QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday, October 24, 1995 (Day 1)

1. Let K be a field of characteristic 0.

a. Find three nonconstant polynomials $x(t), y(t), z(t) \in K[t]$ such that

$$x^2 + y^2 = z^2$$

b. Now let n be any integer, $n \geq 3$. Show that there do not exist three nonconstant polynomials $x(t), y(t), z(t) \in K[t]$ such that

$$x^n + y^n = z^n.$$

2. For any integers k and n with $1 \leq k \leq n$, let

$$S^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

be the n-sphere, and let $D_k \subset \mathbb{R}^{n+1}$ be the closed disc

$$D_k = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_k^2 \leq 1;\ x_{k+1} = \ldots = x_{n+1} = 0\} \subset \mathbb{R}^{n+1}.$$

Let $X_{k,n} = S^n \cup D_k$ be their union. Calculate the cohomology ring $H^*(X_{k,n}, \mathbb{Z})$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any $\mathcal{C}^\infty$ map such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0.$$

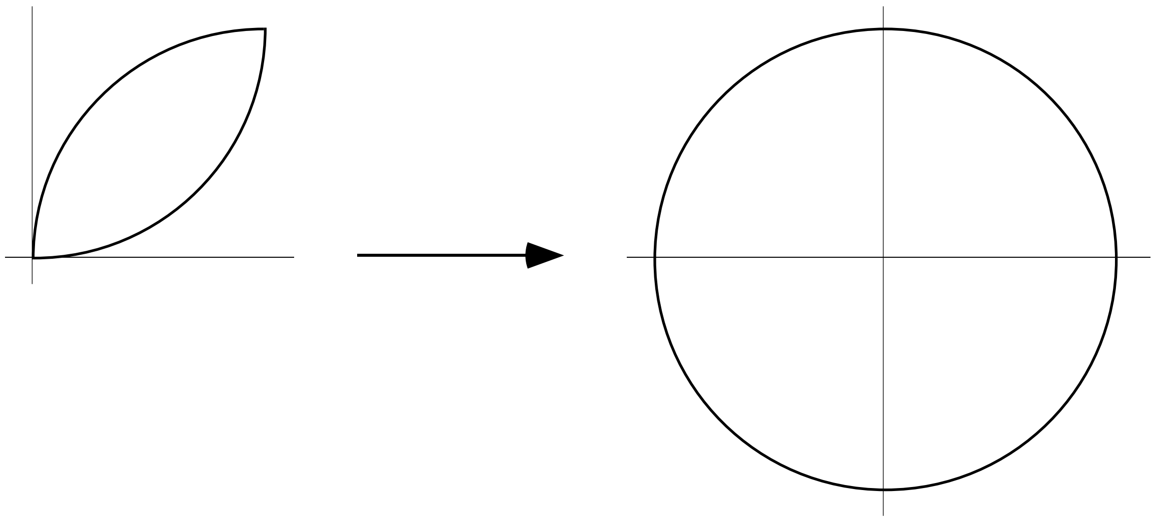
Show that if f is not surjective then it is constant.

4. Let G be a finite group, and let $\sigma, \tau \in G$ be two elements selected at random from G (with the uniform distribution). In terms of the order of G and the number of conjugacy classes of G, what is the probability that σ and τ commute? What is the probability if G is the symmetric group S_5 on 5 letters?

5. Let $\Omega \subset \mathbb{C}$ be the region given by

$$\Omega \;=\; \{z \;:\; |z-1| < 1 \quad \text{and} \quad |z-i| < 1\}.$$

Find a conformal map $f : \Omega \to \Delta$ of Ω onto the unit disc $\Delta = \{z : |z| < 1\}$.



6. Find the degree and the Galois group of the splitting fields over $\mathbb{Q}$ of the following polynomials:

a. $x^6 - 2$

b. $x^6 + 3$

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, October 25, 1995 (Day 2)

1. Find the ring A of integers in the real quadratic number field $K = \mathbb{Q}(\sqrt{5})$. What is the structure of the group of units in A? For which prime numbers $p \in \mathbb{Z}$ is the ideal $pA \subset A$ prime?

2. Let $U \subset \mathbb{R}^2$ be an open set.

a. Define a *Riemannian metric* on U.

b. In terms of your definition, define the *distance* between two points $p, q \in U$.

c. Let $\Delta = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$, and consider the metric on Δ given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Show that Δ is complete with respect to this metric.

3. Let K be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X, Y, Z)$ of degree d modulo scalars ($N = d(d+3)/2$). Let U be the subset of $\mathbb{P}^N$ of polynomials F whose zero loci are smooth plane curves $C \subset \mathbb{P}^2$ of degree d, and let $V \subset \mathbb{P}^N$ be the complement of U in $\mathbb{P}^N$.

a. Show that V is a closed subvariety of $\mathbb{P}^N$.

b. Show that $V \subset \mathbb{P}^N$ is a hypersurface.

c. Find the degree of V in case $d = 2$.

d. Find the degree of V for general d.

4. Let $\mathbb{P}^n_{\mathbb{R}}$ be real projective n-space.

a. Calculate the cohomology ring $H^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$.

b. Show that for $m > n$ there does not exist an *antipodal* map $f : S^m \to S^n$, that is, a continuous map carrying antipodal points to antipodal points.

5. Let V be any continuous nonnegative function on $\mathbb{R}$, and let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$H(f) \;=\; \frac{-1}{2}\frac{d^2 f}{dx^2} + V \cdot f.$$

a. Show that the eigenvalues of H are all nonnegative.

b. Suppose now that $V(x) = \frac{1}{2}x^2$ and f is an eigenfunction for H. Show that the *Fourier transform*

$$\hat{f}(y) \;=\; \int_{-\infty}^{\infty} e^{-ixy} f(x)dx$$

is also an eigenfunction for H.

6. Find the Laurent expansion of the function

$$f(z) \;=\; \frac{1}{z(z+1)}$$

valid in the annulus $1 < |z-1| < 2$.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, October 26, 1995 (Day 3)

1. Evaluate the integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

2. Let p be an odd prime, and let V be a vector space of dimension n over the field $\mathbb{F}_p$ with p elements.

a. Give the definition of a *nondegenerate quadratic form* $Q : V \to \mathbb{F}_p$

b. Show that for any such form Q there is an $\epsilon \in \mathbb{F}_p$ and a linear isomorphism

$$\begin{aligned} \phi\ : V \ &\longrightarrow\ \mathbb{F}_p^n \\ v\ &\longmapsto\ (x_1, \ldots, x_n) \end{aligned}$$

such that Q is given by the formula

$$Q(x_1, x_2, \ldots, x_n) \ = \ x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \epsilon x_n^2$$

c. In what sense is ϵ determined by Q?

3. Let G be a finite group. Define the *group ring* $R = \mathbb{C}[G]$ of G. What is the center of R? How does this relate to the number of irreducible representations of G? Explain.

4. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be any isometry, that is, a map such that the euclidean distance between any two points $x, y \in \mathbb{R}^n$ is equal to the distance between their images $\phi(x), \phi(y)$. Show that ϕ is *affine linear*, that is, there exists a vector $b \in \mathbb{R}^n$ and an orthogonal matrix $A \in O(n)$ such that for all $x \in \mathbb{R}^n$,

$$\phi(x) \ = \ Ax + b.$$

5. Let G be a finite group, $H \subset G$ a proper subgroup. Show that the union of the conjugates of H in G is not all of G, that is,

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

Give a counterexample to this assertion with G a compact Lie group.

6. Show that the sphere S^{2n} is not the underlying topological space of any Lie group.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday, March 12 (Day 1)

1. Let X be a compact n-dimensional differentiable manifold, and $Y \subset X$ a closed submanifold of dimension m. Show that the Euler characteristic $\chi(X \setminus Y)$ of the complement of Y in X is given by

$$\chi(X \setminus Y) \;=\; \chi(X) + (-1)^{n-m-1}\chi(Y).$$

Does the same result hold if we do not assume that X is compact, but only that the Euler characteristics of X and Y are finite?

2. Prove that the infinite sum

$$\sum_{p \text{ prime}} \frac{1}{p} \;=\; \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots$$

diverges.

3. Let $h(x)$ be a $\mathcal{C}^\infty$ function on the real line $\mathbb{R}$. Find a $\mathcal{C}^\infty$ function $u(x,y)$ on an open subset of $\mathbb{R}^2$ containing the x-axis such that

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} \;=\; u^2$$

and $u(x,0) = h(x)$.

4. a) Let K be a field, and let $L = K(\alpha)$ be a finite Galois extension of K. Assume that the Galois group of L over K is cyclic, generated by an automorphism sending α to $\alpha + 1$. Prove that K has characteristic $p > 0$ and that $\alpha^p - \alpha \in K$.

b) Conversely, prove that if K is of characteristic p, then every Galois extension L/K of degree p arises in this way. (Hint: show that there exists $\beta \in L$ with trace 1, and construct α out of the various conjugates of β.)

5. For small positive α, compute

$$\int_0^\infty \frac{x^\alpha \, dx}{x^2 + x + 1}.$$

For what values of $\alpha \in \mathbb{R}$ does the integral actually converge?

6. Let $M \in \mathcal{M}_n(\mathbb{C})$ be a complex $n \times n$ matrix such that M is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that M is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, March 13 (Day 2)

1. Prove the Brouwer fixed point theorem: that any continuous map from the closed n-disc $D^n \subset \mathbb{R}^n$ to itself has a fixed point.

2. Find a harmonic function f on the right half-plane $\{z \in \mathbb{C} \mid \text{Re } z > 0\}$ satisfying

$$\lim_{x \to 0+} f(x + iy) = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases}.$$

3. Let n be any integer. Show that any odd prime p dividing $n^2 + 1$ is congruent to 1 (mod 4).

4. Let V be a vector space of dimension n over a finite field with q elements.
a) Find the number of one-dimensional subspaces of V.
b) For any $k : 1 \leq k \leq n - 1$, find the number of k-dimensional subspaces of V.

