degrees $[E : \mathbb{Q}]$, $[E : \mathbb{Q}(\zeta)]$, and $[E : \mathbb{Q}(\alpha)]$.
(c). Write down elements σ and τ of orders 6 and 7, respectively, in $\mathrm{Gal}(E/\mathbb{Q})$ by explicitly giving the values $\sigma(\zeta)$, $\sigma(\alpha)$, and $\tau(\zeta)$, $\tau(\alpha)$.

Algebra Ph.D. Qualifying Exam
Fall, 1999
Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

Notation:
$\mathbb{Z}$: the ring of ordinary integers
$\mathbb{Q}$: the field of rational numbers
$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers
$\mathbb{F}_q$: the finite field with q elements
$M_n(R)$: the ring of $n \times n$ matrices with entries in the ring R
$\mathrm{GL}_n(R)$: the group of invertible $n \times n$ matrices in $M_n(R)$
$R[t]$: the ring of polynomials with coefficients in the ring R
$\mathbb{Z}/n$: the ring of integers mod n. (Can also be thought of as the cyclic group of order n.)

1. If G is a simple group of order 60, determine, with proof, the number of elements of order 3 in G. (You may not assume there is only one such group.)

2. Let G be the group given by generators and relations $G = \{x, y \mid x^5 = xyx^{-1}y^{-2} = 1\}$.

(a) Prove G is finite.
(b) What is $|G|$?
(c) How many 5-Sylow subgroups are there in G?

3. Let G be the finite group of order 21 defined by generators and relations:

$$\langle x, y \mid x^3 = y^7 = 1, xyx^{-1} = y^2 \rangle.$$

Determine the conjugacy classes of G and construct its character table.

4. Let $\mathbb{K}$ be an arbitrary field and suppose that $T \in M_n(\mathbb{K})$. Prove that there exists a vector $v \in \mathbb{K}^n$ so that the vectors

$$\{v, Tv, T^2v, \ldots, T^{n-1}v\}$$

form a basis for $\mathbb{K}^n$ if and only if the only matrices in $M_n(\mathbb{K})$ which commute with T are expressible as polynomials in T (i.e., A commutes with T if and only if $A = a_0 I + a_1 T + \cdots + a_{n-1}T^{n-1}$ where $I \in M_n(\mathbb{K})$ is the identity matrix).

5.

(a) Determine the Galois group of $x^3 - x + 3$ over $\mathbb{Q}$.
(b) Determine the Galois group of $x^3 - x + 3$ over $\mathbb{F}_5$.
(c) Determine the Galois group of $x^4 + t$ over $\mathbb{R}[t]$.

Algebra Ph.D. Qualifying Exam
Fall, 1999
Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

Notation:
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$M_n(R)$: the ring of $n \times n$ matrices with entries in the ring R
$\mathrm{GL}_n(R)$: the group of invertible $n \times n$ matrices in $M_n(R)$
$R[t]$: the ring of polynomials with coefficients in the ring R
$\mathbb{Z}/n$: the ring of integers mod n. (Can also be thought of as the cyclic group of order n.)

1. Suppose that $\mathbb{K}$ is a non-Galois extension of $\mathbb{Q}$ of degree 5. Let $\mathbb{L}$ be the Galois closure of $\mathbb{K}$ (the smallest Galois extension of $\mathbb{Q}$ containing $\mathbb{K}$), and suppose $\mathbb{L}$ does not contain any quadratic extensions of $\mathbb{Q}$. Prove $\mathrm{Gal}(\mathbb{L}/\mathbb{Q}) = \mathcal{A}_5$, the alternating group on 5 letters.

2. Determine all prime ideals of the ring $\mathbb{Z}[t]/(t^2)$.

3. Suppose that A, B are elements of $M_2(\mathbb{C})$ such that $A^2 = B^3 = I$, $ABA = B^{-1}$ with $A \neq I$, $B \neq I$. If $D \in M_2(\mathbb{C})$ commutes with A and B, show that D is a scalar matrix, i.e., a scalar multiple of I.

4. Let V be a valuation ring, i.e. a commutative ring (with unit) such that for all $a, b \in V$ either $a|b$ or $b|a$. (Here, $a|b$ means that $b = ac$ for some $c \in V$.)

(i) Prove that if I and J are two ideals in V then $I \subset J$ or $J \subset I$.
(ii) Prove that any finitely generated ideal of V is principal, that is, generated by a single element.
(iii) Prove that if V is a Noetherian valuation ring, then there exists an element $t \in V$ such that any proper nonzero ideal of V is (t^n) for some whole number $n \geq 1$.

5. Let G be a finite simple group, and let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation, where $n > 1$. Let χ be its character. If $|\chi(g)| = n$, prove that g is the identity element of G.

QUALIFYING EXAM – ALGEBRA
SPRING 1999
MORNING SESSION

Do all problems. Use a separate blue book for each.

Notation: **Z**, **Q**, **R**, **C**, and $\mathbf{F}_q$ denote the ring of integers, and the fields of rational numbers, real numbers, complex numbers, and q elements, respectively.

1. Classify all groups of order 24 containing a normal subgroup which is cyclic of order 4.

2. Describe all similarity classes (conjugacy classes) of 6×6 matrices with minimal polynomial $x^4 + x^2$:

(i) over **Q**,
(ii) over $\mathbf{F}_5$.

3. Find the Galois group of the splitting field of the polynomial $x^3 - x + 1$:

(i) over **R**,
(ii) over **Q**,
(iii) over $\mathbf{F}_2$.

4. Suppose K is a finite extension of **Q**. Prove that the integral closure of **Z** in K is a free **Z**-module of rank $[K : \mathbf{Q}]$.

5. Suppose G is a *nonabelian* group of order pq, where $p < q$ are primes.

(a) Describe the conjugacy classes in G.
(b) Describe all representations of G (over **C**).

QUALIFYING EXAM – ALGEBRA
SPRING 1999
AFTERNOON SESSION

Do all problems. Use a separate blue book for each.

Notation: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{F}_q$ denote the ring of integers, and the fields of rational numbers, real numbers, complex numbers, and q elements, respectively.

1. Describe all simple left modules over the matrix ring $M_n(\mathbf{Z})$ ($n \times n$ matrices over $\mathbf{Z}$). Recall that a module is simple if it has no proper submodules.

2. Let G be a finite group. Prove that the following are equivalent:

(i) every element of G is conjugate to its inverse,
(ii) every character of G is real-valued.

3. Let R be the ring $\mathbf{C}[x,y]/(y^4 - (x-1)(x-2)(x-3)(x-4))$. (You may assume that $y^4 - (x-1)(x-2)(x-3)(x-4)$ is irreducible.) Let K be the quotient field of R.

(a) Show that K is a Galois extension of $\mathbf{C}(x)$.
(b) Consider R as an extension of $\mathbf{C}[x]$. For every prime $\mathfrak{p}$ of $\mathbf{C}[x]$, find the primes of R above $\mathfrak{p}$ and describe the action of $\mathrm{Gal}(K/\mathbf{C}(x))$ on them.

4. Suppose G is a finite group, F is a field whose characteristic does not divide the order of G, and V is a representation of G over F (i.e., an F-vector space on which G acts F-linearly). Prove that if U is a subspace of V stable under G, then there is a complementary subspace W of V, also stable under G, such that $V = U \oplus W$.

5. Suppose K is an extension of $\mathbf{Q}$ of degree n, and let $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbf{C}$ be the distinct embeddings of K into $\mathbf{C}$. Let $\alpha \in K$. Regarding K as a vector space over $\mathbf{Q}$, let $\phi : K \to K$ be the linear transformation $\phi(x) = \alpha x$. Show that the eigenvalues of ϕ are $\sigma_1(\alpha), \ldots, \sigma_n(\alpha)$.

Algebra Qualifying Exam, Fall 2000: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:
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$\mathbb{C}$: Complex Field
$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)

1. How many distinct isomorphism types are there for groups of order 2525?

2.(a) Suppose K is a field of characteristic zero which contains the p-th roots of 1, where p is a fixed prime. If L/K is a Galois extension of degree p, explain why $L = K[\alpha]$ where $\alpha^p = a \in K$.

(b) If p is odd, show that there is no $\beta \in L$ with $\beta^p = \alpha$. (**Hint**: Use norms.)

(c) Give a counterexample to the assertion in part (b) if $p = 2$.

3. Suppose that W is an even-dimensional real vector space, $T : W \to W$ a linear transformation with $T^m = I$ (the identity transformation) with m odd. Show that there exists a linear transformation $S : W \to W$ with $S^2 = -I$ and $ST = TS$.

4. Let A be a principal ideal domain, M a finitely generated free A-module.

(a) Show that the number of elements in a free basis for M over A is independent of the choice of basis.
(b) Let $N \subset A^m$ be a submodule. Prove that N is free on n generators for some $n \leq m$.
(c) Prove that $n = m$ in part (b) if and only if there is an nonzero $a \in A$ with $aA^m \subset N$.

5. Suppose G is a finite group, K a normal subgroup of G, and that (ρ, V) is an irreducible complex representation of G. Consider the restriction (ρ_K, V) of this representation to K. Show that all K-invariant subspaces of V which are irreducible over K have the same dimension and occur with the same multiplicity in V.

Algebra Qualifying Exam, Fall 2000: Part II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:
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$\mathrm{GL}_n(R)$: Group of $n \times n$ invertible matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)

1. If G is a group, define subgroups $G^{(n)}$ recursively by $G^{(1)} = [G, G]$ (the commutator subgroup) and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. The group G is *solvable* if $G^{(n)} = \{e\}$ for some n.

   (a) If K is a normal subgroup of G such that both K and G/K are solvable, show that G is solvable.

   (b) Show that all groups of order p^n with p prime are solvable.

2. Determine the number of conjugacy classes in the group $\mathrm{GL}_2(\mathbb{F}_q)$ for all finite fields $\mathbb{F}_q$. (**Hint:** Use linear algebra.)

3.(a) If A is a commutative ring with 1 show that the polynomial ring $A[X]$ contains infinitely many distinct maximal ideals.

(b) Describe all maximal ideals in the ring of formal power series $\mathbb{Z}[[X]]$.

4.(a) If G is a non-abelian group of order p^3, show that G has a quotient group isomorphic to $(\mathbb{Z}/p) \times (\mathbb{Z}/p)$. What are the number and dimensions of the irreducible complex representations of G?

(b) If the nonabelian group of order p^3 contains an element x of order p^2 show that G has irreducible p-dimensional representations induced from suitable 1-dimensional representations of $\langle x \rangle \cong \mathbb{Z}/p^2$.

5. Let $\mathbb{Q}[\zeta]$ be the field extension of $\mathbb{Q}$ generated by a primitive 11-th root of unity ζ. The integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\zeta]$ is the ring $\mathbb{Z}[\zeta]$. For each of the following primes $p \in \mathbb{Z}$, describe how the ideal $p\,\mathbb{Z}[\zeta]$ factors in $\mathbb{Z}[\zeta]$.

(a) $p = 11$; (b) $p = 43$; (c) $p = 37$.

Algebra Qualifying Exam, Spring 2000: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your assertions and state precisely any theorems that you quote.

Notation:
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$\mathbb{C}$: Complex Field
$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)
S_n: Symmetric group of degree n

1. How many distinct isomorphism types are there for groups of order 5555?

2. Find the Galois group of $x^4 - 2$ over the fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{F}_3$ and $\mathbb{F}_{27}$.

