Do all 6 problems. All problems are equally weighted. Show all details in your solutions.

**Notation:** \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) are fields of rational, real and complex numbers.

1. Let \( \mathbb{F}_2 \) be the finite field with 2 elements.
   (a) What is the order of \( GL_3(\mathbb{F}_2) \), the group of \( 3 \times 3 \) invertible matrices over \( \mathbb{F}_2 \)?
   (b) Assuming the fact that \( GL_3(\mathbb{F}_2) \) is a simple group, find the number of elements of order 7 in \( GL_3(\mathbb{F}_2) \).

2. Let \( K \subset \mathbb{C} \) be the splitting field of \( f(x) = x^6 + 3 \) over \( \mathbb{Q} \). Let \( \alpha \) be a root of \( f(x) \) in \( K \).
   (a) Show that \( K = \mathbb{Q}(\alpha) \).
   (b) Determine the Galois group \( \text{Gal}(K/\mathbb{Q}) \).

3. Let \( k \) be a field with characteristic 0. Let \( m \geq 2 \) be an integer. Show that \( f(x, m) = x^m + x + 1 \) is irreducible in \( k[x] \).
Notation: Integers: \( \mathbb{Z} \), the integers modulo \( p \): \( \mathbb{Z}/(p) \), rationals: \( \mathbb{Q} \).

(1) (3 points) Let \( H \) be a subgroup of \( G \) with \( a, b \in G \). Prove (without assuming very much) that the right cosets \( Ha \) and \( Hb \) are either equal or disjoint.

(2) (3 points) Prove that if \( H \) is a subgroup of index 2 in \( G \) then \( H \) is normal in \( G \).

(3) (3 points) Working over the integers, calculate (and show your work in a readable fashion) Tor(\( \mathbb{Z}/(p) \), \( \mathbb{Z}/(p) \)).

(4) (3 points) Working over the integers, calculate (and show your work in a readable fashion) Ext(\( \mathbb{Z}/(p) \), \( \mathbb{Z}/(p) \)).

(5) (6 points) Recall \( D_4 = \{1, a, a^2, a^3, ba, ba^2, ba^3\}, |a| = 4, |b| = 2, \ aba = b \). Find the center of \( D_4 \), \( Z(D_4) \), and describe \( D_4/Z(D_4) \).

(6) (3 points) Calculate all the group homomorphisms from \( S_3 \), the symmetric group on 3 elements, to \( \mathbb{Z}/(2) \times \mathbb{Z}/(2) \). Explain your answer.

(7) (1 point) How many monic polynomials of degree 3 are there over \( \mathbb{Z}/(3) \)?

(8) (3 points) How many irreducible monic polynomials of degree 3 are there over \( \mathbb{Z}/(3) \).

(9) (5 points) For every monic polynomial, \( f(x) \), in problem \# 7, we can define the quotient ring \( \mathbb{Z}/(3)[x]/(f(x)) \). How many different rings do we get if we use only the monic polynomials of problem \# 8? Explain your answer and identify your answers as familiar rings.

(10) (3 points) Find all the idempotents not equal to 0 or 1 in the ring \( \mathbb{Z}/(2)[x]/(x^3 + 1) \).

(11) (4 points) Find the minimal polynomial for \( \sqrt{2} + i \) over \( \mathbb{Q} \).

(12) (10 points) Demonstrate your knowledge of Galois theory for the field extension \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{2} + i) \) (in the complex numbers).

(13) (3 points) Prove Cauchy's theorem that if a prime \( p \) divides the order of a group, \( |G| \), then \( G \) has an element of order \( p \). (You can assume the result for Abelian groups.)

(14) (3 points) Give all groups of order 175. Two are pretty easy, it is the rest I care about. Explain your answer.
Do all problems. All problems are equally weighted. Show all details.

1. Let $H$ be a proper subgroup of a finite group $G$. Show that $G$ is not the union of all the conjugates of $H$.

2. Let $N$ be the set of all nilpotent elements in a ring $R$. Assume first that $R$ is commutative.
   (a) Show that $N$ is an ideal in $R$, and $R/N$ contains no non-zero nilpotent elements.
   (b) Show that $N$ is the intersection of all the prime ideals of $R$.
   (c) Give an example with $R$ non-commutative where $N$ is not an ideal in $R$.