5. Let K be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X, Y, Z)$ of degree d modulo scalars ($N = d(d+3)/2$). Let $W \subset \mathbb{P}^N$ be the subset of polynomials F of the form

$$F(X, Y, Z) \; = \; \prod_{i=1}^{d} L_i(X, Y, Z)$$

for some collection of linear forms $L_1, \ldots, L_d$.
a. Show that W is a closed subvariety of $\mathbb{P}^N$.
b. What is the dimension of W?
c. Find the degree of W in case $d = 2$ and in case $d = 3$.

6. a. Suppose that $M \to \mathbb{R}^{n+1}$ is an embedding of an n-dimensional Riemannian manifold (i.e., M is a hypersurface). Define the *second fundamental form* of M.

b. Show that if $M \subset \mathbb{R}^{n+1}$ is a compact hypersurface, its second fundamental form is positive definite (or negative definite, depending on your choice of normal vector) at at least one point of M.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, March 14 (Day 3)

1. In $\mathbb{R}^3$, let S, L and M be the circle and lines

$$\begin{aligned} S &= \{(x,y,z) : x^2+y^2=1;\ z=0\} \\ L &= \{(x,y,z) : x=y=0\} \\ M &= \{(x,y,z) : x=\frac{1}{2};\ y=0\} \end{aligned}$$

respectively.

a. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L)$.

b. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L \cup M)$.

2. Let $L, M, N \subset \mathbb{P}^3_{\mathbb{C}}$ be any three pairwise disjoint lines in complex projective threespace. Show that there is a unique quadric surface $Q \subset \mathbb{P}^3_{\mathbb{C}}$ containing all three.

3. Let G be a compact Lie group, and let $\rho : G \to GL(V)$ be a representation of G on a finite-dimensional $\mathbb{R}$-vector space V.

a) Define the *dual representation* $\rho^* : G \to GL(V^*)$ of V.

b) Show that the two representations V and V^* of G are isomorphic.

c) Consider the action of $SO(n)$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and the corresponding representation of $SO(n)$ on the vector space V of $\mathcal{C}^\infty$ $\mathbb{R}$-valued functions on S^{n-1}. Show that each nonzero irreducible $SO(n)$-subrepresentation $W \subset V$ of V has a nonzero vector fixed by $SO(n-1)$, where we view $SO(n-1)$ as the subgroup of $SO(n)$ fixing the vector $(0,\ldots,0,1)$.

4. Show that if K is a finite extension field of $\mathbb{Q}$, and A is the integral closure of $\mathbb{Z}$ in K, then A is a free $\mathbb{Z}$-module of rank $[K:\mathbb{Q}]$ (the degree of the field extension). (Hint: sandwich A between two free $\mathbb{Z}$-modules of the same rank.)

5. Let n be a nonnegative integer. Show that

$$\sum_{\substack{0\le k\le l\\ k+l=n}} (-1)^l \binom{l}{k} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ -1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3 \end{cases}.$$

(Hint: Use a generating function.)

6. Suppose K is integrable on $\mathbb{R}^n$ and for $\epsilon > 0$ define

$$K_\epsilon(x) = \epsilon^{-n} K(\frac{x}{\epsilon}).$$

Suppose that $\int_{\mathbb{R}^n} K = 1$.

a. Show that $\int_{\mathbb{R}^n} K_\epsilon = 1$ and that $\int_{|x|>\delta} |K_\epsilon| \to 0$ as $\epsilon \to 0$.

b. Suppose $f \in L^p(\mathbb{R}^n)$ and for $\epsilon > 0$ let $f_\epsilon \in L^p(\mathbb{R}^n)$ be the convolution

$$f_\epsilon(x) \;=\; \int_{y\in\mathbb{R}^n} f(y) K_\epsilon(x-y)dy\,.$$

Show that for $1 \le p < \infty$ we have

$$\|f_\epsilon - f\|_p \to 0 \text{ as } \epsilon \to 0.$$

c. Conclude that for $1 \le p < \infty$ the space of smooth compactly supported functions on $\mathbb{R}^n$ is dense in $L^p(\mathbb{R}^n)$.

Extra problems: Let me know if you think these should replace any of the ones above, either for balance or just by preference.

1. Suppose that $M \to \mathbb{R}^N$ is an embedding of an n-dimensional manifold into N-dimensional Euclidean space. Endow M with the induced Riemannian metric. Let $\gamma : (-1,1) \to M$ be a curve in M and $\overline{\gamma} : (-1,1) \to \mathbb{R}^N$ be given by composition with the embedding. Assume that $\|\frac{d\overline{\gamma}}{dt}\| \equiv 1$. Prove that γ is a geodesic iff

$$\frac{d^2\overline{\gamma}}{dt^2}$$

is normal to M at $\gamma(t)$ for all t.

2. Let A be a commutative Noetherian ring. Prove the following statements and explain their geometric meaning (even if you do not prove all the statements below, you may use any statement in proving a subsequent one):

a) A has only finitely many minimal prime ideals $\{\mathbf{p_k} \mid k = 1, \ldots, n\}$, and every prime ideal of A contains one of the $\mathbf{p_k}$.

b) $\bigcap_{k=1}^n \mathbf{p_k}$ is the set of nilpotent elements of A.

c) If A is reduced (i.e., its only nilpotent element is 0), then $\bigcup_{k=1}^n \mathbf{p_k}$ is the set of zero-divisors of A.

4. Let A be the $n \times n$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1/n & 1/n & 1/n & \ldots & 1/n \end{pmatrix}.$$

Prove that as $k \to \infty$, A^k tends to a projection operator P onto a one-dimensional subspace. Find the kernel and image of P.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday, February 25, 1997 (Day 1)

1. Factor the polynomial $x^3 - x + 1$ and find the Galois group of its splitting field if the ground field is:

$$\text{a) } \mathbf{R}, \qquad \text{b) } \mathbf{Q}, \qquad \text{c) } \mathbf{Z}/2\mathbf{Z}\,.$$

2. Let A be the $n \times n$ (real or complex) matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1/n & 1/n & 1/n & \dots & 1/n \end{pmatrix}.$$

Prove that as $k \to \infty$, A^k tends to a projection operator P onto a one-dimensional subspace. Find $\ker P$ and $\operatorname{Image} P$.

3. a. Show that there are infinitely many primes p congruent to 3 mod 4.
b. Show that there are infinitely many primes p congruent to 1 mod 4.

4. a. Let L_1, L_2 and $L_3 \subset \mathbf{P}^3_{\mathbf{C}}$ be three pairwise skew lines. Describe the locus of lines $L \subset \mathbf{P}^3_{\mathbf{C}}$ meeting all three.

b. Now let L_1, L_2, L_3 and $L_4 \subset \mathbf{P}^3_{\mathbf{C}}$ be four pairwise skew lines. Show that if there are three or more lines $L \subset \mathbf{P}^3_{\mathbf{C}}$ meeting all four, then there are infinitely many.

5. a. State the Poincaré duality and Kunneth theorems for homology with coefficients in $\mathbf{Z}$ (partial credit for coefficients in $\mathbf{Q}$).

b. Find an example of a compact 4-manifold M whose first and third Betti numbers are not equal, that is, such that $H^1(M, \mathbf{Q})$ and $H^3(M, \mathbf{Q})$ do not have the same dimension.

6. Compute

$$\int_0^\infty \frac{\log x}{x^2 + b^2}\, dx$$

for b a positive real number.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, February 26, 1997 (Day 2)

1. Define a metric on the unit disc $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$ by the line element

$$ds^2 \;=\; \frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^p}\,.$$

Here (r, θ) are polar coordinates and p is any real number.

a. For which p is the circle $r = 1/2$ a geodesic?

b. Compute the Gaussian curvature of this metric.

2. Let $\mathcal{C}$ be the space $\mathcal{C}[0, 1]$ of continuous real-valued functions on the closed interval $[0, 1]$, with the sup norm

$$||f||_\infty \;=\; \max_{t\in[0,1]} f(t)\,.$$

Let $\mathcal{C}^1$ be the space $\mathcal{C}^1[0, 1]$ of $\mathcal{C}^1$ functions on $[0, 1]$ with the norm

$$||f|| \;=\; ||f||_\infty + ||f'||_\infty\,.$$

Prove that the natural inclusion $\mathcal{C}^1 \subset \mathcal{C}$ is a compact operator.

3. Let M be a compact Riemann surface, and let f and g be two meromorphic functions on M. Show that there exists a polynomial $P \in \mathbf{C}[X, Y]$ such that $P(f(z), g(z)) \equiv 0$.

4. Let $S^3 = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 = 1\}$. Let p be a prime and m an integer relatively prime to p. Let ζ be a primitive p^{th} root of unity, and let the group G of p^{th} roots of unity act on S^3 by letting $\zeta \in G$ send (z, w) to $(\zeta z, \zeta^m w)$. Let $M = S^3/G$.

a. compute $\pi_i(M)$ for $i = 1$, 2 and 3.

b. compute $H_i(M, \mathbf{Z})$ for $i = 1$, 2 and 3.

c. compute $H^i(M, \mathbf{Z})$ for $i = 1$, 2 and 3.

5. Let d be a square-free integer. Compute the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. Give an example where this ring is not a principal ideal domain, and give an example of a non-principal ideal.

6. Prove that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}\,.$$

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, October 17, 1996 (Day 3)

1. Let $\alpha : (0,1) \to \mathbf{R}^3$ be any regular arc (that is, α is differentiable and α' is nowhere zero). Let $\mathbf{t}(u)$, $\mathbf{n}(u)$ and $\mathbf{b}(u)$ be the unit tangent, normal and binormal vectors to α at $\alpha(u)$. Consider the *normal tube of radius* ϵ around α, that is, the surface given parametrically by

$$\phi(u,v) \; = \; \alpha(u) + \epsilon \cos(v)\mathbf{n}(u) + \epsilon \sin(v)\mathbf{b}(u)\,.$$

a. For what values of ϵ is this an immersion?

b. Assuming that α itself has finite length, find the surface area of the normal tube of radius ϵ around α.