3. Prove the following generalization of Nakayama's Lemma to noncommutative rings. Let R be a ring with 1 (not necessarily commutative) and suppose that $J \subset R$ is an ideal contained in every maximal left ideal of R. If M is a finitely generated left R-module such that $JM = M$, prove that $M = 0$.

4. Let S be a set of $n \times n$ nilpotent matrices over a field K that pairwise commute. Show that there is an invertible matrix M such that every matrix MAM^{-1} with $A \in S$ is strictly upper triangular, that is, all entries on or below the main diagonal are zero.

5.(a) Compute $|\,\mathrm{GL}_3(\mathbb{F}_2)|$, the number of invertible 3×3 matrices over the field $\mathbb{F}_2$. If $\mu \in \mathrm{GL}_3(\mathbb{F}_2)$ has order 7 explain why μ must act *transitively* on the non-zero elements of $\mathbb{F}_2^3 = (\mathbb{Z}/2)^3$.

(b) Using (a), show that there is a non-abelian group G of order $56 = 8 \cdot 7$ with a normal 2-Sylow subgroup isomorphic to $(\mathbb{Z}/2)^3$. Find the number of irreducible complex representations of G and their dimensions.

(c) Find the conjugacy classes of G and compute the character values for at least one irreducible complex representation of G of dimension greater than one.

ALGEBRA QUALIFYING EXAM, SPRING 2000: PART II

Directions: Work each problem in a separate bluebook. Give reasons for your assertions and state precisely any theorems that you quote.

Notation:
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$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)
S_n: Symmetric group of degree n

1. Suppose that A is a Noetherian local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{a} \subset A$ is an ideal such that the only prime ideal of A containing $\mathfrak{a}$ is $\mathfrak{m}$, show that $\mathfrak{m}^k \subset \mathfrak{a}$ for some $k \geq 1$.

2. A subgroup $H \subseteq S_n$ is *transitive* if for all i, j with $1 \leq i, j \leq n$, there exists some $\sigma \in H$ with $\sigma(i) = j$. An automorphism of a group G is called *inner* if it is of the form $x \to axa^{-1}$ for some $a \in G$.

(a) Show that S_5 has six 5-Sylow subgroups.
(b) Show that S_6 contains a transitive subgroup isomorphic to S_5.
(c) The subgroup $H \subset S_6$ from part b has six cosets. Show that there is an isomorphism $\alpha : S_6 \to S_6$ such that $\alpha(H) \subset S_6$ is *not* a transitive subgroup of S_6.
(d) Explain why the automorphism in part (c) is not inner.

3. How many similarity classes are there of 10×10 matrices with minimal polynomial $(x^2 + 1)(x^3 - 2)$ over the field $\mathbb{Q}$? Over the field $\mathbb{F}_5$?

4. Let k be a field of characteristic zero.

(a) Suppose K and L are two finite extensions of k, in some fixed algebraic closure of k, such that K is *normal* over k. Prove that $|KL : L|$ divides $|K : k|$.
(b) Suppose that E is a Galois extension of k with $\mathrm{Gal}(E/k) = S_n$, the symmetric group. Show that for any integer j with $1 < j < n$ there are subfields K, $L \subset E$ with $K \cap L = k$, $|K : k| = n$, $|KL : L| = j$ and $|L : k| = n!/j!$. [**Hint:** Galois correspondence.]

5. Let G be a group of odd order.

(a) Show that the only irreducible complex character of G which is real valued is the trivial character χ_1. [**Hints:** Assume χ_V is a counterexample and get a contradiction from $0 = \langle \chi_1, \chi_V \rangle$. Make use of algebraic integers and the fact that $g \neq g^{-1}$ for $g \neq 1$.]

(b) Using (a), explain why the real group ring $\mathbb{R}[G]$ has structure

$$\mathbb{R} \times \prod_{i=1}^{\frac{s-1}{2}} \mathrm{Mat}_{r_i}(\mathbb{C}),$$

where s is the number of conjugacy classes of G and $\mathrm{Mat}_{r_i}(\mathbb{C})$ is the ring of $r_i \times r_i$ matrices with entries in $\mathbb{C}$.

Fall 2001

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Determine the number of isomorphism classes of groups of order $1705 = 5 \cdot 11 \cdot 31$.

2. If A is a commutative Noetherian ring with 1 prove that $(0) = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_k$ for some finite collection of (not necessarily distinct) prime ideals $\mathfrak{p}_i \subset A$.

[Hint: Consider the set of all ideals of A which do not contain a finite product of prime ideals.]

3. Determine the number of similarity classes of matrices over $\mathbb{C}$ and which have characteristic polynomial $(X^4 - 1)(X^8 - 1)$. Do the same thing over $\mathbb{Q}$.

4. Let F be a field. Consider the polynomial $f(X) = X^4 - a$ where $a \in F$. Determine (with explanation) all possible Galois groups of $f(X)$ as the field F and the element $a \in F$ vary. Give an example for every possible Galois group.

5. Let G be a finite group and let $z \in G$. Suppose for every irreducible complex character χ of G we have $|\chi(z)| = |\chi(1)|$. Prove that z is in the center of G.

Fall 2001

Ph.D. Qualifying Examination

Algebra

Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. If A is a finite abelian group and m is a positive integer show that every automorphism of the subgroup mA of A can be extended to an automorphism of A.

[Hint: Reduce to the case where m is prime. Then use the structure theorem for finite abelian groups.]

2. Let k be a field and let A and B be k-algebras with unit having centers $Z(A)$ and $Z(B)$. Prove that the center of the k-algebra $A \otimes_k B$ is $Z(A) \otimes_k Z(B)$.

[Hint: First express $z \in A \otimes B$ as $\sum_{i=1}^{n} a_i \otimes b_i$ where a_i are linearly independent over k. Show all $b_i \in Z(B)$.]

3. Suppose V is a finite dimensional vector space over a field k and $T : V \to V$ is a linear tranformation. Let $\wedge^2 T : \wedge^2 V \to \wedge^2 V$ be the induced endomorphism of the second exterior power of V. Explain why the characteristic polynomial of $\wedge^2 T$ depends only on the characteristic polynomial of T, and express the characteristic polynomial of $\wedge^2 T$ in terms of the eigenvalues of T.

4. Let G be the group of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

in $GL(3, \mathbb{F}_3)$. Find the conjugacy classes in G and compute it character table.

5. Suppose F is a field and K and E are finite extensions of F in some algebraic closure of F. Suppose that E is Galois over F (normal and separable). Show that $L = KE$ is Galois over K with $[L : K] = [E : E \cap K]$.

Spring 2001

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. If $p < q < r$ are primes and G is a group of order pqr, show that G contains a normal subgroup of order r. [Hint: First show that G contains some normal Sylow subgroup.]

2. (a) If $I \subset A$ is an ideal in a commutative Noetherian ring and if $ab \in I$ for some a, b with $a \notin I$ and $b^n \notin I$ for all n, show that $I = (I, b^m) \cap (I, a)$ for some m. [Hint: first show $xb^{m+1} \in I$ implies that $xb^m \in I$ for some m.

(b) Let A be a commutative ring and E be a finitely generated A-module. If $\{e_1, \cdots, e_r\} \subset E$ is a finite subset whose images span E/mE as an A/m vector space for all maximal ideals $m \subset A$, show that $\{e_1, \cdots, e_r\}$ generate E as an A-module.

3. (a) How many similarity classes of 10×10 matrices over $\mathbb{Q}$ are there with minimal polynomial $(x+1)^2(x^4+1)$?

(b) Give an example of a 10×10 matrix over $\mathbb{R}$ with minimal polynomial $(x+1)^2(x^4+1)$ which is not similar to a matrix with rational coefficients.

4. Let G be a finite group and H be a subgroup of index k. Let (π, V) be an irreducible complex representation of G, and let U be a nonzero H-invariant subspace. Prove that the dimension of U is at least $\frac{1}{k}\dim(V)$. If its dimension is exactly $\frac{1}{k}\dim(V)$, prove U is irreducible over H and that there is no other H-invariant subspace of V isomorphic to U as an H-module.

5. (a) Find $[E : \mathbb{Q}]$ where E is the splitting field of $x^6 - 4x^3 + 1$ over $\mathbb{Q}$.

(b) Show that $\mathrm{Gal}(E/\mathbb{Q})$ is nonabelian and contains an element of order 6.

Spring 2001

Ph.D. Qualifying Examination

Algebra

Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Determine the number of isomorphism classes of groups of order $273 = 3 \cdot 7 \cdot 13$.

2. Suppose that E/F is an algebraic extension of fields of characteristic zero. Suppose that every polynomial in $F[x]$ has at least one root in E.

(a) Show that E/F is normal.

(b) Show that E is algebraically closed.

3. Suppose that E is the degree three field extension of the rational function field $\mathbb{Q}(x)$ defined by $E = \mathbb{Q}(x)[Y]/(Y^3 + x^2 - 1)$. Let y be the image of Y in E and let $B \subset E$ denote the integral closure of $A = \mathbb{Q}[x]$ in E. It is known—and you may assume—that B is the ring $\mathbb{Q}[x, y]$ generated by x and y over $\mathbb{Q}$. For each of the prime ideals P of A below, describe the factorization of the ideal PB of B.

(i) $P = (x)$.

(ii) $P = (x - 1)$.

(iii) $P = (x^2 + 3)$.

4. Suppose k is a field and V is a module over the polynomial ring $k[T]$ which is finite dimensional as a vector space over k. Define a $k[T]$ module structure on the dual vector space V^* by $(T\alpha)v = \alpha(Tv)$, $\alpha \in V^*$, $v \in V$. Show that $V \cong V^*$ as $k[T]$ modules.

5. Let G be the nonabelian group of order 39 with generators and relations

$$\left\langle x, y | x^3 = y^{13} = 1, xyx^{-1} = y^3 \right\rangle.$$

Find its conjugacy classes and compute its character table.

Notation:

$\mathbb{Q}$ denotes the field of rational numbers,

$\mathbb{Z}$ denotes the ring of ordinary integers,

$\mathbb{R}$ denotes the field of real numbers,

$\mathbb{C}$ denotes the field of complex numbers,

$\mathbb{F}_q$ denotes the finite field with q elements.

If R is any ring then $\mathrm{Mat}_n(R)$ denotes the ring of $n \times n$ matrices with coefficients in R.

If R is any ring then $GL_n(R)$ denotes the group of invertible $n \times n$ matrices in $\mathrm{Mat}_n(R)$.

If A is any ring then $A[t]$ denotes the ring of polynomials with coefficients in A.

Fall 2002

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. (a) Let F be a field, V a finite-dimensional vector space over F and $T : V \to V$ a linear transformation. Suppose that all roots of the characteristic polynomial of T are in F. Show that with respect to some basis of V the matrix of T is upper triangular.

(b) Suppose that V is a four dimensional vector space over the field $\mathbb{R}$ of real numbers and $T : V \to V$ a linear transformation. Show that with respect to some basis of V the matrix of T has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

2. Classify those finite groups of order $351 = 3^3 \cdot 13$ that have an abelian 3-Sylow subgroup containing no elements of order 9.

3. Let K be the splitting field of the polynomial $x^6 - 3 = 0$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(K/\mathbb{Q})$.