3. Let $f(x) = x^5 - 9x + 3$. Determine the Galois group of $f$ over $\mathbb{Q}$.

4. Let $\lambda_1, \ldots, \lambda_n$ be roots of unity, with $n \geq 2$. Assume that $\frac{1}{n} \sum_{i=1}^{n} \lambda_i$ is integral over $\mathbb{Z}$. Show that either $\sum_{i=1}^{n} \lambda_i = 0$ or $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

5. Consider the ideal $I = (2, x)$ in $R = \mathbb{Z}[x]$.
   (a) Construct a non-trivial $R$-module homomorphism $I \otimes_R I \to R/I$, and use that to show that $2 \otimes x - x \otimes 2$ is a non-zero element in $I \otimes_R I$.
   (b) Determine the annihilator of $2 \otimes x - x \otimes 2$.

6. Let $D_8$ be the dihedral group of order 8, given by generators and relations
   $$< r, s \mid r^4 = 1 = s^2, rs = sr^{-1} >$$
   (a) Determine the conjugacy classes of $D_8$.
   (b) Determine the commutator subgroup $D'_8$ of $D_8$. Determine the number of distinct degree one characters of $D_8$.
   (c) Write down the complete character table of $D_8$. 
Directions: Solve four problems from the following list of five and clearly indicate which problems you chose as only those will be graded. Show all your work. In general, it is permissible to use earlier parts of a problem in order to solve a later part even if you have not solved the earlier parts.

1. Let $G$ be a finite group and let $N$ be a normal subgroup of $G$ such that $N$ and $G/N$ have relatively prime orders.
   (a) Assume that there exists a subgroup $H$ of $G$ having the same order as $G/N$. Show that $G = HN$. (Here $HN$ denotes the set $\{xy \mid x \in H, y \in N\}$.)
   (b) Show that $\phi(N) = N$, for all automorphisms $\phi$ of $G$.

2. Let $S$ denote the ring $\mathbb{Z}[X]/X^2\mathbb{Z}[X]$, where $X$ is a variable.
   (a) Show that $S$ is a free $\mathbb{Z}$-module and find a $\mathbb{Z}$-basis for $S$.
   (b) Which elements of $S$ are units (i.e. invertible with respect to multiplication)?
   (c) List all the ideals of $S$.
   (d) Find all the nontrivial ring morphisms defined on $S$ and taking values in the ring of Gaussian integers $\mathbb{Z}[i]$.

3. Assume that $A$ is an $n \times n$ matrix with entries in the field of complex numbers $\mathbb{C}$ and $A^m = 0$ for some integer $m > 0$.
   (a) Show that if $\lambda$ is an eigenvalue of $A$, then $\lambda = 0$.
   (b) Determine the characteristic polynomial of $A$.
   (c) Prove that $A^n = 0$.
   (d) Write down a $5 \times 5$ matrix $B$ for which $B^3 = 0$ but $B^2 \neq 0$.
   (e) If $M$ is any $5 \times 5$ matrix over $\mathbb{C}$ with $M^3 = 0$ and $M^2 \neq 0$, must $M$ be similar to the matrix $B$ you found in part (d)? Justify your answer.

4. Let $K := \mathbb{Q}(\sqrt{3} + \sqrt{5})$.
   (a) Show that $K$ is the splitting field of $X^4 - 6X^2 + 4$.
   (b) Find the structure of the Galois group of $K/\mathbb{Q}$.
   (c) List all the fields $k$, satisfying $\mathbb{Q} \subseteq k \subseteq K$.

5. Let $\rho : G \rightarrow \text{Gl}_n(\mathbb{C})$ be a complex irreducible representation of degree $n$ of a finite group $G$. Let $\chi$ be its associated character and let $C$ be the center of $G$.
   (a) Show that, for all $s \in C$, $\rho(s)$ is a scalar multiple of the identity matrix $I_n$.
   (b) Use (a) to show that $|\chi(s)| = n$, for all $s \in C$.
   (c) Prove the inequality $n^2 \leq |G : C|$, where $|G : C|$ denotes the index of $C$ in $G$.
   (d) Show that, if $\rho$ is faithful (i.e. an injective group morphism), then the group $C$ has to be cyclic.
1. How many elements of order 7 are there in a simple group of order 168?

2. Let \( m \) be an integer \( \geq 2 \) and \( \mathbb{Z}[X] \) be the polynomial ring over \( \mathbb{Z} \). Find a condition on \( m \) so that the ideal \((m, X)\) in the ring is maximal.

3. Prove that the group of automorphisms of the field \( \mathbb{R} \) of real numbers is trivial.

4. For a field \( K \), let \( SL_2(K) \) be the special linear group over \( K \), i.e. the group of \( 2 \times 2 \) matrices over \( K \) with determinant 1, and let \( PSL_2(K) \) be the quotient of \( SL_2(K) \) by its center, i.e. the projective special linear group. Find the order of \( PSL_2(F_7) \) where \( F_7 \) denotes the finite field of 7 elements.