The answers to both questions should be expressed in terms of the curvature $\kappa(u)$ and torsion $\tau(u)$ of α.

2. Recall that a *fundamental solution* of a linear partial differential operator P on $\mathbf{R}^n$ is a distribution E on $\mathbf{R}^n$ such that $PE = \delta$ in the distribution sense, where δ is the unit Dirac measure at the origin. Find a fundamental solution E of the Laplacian on $\mathbf{R}^3$

$$\Delta \; = \; \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

that is a function of $r = |x|$ alone. Prove that your fundamental solution indeed satisfies $\Delta E = \delta$.

Hint: Use the appropriate form of Green's theorem.

3. The group of rotations of the cube in $\mathbf{R}^3$ is the symmetric group S_4 on four letters. Consider the action of this group on the set of 8 vertices of the cube, and the corresponding permutation representation of S_4 on $\mathbf{C}^8$. Describe the decomposition of this representation into irreducible representations.

4. Suppose a_i, $i = i, \ldots, n$ are positive real numbers with $a_1 + \ldots + a_n = 1$. Prove that for any nonnegative real numbers $\lambda_1, \ldots, \lambda_n$,

$$\sum_{i=1}^{n} a_i \lambda_i^2 \geq \left(\sum_{i=1}^{n} a_i \lambda_i \right)^2$$

with equality holding only if $\lambda_1 = \ldots = \lambda_n$.

5. a. For which natural numbers n is it the case that every continuous map from $\mathbf{P}^n_{\mathbf{C}}$ to itself has a fixed point?

b. For which n is it the case that every continuous map from $\mathbf{P}^n_{\mathbf{R}}$ to itself has a fixed point?

6. Fermat proved that the number $2^{37} - 1 = 137438953471$ was composite by finding a small prime factor p. Suppose you know that $200 < p < 300$. What is p?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday October 2, 2001 (Day 1)

Each question is worth 10 points, and parts of questions are of equal weight.

1a. Let X be a measure space with measure μ. Let $f \in L^1(X, \mu)$. Prove that for each $\epsilon > 0$ there exists $\delta > 0$ such that if A is a measurable set with $\mu(A) < \delta$, then
$$\int_A |f| d\mu < \epsilon.$$

2a. Let P be a point of an algebraic curve C of genus g. Prove that any divisor D with $\deg D = 0$ is equivalent to a divisor of the form $E - gP$, where $E > 0$.

3a. Let f be a function that is analytic on the annulus $1 \leq |z| \leq 2$ and assume that $|f(z)|$ is constant on each circle of the boundary of the annulus. Show that f can be meromorphically continued to $\mathbb{C} - \{0\}$.

4a. Prove that the rings $\mathbb{C}[x, y]/(x^2 - y^m)$, $m = 1, 2, 3, 4$, are all non-isomorphic.

5a. Show that the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ is not isometric to any sphere $x^2 + y^2 + z^2 = r$.

6a. For each of the properties P_1 through P_4 listed below either show the existence of a CW complex X with those properties or else show that there doesn't exist such a CW complex.

- P1. The fundamental group of X is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$.
- P2. The cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to the graded ring freely generated by one element in degree 2.
- P3. The CW complex X is "finite" (i.e., is built out of a finite number of cells) and the cohomology ring of its universal covering space is not finitely generated.
- P4. The cohomology ring $H^*(X, \mathbb{Z})$ is generated by its elements of degree 1 and has nontrivial elements of degree 100.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday October 3, 2001 (Day 2)

Each question is worth 10 points, and parts of questions are of equal weight.

1b. Prove that a general surface of degree 4 in $\mathbb{P}^3_{\mathbb{C}}$ contains no lines.

2b. Let R be a ring. We say that Fermat's last theorem is false in R if there exists $x, y, z \in R$ and $n \in \mathbb{Z}$ with $n \geq 3$ such that $x^n + y^n = z^n$ and $xyz \neq 0$. For which prime numbers p is Fermat's last theorem false in the residue class ring $\mathbb{Z}/p\mathbb{Z}$?

3b. Compute the integral

$$\int_0^\infty \frac{\cos(x)}{1+x^2}\,dx.$$

4b. Let $R = \mathbb{Z}[x]/(f)$, where $f = x^4 - x^3 + x^2 - 2x + 4$. Let $I = 3R$ be the principal ideal of R generated by 3. Find all prime ideals $\wp$ of R that contain I. (Give generators for each $\wp$.)

5b. Let $\mathfrak{S}_4$ be the symmetric group on four letters. Give the character table of $\mathfrak{S}_4$, and explain how you computed it.

6b. Let $X \subset \mathbb{R}^2$ and let $f : X \to \mathbb{R}^2$ be distance non-increasing. Show that f extends to a distance non-increasing map $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{f}|_X = f$. Does your construction of $\hat{f}$ necessarily use the Axiom of Choice?

(Hint: Imagine that X consists of 3 points. How would you extend f to $X \cup \{p\}$ for any 4th point p?)

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday October 4, 2001 (Day 3)

Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be the surface defined by the equation $XY - ZW = 0$. Find two skew lines on S. Prove that S is nonsingular, birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$, but not isomorphic to $\mathbb{P}^2_{\mathbb{C}}$.

2c. Let $f \in \mathbb{C}[z]$ be a degree n polynomial and for any positive real number R, let $M(R) = \max_{|z|=R} |f(z)|$. Show that if $R_2 > R_1 > 0$, then

$$\frac{M(R_2)}{R_2^n} \leq \frac{M(R_1)}{R_1^n},$$

with equality being possible only if $f(z) = Cz^n$, for some constant C.

3c. Describe, as a direct sum of cyclic groups, the cokernel of $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by left multiplication by the matrix

$$\begin{bmatrix} 3 & 5 & 21 \\ 3 & 10 & 14 \\ -24 & -65 & -126 \end{bmatrix}.$$

4c. Let X and Y be compact orientable 2-manifolds of genus g and h, respectively, and let $f : X \to Y$ be any continuous map. Assuming that the degree of f is nonzero (that is, the induced map $f^* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is nonzero), show that $g \geq h$.

5c. Use the Rouché's theorem to show that the equation $ze^{\lambda - z} = 1$, where λ is a given real number greater than 1, has exactly one root in the disk $|z| < 1$. Show that this root is real.

6c. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function such that for all x and $y \neq 0$,

$$\frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} \leq B,$$

for some finite constant B. Prove that for all $x \neq y$,

$$|f(x) - f(y)| \leq M \cdot |x - y| \cdot \left(1 + \log^+\left(\frac{1}{|x-y|}\right)\right),$$

where M depends on B and $\|f\|_\infty$, and $\log^+(x) = \max(0, \log x)$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday October 1, 2002 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1a. Exhibit a polynomial of degree three with rational coefficients whose Galois group over the field of rational numbers is cyclic of order three.

2a. The Catenoid C is the surface of revolution in $\mathbb{R}^3$ of the curve $x = \cosh(z)$ about the z axis. The Helicoid H is the surface in $\mathbb{R}^3$ generated by straight lines parallel to the xy plane that meet both the z axis and the helix

$$t \longmapsto [\cos(t), \sin(t), t].$$

(Recall that $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.)

(i) Show that both C and H are manifolds by exhibiting natural coordinates on each.

(ii) In the coordinates above, write the local expressions for the metrics g_C and g_H, induced by $\mathbb{R}^3$, on C and H, respectively.

(iii) Is there a covering map from H to C that is a local isometry?

3a. In $\mathbb{R}^n$, consider the Laplace equation

$$u_{11} + u_{22} + \cdots + u_{nn} = 0.$$

Show that the equation is invariant under orthonormal transformations. Find all rotationally symmetric solutions to this equation. (Here u_{ii} denotes the second derivative in the ith coordinate of a function u.)

4a. Let C denote the unit circle in $\mathbb{C}$. Evaluate

$$\oint_C \frac{e^{1/z}}{1-2z}$$

5a. Let $\mathbb{G}(1,3)$ be the Grassmannian variety of lines in $\mathbb{C}P^3$.

(i) Show that the subset $I \subset \mathbb{G}(1,3)^2$

$$I = \{(l_1, l_2) \mid l_1 \cap l_2 \neq \emptyset\}$$

is irreducible in the Zariski topology. (Hint: Consider the space of triples $(l_1, l_2, p) \in \mathbb{G}(1,3)^2 \times \mathbb{C}P^3$ such that $p \in l_1 \cap l_2$, and consider two appropriate projections.)

(ii) Show that the subset $J \subset \mathbb{G}(1,3)^3$

$$J = \{(l_1, l_2, l_3) \mid l_1 \cap l_2 \neq \emptyset,\ \ l_2 \cap l_3 \neq \emptyset,\ \ l_3 \cap l_1 \neq \emptyset\}$$

is reducible. How many irreducible components does it have?

6a. For the purposes of this problem, a manifold is a CW complex which is locally homeomorphic to $\mathbb{R}^n$. (In particular, it has no boundary.)

(i) Show that a connected simply-connected compact 2-manifold is homotopy equivalent to S^2. (Do not use the classification of surfaces.)

(ii) Let M be a connected simply-connected compact orientable 3-manifold. Compute $\pi_3(M)$.

(iii) Show that a connected simply-connected compact orientable 3-manifold is homotopy equivalent to S^3.

(iv) Find a simply-connected compact 4-manifold that is not homotopy equivalent to S^4.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday October 2, 2002 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1b. Let $\mathbb{C}[S_4]$ be the complex group ring of the symmetric group S_4. For $n \geq 1$ let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ matrices with complex entries. Prove that the algebra $\mathbb{C}[S_4]$ is isomorphic to a direct sum

$$\bigoplus_{i=1,\ldots,t} M_{n_i}(\mathbb{C})$$

and calculate the n_i's.

2b. (i) Show that the 2 dimensional sphere S^2 is an analytic manifold by exhibiting an atlas for which the change of coordinate functions are analytic functions. Write the local expression of the standard metric on S^2 in the above coordinates.

(ii) Put a metric on $\mathbb{R}^2$ such that the corresponding curvature is equal to 1. Is this metric complete?