4. (*Chinese remainder theorem.*) Let A be a commutative ring with unit and let I, J be ideals of A such that $A = I + J$. Prove that $IJ = I \cap J$ and that there is a ring isomorphism

$$A/IJ \cong (A/I) \times (A/J).$$

5. A nonabelian group G of order 36 has generators x, y and z subject to the relations:

$$x^3 = y^3 = 1, \qquad xy = yx, \qquad z^4 = 1, \qquad zxz^{-1} = y, \qquad zyz^{-1} = x^2.$$

Find the conjugacy classes of G and compute its character table.

Fall 2002

Ph.D. Qualifying Examination

Algebra, Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Let p be a prime.

(a) Consider the action of $GL(4, \mathbb{F}_p)$ on the set of two-dimensional vector subspaces of $\mathbb{F}_p^4$. Let $U = \{(0, 0, x, y) \,|\, x, y \in \mathbb{F}_p\}$. Describe the subgroup of $g \in GL(4, \mathbb{F}_p)$ such that $gU = U$ and compute its order.

(b) Compute the number of two dimensional vector subspaces of $\mathbb{F}_p^4$.

2. Let p and q be primes with $q \geq p$. Prove that there exists a nonabelian group of order pq^2 if and only if p divides one of $q - 1$, q or $q + 1$.

3. If A is a commutative ring with unit and $I \subset A$ is a proper ideal, prove that there exists a prime ideal $P \subset A$ which is "minimal over I." This means that $I \subseteq P$ and if Q is prime with $I \subseteq Q \subseteq P$ then $Q = P$. [**Hint:** Zorn's Lemma.]

4. Let p be a prime and let E/F be a cyclic Galois extension of degree p. Let σ be a generator of $\mathrm{Gal}(E/F)$.

(a) Suppose the characterisic of E and F is p. Show that there exists $\alpha \in E$ such that $\alpha \notin F$ but $\sigma(\alpha) - \alpha \in F$.

(b) Show that if the characteristic of E and F is *not* p then $\sigma(\alpha) - \alpha \in F$ if and only if $\alpha \in F$.

Hint for both parts: It may help to think of E as a vector space over F, and σ as a linear transformation.

5. Let R be a commutative ring containing $\mathbb{C}$, and let M be a simple R-module. (Recall that this means that M has no submodules except $\{0\}$ and M itself.) Suppose that $\dim_{\mathbb{C}}(M) < \infty$.

(a) Prove that if $r \in R$ there exists $\alpha \in \mathbb{C}$ such that $rm = \alpha m$ for all $m \in M$. (**Remark:** If you use some version of Schur's Lemma, you must prove it.)

(b) Prove that $\dim_{\mathbb{C}}(M) = 1$.

Spring 2002

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Let p and q be primes, $q > 2$. Let $G = SL(2, \mathbb{F}_p)$. (Here $\mathbb{F}_p$ is the finite field with p elements.) Suppose that q divides $|G| = p(p^2 - 1)$. Show that a q-Sylow subgroup of G is cyclic. (**Hint:** first show that G has cyclic subgroups of orders p, $p - 1$ and $p + 1$.)

2. Let R be a commutative ring with unit.

(i) Let S be a *saturated* multiplicative set of R. This means that $1 \in S$, $0 \notin S$, and $xy \in S$ if and only if $x \in S$ and $y \in S$. Show that $R - S$ is a union of prime ideals. [Hint: If $a \in R - S$ consider ideals J with $a \in J \subset R - S$.]

(ii) An element $a \in R$ is a *zero divisor* if $ab = 0$ for some $b \neq 0$. Apply (i) to show that the set of zero divisors is a union of prime ideals of R.

3. Let p be prime. Show that there exists $\alpha \in \mathbb{C}$ such that $K = \mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order p. Exhibit such an α when $p = 5$.

4. Let G be a finite group and let $H \subset G$ be an abelian subgroup of prime index p. Let χ be an irreducible character of G such that $\chi(1) = p$. Prove that there exists a character ψ of H such that χ is the character of G induced from ψ.

5. (i) Let R be a principal ideal domain, and let $f, g \in R$ be coprime elements. Show that

$$R/(fg) \cong R/(f) \oplus R/(g)$$

as R-modules.

(ii) Let F be a field, and let $f(X) = X^2 + aX + b$, $g(X) = X^2 + cX + d$ be distinct irreducible polynomials over F. Let $fg = X^4 + tX^3 + uX^2 + vX + w$. Show that the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -b & -a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -d & -c \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -w & -v & -u & -t \end{pmatrix}$$

are conjugate in $GL(4, F)$, the group of 4×4 invertible matrices with coefficients in F.

Spring 2002

Ph.D. Qualifying Examination

Algebra

Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

Notation: Here $\mathbb{F}_p$ denotes the finite field with p elements, and S_n denotes the symmetric group of degree n.

1. Let H be the subgroup of S_6 generated by (16425) and (16)(25)(34). Let H act on S_6 by conjugation. Show that the set

$$\Sigma = \{(12)(35)(46),\ (13)(24)(56),\ (14)(25)(36),\ (15)(26)(34),\ (16)(23)(45)\}$$

is invariant under H. Hence obtain a homomorphism $\phi : H \to S_5$. Show that ϕ is an isomorphism.

2. Let Q be the group of order 8 having generators x and y such that $x^4 = y^4 = 1$, $x^2 = y^2$ and $xyx^{-1} = y^3$. Find the conjugacy classes of Q and compute its character table.

3. Let p and q be distinct primes. Show that the polynomial

$$\Phi(X) = X^{p-1} + X^{p-2} + \ldots + 1$$

has a root in $\mathbb{F}_{q^2}$ if and only if $q \equiv \pm 1$ modulo p.

4. Let V be a finite dimensional complex vector space endowed with an inner product, that is, a positive definite Hermitian form $\langle\ ,\ \rangle$. Let $T : V \to V$ be a linear transformation which commutes with its adjoint T^*, defined by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle .$$

Prove that V has a basis consisting of eigenvectors of T.

5. Let A be a commutative Noetherian ring with unit. An ideal $J \subset A$ is called a *radical ideal* if $x^n \in J$ implies that $x \in J$. Show that every proper radical ideal is a finite intersection of prime ideals. [Hint: Among counterexamples, a maximal one couldn't be prime.]

Stanford University Mathematics Department
Algebra Qualifying Exam, Fall 2003
Part I

1. Classify finite groups of order $2p^2$ up to isomorphism, where p is an odd prime.

2. Let $A \in M(n, K)$ be an $n \times n$ matrix over a field K such that the minimal polynomial of A has degree n. Show that every matrix in $M(n, K)$ that commutes with A is a K-linear combination of the identity matrix and powers of A.

3(a). Suppose $\rho : G \to \mathrm{GL}(V)$ is an irreducible complex representation of a finite group G. Show that if $\mathrm{Z}(G) \subset G$ is the center of G, then there is a homomorphism $\chi : \mathrm{Z}(G) \to \mathbf{C}^*$ such that $\rho(g)v = \chi(g)v$ for all $g \in \mathrm{Z}(G)$ and $v \in V$.

(b). Conversely, show that for any homomorphism $\chi : \mathrm{Z}(G) \to \mathbf{C}^*$, there exists an irreducible $\rho : G \to \mathrm{GL}(V)$ such that $\rho(g)v = \chi(g)v$ for all $g \in \mathrm{Z}(G)$ and $v \in V$.

4. Suppose $g(T) \in \mathbf{Z}[T]$ is a monic polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{C}$. If $|\alpha_j| = 1$ for all j, show that each α_j is a root of unity. Show by example that this conclusion may fail to hold if $g(T)$ is not assumed monic.

5. Let R be a ring with identity, and let M be a left R-module. Prove there exist submodules $M = M_0 \supset M_2 \supset \cdots \supset M_n = (0)$ so that M_j/M_{j+1} is a simple R-module (for all j) if and only if M satisfies both the ascending chain condition and the descending chain condition for submodules. [Hints: For the "if" direction, begin by using one chain condition to get a maximal or a minimal proper submodule. For the "only if" direction, use an induction on n.]

Stanford University Mathematics Department
Algebra Qualifying Exam, Fall 2003
Part II

1. If $p < q < r$ are primes and G is a finite group with $|G| = pqr$, prove that the Sylow-r subgroup of G is normal. [Hint: First get some normal Sylow subgroup.]

2. Suppose V is a vector space of dimension 20 over the field $\mathbf{Q}$, and $A : V \to V$ is a linear transformation with minimal polynomial $(T^2+1)^2(T^3+2)^2$.

(a). How many distinct similarity classes of such A exist?

(b). If V is generated by two elements as a $\mathbf{Q}[T]$ module, where T acts on V as the linear transformation A, and if $p(T) \in \mathbf{Q}[T]$, what integers can occur as the dimension of the kernel of $p(T) : V \to V$ as a rational vector space?

3. Find the Galois groups of the polynomials $X^6 + 3$ and $X^6 + X^3 + 1$ over the fields $\mathbf{Q}$ and $\mathbf{F}_7$.

4. Suppose k is a field of characteristic $\neq 2$, and let R denote the polynomial ring $k[x_1, x_2, \ldots, x_n]$. Suppose $f(x_1, x_2, \ldots, x_n) \in R$ is a non-constant polynomial that is not divisible by the square of any non-constant polynomial in R. Show that the ring $S = R[T]/(T^2 f)$ is the integral closure of R in the field of fractions of S.

5. Suppose G is the group of order 12 with presentation

$$G = \left\langle\, x, y \mid x^4 = y^3 = 1,\ xyx^{-1} = y^2 \,\right\rangle.$$

Find the complex character table of G.

Algebra Qualifying Exam, Spring 2003: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)

1. Classify all finite groups of order 140 up to isomorphism.

2. Find the degree $|E : \mathbb{Q}|$ if E is the splitting field of the polynomial $X^{10} - 5 \in \mathbb{Q}[X]$. How many distinct intermediate fields K exist with $\mathbb{Q} \subsetneq K \subsetneq E$?

3(a). Find all positive integers that can occur as the order of some element of $\mathrm{GL}(2, \mathbb{R})$. Exhibit an element of order 5.

(b). Find all positive integers that can occur as the order of some element of $\mathrm{GL}(3, \mathbb{F}_7)$.

(c). Find all positive integers that can occur as the order of some element of $\mathrm{GL}(4, \mathbb{Q})$. Exhibit an element of order $\neq 1$ or 2.

4(a). Find the integral closure B of the integers $\mathbb{Z}$ in the field $\mathbb{Q}\left[\sqrt{-39}\right]$.

(b). Show that there are two distinct prime ideals of B that contain the ideal $5B \subset B$. Give generators for these two prime ideals and show that neither is principal.

(c). How does the ideal $3B \subset B$ factor as a product of prime ideals?

5. Let $\rho : G \to \mathrm{GL}(3, \mathbb{C})$ be a 3-dimensional complex representation of a finite group G. Let V be the vector space of all 3×3 matrices over $\mathbb{C}$. Define the adjoint representation $\rho^\frown : G \to \mathrm{GL}(V)$ by

$$\rho^\frown(g)A = \rho(g)\, A\, \rho(g^{-1})$$

for $g \in G$ and $A \in V$. Which integers can occur as the multiplicity of the trivial one dimensional representation in $\rho^\frown$?