5. Let \( \zeta = e^{\frac{2\pi i}{5}} \) and \( K = \mathbb{Q}(\zeta) \) the field generated by \( \zeta \) over the field of rational numbers. Prove that \( K \) contains \( \sqrt{5} \).
Do all 6 problems. All problems are equally weighted. Show all details in your proofs.

1. Let $k$ be a field. Let $G = GL_n(k)$ be the general linear group. Here $n > 0$. Let $D$ be the subgroup of diagonal matrices. Let $N = N_G(D)$ be the normalizer of $D$. Determine the quotient group $N/D$.

2. Let $F_p$ be the field with $p$ elements, where $p$ is a prime number. Let $f_{n,p}(x) = x^{p^n} - x + 1$, and suppose that $f_{n,p}(x)$ is irreducible in $F_p[x]$. Let $\alpha$ be a root of $f_{n,p}(x)$.
   (a) Show that $F_p^n \subset F_p(\alpha)$ and $[F_p(\alpha) : F_p^n] = p$.
   (b) Determine all pairs $(n, p)$ for which $f_{n,p}(x)$ is irreducible.

3. Let $\xi$ be a primitive $p^n$-th root of unity. Here $p$ is prime and $n > 0$. Let $f(x)$ be the minimal polynomial of $\xi$ over $\mathbb{Q}$, and let $m$ be its degree.
   (a) Determine $f(x)$.
   (b) Let $\alpha_1, \ldots, \alpha_m$ be all the roots of $f(x)$. Define the discriminant of $\xi$ as:
   \[ D(\xi) = [\det(\alpha_i^{-1})_{ij}]^2, \quad i, j = 1, \ldots m. \]
   Show that $D(\xi) = (-1)^{m(m-1)/2} N_{\mathbb{Q}(\xi)}^\mathbb{Q}(f'(\xi))$.
   (c) Take $n = 2$. Compute $D(\xi)$ in this case.

4. Let $R$ be a ring. Let $L$ be a minimal left ideal of $R$ (i.e., $L$ contains no nonzero proper left ideal of $R$). Assume $L^2 \neq 0$. Show that $L = Re$ for some non-zero idempotent $e \in R$.

5. Let $A$ be an integral domain and let $K$ be its field of fractions. Let $A'$ be the integral closure of $A$ in $K$. Let $P \subset A$ be a prime ideal and let $S = A - P$. (Note that $A_P = S^{-1}A$ is contained in $K$.) Show that $A_P$ is integrally closed in $K$ if and only if $(A'/A) \otimes_A A_P = 0$.

6. Let $V$ be a finite dimensional vector space over a field $k$. Let $G$ be a finite group. Let $\varphi : G \to GL(V)$ be an irreducible representation of $G$. Suppose that $H$ is a finite abelian subgroup of $GL(V)$ such that $H$ is contained in the centralizer of $\varphi(G)$. Show that $H$ is cyclic.
Qualifying Exam in Algebra, Fall 2003

Directions: This is a closed book exam. You have two hours to do all five of the (equally weighted) problems.

1. In a group $G$, let $1$ denote the identity element and let $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of the elements $x, y \in G$.
   a) Express $[z, xy]x$ in terms of $x, [z, x]$ and $[z, y]$.
   b) Prove that if the identity $[[x, y], z] = 1$ holds in a group $G$, then the identities

   \[
   [x, yz] = [x, y][x, z] \quad \text{and} \quad [xy, z] = [x, z][y, z]
   \]

   hold in $G$.

2. Let $k$ be a field of characteristic $p$ and let $t, u$ be algebraically independent over $k$. Prove the following:
   a) $k(t, u)$ has degree $p^2$ over $k(t^p, u^p)$.
   b) There exist infinitely many fields between $k(t, u)$ and $k(t^p, u^p)$.

3. Obtain a factorization into irreducible factors in $\mathbb{Z}[x]$ of the polynomial $x^{10} - 1$.

4. Verify the isomorphism of algebras over a field $K$:
\[
\mathbb{M}_n(K) \otimes_K \mathbb{M}_m(K) \simeq \mathbb{M}_{mn}(K).
\]
[Note: $\mathbb{M}_n(K)$ denotes the algebra of $n \times n$ matrices over $K$.]