3b. Let $C \in \mathbb{C}P^2$ be a smooth projective curve of degree $d \geq 2$. Let $\mathbb{C}P^{2*}$ be the dual space of lines in $\mathbb{C}P^2$ and $C^* \subset \mathbb{C}P^{2*}$ the dual curve of lines tangent to C. Find the degree of C^*. (Hint: Project from a point.)

4b. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set of points $x \in \mathbb{R}$ where f is continuous is a countable intersection of open sets.

5b. Prove that the only meromorphic functions $f(z)$ on $\mathbb{C} \cup \{\infty\}$ are rational functions.

6b. (i) Show that the fundamental group of a Lie group is abelian.

(ii) Find $\pi_1(\mathrm{SL}_2(\mathbb{R}))$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday October 3, 2002 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let

$$H = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

For $e_2 = (0, 1) \in \mathbb{R}^2$, map H to B by the following diffeomorphism.

$$\mathbf{v} \longmapsto \mathbf{x} = -e_2 + \frac{2(\mathbf{v} + e_2)}{\|\mathbf{v} + e_2\|^2}.$$

(i) Verify that the image of the above map is indeed B. (Hint: Think of the standard inversion in the circle.)

(ii) Consider the following metric on B:

$$g = \frac{dx^2 + dy^2}{(1 - \|\mathbf{x}\|^2)^2}.$$

Put a metric on H such that the above map is an isometry.

(iii) Show that H is complete.

2c. Let $C \subset \mathbb{C}P^2$ be a smooth projective curve of degree 4.

(i) Find the genus of C and give the Riemann-Roch formula for the dimension of the space of sections of a line bundle M of degree d on the curve C.

(ii) If $l \in \mathbb{C}P^2$ is a line meeting C at four distinct points $p_1, \ldots, p_4$, prove that there exists a nonzero holomorphic differential form on C vanishing at the four points p_i. (Hint: Note that $\mathcal{O}_{\mathbb{C}P^2}(1)$ restricted to C is a line bundle of degree 4. Use the Riemann-Roch formula to prove that this restriction is the canonical line bundle K_C.)

3c. Let A be the ring of real-valued continuous functions on the unit interval $[0, 1]$. Construct (with proof) an ideal in A which is not finitely generated.

4c. Construct a holomorphic function $f(z)$ on $\mathbb{C}$ satisfying the following two conditions:

(i) For every algebraic number z, the image $f(z)$ is algebraic.

(ii) $f(z)$ is not a polynomial.

(Hint: The algebraic numbers are countable.)

5c. Let $q < p$ be two prime numbers and $N(q, p)$ the number of distinct isomorphy types of groups of order pq. What can you say, more concretely, about the number $N(q, p)$?

6c. Let $i : S^1 \hookrightarrow S^3$ be a smooth embedding of S^1 in S^3. Let X denote the complement of the image of i. Compute the homology groups $H_*(X; \mathbb{Z})$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 30 September 2003 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1a. Let S be an embedded closed surface in $\mathbf{R}^3$ with the position vector $\vec{X}(p)$ and the unit outward normal vector $\vec{N}(p)$ for $p \in S$. For a fixed (small) t, define a surface S_t to be the set

$$S_t = \left\{ \vec{X}(p) + t\vec{N}(p) \in \mathbf{R}^3 \mid p \in S \right\}.$$

Let κ_1, κ_2 be the principal curvatures of S at the point p with respect to the outward normal vector. Let H_t be the mean curvature of S_t at the point $\vec{X}(p) + t\vec{N}(p)$ with respect to the outward normal vector (mean curvature is defined to be the sum of the two principal curvatures). Show that

$$H_t = \frac{\kappa_1}{1 - t\kappa_1} + \frac{\kappa_2}{1 - t\kappa_2}.$$

2a. Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk in $\mathbf{C}$ and $\bar{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ be the closed unit disk. Suppose $f : \bar{D} \to \bar{D}$ is analytic, one-to-one in D and continuous in $\bar{D}$. Also suppose $g : \bar{D} \to \bar{D}$ is analytic in D and continuous in $\bar{D}$, with $g(0) = f(0)$ and $g(D) \subset f(D)$. Prove $|g'(0)| \leq |f'(0)|$.

3a. Use the Riemann-Hurwitz (or any other) method to compute the genus of the Fermat curve, which is given in $\mathbf{CP}^2$ with homogeneous coordinates $(x : y : z)$ by the equation $x^n + y^n = z^n$ (assume that the base field is $\mathbf{C}$).

4a. Let k be a finite field with q elements and let $\Gamma = \mathrm{GL}(2, k)$ denote the group of invertible 2×2 matrices over k.

(i) How many elements are there in Γ?

(ii) How many complex irreducible representations does Γ have?

(iii) Consider the representation of Γ on the space of complex-valued functions on $\mathbf{P}^1$ over k (induced by the natural action of Γ on $\mathbf{P}^1$). Let V be the quotient of this space by the subspace of constant functions. Prove that V is an irreducible representation of Γ.

5a. Let V be a Hilbert space, and W a vector subspace of V. Show that

$$V = \overline{W} \oplus W^{\perp}.$$

6a. What is $\pi_1(\mathbf{RP}^3)$? Show that any continuous map $f : \mathbf{RP}^3 \to S^1$ is null-homotopic.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday 1 October 2003 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1b. Let $(\mathbf{H}^2, g)$ be the two-dimensional hyperbolic space, where

$$\mathbf{H}^2 = \{(x, y) \in \mathbf{R}^2 : y > 0\}$$

is the upper half plane of $\mathbf{R}^2 = \mathbf{C}$ and the metric g is given by

$$g = \frac{dx^2 + dy^2}{y^2}.$$

(i) Suppose a, b, c and d are real numbers such that $ad - bc = 1$. Define

$$\varphi(z) = \frac{az + b}{cz + d}$$

for any $z = x + \sqrt{-1}y$. Prove that φ is an isometry for $(\mathbf{H}^2, g)$.

(ii) Prove that $(\mathbf{H}^2, g)$ has constant Gaussian curvature.

2b. Prove the open mapping theorem for analytic functions of one complex variable: "if U is a connected open subset of $\mathbf{C}$ and $f : U \to \mathbf{C}$ is holomorphic and non-constant, then $f(U)$ is open." You may assume that a holomorphic function that is constant on an open subset of U is constant on U.

3b. Prove that if k is a field of characteristic p and $f(x) \in k[x]$ is a polynomial, then the map from the curve $y^p + y = f(x)$ to $\mathbf{A}^1_k$ sending (x, y) to x is everywhere unramified.

4b. (i) Let k be an algebraically closed field. Assume that k is *uncountable*. Let now V be a vector space over k of at most countable dimension and $A : V \to V$ be a linear operator. Prove that there exists $\lambda \in k$ such that the operator $A - \lambda\,\mathrm{id}_V$ is not invertible. (Hint: show first that in the field $k(t)$ of rational functions over k the elements $\frac{1}{t-\lambda}$ are linearly independent (for different values of λ) and then use this fact.)

(ii) Show that (i) is not necessarily true if k is countable.

(iii) Use (i) to show that for k uncountable every maximal ideal in the ring $k[x_1, \ldots, x_n]$ is generated by $(x-\lambda_1, \ldots, x-\lambda_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in k^n$.

5b. Give an example or show that none exist.

(i) A function $f : \mathbf{R} \to \mathbf{R}$ whose set of discontinuities is precisely the set $\mathbf{Q}$ of rational numbers.

(ii) A function $f : \mathbf{R} \to \mathbf{R}$ whose set of discontinuities is precisely the set $\mathbf{R} \setminus \mathbf{Q}$ of irrational numbers.

6b. Let X be the manifold-with-boundary $D^2 \times S^2$. Calculate $H_2(X;\mathbf{Z})$, $H^2(X;\mathbf{Z})$ and $H^2(X, \partial X;\mathbf{Z})$, using any techniques you choose. Calculate the map $j_* : H^2(X, \partial X;\mathbf{Z}) \to H^2(X;\mathbf{Z})$ that arises from the inclusion $j : (X, \emptyset) \to (X, \partial X)$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 2 October 2003 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1c. Let Σ be an embedded, compact surface without boundary in $\mathbf{R}^3$. Show that there exists at least one point p in Σ which has strictly positive Gaussian curvature.

2c. Determine for which $x \in \mathbf{Q}_p$ the exponential power series $\sum x^n/n!$ converges. Do the same for the logarithmic power series $\sum x^n/n$.

3c. Let V be a variety over an algebraically closed field k, and suppose V is also a group, i.e., there are morphisms $\varphi : V \times V \to V$ (multiplication or addition), and $\psi : V \to V$ (inverse) that satisfy the group axioms. Then V is called an *algebraic group*.

(i) Suppose that V is a nonsingular plane cubic. Describe a way to put a group structure on V. You do not have to prove that the maps you define are morphisms, but you do have to prove that they satisfy the axioms of a group.

(ii) Let V be defined by $y^2 z = x^3$ in $\mathbf{P}^2$. Prove that $V - \{(0,0,1)\}$ can be equipped with the structure of algebraic group.

(iii) Let V be defined by $x^3 + y^3 = xyz$ in $\mathbf{P}^2$. Prove that $V - \{(0,0,1)\}$ can be equipped with the structure of algebraic group.

4c. Compute the following integral:

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\ \mathrm{d}x.$$

5c. (i) Let a, b be nonnegative numbers, and p, q such that $1 < p < \infty$ and $1/p + 1/q = 1$. Establish Young's inequality:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

(ii) Using Young's inequality, prove the Hölder inequality: If $f \in L^p[0,1]$ and $g \in L^q[0,1]$, where p and q are as above, then $fg \in L^1[0,1]$, and

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

(iii) For $1 < p < \infty$, and $g \in L^q$, consider the linear functional F on L^p given by

$$F(f) = \int fg.$$

Show that $\|F\| = \|g\|_q$. (Recall that $\|F\| = \sup\{|F(f)|/\|f\| : f \in L^p\}$.)