Algebra Qualifying Exam, Spring 2003: Part II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: Fields of rational, real, and complex numbers, respectively
$\mathrm{GL}_n(R)$: Group of $n \times n$ invertible matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)

1(a). Find all abelian groups G that contain a subgroup H isomorphic to $\mathbb{Z}/72\mathbb{Z}$ for which the quotient group G/H is also isomorphic to $\mathbb{Z}/72\mathbb{Z}$.

(b). Find the invariant factors of the abelian group $(\mathbb{Z}/44,000\mathbb{Z})^*$, i.e., the multiplicative group of invertible elements in the ring $\mathbb{Z}/44,000\mathbb{Z}$.

2. Suppose $I \subset \mathbb{Q}[x_1, x_2, \ldots, x_n]$ is an ideal such that the set of zeroes

$$V(I) = \{\mathbf{x} \in \mathbb{C}^n : f(\mathbf{x}) = 0 \text{ for all } f \in I\}$$

is a finite set. Show that the ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]/I$ is a finite dimensional vector space over $\mathbb{Q}$.

[Hint: first show that $\mathbb{Q}[x_1, x_2, \ldots, x_n]/J$ is finite dimensional, where $J = \sqrt{I}$ is the radical of I. Then consider powers $J \supset J^2 \supset \ldots$.]

3(a). Factor $X^5 + 7X^3 + 6X^2 + X + 5$ over the fields $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$.

[You may assume the (true) result that this polynomial has no irreducible quadratic factors over $\mathbb{F}_3$.]

(b). What are the Galois groups of $X^5 + 7X^3 + 6X^2 + X + 5$ over $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$?

(c). What are the Galois groups of $X^5 + 7X^3 + 6X^2 + X + 5$ over $\mathbb{Q}$?

4. Suppose V is a finite dimensional vector space over a field k and suppose $A : V \to V$ is a k-linear endomorphism whose minimal polynomial is *not* equal to its characteristic polynomial. Show that there exist k-linear endomorphisms $B, C : V \to V$ such that $AB = BA$, $AC = CA$, but $BC \neq CB$.

5(a). Produce a complex character table for the symmetric group S_4.

(b). The rotation group of the cube is isomorphic to S_4 as a permutation group of the four diagonals of the cube. Let $\rho : S_4 \to \mathrm{GL}(8, \mathbb{C})$ be the permutation representation of S_4 defined by the action of the rotation group on the eight vertices of the cube. Find the character of ρ.

(c). Decompose ρ as a direct sum of irreducible representations of S_4.

Stanford PhD Qualifying Exam in Algebra
Fall 2004 (Morning Session)

General Instructions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Classify all finite groups of order $147 = 3 \cdot 7^2$.

2. Let $A, B \in \mathrm{Mat}_n(\mathbb{R})$ be a pair of commuting matrices.

(a) Suppose that A and B are both nilpotent. Show that they have a nonzero common nullvector.

(b) Suppose that n is odd. Show that A and B have a common eigenvector. (It is no longer assumed that they are nilpotent.)

3. (a) Find the minimal polynomial of $\sqrt{4+\sqrt{7}}$ over $\mathbb{Q}$.

(b) Find the Galois group of that polynomial's splitting field over $\mathbb{Q}$.

Hint: Check that $\sqrt{4+\sqrt{7}} = \frac{1}{2}(\sqrt{2}+\sqrt{14})$.

4. Recall the following definitions: If R is a commutative ring and $\mathfrak{a}$ is an ideal, then the *radical* $r(\mathfrak{a}) = \{x \in R \,|\, x^n \in \mathfrak{a}\}$. An ideal $\mathfrak{q}$ is *primary* if $xy \in \mathfrak{q}$ implies that either $x \in \mathfrak{q}$ or $y \in r(\mathfrak{q})$. Prove that if $r(\mathfrak{q})$ is a maximal ideal, then $\mathfrak{q}$ is primary.

5. Let G be a nonabelian group of order pq where p and q are distinct primes such that $p < q$.

(a) Show that p divides $q-1$, and show that the number of conjugacy classes of G is exactly $p + \frac{q-1}{p}$.

(b) Determine the number and degrees of the irreducible complex characters of G.

Stanford PhD Qualifying Exam in Algebra
Fall 2004 (Afternoon Session)

General Instructions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Let $G = \langle x \rangle$ be a cyclic group of order 2^n, and let $R = \mathbb{F}_2[G]$ be the group algebra.

(a) Show that

$$J = \left\{ \sum_{i=0}^{2^n-1} a_i x^i \mid \sum a_i = 0 \right\}$$

is a nilpotent ideal in the commutative ring R, and deduce that

$$\Gamma = 1 + J = \left\{ \sum_{i=0}^{2^n-1} a_i x^i \mid \sum a_i = 1 \right\}$$

is an abelian group of order 2^{2^n-1}.

(b) Consider $\Gamma^{2^k} = \{u^{2^k} \mid u \in \Gamma\}$. Show that

$$|\Gamma^{2^k}| = \begin{cases} 2^{2^{n-k}-1} & \text{if } k \leqslant n; \\ 1 & \text{if } k \geqslant n. \end{cases}$$

(c) There is enough information in this fact to determine the structure of Γ. Illustrate this by determining the structure of Γ when $n = 4$.

2. Let $r > 0$, and let q be a prime power. If $a \in \mathbb{F}_{q^r}$ let $T(a) : \mathbb{F}_{q^r} \longrightarrow \mathbb{F}_{q^r}$ be the map $T(a)x = ax$. Regarding $\mathbb{F}_{q^r}$ as a r-dimensional vector space over $\mathbb{F}_q$, we may think of $T(a)$ as an element of $\mathrm{GL}(r, \mathbb{F}_q)$.

(a) Show that the composite $\det \circ T$ coincides with the norm map $\mathbb{F}_{q^r} \longrightarrow \mathbb{F}_q$.

(b) Show that if $b \in \mathbb{F}_q^\times$, then there exists $a \in \mathbb{F}_{q^r}^\times$ such that $\det\, T(a) = b$.

3. Let G be a finite group and H a subgroup. Let $\rho : H \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation. Show that if ρ_1 and ρ_2 are extensions of ρ to G, and if the characters χ_1 and χ_2 of ρ_1 and ρ_2 are the same, then $\rho_1(g) = \rho_2(g)$ for all $g \in G$.

4. Let A be a Noetherian local commutative ring with maximal ideal $\mathfrak{m}$. Assume that $\mathfrak{m}$ is principal. Show that every nonzero ideal of A is of the form $\mathfrak{m}^k$ for some k.

5. Let $\zeta = e^{2\pi i/40}$ and let $K = \mathbb{Q}(\zeta)$. (a) Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

(b) Find all quadratic extensions of $\mathbb{Q}$ contained in K. Express them in the form $\mathbb{Q}(\sqrt{D})$ for $D \in \mathbb{Z}$.

Stanford Math PhD Qualifying Exam, Part I
Spring, 2004

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Classify the finite groups of order $333 = 3^2 \cdot 37$.

2. (a) If $\mathbb{F}_q$ is the finite field with q elements, show that $X^{q^r} - X \in \mathbb{F}_q[X]$ is exactly the product of all irreducible polynomials $f(X) \in \mathbb{F}_q[X]$ whose degree divides r.

(b) Prove that the nmber of irreducible polynomials of degree r in $\mathbb{F}_q[X]$ is

$$\frac{1}{r}\sum_{d|r} \mu\left(\frac{r}{d}\right) q^d,$$

where μ is the Moebius function:

$$\mu(d) = \begin{cases} (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes;} \\ 0 & \text{otherwise}. \end{cases}$$

3. Let A be an integral domain with field of fractions F. Assume that for every prime ideal $\mathfrak{p} \subset A$ the localization $A_{\mathfrak{p}}$ is integrally closed (in F). Prove that A is integrally closed (in F).

4. Let A and B be nilpotent complex $n \times n$ matrices. Suppose that $\text{rank}(A^k) = \text{rank}(B^k)$ for all k. Prove that $A = MBM^{-1}$ for some $M \in \text{GL}(n, \mathbb{C})$.

5. Here is a partial character table of A_5.

	1	(123)	(12)(34)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3				
χ_5	3				

Complete this character table by constructing χ_4 and χ_5.

Stanford Math PhD Qualifying Exam, Part II
Spring, 2004

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. An abelian group G (written additively) is called *divisible* if the homomorphism $x \longmapsto nx = x + \ldots + x$ (n terms) is surjective for all $n \geqslant 1$. The abelian group G is called *injective* if whenever A and B are abelian groups with $A \subset B$, a homomorphism $\varphi : A \longrightarrow G$ can be extended to a homomorphism $\Phi : B \longrightarrow G$. Assume that G is divisible. Prove that G is injective. [**Hint:** Use Zorn's Lemma.]

2. Show that if G is a finite abelian group, then there exists a finite extension F of $\mathbb{Q}$ such that $\mathrm{Gal}(F/\mathbb{Q}) \cong G$. [**Hint:** Think about roots of unity.]

3. Suppose that A is a commutative Noetherian ring.

(a) Prove that every ideal $I \subset A$ contains a finite product of prime ideals.

(b) Prove that A has only finitely many minimal prime ideals. [**Hint:** Think about the zero ideal.]

(c) Prove that if A has no nilpotent elements then the set of zero divisors in A is exactly the union of the minimal prime ideals of A.

4. Let (π, V) be a nontrivial irreducible complex representation of the finite group G with character χ. Suppose that $1 \neq g \in G$ is such that $|\chi(g)| = \chi(1)$. Show that $\pi(g)$ is a scalar endomorphism of V and deduce that G is not a nonabelian simple group.

5. Determine the number of conjugacy classes of elements of orders 3, 5 and 11 in $\mathrm{GL}(2, \mathbb{F}_{11})$.

Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2005
Morning Session

1. Let p and q be primes with $p, q \neq 2$ and suppose that p divides $q+1$.

(a) Show that there exists a nonabelian group G of order pq^2 whose Sylow q-subgroup is not cyclic.

(b) Show that if G is a nonabelian group of order pq^2 then it has a normal q-Sylow subgroup Q, and if Q is not cyclic then a p-Sylow subgroup of $\mathrm{Aut}(Q)$ is cyclic.

(c) Show that any two nonabelian groups of order pq^2 with noncyclic Sylow q-subgroups are isomorphic.

2. Suppose that $f(t) \in \mathbb{Q}[t]$ is an irreducible polynomial of degree 5 with exactly 3 real roots. Let K be the splitting field of f over $\mathbb{Q}$. Show that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_5$. Prove any nonobvious facts you use about S_5.

3. Let A be a commutative ring. The ring A is called *Artinian* if it satisfies the *decreasing chain condition:* if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a sequence of ideals then for some N we have $I_N = I_{N+1} = I_{N+2} = \dots$.

(a) If A is an Artinian integral domain show that A is a field.

(b) If A is an Artinian commutative ring, show that every prime ideal in A is maximal.

4. Let G be a group of odd order. Prove that if χ is a complex irreducible character of G and $\chi(g)$ is real for all $g \in G$ then $\chi = 1$. (**Hint:** Consider the value of $\sum_{g \neq 1} \chi(g)$ and the fact that $g \neq g^{-1}$ when $g \neq 1$. Think about algebraic integers.)

5. Let T, $U \in \mathrm{Mat}_n(F)$ where F is any field. Prove that if T and U are nilpotent matrices and $\mathrm{rank}(T^k) = \mathrm{rank}(U^k)$ for all k, then $T = AUA^{-1}$ for some $A \in \mathrm{Mat}_n(F)$.

Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2005
Afternoon Session

1. Let G be the group of order 18 with generators x, y, z subject to relations

$$x^3 = y^3 = z^2 = 1, \qquad xy = yx, \qquad zxz^{-1} = y, \qquad zyz^{-1} = x.$$

Determine the conjugacy classes of G and compute its character table.

2. Let p be an odd prime and $\zeta = e^{2\pi i/p}$. Show that there exists a unique subfield K of $\mathbb{Q}(\zeta)$ such that $[K\colon \mathbb{Q}] = 2$. Let $\chi\colon (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \{\pm 1\}$ be the unique nontrivial homomorphism and let

$$\alpha = \sum_{a=1}^{p-1} \chi(a)\,\zeta^a.$$

Show that $\alpha^2 = (-1)^{(p-1)/2} p$ and conclude that $K = \mathbb{Q}(\alpha)$.

3. Let A be an Noetherian integral domain with field of fractions F. If $f \in A$ is not a unit, prove that the ring $A[f^{-1}]$ generated by f^{-1} and A is not a finitely-generated A-module.

4. Let G be a finite group of odd order. Prove that if $g \in G$ is conjugate to g^{-1} then $g = 1$.

5. Let $n > 1$ be odd. Let A and B be matrices in $\mathrm{GL}_2(\mathbb{C})$ such that $A^n = 1$, $BAB^{-1} = A^{-1}$ and $A \neq I$. Suppose that X commutes with both A and B. Prove that X is a scalar matrix.

Stanford Mathematics PhD Qualifying Exam
Algebra – Spring 2005
Morning Session

1. Suppose p and q are odd primes and $p < q$. Let G be a finite group of order $p^3 q$.

(a) Prove that G has a normal Sylow subgroup.

(b) Let n_p and n_q denote the number of p-Sylow and q-Sylow subgroups of G. Determine, with proof, all ordered pairs (n_p, n_q) that are possible for groups of order $p^3 q$.

2. Let $f(X) \in \mathbb{Q}[X]$ be a monic irreducible polynomial of degree 4 with roots $\alpha, \beta, \gamma, \delta$.

The *discriminant* of a polynomial with roots $r_1, \cdots, r_n$ is $\prod_{i<j} (r_i - r_j)^2$.

(a) Prove that $\lambda = \alpha\beta + \gamma\delta$ is the root of a monic cubic polynomial $g(X) \in \mathbb{Q}[X]$ whose discriminant is the same as the discriminant of f.

(b) If $f \in \mathbb{Z}[X]$ prove that $g \in \mathbb{Z}[X]$.

3. Let M be a finitely-generated module over the Noetherian commutative ring R. Prove that if $f: M \longrightarrow M$ is an R-module homomorphism, and if f is surjective, then f is also injective. *Hint*: consider the submodules $\ker(f^n)$.

4. Let G be the nonabelian group of order 16 with generators x and y subject to the relations

$$x^8 = y^2 = 1, \quad yxy^{-1} = x^3.$$

Determine the conjugacy classes of G and compute its character table.

5. If B is a positive-definite symmetric real matrix, show that there exists a unique positive-definite symmetric real matrix C such that $C^2 = B$.

Stanford Mathematics PhD Qualifying Exam

Algebra – Spring 2005

Afternoon Session

1. Suppose that $A \subset B$ is an integral extension of commutative rings with unit.

(a) If $\mathfrak{q}$ is a maximal ideal of B, prove that $\mathfrak{p} = \mathfrak{q} \cap A$ is a maximal ideal of A.

(b) Outline the proof that for any prime ideal $\mathfrak{p} \subset A$ there exists a prime ideal $\mathfrak{q}$ of B with $\mathfrak{p} = \mathfrak{q} \cap A$.

2. Let $F = \mathbb{Z}/2\mathbb{Z}$, and let $F[X, Y]$ be the polynomial ring in two variables. Let I be the ideal generated by $X^5 + X^3 + X$ and $Y^3 + (X^3 + 1)Y + 1$, and let R be the quotient ring $F[X, Y]/I$. Determine the number of maximal ideals in the ring R.

Hint: if $a \in \mathbb{F}_4$, what is a^3?

3. If G is a permutation group acting on a set S we say G is *n-transitive* if $|S| \geqslant n$ and whenever $x_1, \cdots, x_n$ are distinct elements of S and $y_1, \cdots, y_n$ are distinct elements of S there exists $g \in G$ such that $g(x_i) = y_i$. We will denote by $\chi(g)$ the number of fixed points of g. Prove that a necessary and sufficient condition for G to be 3-transitive is that

$$\frac{1}{|G|} \sum \chi(g)^3 = 5.$$

4. Suppose that A is an $n \times n$ matrix over $\mathbb{C}$ with minimal polynomial $(X - \lambda)^n$ where $\lambda \neq 0$. Find the Jordan form of A^2. What if $\lambda = 0$?

5. Find the Galois group of the polynomial $X^5 + 99X - 1$ over the fields $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Q}$.

Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2006
Morning Session

1. In parts (a) and (c), let G be a nonabelian group of order 56.

(a) Prove that G has a normal 2-Sylow subgroup or a normal 7-Sylow subgroup.

(b) Let Z_n denote a cyclic group of order n. Compute the order of $\mathrm{Aut}(Q)$ when $Q = Z_8$, $Z_4 \times Z_2$ and $Z_2 \times Z_2 \times Z_2$.

(c) How many isomorphism classes are there of nonabelian groups of order 56 with normal abelian 2-Sylow subgroup? Explain. (**Hint:** use part (b).)

2. Let G be the following group of order 42.

$$G = \langle x, y \,|\, x^7 = y^6 = 1,\, yxy^{-1} = x^2 \rangle.$$

Determine the conjugacy classes of G and the degrees of its irreducible characters. Compute the values of at least one irreducible character of degree > 1.

3. Find an extension E of $\mathbb{Q}$ with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$.

4. Suppose J is an $n \times n$ matrix over an algebraically closed field of characteristic $\neq 3$ and minimal polynomial $(T - \lambda)^n$ where $\lambda \neq 0$. Find the Jordan canonical form of J^3.

5. In this exercise, "commutative ring" means "commutative ring with unit," and it is assumed that a ring homomorphism $f: A \longrightarrow B$ satisfies $f(1_A) = 1_B$. If A, B and C are commutative rings, we say that C is a *coproduct* of A and B if there exist ring homomorphisms $\alpha: A \longrightarrow C$ and $\beta: B \longrightarrow C$ such that if D is any commutative ring, and $f: A \longrightarrow D$ and $g: B \longrightarrow D$ are ring homomorphisms, there exists a unique ring homomorphism $\phi: C \longrightarrow D$ such that $f = \phi \circ \alpha$ and $g = \phi \circ \beta$.

(a) If A and B are commutative rings, regard them as $\mathbb{Z}$-modules, and let $A \otimes B = A \otimes_{\mathbb{Z}} B$. Explain briefly why $A \otimes_{\mathbb{Z}} B$ naturally has the structure of a commutative ring.

(b) Prove that C is a coproduct of A and B if and only if $C \cong A \otimes B$.

Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2006
Afternoon Session

1. Let G be a p-group and H a nontrivial normal subgroup. Show that $H \cap Z(G)$ has at least p elements.

2. Let $A \subset B$ be finite abelian groups, and let $\chi\colon A \longrightarrow \mathbb{C}^\times$ be a homomorphism (linear character). Show that χ can be extended to B, and that the number of such extensions equals $[B\colon A]$.

3. Let $q = p^n$ where p is a prime, and let $\mathbb{F}_q$ denote the finite field with q elements. Show that the Frobenius automorphism $\sigma\colon \mathbb{F}_q \longrightarrow \mathbb{F}_q$ defined by $\sigma(x) = x^p$ is diagonalizable as a linear transformation *over* $\mathbb{F}_p$ if and only if n divides $p-1$.

4. Determine the splitting field K and Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of the polynomial $x^4 - 2$ over the field $\mathbb{Q}$. Find all quadratic extensions of $\mathbb{Q}$ contained in K.

5. Let $F \subset K$ be fields. Let R be the polynomial ring $F[X]$, where X is an indeterminate, and similarly let $S = K[X]$.

(a) Show that if f and g are monic polynomials in S, and $S/fS \cong S/gS$ as S-modules, then $f = g$.

(b) Show that if $x \in R$ then $S \otimes_R (R/xR) \cong S/xS$.

(c) Suppose that M, N be finitely generated R-modules. Show that if $S \otimes_R M \cong S \otimes_R N$ as S-modules then $M \cong N$ as R-modules.

Stanford Mathematics PhD Qualifying Exam

Algebra – Spring 2006

Morning Session

1. Suppose that H is a subgroup of a group G of index n. Show that G has a normal subgroup of index $\leqslant n!$. Use this to prove that there is no finite simple group of order $2430 = 2 \cdot 3^5 \cdot 5$.

2. Let G be the following group of order 28.

$$G = \langle x, y | x^7 = y^4 = 1, yxy^{-1} = x^{-1} \rangle.$$

Determine the conjugacy classes of G and compute its character table.

3. (i) Suppose that $d > 1$ is a square-free integer and $d \equiv 1$ modulo 4. Determine (with proof) the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$.

(ii) Explain how the principal ideals (2), (3) and (13) factor into prime ideals in the ring of algebraic integers in $\mathbb{Q}(\sqrt{13})$.

4. Find the Galois group of $x^4 + 1$ over $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_7$.

5. How many similarity classes are there of rational matrices with characteristic polynomial $(x^3 + 1)^3(x^2 + 1)^3$ and a minimal polynomial of degree 10?

Stanford Mathematics PhD Qualifying Exam
Algebra – Spring 2006
Afternoon Session

1. Let V be a finite-dimensional vector space over a field F of characteristic p and let T: $V \longrightarrow V$ be a linear transformation such that $T^p = I$ is the identity map.

(i) Show that T has an eigenvector in V.

(ii) Show that T is upper triangular with respect to a suitable basis of V.

2. Let G be a finite group, and let H be a subgroup of index two. Let $x \in H$ and let $C_G(x)$ be the centralizer of x in G.

(i) Prove that if $C_G(x) \not\subset H$ then the conjugacy class of x in G agrees equals the H-conjugacy class of x; on the other hand, if $C_G(x) \subset H$ then the conjugacy class of x in G is contained in H but splits into two H-conjugacy classes.

(ii) Let G be the symmetric group S_9 and H be A_9. Determine the G-conjugacy classes of even permutations that split into two conjugacy classes in H. **Hint:** there are two.

3. Let A be a Noetherian commutative ring containing a field k and an ideal I such that if $J = \sqrt{I}$ is the radical of I then A/J is a finite-dimensional k-vector space. Prove that A/I is also a finite-dimensional k-vector space.

4. Let G be a finite p-group, and $\lambda: G \longrightarrow \mathbb{C}^\times$ a homomorphism. Assume that the order of λ is a prime p, so that $H = \ker(\lambda)$ is a subgroup of index p. Let θ be an irreducible character of G such that $\lambda\theta = \theta$. Show that $\langle \theta, \theta \rangle_H = p$ and deduce that θ is induced from a character of H.

5. Let $\zeta = e^{2\pi i/7}$. Find an element α of $\mathbb{Q}(\zeta)$ such that $[\mathbb{Q}(\alpha): \mathbb{Q}] = 3$. Show that there does not exist $\beta \in \mathbb{Q}(\alpha)$, $\beta \notin \mathbb{Q}$ such that $\beta^3 \in \mathbb{Q}$.

Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2007
Morning Session

1. Let G be a group and X and Y two sets with (left) actions of G. We say the actions are *equivalent* if there is a bijection $\phi\colon X \longrightarrow Y$ such that $\phi(g \cdot x) = g \cdot \phi(x)$. Fix elements $x_0 \in X$ and $y_0 \in Y$. Let $H = \{g \in G \,|\, g \cdot x_0 = x_0\}$ and $K = \{g \in G \,|\, g \cdot y_0 = y_0\}$ denote the isotropy subgroups of x_0 and y_0, respectively.

(a) If the actions on X and Y are transitive, show the actions are equivalent if and only if H and K are conjugate.

(b) Let $F = \mathbb{F}_p$ where p is prime, and let $G = \mathrm{GL}(2, \mathbb{F}_p)$. Here are two sets X and Y with actions of G. The set X is the projective line, consisting of all one-dimensional subspaces of the two-dimensional vector space $\mathbb{F}_p^2$; and Y is the set of p-Sylow subgroups of G. Here the action of G on X is by matrix multiplication, and the action of G on Y is by conjugation. Show that these two actions are equivalent. [**Hint:** Let $V = F\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in X$. What is the isotropy subgroup of V?]

2. Show that there is a nonabelian group G of order $3 \cdot 13 = 39$. Describe G by generators and relations, find the conjugacy classes and construct the character table.

3. Suppose that A is a commutative Noetherian ring.

(a) Prove that every ideal $\mathfrak{a}$ of A contains a finite product $\mathfrak{p}_1 \cdots \mathfrak{p}_r$ of prime ideals.

(b) Prove that A has only finitely many minimal prime ideals, and that every prime ideal of A contains at least one of these.

4. Let $F = \mathbb{F}_p$ where p is prime. How many irreducible polynomials over F are there of degrees $2, 3$ and 6?

5. Consider the field of rational functions $E = \mathbb{Q}(s, t, u, v)$ in four variables. Let $F \subset E$ denote the fixed field of the obvious action of S_4 on E permuting $\{s, t, u, v\}$. Show that $w = st + uv \in E$ has degree 3 over F. What is the Galois group $\mathrm{Gal}(E/F(w))$?

Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2007
Afternoon Session

1. Classify all groups of order 225 up to isomorphism.

2. Let G be a finite group with center $Z(G)$. Show that the number of irreducible complex representations of G is at least $|Z(G)|$. (**Hint:** first prove that if θ: $Z(G) \longrightarrow \mathbb{C}^\times$ there is an irreducible complex representation π: $G \longrightarrow \mathrm{GL}(V)$ such that $\pi(zg) = \theta(z)\pi(g)$ for $z \in Z(G)$.)

3. (a) Suppose that g is a complex matrix such that $\mathbb{C}^n$ has a basis of eigenvectors $v_1, \cdots, v_n$ such that $v_i \in \mathbb{R}^n$ and the v_i are orthogonal with respect to the usual dot product, which is the symmetric bilinear form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i.$$

Prove that there exists a real orthogonal matrix $k = (k^t)^{-1}$ such that $kgk^{-1} = kgk^t$ is diagonal.

(b) Let $g \in \mathrm{GL}(n, \mathbb{C})$ be a unitary symmetric matrix; that is, $g = g^t$ and $g \cdot \bar{g}^t = I$. Prove that there exists a real orthogonal matrix $k = (k^t)^{-1}$ such that $kgk^{-1} = kgk^t$ is diagonal. (**Hint:** Let λ be an eigenvalue of g. Observe that $|\lambda| = 1$ and show that $V_\lambda = \{v \in \mathbb{C}^n | gv = \lambda g\}$ is stable under complex conjugation.)

4. (a) Suppose k is an infinite field, $k(\alpha, \beta)$ an algebraic extension of k such that β is separable over k (α is not assumed separable). Let $f(x)$ and $g(x)$ denote the minimal polynomials of α and β respectively over k. Show that there exists a $c \in k$ such that if $\theta = \alpha + c\beta$ then the polynomials $g(x)$ and $h(x) = f(\theta - cx)$ have exactly one root in common in an algebraic closure of k, namely $x = \beta$.

(b) Deduce that $k(\alpha, \beta) = k(\theta)$. [**Hint:** What is the greatest common divisor of the polynomials $g(x)$ and $h(x)$ in the polynomial ring $k(\theta)[x]$?]

5. Let A be an integral domain with field of fractions K.

(a) If $\mathfrak{m}$ is a maximal ideal of A then the localization $A_\mathfrak{m}$ can be regarded as a subring of K. Prove that

$$A = \bigcap_{\text{maximal } \mathfrak{m}} A_\mathfrak{m}.$$

(b) Show that if $A_\mathfrak{m}$ is a unique factorization domain for all maximal ideals $\mathfrak{m}$ then A is integrally closed in K.

Stanford Mathematics PhD Qualifying Exam

Algebra – Spring 2007

Morning Session

1. Let V be a finite-dimensional vector space over a field F, and let S be a set of commuting linear transformations of V. Assume that for every $T \in S$ the characteristic polynomial of T factors into linear factors over F, not necessarily distinct. Prove that V has a basis with respect to which every $T \in S$ is represented by an upper-triangular matrix.

2. Let K be a field, p a prime. Let $\mu \subset K$ be the group of p-th roots of unity in K. Assume that $|\mu| = p$.

(i) Prove that the characteristic of K is not equal to p.

(ii) Let $\sigma: K \longrightarrow K$ be an automorphism of order p, and let F be the fixed field of σ. Show that $\mu \subset F$ and deduce that $K = F(\alpha)$ for some α such that $\alpha^p \in F$. State any theorems you quote.

3. Let A be a commutative ring and I an ideal. Recall that the *radical* $\sqrt{I}$ is the ideal $\{x \in A | x^n \in I \text{ for some } n\}$, and that I is called *primary* if whenever $ab \in I$ we have either $a \in I$ or $b \in \sqrt{I}$.

(a) Let A be a commutative ring, and let $I \in A$ be an ideal such that the radical $\sqrt{I} =$ is maximal. Prove that I is primary.

(b) Show that if $A = \mathbb{Q}[x, y]$ is a polynomial ring in two variables and $I = (x^3, x^2 + xy)$, then $\sqrt{I}$ is prime, but I is not primary.

4. Classify the finite groups of order $2007 = 3^2 \cdot 223$. (**Hint:** 223 is prime; $222 = 2 \cdot 3 \cdot 37$.)

5. Let G be the finite group of order 16 with generators and relations

$$G = \left\langle x, y | x^8 = y^2 = 1, yxy^{-1} = x^3 \right\rangle.$$

Find the conjugacy classes and construct the character table of G.

Stanford Mathematics PhD Qualifying Exam
Algebra – Spring 2007
Afternoon Session

1. Let G be a nonabelian group of order $117 = 3^2 \cdot 13$. Show that G has a normal cyclic subgroup of index 3. Find the degrees of the irreducible representations, and deduce number of conjugacy classes. (**Hint:** there are two possible G but the answer is the same for both. You can do both cases simultaneously.)

2. Let G be a finite group of odd order, and let p be the smallest prime dividing $|G|$. Suppose that G has a normal p-Sylow subgroup P of order $\leqslant p^2$. Prove that P is contained in the center of G.

3. Suppose that $A \subset B$ are integral domains so that the field of fractions of B is algebraic over the field of fractions of A.

(i) If $Q \subset B$ is a nonzero prime ideal, prove that $A \cap Q \neq 0$.

(ii) Assume that A is a principal ideal domain. Prove that every non-zero prime ideal of B is maximal.

4. Let F be a finite field with q elements, and let $W = F^6$. Count the number of pairs (U, V) where $W \supset V \supset U$, with U a 2-dimensional subspace and V a 4-dimensional subspace.

5. Let p be a prime, $a \in \mathbb{Q}$.

(a) Prove that either $x^p - a$ is irreducible, or it has a root in $\mathbb{Q}$. [**Hint:** what is the factorization of $x^p - a$ over $\mathbb{C}$?]

(b) Show that the splitting field of $x^p - a$ over $\mathbb{Q}$ contains no primitive p^2 root of 1.

Stanford Mathematics PhD Qualifying Exam

Algebra – Fall 2008

Morning Session

1. Let $G = \mathrm{SL}_2(\mathbb{F}_q)$ where q is an odd prime power. If ℓ is a prime dividing either $q-1$ or $q+1$ prove that the ℓ-Sylow subgroup of G is cyclic.

2. Let F be a field and let K be an algebraic closure of F. Let $G = \mathrm{Gal}(K/F)$ denote the group, possibly infinite, of automorphisms of K that are trivial on F. Use the Nullstellensatz, stating the version that you use, to exhibit a bijection between the orbits of G on K^n and maximal ideals in the polynomial ring $F[x_1, \cdots, x_n]$. (Prove your answer.)

3. Let A be a commutative ring with unit such that (i) for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_\mathfrak{m}$ is Noetherian and (ii) for every $0 \neq x \in A$ the set of maximal ideals of A which contain x is finite. Show that A is Noetherian.

4. Let V be and n-dimensional vector space over $\mathbb{F}_q$. For $1 \leqslant j \leqslant n$ determine the number of j-dimensional vector subspaces of V.

5. Let G be a cyclic group of order p^n where p is prime. Let V be a finite-dimensional vector space of dimension n over $\mathbb{F}_p$. We say that a representation $\sigma : G \longrightarrow \mathrm{GL}(W)$ is *indecomposable* if it is not a direct sum of nontrivial invariant subspaces. Show that if $\pi : G \longrightarrow \mathrm{GL}(V)$ is any representation on a finite dimensional vector space over $\mathbb{F}_p$ then V is a direct sum of invariant subspaces W_i such that the restriction of π to W_i is an indecomposable representation. Show that the number and dimensions of the W_i are determined by π. Make explicit what an indecomposable W_i looks like. [*Hint:* this is essentially a question of linear algebra. If x is a generator of G, consider $\pi(x)$ as a linear transformation.]

Stanford Mathematics PhD Qualifying Exam

Algebra – Fall 2008

Afternoon Session

1. Let G be a group of order p^r where p is prime and $r \geqslant 3$. Show that p divides $|\operatorname{Aut}(G)|$ and that $|\operatorname{Aut}(G)| \geqslant p^2$.

2. (a) Find (with proof) the Galois group of $x^9 - 2$ over $\mathbb{Q}$.

(b) Find (with proof) the Galois group of $x^9 - 2$ over $\mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_7$.

3. Find all algebraic integers in the field $\mathbb{Q}[\sqrt{10}]$. For which primes p in $\mathbb{Z}$ does p generate a prime ideal in this ring of integers? (Prove your answer.)

4. Let F be a field and $E \supset F$ an extension field.

(i) If $M, N \in \mathrm{GL}_n(F)$ are conjugate in $\mathrm{GL}_n(E)$ show that they are conjugate in $\mathrm{GL}_n(F)$.

(ii) Give an example of two matrices in $\mathrm{SL}_2(\mathbb{R})$ that are conjugate in $\mathrm{SL}_2(\mathbb{C})$ but not in $\mathrm{SL}_2(\mathbb{R})$.