5. Let $S_4$ be the symmetric group on 4 elements.
   a) Give an example of a non-trivial 8-dimensional representation of the group $S_4$.
   b) Prove for any 8-dimensional complex representation of $S_4$ the existence of a 2-dimensional invariant subspace.

Directions: Solve five problems from the following list of seven and clearly indicate which problems you chose as only those will be graded. Show all your work.

In general, it is permissible to use earlier parts of a problem in order to solve a later part even if you have not solved the earlier parts.

1. Let $G$ be a non-abelian group of order $p^3$, where $p$ is a prime number. Let $Z(G)$ be its center and $G'$ its commutator subgroup.
   (a) Show that $Z(G) = G'$ and this is the unique normal subgroup of $G$ of order $p$.
   (b) What is the exact number of distinct conjugacy classes of $G$?

2. (a) Let $p$ be a prime number. Show that $f(X) = X^p - pX - 1$ is irreducible in $Q[X]$. (Hint: use Eisenstein’s criterion of irreducibility for the image of $f(X)$ via a ring automorphism of $Q[X]$.)
   (b) Let $R$ be the ring $Z[X]/(X^4 - 3X^2 - X)$, where $(X^4 - 3X^2 - X)$ is the ideal generated by $X^4 - 3X^2 - X$ in $Z[X]$. Find all the prime ideals of $R$ containing 3 (the image of 3 in $Z[X]$ via the canonical surjection $Z[X] \rightarrow R$).

3. Let $K/k$ be a finite, separable field extension of degree $n$. Let
   $\rho, \rho' : K \rightarrow M_n(k)$
be two morphisms of $k$-algebras, where $M_n(k)$ is the ring of $n \times n$ matrices with entries in $k$. Show that there exists an invertible matrix $A$ in $M_n(k)$ such that
   $\rho'(x) = A \cdot \rho(x) \cdot A^{-1}$, for all $x \in K$.

4. Let $G$ be a finite group. Show that there exists a Galois field extension $K/k$ whose Galois group is isomorphic to $G$.

5. Let $k$ be a field of characteristic $p$, where $p$ is a prime number. Let $X$ and $T$ be two (algebraically) independent variables over $k$.
   (a) Show that the degree of the field extension $k(X,T)/k(X^p,T^p)$ is $p^2$.
   (b) Show that there exist infinitely many distinct fields $F$ such that
      $k(X^p,T^p) \subseteq F \subseteq k(X,T)$.

(see over, please)
6. Let $R$ be the ring $\mathbb{Q}[X]/(X^7 - 1)$, where $(X^7 - 1)$ is the ideal generated by $X^7 - 1$ in $\mathbb{Q}[X]$. Give an example of a finitely generated projective $R$-module which is not $R$-free. (We remind you that an $R$-module is called projective if it is a direct summand of a free $R$-module.)

7. Let $D_{10}$ be the dihedral group of order 10, given by the usual generators and relations

$$D_{10} = \langle r, s \mid r^5 = 1 = s^2, rs = sr^{-1} \rangle$$

(1) Compute the conjugacy classes of $D_{10}$.
(2) Compute the commutator subgroup $D'_{10}$ of $D_{10}$.
(3) Show that $D_{10}/D'_{10}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and conclude that $D_{10}$ has precisely two distinct characters of degree 1.
(4) Write down the complete character table of $D_{10}$. 
Directions: Solve five problems from the following list of six and clearly indicate which problems you chose as only those will be graded. Show all your work.

In general, it is permissible to use earlier parts of a problem in order to solve a later part even if you have not solved the earlier parts.

1. Let $G$ be a finite group and $p$ the smallest prime number dividing the cardinality $|G|$ of $G$. Let $H$ be a subgroup of $G$ of index $p$ in $G$. Show that $H$ is necessarily a normal subgroup of $G$.

2. Let $K$ be the splitting field of $f(X) = X^3 - 2$ over $\mathbb{Q}$.
   (a) Determine an explicit set of generators for $K$ over $\mathbb{Q}$.
   (b) Show that the Galois group $G(K/\mathbb{Q})$ of $K$ over $\mathbb{Q}$ is isomorphic to the symmetric group $S_3$.
   (c) Provide the complete list of intermediate fields $k$, $\mathbb{Q} \subseteq k \subseteq K$, satisfying $[k : \mathbb{Q}] = 3$.
   (d) Which of the fields determined in (c) are normal extensions of $\mathbb{Q}$?

3. Calculate the complete character table for $\mathbb{Z}/3\mathbb{Z} \times S_3$, where $S_3$ is the symmetric group in 3 letters.