(iv) Establish similar results for $p = 1$ and $p = \infty$.

6c. (i) Prove that every continuous map $f : \mathbf{CP}^6 \to \mathbf{CP}^6$ has a fixed point.

(ii) Exhibit a continuous map $f : \mathbf{CP}^3 \to \mathbf{CP}^3$ without a fixed point. (Hint: Try the case of $\mathbf{CP}^1$ first and write your answer in terms of homogeneous coordinates.)

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 21, 2004 (Day 1)

Each of the six questions is worth 10 points.

1) Let H be a (real or complex) Hilbert space. We say that a set of vectors $\{\phi_n\} \subset H$, $n = 1, 2, \ldots$, has "property D" provided H is the closure of the space of all finite linear combinations of the ϕ_n. Now let $\{\phi_n\}$, $n = 1, 2, \ldots$, be an orthonormal set having property D, and $\{\psi_n\}$ a set of vectors satisfying

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 \ < \ 1 \,,$$

where $\| \ \ \|$ refers to the Hilbert space norm. Show that $\{\psi_n\}$ also has property D.

2) Let K be the splitting field of the polynomial $x^4 - x^2 - 1$. Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_8, and compute the lattice of subfields of K.

3) Let S be a smooth surface in $\mathbb{R}^3$ defined by $\mathbf{r}(u,v)$, where $\mathbf{r}$ is the radius vector of $\mathbb{R}^3$ and (u,v) are curvilinear coordinates on S. Let H and K be respectively the mean curvature and the Gaussian curvature of S. Let A and B be respectively the supremum of the absolute value of H and K on S. Let a be a positive number and $\mathbf{n}$ be the unit normal vector of S. Consider the surface $\tilde{S}$ defined by $\vec{\rho}(u,v) = \mathbf{r}(u,v) + a\,\mathbf{n}(u,v)$. Let C be a curve in S defined by $u = u(t)$ and $v = v(t)$. Let $\tilde{C}$ be the curve in $\tilde{S}$ defined by $t \mapsto \vec{\rho}(u(t), v(t))$. Show that the length of $\tilde{C}$ is no less than the length of C multiplied by $1 - a\left(A + \sqrt{A^2 + 4B}\right)$. (*Hint:* compare the first fundamental form of $\tilde{S}$ with the difference of the first fundamental form of S and $2a$ times the second fundamental form of S.)

4) Compute the integral

$$\int_0^{\infty} \frac{x^{a-1}}{1+x^4}\, dx \qquad \left(0 < a < 4\right).$$

5) The *Grassmann manifold* $\mathbf{G}(2,4)$ is the set of all 2-dimensional planes in $\mathbb{R}^4$. More precisely,

$$\mathbf{G}(2,4) = \left\{ M = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \;\middle|\; a, \ldots, h \in \mathbb{R}\,,\ M \text{ has rank } 2 \right\} / \sim$$

where $M_1 \sim M_2$ if and only if $M_1 = AM_2$ for some invertible 2×2 real matrix A. We equip $\mathbf{G}(2,4)$ with the quotient topology; it is a compact orientable manifold.

a) Compute $\pi_1(\mathbf{G}(2,4))$ (*You may want to use the fact that given any* 2×4 *real matrix* M *there exists an invertible* 2×2 *real matrix* A *such that* AM *is in reduced row-echelon form. This gives a cell decomposition of* $\mathbf{G}(2,4)$ *with one cell for each possible reduced row-echelon form.*)

b) Compute the homology and cohomology groups of $\mathbf{G}(2,4)$ (with integer coefficients), stating carefully any theorems that you use.

6) a) What is the dimension of the space of hyperplanes in $\mathbf{P}^{n+1}$ containing a fixed linear subspace L of dimension k?

b) Let $Q \subset \mathbf{P}^{n+1}$ be a smooth quadric over $\mathbb{C}$. Show that the map $Q \to (\mathbf{P}^{n+1})^*$ associating to a point $x \in Q$ the tangent hyperplane $T_xQ \subset \mathbf{P}^{n+1}$ is an isomorphism onto its image. (Here $(\mathbf{P}^{n+1})^*$ is the dual projective space parameterizing hyperplanes in $\mathbf{P}^{n+1}$.)

c) Show that if L is a linear subspace contained in a quadric Q as in b), then $\dim L \leq n/2$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday September 22, 2004 (Day 2)

Each of the six questions is worth 10 points.

1) Show that if $Y \subset \mathbf{P}^n$ is a projective subvariety of dimension at least 1, over an algebraically closed field, and $H \subset \mathbf{P}^n$ is a hypersurface, then $Y \cap H \neq \emptyset$. (Justify any intermediate statement you may use.)

2) Let $u \mapsto \vec{\rho}(u)$ be a smooth curve in $\mathbb{R}^3$. Let S be the surface in $\mathbb{R}^3$ defined by $(u, v) \mapsto \vec{\rho}(u) + v\,\vec{\rho}\,'(u)$, where $\vec{\rho}\,'(u)$ means the first-order derivative of $\vec{\rho}(u)$ with respect to u. Assume that the two vectors $\vec{\rho}\,'(u)$ and $\vec{\rho}\,'(u) + v\vec{\rho}\,''(u)$ are $\mathbb{R}$-linear independent at the point $(u, v) = (u_0, v_0)$, where $\vec{\rho}\,''(u)$ means the second-order derivative of $\vec{\rho}(u)$ with respect to u. Verify directly from the definition of Gaussian curvature that the Gaussian curvature of S is zero at the point $(u, v) = (u_0, v_0)$.

3) a) Show that every linear fractional transformation which maps the upper half plane onto itself is of the form

$$F(z) \;=\; \frac{a\,z + b}{c\,z + d}, \quad \text{with} \;\; a,\, b,\, c,\, d \in \mathbb{R}\,, \quad a\,d - b\,c \;=\; 1\,.$$

b) Show that every linear fractional transformation which maps the unit disk onto itself is of the form

$$F(z) \;=\; \frac{\alpha\,z + \beta}{\bar{\beta}\,z + \bar{\alpha}}, \quad \text{with} \;\; \alpha,\, \beta \in \mathbb{C}\,, \quad |\alpha|^2 - |\beta|^2 \;=\; 1\,.$$

4) a) State van Kampen's Theorem. Use it to exhibit a topological space X such that $\pi_1(X)$ is isomorphic to the free group on 2 generators.

b) Show that the free group on 2 generators contains the free group on n generators as a subgroup of finite index.

c) Show that every subgroup of a free group is free.

5) Let $v_1, \dots v_n$ be complex numbers, and let A be the matrix:

$$A \;=\; \begin{pmatrix} v_1 & v_2 & v_3 & \dots & v_n \\ v_n & v_1 & v_2 & \dots & v_{n-1} \\ v_{n-1} & v_n & v_1 & \dots & v_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ v_2 & v_3 & v_4 & \dots & v_1 \end{pmatrix}.$$

Compute the eigenvalues and eigenvectors of A.

6) Given $f \in C_0^\infty(\mathbb{R})$ and $\epsilon > 0$, consider the function

$$g_\epsilon(x) \;:=\; \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy \;.$$

a) Show that

$$Hf(x) \;:=\; \lim_{\epsilon\to 0}\, g_\epsilon(x)$$

exists for each $x \in \mathbb{R}$, and that $Hf \in C^\infty(\mathbb{R})$.

b) Exhibit a universal constant C such that

$$\|Hf\|_{L^2} \;=\; C\,\|f\|_{L^2}\,.$$

Show how to extend the operator H to an isomorphism from $L^2(\mathbb{R})$ to itself.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 23, 2004 (Day 3)

Each of the six questions is worth 10 points.

1) a) Let G be a group of order n, acting on a finite set S. Show that the number of orbits of this action equals

$$\frac{1}{n}\sum_{g\in G} \#\{x \in S \mid gx = x\}.$$

b) Let S be the set of integer points in the rectangle $[0,3] \times [0,2]$. We consider two subsets of S equivalent if one can be transformed into the other by a series of reflections around the horizontal and vertical axes of symmetry of the rectangle. How many equivalence classes of four-element subsets of S are there?

2) Let $U \subset \mathbb{C}$ be a connected open subset. Carefully define the topology of locally uniform convergence on $\mathcal{O}(U)$, the space of holomorphic functions on U. Show that $\mathcal{O}(U)$, equipped with this topology, is a Fréchet space.

3) Consider the two dimensional torus $\mathbf{T}^2 = S^1 \times S^1$, where $S^1 = \mathbb{R}/2\pi\,\mathbb{Z}$. For any fixed $\alpha \in \mathbb{R}$, find all functions $f \in L^2(\mathbf{T}^2)$ with the property

$$f(x_1+\alpha, x_1+x_2) \;=\; f(x_1,x_2)\,.$$

4) Let Γ be a set of seven points in $\mathbb{CP}^3$, no four of them lying in a plane. What is the dimension of the subspace of homogeneous quadratic polynomials in $\mathbb{C}[X_0, X_1, X_2, X_3]$ vanishing along any subset $\{p_1, \ldots, p_m\} \subset \Gamma$, $m \leq 7$?