5. Let G be the group of order 24 with the following generators and relations:

$$\left\langle x^2 = y^2 = (xy)^2 = 1, z^6 = 1, zxz^{-1} = y, zyz^{-1} = xy \right\rangle.$$

Find the eight conjugacy classes of G and compute its character table.

Stanford Mathematics PhD Qualifying Exam

Algebra – Fall 2008

Morning Session

1. Let $G = \mathrm{SL}_2(\mathbb{F}_q)$ where q is an odd prime power. If ℓ is a prime dividing either $q-1$ or $q+1$ prove that the ℓ-Sylow subgroup of G is cyclic.

2. Let F be a field and let K be an algebraic closure of F. Let $G = \mathrm{Gal}(K/F)$ denote the group, possibly infinite, of automorphisms of K that are trivial on F. Use the Nullstellensatz, stating the version that you use, to exhibit a bijection between the orbits of G on K^n and maximal ideals in the polynomial ring $F[x_1, \cdots, x_n]$. (Prove your answer.)

3. Let A be a commutative ring with unit such that (i) for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is Noetherian and (ii) for every $0 \neq x \in A$ the set of maximal ideals of A which contain x is finite. Show that A is Noetherian.

4. Let V be and n-dimensional vector space over $\mathbb{F}_q$. For $1 \leqslant j \leqslant n$ determine the number of j-dimensional vector subspaces of V.

5. Let G be a cyclic group of order p^n where p is prime. Let V be a finite-dimensional vector space of dimension n over $\mathbb{F}_p$. We say that a representation $\sigma : G \longrightarrow \mathrm{GL}(W)$ is *indecomposable* if it is not a direct sum of nontrivial invariant subspaces. Show that if $\pi : G \longrightarrow \mathrm{GL}(V)$ is any representation on a finite dimensional vector space over $\mathbb{F}_p$ then V is a direct sum of invariant subspaces W_i such that the restriction of π to W_i is an indecomposable representation. Show that the number and dimensions of the W_i are determined by π. Make explicit what an indecomposable W_i looks like. [*Hint:* this is essentially a question of linear algebra. If x is a generator of G, consider $\pi(x)$ as a linear transformation.]

Stanford Mathematics PhD Qualifying Exam

Algebra – Fall 2008

Afternoon Session

1. Let G be a group of order p^r where p is prime and $r \geqslant 3$. Show that p divides $|\operatorname{Aut}(G)|$ and that $|\operatorname{Aut}(G)| \geqslant p^2$.

2. (a) Find (with proof) the Galois group of $x^9 - 2$ over $\mathbb{Q}$.

(b) Find (with proof) the Galois group of $x^9 - 2$ over $\mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_7$.

3. Find all algebraic integers in the field $\mathbb{Q}[\sqrt{10}]$. For which primes p in $\mathbb{Z}$ does p generate a prime ideal in this ring of integers? (Prove your answer.)

4. Let F be a field and $E \supset F$ an extension field.

(i) If $M, N \in \mathrm{GL}_n(F)$ are conjugate in $\mathrm{GL}_n(E)$ show that they are conjugate in $\mathrm{GL}_n(F)$.

(ii) Give an example of two matrices in $\mathrm{SL}_2(\mathbb{R})$ that are conjugate in $\mathrm{SL}_2(\mathbb{C})$ but not in $\mathrm{SL}_2(\mathbb{R})$.

5. Let G be the group of order 24 with the following generators and relations:

$$\left\langle x^2 = y^2 = (xy)^2 = 1, z^6 = 1, zxz^{-1} = y, zyz^{-1} = xy \right\rangle .$$

Find the eight conjugacy classes of G and compute its character table.

ALGEBRA QUALIFYING EXAM, FALL 2009, PART I

1. Let k be a finite field of size q.

(a) Prove that the number of 2×2 matrices over k satisfying $T^2 = 0$ is q^2.

(b) Prove that the number of 3×3 matrices over k satisfying $T^3 = 0$ is q^6.

2. (a) Prove that if K is a field of finite degree over $\mathbb{Q}$ and $x_1, \ldots, x_n$ are finitely many elements of K then the subring $\mathbb{Z}[x_1, ..., x_n]$ they generate over $\mathbb{Z}$ is not equal to K. (*Hint:* Show they all lie in $\mathcal{O}_K[1/a]$ for a suitable nonzero a in $\mathcal{O}_K$, where $\mathcal{O}_K$ denotes the integral closure of $\mathbb{Z}$ in K.)

(b) Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{Z}[x_1, ...x_n]$ and $F = \mathbb{Z}[x_1, ..., x_n]/\mathfrak{m}$. Use (a) and the Nullstellensatz to show that F cannot have characteristic 0, and then deduce for $p = \text{char}(F)$ that F is of finite degree over $\mathbb{F}_p$ (so F is actually finite).

3. Let E be the splitting field of

$$f(x) = (x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x^1 + 1$$

over $\mathbb{Q}$. Let ζ be a zero of $f(x)$, i.e. a primitive seventh root of 1.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$. (*Hint:* consider $f(y + 1)$ and use Eisenstein's criterion.)
(b) Show that the Galois group of $E/\mathbb{Q}$ is cyclic, and find an explicit generator.
(c) Let $\beta = \zeta + \zeta^2 + \zeta^4$. Show that the intermediate field $\mathbb{Q}(\beta)$ is actually $\mathbb{Q}(\sqrt{-7})$. (*Hint:* first show that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$ by finding a linear dependence over $\mathbb{Q}$ among $\{1, \beta, \beta^2\}$.)
(d) Let $\gamma_q = \zeta + \zeta^q$. Find (with proof) a q such that $\mathbb{Q}(\gamma_q)$ is a degree 3 extension of $\mathbb{Q}$. (*Hint:* use (b).) Is this extension Galois?

4. Let G be a nontrivial finite group and p be the smallest prime dividing the order of G. Let H be a subgroup of index p. Show that H is normal. (*Hint:* If H isn't normal, consider the action of G on the conjugates of H.)

5. Let G be a finite group and $\pi : G \to GL(V)$ a finite-dimensional complex representation. Let χ be the character of π. Show that the characters of the representations on $V \otimes V$, $\text{Sym}^2(V)$ and $\wedge^2(V)$ are $\chi(g)^2$, $(\chi(g)^2 + \chi(g^2))/2$ and $(\chi(g)^2 - \chi(g^2))/2$. (*Hint:* Express $\chi(g)^2$, $(\chi(g)^2 + \chi(g^2))/2$ and $(\chi(g)^2 - \chi(g^2))/2$ in terms of the eigenvalues of $\pi(g)$.)

Date: Thursday, September 17, 2009.

ALGEBRA QUALIFYING EXAM, FALL 2009, PART II

This part has five problems on two pages.

1. Let V be a vector space over a field F, and let $B : V \times V \longrightarrow F$ be a symmetric bilinear form. This means that B is bilinear and $B(x, y) = B(y, x)$. Let $q(v) = B(v, v)$.

(a) Show that if the characteristic of F is not 2 then $B(v, w) = \frac{1}{2}(q(v+w) - q(v) - q(w))$. (This obviously implies that if $q = 0$ then $B = 0$.)

(b) Give an example where the characteristic of F is 2 and $q = 0$ but $B \neq 0$.

(c) Show that if the characteristic of F is not 2 or 3 and if $B(u, v, w)$ is a symmetric trilinear form, and if $r(v) = B(v, v, v)$, then $r = 0$ implies $B = 0$.

2. Let G be a finite group.

(a) Let $\pi : G \to GL(V)$ be an irreducible complex representation, and let χ be its character. If $g \in G$, show that $|\chi(g)| = \dim(V)$ if and only if there is a scalar $c \in \mathbb{C}$ such that $\pi(g)v = cv$ for all $v \in V$.

(b) Show that g is in the center $Z(G)$ if and only if $|\chi(g)| = \chi(1)$ for every irreducible character χ of G.

3. Let V be a vector space of finite dimension $d \geq 1$ over a field k of arbitrary characteristic. Let V^* denote the dual space.

(a) For any $n \geq 1$, prove that there is a unique bilinear pairing $V^{\otimes n} \times (V^*)^{\otimes n} \to k$ satisfying

$$(v_1 \otimes \cdots \otimes v_n, \ell_1 \otimes \cdots \otimes \ell_n) \mapsto \prod \ell_i(v_i),$$

and by using bases show that it is a perfect pairing (i.e., identifies $(V^*)^{\otimes n}$ with $(V^{\otimes n})^*$).

(b) For any $1 \leq n \leq d$, do similarly with $\wedge^n(V)$ and $\wedge^n(V^*)$ using the requirement

$$(v_1 \wedge \cdots \wedge v_n, \ell_1 \wedge \cdots \wedge \ell_n) \mapsto \det(\ell_i(v_j)).$$

4. Let K/k be a finite extension of fields with $\alpha \in K$ as a primitive element over k. Let $f \in k[x]$ be the minimal polynomial of α over k.

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(a) Explain why $K \simeq k[x]/(f)$ as k-algebras, and use this to relate the local factor rings of $K \otimes_k F$ to the irreducible factors of f in $F[x]$, with F/k a field extension.

(b) Assume K/k is Galois with Galois group G. Prove that the natural map $K \otimes_k K \to \prod_{g \in G} K$ defined by $a \otimes b \mapsto (g(a)b)$ is an isomorphism.

5. Let G be a finite abelian group, $\omega : G \times G \to \mathbb{R}/\mathbb{Z}$ a bilinear mapping so that

(i) $\omega(g, g) = 0$ for all g in G;
(ii) $\omega(x, g) = 0$ for all g if and only if x is the identity element.

Prove that the order of G is a square. Give an example of G of square order for which no such ω exists.

Hint: Consider a subgroup A of G which is maximal for the property that $\omega(x, y) = 0$ for x, y in A. You may use the following fact without proof: any finite abelian group X admits $|X|$ distinct homomorphisms to $\mathbb{R}/\mathbb{Z}$.

ALGEBRA QUAL, SPRING 2009, PART I

1. (a) [5 points] Prove that if A is a commutative noetherian ring then the polynomial ring $A[T]$ is noetherian. (That is, prove the 'Hilbert Basis Theorem'.)

(b) [3 points] Suppose k is a field and B is a commutative ring finitely generated over k. Let $S \subset B$ be a multiplicative set. Explain why the localization $S^{-1}B$ is a noetherian ring.

(c) [2 points] Give an example (with proof) of the situation in part (b) where $S^{-1}B$ is *not* finitely generated over k as a ring.

2. Let k be a field, $f(x) \in k[x]$ a monic, non-constant polynomial.

(a) [2 points] Define what it means for a field $K \supset k$ to be a *splitting field* of $f(x)$ over k.

(b) [8 points] Prove the existence of such a splitting field K, and the uniqueness of K up to isomorphism over k.

3. Let K be a splitting field for $x^4 - 7$ over $\mathbb{Q}$.

(a) [5 points] Determine $|K : \mathbb{Q}|$ and give field generators for K over $\mathbb{Q}$. Describe $G = Gal(K/\mathbb{Q})$ in terms of generators and relations, and describe how the group generators of G act on the field generators of K.

(d) [5 points] List all intermediate fields L with $\mathbb{Q} \subsetneq L \subsetneq K$, their degrees over $\mathbb{Q}$, and the inclusion relations that hold between the fields L. The fields L should be named in terms of generators over $\mathbb{Q}$.