4. Let $p$ be a prime number, $F_p$ the prime field of $p$ elements, $X$ and $Y$ algebraically independent variables over $F_p$, $K = F_p(X,Y)$, and $F = F_p(X^p - X, Y^p - X)$.
   (a) Show that $[K : F] = p^2$ and the separability and inseparability degrees of $K/F$ are both equal to $p$.
   (b) Show that there exists a field $E$, such that $F \subseteq E \subseteq K$, which is a purely inseparable extension of $F$ of degree $p$.

5. (a) Prove that an $n \times n$ matrix $A$ with entries in the field $\mathbb{C}$ of complex numbers, satisfying $A^3 = A$, can be diagonalized over $\mathbb{C}$.
   (b) Does the statement in (a) remain true if one replaces $\mathbb{C}$ by an arbitrary algebraically closed field $F$? Why or why not?

6. Let $R$ be the ring $\mathbb{Z}[X,Y]/(YX^2 - Y)$, where $X$ and $Y$ are two algebraically independent variables, and $(YX^2 - Y)$ is the ideal generated by $YX^2 - Y$.
in \( \mathbb{Z}[X,Y] \).
(a) Show that the ideal \( I \) generated by \( Y - 4 \) in \( R \) is not prime.
(b) Provide the complete list of prime ideals in \( R \) containing the ideal \( I \) described in question (a).
(c) Which of the ideals found in (b) are maximal?
1. Show that every group of order $p^2$, $p$ a prime, is Abelian. Show that up to isomorphism there are only two such groups.

2. Let $K$ be a field. A polynomial $f(x) \in K[x]$ is called separable if, in any field extension, it has distinct roots. Prove that:
   (a) if $K$ has characteristic 0, then each irreducible polynomial in $K[x]$ is separable; and
   (b) if $K$ has characteristic $p \neq 0$, then an irreducible polynomial $f(x) \in K[x]$ is separable if and only if has no form $g(x^p)$ where $g(x) \in K[x]$.
   Give an example of an inseparable irreducible polynomial.

3. Prove that if a linear operator on a complex vector space is diagonal in some basis, then its restriction on any invariant subspace $L$ is also diagonal in some basis of $L$.

4. A differentiation of a ring $R$ is a mapping $D: R \rightarrow R$ such that, for all $x, y \in R$,
   (1) $D(x + y) = D(x) + D(y)$; and
   (2) $D(xy) = D(x)y + xD(y)$.
   If $K$ is a field and $R$ is a $K$-algebra, then its differentiations are supposed to be over $K$, that is,
   (3) $D(x) = 0$ for any $x \in K$.
   Let $D$ be a differentiation of the $K$-algebra $M_n(K)$ of $n \times n$-matrices. Find a matrix $A \in M_n(K)$ such that $D(X) = AX -XA$ for all $X \in M_n(K)$.

5. Prove the existence of a 1-dimensional invariant subspace for any 5-dimensional representation of the group $A_4$ (the alternating group of degree 4).
Qualifying Exam in Algebra, Fall 2003

Directions: This is a closed book exam. You have two hours to do all five of the (equally weighted) problems.

1. In a group $G$, let $1$ denote the identity element and let $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of the elements $x, y \in G$.
   a) Express $[z, xy]x$ in terms of $x, [z, x]$ and $[z, y]$.
   b) Prove that if the identity $[[x, y], z] = 1$ holds in a group $G$, then the identities
   \[ [x, yz] = [x, y][x, z] \quad \text{and} \quad [xy, z] = [x, z][y, z] \]
   hold in $G$.

2. Let $k$ be a field of characteristic $p$ and let $t, u$ be algebraically independent over $k$. Prove the following:
   a) $k(t, u)$ has degree $p^2$ over $k(t^p, u^p)$.
   b) There exist infinitely many fields between $k(t, u)$ and $k(t^p, u^p)$.

3. Obtain a factorization into irreducible factors in $\mathbb{Z}[x]$ of the polynomial $x^{10} - 1$.

4. Verify the isomorphism of algebras over a field $K$:
   \[ \mathcal{M}_n(K) \otimes_K \mathcal{M}_m(K) \cong \mathcal{M}_{mn}(K) \]
   [Note: $\mathcal{M}_n(K)$ denotes the algebra of $n \times n$ matrices over $K$.]

5. Let $S_4$ be the symmetric group on 4 elements.
   a) Give an example of a non-trivial 8-dimensional representation of the group $S_4$.
   b) Prove for any 8-dimensional complex representation of $S_4$ the existence of a 2-dimensional invariant subspace.