5) (Smooth Version of Michael Artin's Generalization of the Implicit Function Theorem.) Let a and b be positive numbers. Let $\mathcal{R}$ be the ring of all $\mathbb{R}$-valued infinitely differentiable functions on the open interval $(-a,a)$. For elements F, G, H in $\mathcal{R}$ we say that F is congruent to G modulo H in $\mathcal{R}$ if there exists some element Q of $\mathcal{R}$ such that $F - G = QH$ as functions on $(-a,a)$. Let $f(x,y)$ be an $\mathbb{R}$-valued infinitely differentiable function on $\{|x|<a,\ |y|<b\}$ with $f(0,0)=0$. Denote by

$f_y(x, y)$ the first-order partial derivative of $f(x, y)$ with respect to y. Let $g(x)$ be an element of $\mathcal{R}$ such that $g(0) = 0$ and $\sup_{|x|<a} |g(x)| < b$. Assume that $f(x, g(x))$ is congruent to 0 modulo $(f_y(x, g(x)))^2$ in $\mathcal{R}$. Prove that there exists an $\mathbb{R}$-valued infinitely differentiable function $q(x)$ on $|x| < \eta$ for some positive number $\eta \le a$ such that

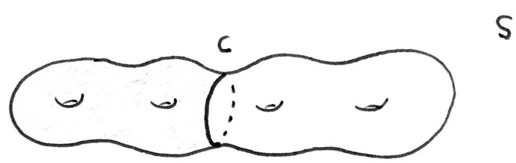
(i) $q(0) = 0$,

(ii) $f(x, q(x)) \equiv 0$ on $|x| < \eta$,

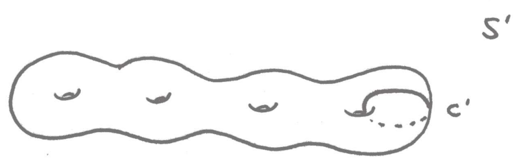
(iii) $q(x)$ is congruent to $g(x)$ modulo $f_y(x, g(x))$ in the ring of all $\mathbb{R}$-valued infinitely differentiable functions on the open interval $(-\eta, \eta)$.

(Note that the usual implicit function theorem is the special case where $g(x) \equiv 0$ and $f_y(x, g(x))$ is nowhere zero on $(-a, a)$ and is therefore a unit in the ring $\mathcal{R}$.)
Hint: Let $q(x) = g(x) + f_y(x, g(x))h(x)$ and solve for $h(x)$ by the usual implicit function theorem after using an appropriate Taylor expansion of the equation.

6) For each of the following, either exhibit an example or show that no such example exists:
a) A space X and a covering map $f : \mathbb{CP}^2 \to X$.
b) A retract from the surface S to the curve C



c) A retract from the surface S' onto the curve C'



QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 20 September 2005 (Day 1)

1. Let X be the CW complex constructed as follows. Start with $Y = S^1$, realized as the unit circle in $\mathbb{C}$; attach one copy of the closed disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$ to Y via the map $\partial D \to S^1$ given by $e^{i\theta} \mapsto e^{4i\theta}$; and then attach another copy of the closed disc D to Y via the map $\partial D \to S^1$ given by $e^{i\theta} \mapsto e^{6i\theta}$.

(a) Calculate the homology groups $H_*(X, \mathbb{Z})$.

(b) Calculate the homology groups $H_*(X, \mathbb{Z}/2\mathbb{Z})$.

(c) Calculate the homology groups $H_*(X, \mathbb{Z}/3\mathbb{Z})$.

2. Show that if a curve in $\mathbb{R}^3$ lies on a sphere and has constant curvature then it is a circle.

3. Let $X \cong \mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2_{\mathbb{C}}$; that is, the space of nonzero homogeneous quadratic polynomials $F \in \mathbb{C}[A, B, C]$ up to scalars. Let $Y \subset X$ be the set of quadratic polynomials that factor as the product of two linear polynomials; and let $Z \subset X$ be the set of quadratic polynomials that are squares of linear polynomials.

(a) Show that Y is a closed subvariety of $X \cong \mathbb{P}^5$, and find its dimension and degree.

(b) Show that Z is a closed subvariety of $X \cong \mathbb{P}^5$, and find its dimension and degree.

4. We say that a linear functional F on $\mathcal{C}([0, 1])$ is *positive* if $F(f) \geq 0$ for all non-negative functions f. Show that a positive F is continuous with the norm $||F|| = F(1)$, where 1 means the constant function 1 on $[0, 1]$.

5. Let D_8 denote the dihedral group with 8 elements.

(a) Calculate the character table of D_8.

(b) Let V denote the four dimensional representation of D_8 corresponding to the natural action of the dihedral group on the vertices of a square. Decompose $\mathrm{Sym}^2 V$ as a sum of irreducible representations.

6. Let f be a holomorphic function on $\mathbb{C}$ with no zeros. Does there always exist a holomorphic function g on $\mathbb{C}$ such that $\exp(g) = f$?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday 21 September 2005 (Day 2)

1. Let n be a positive integer. Using Cauchy's Integral Formula, calculate the integral

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z}$$

where C is the unit circle in $\mathbb{C}$. Use this to determine the value of the integral

$$\int_0^{2\pi} \sin^n t \, dt$$

2. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve of degree $d > 1$. Let $\mathbb{P}^{2^*}$ be the dual projective plane, and $C^* \subset \mathbb{P}^{2^*}$ the set of tangent lines to C.

(a) Show that C^* is a closed subvariety of $\mathbb{P}^{2^*}$.

(b) Find the degree of C^*.

(c) Show that not every tangent line to C is bitangent, i.e., that a general tangent line to C is tangent at only one point. (Note: this is false if $\mathbb{C}$ is replaced by a field of characteristic $p > 0$!)

3. Find all surfaces of revolution $S \subset \mathbb{R}^3$ such that the mean curvature of S vanishes identically.

4. Find all solutions to the equation $y''(t) + y(t) = \delta(t+1)$ in the space $\mathcal{D}'(\mathbb{R})$ of distributions on $\mathbb{R}$. Here $\delta(t)$ is the Dirac delta-function.

5. Calculate the Galois group of the splitting field of $x^5 - 2$ over $\mathbb{Q}$, and draw the lattice of subfields.

6. A covering space $f : X \to Y$ with X and Y connected is called *normal* if for any pair of points $p, q \in X$ with $f(p) = f(q)$ there exists a deck transformation (that is, an automorphism $g : X \to X$ such that $g \circ f = f$) carrying p to q.

(a) Show that a covering space $f : X \to Y$ is normal if and only if for any $p \in X$ the image of the map $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ is a normal subgroup of $\pi_1(Y, f(p))$.

(b) Let $Y \cong S^1 \vee S^1$ be a figure 8, that is, the one point join of two circles. Draw a normal 3-sheeted covering space of Y, and a non-normal three-sheeted covering space of Y.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 22 September 2005 (Day 3)

1. Let $f : \mathbb{CP}^m \to \mathbb{CP}^n$ be any continuous map between complex projective spaces of dimensions m and n.

(a) If $m > n$, show that the induced map $f_* : H_k(\mathbb{CP}^m, \mathbb{Z}) \to H_k(\mathbb{CP}^n, \mathbb{Z})$ is zero for all $k > 0$.

(b) If $m = n$, the induced map $f_* : H_{2m}(\mathbb{CP}^m, \mathbb{Z}) \cong \mathbb{Z} \to H_{2m}(\mathbb{CP}^m, \mathbb{Z}) \cong \mathbb{Z}$ is multiplication by some integer d, called the *degree* of the map f. What integers d occur as degrees of continuous maps $f : \mathbb{CP}^m \to \mathbb{CP}^m$? Justify your answer.

2. Let $a_1, a_2, \ldots, a_n$ be complex numbers. Prove there exists a real $x \in [0, 1]$ such that

$$\left|1 - \sum_{k=1}^{n} a_k e^{2\pi i k x}\right| \geq 1.$$

3. Suppose that ∇ is a connection on a Riemannian manifold M. Define the torsion tensor τ via

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where X, Y are vector fields on M. ∇ is called symmetric if the torsion tensor vanishes. Show that ∇ is symmetric if and only if the Christoffel symbols with respect to any coordinate frame are symmetric, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$. Remember that if $\{E_i\}$ is a coordinate frame, and ∇ is a connection, the Christoffel symbols are defined via

$$\nabla_{E_i} E_j = \sum_k \Gamma^k_{ij} E_k.$$

4. Recall that a commutative ring is called *Artinian* if every strictly descending chain of ideals is finite. Let A be a commutative Artinian ring.

(a) Show that any quotient of A is Artinian.

(b) Show that any prime ideal in A is maximal.

(c) Show that A has only finitely many prime ideals.

5. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 1$, and let $\Lambda \cong \mathbb{P}^k \subset X$ a k-dimensional linear subspace of $\mathbb{P}^n$ contained in X. Show that $k \leq (n-1)/2$.

6. Let $l^\infty(\mathbb{R})$ denote the space of bounded real sequences $\{x_n\}$, $n = 1, 2, \ldots$. Show that there exists a continuous linear functional $L \in l^\infty(\mathbb{R})^*$ with the following properties:

a) $\inf x_n \leq L(\{x_n\}) \leq \sup x_n$,
b) If $\lim_{n\to\infty} x_n = a$ then $L(\{x_n\}) = a$,
c) $L(\{x_n\}) = L(\{x_{n+1}\})$.

Hint: Consider subspace $V \subset l^\infty(\mathbb{R})$ generated by sequences $\{x_{n+1} - x_n\}$. Show that $\{1, 1, \ldots\} \notin \bar{V}$ and apply Hahn-Banach.

Qualifying exam, Fall 2006, Day 1

(1) Let G_1 and G_2 be finite groups, and let V_i be a finite dimensional complex representation of G_i, for $i = 1, 2$. Give $V_1 \otimes_{\mathbb{C}} V_2$ the structure of a representation of the direct product $G_1 \times G_2$ by the rule

$$(g_1, g_2)(v_1 \otimes v_2) := (g_1 v_1) \otimes (g_2 v_2).$$

(a) Show that if V_1 and V_2 are irreducible representations of G_1 and G_2, respectively, then $V_1 \otimes V_2$ is an irreducible representation of $G_1 \times G_2$.

(b) Show that every irreducible representation of $G_1 \times G_2$ arises in this way.

(2) Let R be the polynomial ring on 9 generators $\mathbb{C}[a_{11}, a_{21}, ..., a_{23}, a_{33}]$, and let A be a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with values in R. Let I be the ideal in R generated by the entries of A^3.

(a) Show that the subvariety X of $\mathbb{A}^9$ defined by I is irreducible.

(b) Let J be the ideal of polynomials in R that vanish identically on X. Does J equal I?

(3) Prove that for $n = 1, 2, 3, \ldots$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - 2\sin\theta) d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}.$$

Hint: consider the function $z \mapsto e^{z-1/z}$.