4. Consider the finite groups $SL(2, \mathbb{F}_5)$ and $PSL(2, \mathbb{F}_5) = SL(2, \mathbb{F}_5)/\{\pm Id\}$. In the following problem, you are not allowed to use the fact that $PSL(2, \mathbb{F}_5)$ is isomorphic to a more familiar group, unless you give a complete proof of that fact.

(a) [2 points] Calculate $|SL(2, \mathbb{F}_5)|$. Explain why $SL(2, \mathbb{F}_5) \not\cong S_5$, the symmetric group.

(b) [3 points] Show that there are no elements of order 15 in $PSL(2, \mathbb{F}_5)$.

[Hint: Work in $SL(2, \mathbb{F}_5)$.]

(c) [5 points] Exhibit a 3-Sylow subgroup and a 5-Sylow subgroup of $PSL(2, \mathbb{F}_5)$ and determine (with proof) the number of distinct 3-Sylow subgroups and 5-Sylow subgroups.

[Hint: Part (b) is useful for (c), but there are various approaches.]

5. Let $\mathbb{Q}[x_1, \cdots, x_k]$ be the polynomial ring in k variables over $\mathbb{Q}$, and let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, say inside $\mathbb{C}$. A special case of the *weak Nullstellensatz* states that if $I \subset \mathbb{Q}[x_1, \cdots, x_k]$ is any proper ideal, then

$$V(I) = \{\vec{\gamma} = (\gamma_1, \cdots, \gamma_k) \in (\overline{\mathbb{Q}})^k \mid f(\vec{\gamma}) = 0 \text{ for all } f \in I\} \neq \varnothing.$$

(a) [3 points] Use the weak Nullstellensatz in the form stated above to prove the *strong Nullstellensatz*, in the form that for any proper ideal $J \subset \mathbb{Q}[x_1, \cdots, x_n]$ the radical $\sqrt{J}$ is given by

$$\sqrt{J} = \{g \in \mathbb{Q}[x_1, \cdots, x_n] \mid g(\vec{\gamma}) = 0 \text{ for all } \vec{\gamma} \in V(J) \subset (\overline{\mathbb{Q}})^n\}.$$

[Hint: Make use of k = n+1 in the weak Nullstellensatz.]

(b) [2 points] Explain why $\sqrt{J}$ is the intersection of all *maximal* ideals $Q \subset \mathbb{Q}[x_1, \cdots, x_n]$ with $J \subset Q$.

(c) [5 points] If $P \subset \mathbb{Q}[x_1, \cdots, x_n]$ is a *minimal nonzero* prime ideal, prove that $P = (f)$, where $f \in \mathbb{Q}[x_1, \cdots, x_n]$ is irreducible. Then prove that there is a j, $1 \leq j \leq n$, so that the $n-1$ elements $\{\bar{x}_i = x_i \mod P \mid i \neq j\}$ are algebraically independent over $\mathbb{Q}$ in the integral domain $\mathbb{Q}[x_1, \cdots, x_n]/P$.

ALGEBRA QUAL, SPRING 2009, PART II

1. (a) [2 points] Prove that every finite field $\mathbb{F}$ has order $q = p^n$ for some prime integer p and some integer $n \geq 1$.

(b) [5 points] Prove that for each such $q = p^n$ there is up to isomorphism exactly one field $\mathbb{F}_q$ of order q.

[You may use the existence and uniqueness of splitting fields of polynomials.]

(c) [3 points] Prove that $K = \mathbb{F}_3[x]/(x^2 + x - 1)$ and $K' = \mathbb{F}_3[y]/(y^2 + 1)$ are fields and exhibit an *explicit* isomorphism between them.

2. Suppose G_1 and G_2 are groups and $H \subset G_1 \times G_2$ is a subgroup so that the two compositions

$$\begin{array}{lcll} p_1 : H & \subset & G_1 \times G_2 & \to G_1 \\ p_2 : H & \subset & G_1 \times G_2 & \to G_2 \end{array}$$

are *surjections*. Let $N_1 = ker(p_2)$ and $N_2 = ker(p_1)$. Thus, if $e_1 \in G_1$ and $e_2 \in G_2$ are the identity elements then

$$\begin{array}{lcll} N_1 & = & H \cap (G_1 \times \{e_2\}) & \subset \ G_1 \times \{e_2\} \\ N_2 & = & H \cap (\{e_1\} \times G_2) & \subset \ \{e_1\} \times G_2. \end{array}$$

(a) [5 points] Show that $N_1 \lhd G_1 \times \{e_2\}$ and $N_2 \lhd \{e_1\} \times G_2$ are normal subgroups.

(b) [5 points] Show that
$$\frac{G_1 \times \{e_2\}}{N_1} \simeq \frac{\{e_1\} \times G_2}{N_2}.$$

3. Let $T : V \to V$ be a linear endomorphism of a non-zero finite dimensional vector space over $\mathbb{C}$.

(a) [4 points] State precisely the theorem on the *existence* and *uniqueness* of a Jordan canonical form for T, and prove it using the structure theorem for modules over a PID.

(b) [2 points] Using the Jordan form, prove that $T = T_s + T_n$, where $T_s : V \to V$ is diagonalizable and $T_n : V \to V$ is nilpotent, and where $T_sT_n = T_nT_s$.

(c) [4 points] It is a fact that the T_s and T_n from part (b) can be expressed as polynomials in T with coefficients in $\mathbb{C}$. You don't need to prove this fact, but assuming it, prove that there is a *unique* decomposition $T = T'_s + T'_n$, where T'_s is diagonalizable, T'_n is nilpotent, and $T'_sT'_n = T'_nT'_s$.

4. Let $\mathbb{Q} \subset E$ be a finite Galois extension and let $B \subset E$ be the ring of algebraic integers in E. Suppose $P \subset B$ is a non-zero prime ideal with $P \cap \mathbb{Z} = (p)$, a prime ideal in $\mathbb{Z}$. Set $\overline{E} = B/P$ and suppose $\xi \in \overline{E}$ is a primitive generator for $\overline{E}$ over $\mathbb{F}_p = \mathbb{Z}/p$.

(a) [3 points] Explain why there exists $x \in B$ such that $\xi = x \mod P \in B/P = \overline{E}$ and such that $x \in \tau P \subset B$ for all $\tau \in Gal(E/\mathbb{Q})$ with $\tau P \neq P$.

(b) [7 points] If $G_P = \{\sigma \in Gal(E/\mathbb{Q}) \mid \sigma P = P\} \subset Gal(E/\mathbb{Q})$, prove that the obvious homomorphism $G_P \to Gal(\overline{E}/\mathbb{F}_p)$ is surjective.

5. Suppose that A is a noetherian integral domain. Suppose further that for every *maximal* ideal $Q \subset A$, the quotient Q/Q^2 is a one dimensional vector space over the field A/Q.

(a) [5 points] Prove that every non-zero prime ideal of A is maximal.

[Hint: Prove something about the localizations $A_{(Q)}$ for *maximal* ideals Q.]

(b) [5 points] Prove that A is integrally closed.

[In both (a) and (b), give precise statements of any lemmas you use.]

1. Part I

(1) (a) Explicitly exhibit an element σ of $G = GL(2, \mathbb{Z}/7\mathbb{Z})$ of order 8.

(b) Describe, with proof, the structure of the 2-Sylow subgroup of G.

Hint: think about the multiplicative group of the field of size 49, and the action of the nontrivial automorphism of this field.

(2) Let V, W be vector spaces over an algebraically closed field k, with $\dim(V) = 6$ and $\dim(W) = 9$. Suppose $T : V \to V, S : W \to W$ are linear transformations whose minimal polynomials are, respectively, $T^6 = 0$ and $S^9 = 0$.

Consider the linear transformation $S \otimes T : W \otimes V \to W \otimes V$.

(i) What is the minimal polynomial of $S \otimes T$?

(ii) What is the dimension of $\ker(S \otimes T)$?

(iii) Describe the Jordan normal form of $S \otimes T$ (i.e., number of blocks, and their sizes).

Hint: You should not need to write down any matrices.

(3) Suppose A and B are commutative rings containing a field k, with B finitely generated over k as a ring. If $\phi : A \to B$ is a ring homomorphism with $\phi|_k = \mathrm{Id}$ and if $Q \subset B$ is a maximal ideal, prove that $\phi^{-1}(Q) \subset A$ is a maximal ideal.

(4) Let α be a root of $x^7 - 12$ and ζ a primitive 7^{th} root of unity, both in **C**.

(a) Explain why the powers $\{\alpha^j\}_{0 \le j \le 6}$ are linearly independent over the field $\mathbf{Q}[\zeta]$.

(b) If $\beta \in \mathbf{Q}[\alpha]$ has a conjugate of the form $\zeta^i \beta \ \ (0 < i < 7)$ in the algebraic closure of **Q**, explain why $\beta = c\alpha^j$ for some rational number c and some j with $0 < j < 7$.

(c) Show, using the results of the prior parts, that $x^7 - 11$ has no root in $\mathbf{Q}(\alpha)$.

(5) Let R be a ring and

$$\cdots F_j \xrightarrow{d} F_{j-1} \xrightarrow{d} \cdots \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{d} 0 \xrightarrow{d} 0 \cdots$$

a complex of free R-modules.

(a) Show that this complex is exact (i.e., has vanishing homology) if and only if there exists degree 1 homomorphism $h : F_* \to F_*$ (i.e., a collection of R-module homomorphisms $h_j : F_j \to F_{j+1}$) so that $dh + hd$ is the identity on the complex F_*.

(b) In this case, show that $\mathrm{Hom}(F_*, M)$ has vanishing cohomology for any module M.

(c) Give counterexamples to both statements if F_* is exact but not free.

2. Part II

(1) Find a root of unity ζ so that $\mathbf{Q}(\zeta)$ contains a subfield K which is Galois over $\mathbf{Q}$ with Galois group $\mathbf{Z}/3\mathbf{Z}$. Compute the minimal polynomial over $\mathbf{Q}$ of an element that generates K over $\mathbf{Q}$.

(2) (i) Prove that if a nonzero ideal I in a domain R is free as an R-module then I is principal. As an application, for $R = \mathbb{Z}[\sqrt{-5}]$ prove that neither of the ideals $P = (3, 1+\sqrt{-5})$ nor $Q = (3, 1-\sqrt{-5})$ is free. *Hint:* Use norms!

(ii) Prove that $P \cap Q = 3R$, and that the addition map $P \oplus Q \to R$ defined by $(a, b) \mapsto a + b$ is surjective.

(iii) Deduce that $P \oplus Q \simeq R^2$ as R-modules, so a direct summand of a free module need not be free as an R-module!

(3) Let G be a finite group and H a subgroup whose index is prime to p. Suppose V is a finite-dimensional representation of G over $\mathbb{F}_p$ whose restriction to H is semisimple. Prove that V is semisimple. *Hint:* Imitate the proof of Maschke's theorem.

(4) (i) If A is a commutative Noetherian ring, prove the power series ring $A[[x]]$ is Noetherian.

(ii) If A is a commutative Artin ring, that is, if the ideals satisfy the descending chain condition, prove that every prime ideal $P \subset A$ is maximal and that there are only finitely many prime ideals. *Hints:* If $x \notin P$, consider the ideals (x^n); consider also finite products of prime ideals.

(5) Let K be an algebraically closed field and let $V \subset K^n, W \subset K^m$ be irreducible algebraic sets. Prove that $V \times W \subset K^{n+m}$ is an irreducible algebraic set.