(4) Prove that π_1 of a topological group is abelian.

(5) Let $f : Y \to X$ be a smooth embedding of a manifold Y into a manifold X. Let X be equipped with a Riemannian metric $\overline{g}$ with the associated Levi-Cevita connection $\overline{\nabla}$ on TX. Let $g = f^*\overline{g}$ be the induced metric on Y, with Levi-Cevita connection ∇. For $\eta, \xi \in TY$ define

$$\Psi(\eta, \xi) = \overline{\nabla}_{(f_*\eta)}(f_*\xi) - f_*(\nabla_\eta \xi) \in TX|_Y.$$

Show that Ψ is a well-defined tensor field in $Sym^2(T^*Y) \otimes \mathcal{N}_{Y/X}$, where $\mathcal{N}_{Y/X}$ is the normal to Y in X, i.e., $\mathcal{N}_{Y/X} := TY^\perp \subset TX|_Y$.

(6) Let B be the unit ball in $\mathbb{R}^n$. Prove that the embedding $C^{k+1}(B) \to C^k(B)$ is a compact operator.

Qualifying exam, Fall 2006, Day 2

*All problems are worth 10 points. Problems marked with * will give extra bonus*

(1) Let R be a Noetherian commutative domain, and let M be a torsion-free R-module. (I.e., for $0 \neq r \in R$ and $0 \neq m \in M$ implies $r \cdot m \neq 0$.)
(a) Show that if R is a Dedekind domain and M is finitely generated, then M is a projective R-module.
(b) Give examples showing that M may not be projective if either R is not Dedekind or M is not finitely generated.

(2) Let X be the blow-up of $\mathbb{A}^2$ at 0, and let $Y \subset X$ be the exceptional divisor (i.e., the preimage of 0). Consider the line bundles $\mathcal{L}_n := \mathcal{O}_X(n \cdot Y)$ for $n \in \mathbb{Z}$. Calculate $\Gamma(X, \mathcal{L}_n)$.

(3) Does there exist a nonconstant holomorphic function f on $\mathbb{C}$ such that $f(z)$ is real whenever $|z| = 1$?

(4) Let X be the union of the unit sphere in $\mathbb{R}^3$ and the straight line segment connecting the south and north poles.
(a) Calculate $\pi_1(X)$.
(b*) Calculate $\pi_2(X)$, and describe $\pi_2(X)$ as a $\mathbb{Z}[\pi_1(X)]$-module.

(5) Show that a curve in $\mathbb{R}^3$ lies in a plane if and only if its torsion τ vanishes identically. Identify those curves with vanishing torsion *and* constant curvature k.

(6) Let B be the unit ball in $\mathbb{R}^n$. Recall that if $f : B \to \mathbb{C}$ is a measurable function we define, for $0 < p < \infty$, the $L^p(B)$ norm of f by

$$\|f\|_p = \left(\int_B |f|^p dx \right)^{1/p},$$

and the L^∞ norm of f by

$$\|f\|_\infty = \inf \{a \geq 0 : \{x \in B : |f(x)| > a\} \text{ has Lebesgue measure } 0 \}.$$

The spaces $L^p(B)$ and $L^\infty(B)$ are the spaces of measurable functions on B with finite L^p and L^∞ norms, respectively. Show that if $f \in L^\infty$ then

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q.$$

Qualifying exam, Fall 2006, Day 3

*All problems are worth 10 points. Problems marked with * will give extra bonus*

(1) Let G be a finite p-group, N a normal subgroup, Z the center of G. Prove that $Z \cap N$ is non-trivial.

(2) Let $\mathrm{Gr}(k,n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$, and let W be a fixed d-plane in $\mathbb{C}^n$ with $k+d \geq n$. Let S_i be the subset of $\mathrm{Gr}(k,n)$, consisting of k-planes V, for which $\dim(V+W) \leq n-i$.
(a) Show that S_i is a closed subvariety of $\mathrm{Gr}(k,n)$.
(b) Find the dimension of S_i.
(c*) Show that the singular locus of S_i is contained in S_{i+1}.

(3) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+x+1} dx.$$

(4) Formulate the Poincaré duality theorem for orientable compact manifolds with boundary.

(5) Let G be a Lie group. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g} \subset \mathrm{Lie(G)}$. Show that there exists a unique Lie subgroup $H \subset G$ with $\mathfrak{h} = \mathrm{Lie(H)}$.

(6) Let $f \in L_1(\mathbb{R})$ and $f_\epsilon := \epsilon^{-1} f(x/\epsilon)$. Prove that $\lim_{\epsilon \to +0} f_\epsilon$ exists in the space $\mathcal{D}'(\mathbb{R})$ and find it. Calculate the following limits in $\mathcal{D}'(\mathbb{R})$:

$$\lim_{\epsilon \to +0} \frac{1}{\sqrt{\epsilon}} e^{-\frac{x^2}{\epsilon}}, \quad \lim_{\epsilon \to +0} \frac{\epsilon}{x^2+\epsilon^2}.$$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 18 2007 (Day 1)

1. Let $f(x) = x^4 - 7 \in \mathbb{Q}[x]$.

(a) Show that f is irreducible in $\mathbb{Q}[x]$.

(b) Let K be the splitting field of f over $\mathbb{Q}$. Find the Galois group of $K/\mathbb{Q}$.

(c) How many subfields $L \subset K$ have degree 4 over $\mathbb{Q}$? How many of them are Galois over $\mathbb{Q}$?

2. A real-valued function f defined on an interval $(a, b) \subset \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \;\leq\; \lambda f(x) + (1-\lambda) f(y)$$

whenever $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Prove that every convex function is continuous.

3. Let $\tau_n : S^n \to S^n$ be the antipodal map, and let X be the quotient of $S^n \times S^m$ bythe involution (τ_n, τ_m)—that is,

$$X \;=\; S^n \times S^m/(x, y) \sim (-x, -y)\;\forall\,(x, y)\,.$$

(a) What is the Euler characteristic of X?

(b) Find the homology groups of X in case $n = 1$.

4. Construct a surjective conformal mapping from the pie wedge

$$A = \{z = re^{i\theta} \;:\; \theta \in (0, \pi/4),\; r < 1\}$$

to the unit disk

$$D = \{z \;:\; |z| < 1\}.$$

5. Let $\mathbb{P} \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero $m \times n$ matrices mod scalars, and let $M_k \subset \mathbb{P}$ be the locus of matrices of rank k or less.

(a) Show that M_k is an irreducible algebraic subvariety of $\mathbb{P}$.

(b) Find the dimension of M_k.

(c) In case $k = 1$, find the degree of M_1.

6. Compute the curvature and the torsion of the curve

$$\rho(t) = (t, t^2, t^3)$$

in $\mathbb{R}^3$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday September 19 2007 (Day 2)

1. Evaluate the integral

$$\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 4} dx.$$

2. Consider the paraboloid $S \subset \mathbb{R}^3$ given by the equation $z = x^2 + y^2$. Let g be the metric on S induced by the one on $\mathbb{R}^3$.

(a) Write down the metric g in the coordinate system (x, y).

(b) Compute the Gaussian and the mean curvature of M.

3. Let D_5 denote the group of automorphisms of a regular pentagon. Let V be the 5 dimensional complex representation of D_5 corresponding to the action on the five edges of the pentagon. Decompose V as a sum of irreducible representations.

4. Consider the following three topological spaces:

$$A = \mathbb{CP}^3 \qquad B = S^2 \times S^4 \qquad \text{and} \qquad C = S^2 \vee S^4 \vee S^6$$

where $\mathbb{CP}^3$ is complex projective 3-space, S^n is an n-sphere and $\vee$ denotes connected sum.

(a) Calculate the cohomology groups (with integer coefficients) of all three

(b) Show that A and B are not homotopy equivalent

(c) Show that C is not homotopy equivalent to any compact manifold

5. Let $\mathcal{C}$ be the space $\mathcal{C}[0, 1]$ with the sup norm $||f||_\infty$, and let $\mathcal{C}^1$ be the space $\mathcal{C}^1[0, 1]$ with the sup norm $||f||_\infty + ||f'||_\infty$. Prove that the inclusion $\mathcal{C}^1 \subset \mathcal{C}$ is a compact operator.

6. Let K be a field of characteristic 0.

(a) Find two nonconstant rational functions $f(t), g(t) \in K(t)$ such that

$$f^2 = g^2 + 1.$$

(b) Now let n be any integer, $n \geq 3$. Show that there do not exist two nonconstant rational functions $f(t), g(t) \in K(t)$ such that

$$f^2 = g^n + 1.$$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 20 2007 (Day 3)

1. Let R be the ring

$$R \;=\; \mathbb{C}[x, y, z]/(xy - z^2).$$

Find examples of the following ideals in R:

(a) a minimal prime ideal that is principal;

(b) a minimal prime ideal that is not principal;

(c) a maximal prime ideal than can be generated by two elements; and

(d) a maximal prime ideal than can not be generated by two elements

2. Find the Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

around 0 of the function

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

(a) valid in the open unit disc $\{z : |z| < 1\}$; and

(b) valid in the annulus $\{z : 1 < |z| < 2\}$.

3. (a) Show that any continuous map from the 2-sphere S^2 to a compact orientable 2-manifold of genus $g \geq 1$ is homotopic to a constant map.

(b) Recall that if $f : X \to Y$ is a map between compact, oriented n-manifolds, the induced map $f_* : H_n(X) \to H_n(Y)$ is multiplication by some integer d, called the *degree* of the map f. Now let S and T be compact oriented 2-manifolds of genus g and h respectively, and $f : S \to T$ a continuous map. Show that if $g > h$, then the degree of f is zero.

4. Let H be a (non-trivial) Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on H. Recall that a linear operator $S : H \to H$ is called an adjoint to $T : H \to H$ if

$$(Tx, y) = (x, Sy) \tag{1}$$

holds for all $x, y \in H$.

(a) Prove that any $T \in \mathcal{B}(H)$ has a unique adjoint in $\mathcal{B}(H)$.

(b) Given $T \in \mathcal{B}(H)$, let T^* denote its adjoint. Prove that $(TS)^* = S^*T^*$ for $T, S \in \mathcal{B}(H)$.

(c) Prove that $\|Tx\| = \|T^*x\|$ for all $x \in H$ if and only if $TT^* = T^*T$.

(d) Prove that if $TT^* = T^*T$ then the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal.

5. Prove that every group of order p^2q, where p and q are distinct primes, is solvable.

6. Let $\Gamma = \{p_1, \ldots, p_5\} \subset \mathbb{P}^2$ be a collection of five points in the plane.

(a) What is the Hilbert polynomial of the subvariety $\Gamma \subset \mathbb{P}^2$?

(b) How many different Hilbert functions can Γ have? List them all.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 16 2008 (Day 1)

1. (a) Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbb{Q}$ is of order 6.

(b) Show that in fact G is isomorphic to the symmetric group S_3.

(c) Is there a prime number p such that $X^6 + 3$ is irreducible over the finite field of order p?

2. Evaluate the integral

$$\int_0^\infty \frac{\sqrt{t}}{(1+t)^2} dt.$$

3. For $X \subset \mathbb{R}^3$ a smooth oriented surface, we define the *Gauss map* $g : X \to S^2$ to be the map sending each point $p \in X$ to the unit normal vector to X at p. We say that a point $p \in X$ is *parabolic* if the differential $dg_p : T_p(X) \to T_{g(p)}(S^2)$ of the map g at p is singular.

(a) Find an example of a surface X such that every point of X is parabolic.

(b) Suppose now that the locus of parabolic points is a smooth curve $C \subset X$, and that at every point $p \in C$ the tangent line $T_p(C) \subset T_p(X)$ coincides with the kernel of the map dg_p. Show that C is a planar curve, that is, each connected component lies entirely in some plane in $\mathbb{R}^3$.

4. Let $X = (S^1 \times S^1) \setminus \{p\}$ be a once-punctured torus.

(a) How many connected, 3-sheeted covering spaces $f : Y \to X$ are there?

(b) Show that for any of these covering spaces, Y is either a 3-times punctured torus or a once-punctured surface of genus 2.

5. Let X be a complete metric space with metric ρ. A map $f : X \to X$ is said to be *contracting* if for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < \rho(x, y).$$

The map f is said to be *uniformly contracting* if there exists a constant $c < 1$ such that for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < c \cdot \rho(x, y).$$

(a) Suppose that f is uniformly contracting. Prove that there exists a unique point $x \in X$ such that $f(x) = x$.

(b) Give an example of a contracting map $f : [0, \infty) \to [0, \infty)$ such that $f(x) \neq x$ for all $x \in [0, \infty)$.

6. Let K be an algebraically closed field of characteristic other than 2, and let $Q \subset \mathbb{P}^3$ be the surface defined by the equation

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

(a) Find equations of all lines $L \subset \mathbb{P}^3$ contained in Q.

(b) Let $\mathbb{G} = \mathbb{G}(1, 3) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$, and $F \subset \mathbb{G}$ the set of lines contained in Q. Show that $F \subset \mathbb{G}$ is a closed subvariety.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 17 2008 (Day 2)

1. (a) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(b) What are the units in $\mathbb{Z}[i]$?

(c) What are the primes in $\mathbb{Z}[i]$?

(d) Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

2. Let $U \subset \mathbb{C}$ be the open region

$$U = \{z : |z-1| < 1 \text{ and } |z-i| < 1\}.$$

Find a conformal map $f : U \to \Delta$ of U onto the unit disc $\Delta = \{z : |z| < 1\}$.

3. Let n be a positive integer, A a symmetric $n \times n$ matrix and Q the quadratic form

$$Q(x) = \sum_{1 \le i,j \le n} A_{i,j} x_i x_j.$$

Define a metric on $\mathbb{R}^n$ using the line element whose square is

$$ds^2 = e^{Q(x)} \sum_{1 \le i \le n} dx^i \otimes dx^i.$$

(a) Write down the differential equation satisfied by the geodesics of this metric

(b) Write down the Riemannian curvature tensor of this metric at the origin in $\mathbb{R}^n$.

4. Let H be a separable Hilbert space and $b : H \to H$ a bounded linear operator.

(a) Prove that there exists $r > 0$ such that $b + r$ is invertible.

(b) Suppose that H is infinite dimensional and that b is compact. Prove that b is not invertible.

5. Let $X \subset \mathbb{P}^n$ be a projective variety.

(a) Define the *Hilbert function* $h_X(m)$ and the *Hilbert polynomial* $p_X(m)$ of X.

(b) What is the significance of the degree of p_X? Of the coefficient of its leading term?

(c) For each m, give an example of a variety $X \subset \mathbb{P}^n$ such that $h_X(m) \neq p_X(m)$.

6. Let $X = S^2 \vee \mathbb{RP}^2$ be the wedge of the 2-sphere and the real projective plane. (This is the space obtained from the disjoint union of the 2-sphere and the real projective plane by the equivalence relation that identifies a given point in S^2 with a given point in $\mathbb{RP}^2$, with the quotient topology.)

(a) Find the homology groups $H_n(X, \mathbb{Z})$ for all n.

(b) Describe the universal covering space of X.

(c) Find the fundamental group $\pi_1(X)$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 31 2008 (Day 3)

1. For $z \in \mathbb{C} \setminus \mathbb{Z}$, set

$$f(z) = \lim_{N \to \infty} \left(\sum_{n=-N}^{N} \frac{1}{z+n} \right)$$

(a) Show that this limit exists, and that the function f defined in this way is meromorphic.

(b) Show that $f(z) = \pi \cot \pi z$.

2. Let p be an odd prime.

(a) What is the order of $GL_2(\mathbb{F}_p)$?

(b) Classify the finite groups of order p^2.

(c) Classify the finite groups G of order p^3 such that every element has order p.

3. Let X and Y be compact, connected, oriented 3-manifolds, with

$$\pi_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad \pi_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(a) Find $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ for all n.

(b) Find $H_n(X \times Y, \mathbb{Q})$ for all n.

4. Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the space of differentiable functions on $\mathbb{R}$ with compact support, and let $L^1(\mathbb{R})$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ with respect to the L^1 norm. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{h \to 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)| dy = 0$$

for almost every x.

5. Let $\mathbb{P}^5$ be the projective space of homogeneous quadratic polynomials $F(X, Y, Z)$ over $\mathbb{C}$, and let $\Phi \subset \mathbb{P}^5$ be the set of those polynomials that are products of linear factors. Similarly, let $\mathbb{P}^9$ be the projective space of homogeneous cubic polynomials $F(X, Y, Z)$, and let $\Psi \subset \mathbb{P}^9$ be the set of those polynomials that are products of linear factors.

(a) Show that $\Phi \subset \mathbb{P}^5$ and $\Psi \subset \mathbb{P}^9$ are closed subvarieties.

(b) Find the dimensions of Φ and Ψ.

(c) Find the degrees of Φ and Ψ.

6. Realize S^1 as the quotient $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and consider the following two line bundles over S^1:

L is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$L = \{(\theta, (x, y)) : \cos(\theta) \cdot x + \sin(\theta) \cdot y = 0\}; \text{and}$$

M is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$M = \{(\theta, (x, y)) : \cos(\theta/2) \cdot x + \sin(\theta/2) \cdot y = 0\}.$$

(You should verify for yourself that M is well-defined.) Which of the following are trivial as vector bundles on S^1?

(a) L

(b) M

(c) $L \oplus M$

(d) $M \oplus M$

(e) $M \otimes M$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2009 (Day 1)

1. (RA) Let H be a Hilbert space and $\{u_i\}$ an orthonormal basis for H. Assume that $\{x_i\}$ is a sequence of vectors such that

$$\sum ||x_n - u_n||^2 < 1.$$

Prove that the linear span of $\{x_i\}$ is dense in H.

2. (T) Let $\mathbb{CP}^n$ be complex projective n-space.

(a) Describe the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ and, using the Kunneth formula, the cohomology ring $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$.

(b) Let $\Delta \subset \mathbb{CP}^n \times \mathbb{CP}^n$ be the diagonal, and $\delta = i_*[\Delta] \in H_{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ the image of the fundamental class of Δ under the inclusion $i : \Delta \to \mathbb{CP}^n \times \mathbb{CP}^n$. In terms of your description of $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ above, find the Poincaré dual $\delta^* \in H^{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ of δ.

3. (AG) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety, $\mathbb{G}(1, n)$ the Grassmannian of lines in $\mathbb{P}^n$, and $F \subset \mathbb{G}(1, n)$ the variety of lines contained in X.

(a) If X has dimension k, show that

$$\dim F \leq 2k - 2,$$

with equality holding if and only if $X \subset \mathbb{P}^n$ is a k-plane.

(b) Find an example of a projective variety $X \subset \mathbb{P}^n$ with $\dim X = \dim F = 3$.

4. (CA) Let $\Omega \subset \mathbb{C}$ be the open set

$$\Omega = \{z : |z| < 2 \ \text{ and } \ |z - 1| > 1\}.$$

Give a conformal isomorphism between Ω and the unit disc $\Delta = \{z : |z| < 1\}$.

5. (A) Suppose ϕ is an endomorphism of a 10-dimensional vector space over $\mathbb{Q}$ with the following properties.

1. The characteristic polynomial is $(x - 2)^4(x^2 - 3)^3$.
2. The minimal polynomial is $(x - 2)^2(x^2 - 3)^2$.
3. The endomorphism $\phi - 2I$, where I is the identity map, is of rank 8.

Find the Jordan canonical form for ϕ.

6. (DG) Let $\gamma : (0,1) \to \mathbb{R}^3$ be a smooth arc, with $\gamma' \neq 0$ everywhere.

(a) Define the *curvature* and *torsion* of the arc.

(b) Characterize all such arcs for which the curvature and torsion are constant.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 2, 2009 (Day 2)

1. (CA) Let $\Delta = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disc, and $\Delta^* = \Delta \setminus \{0\}$ the punctured disc. A holomorphic function f on Δ^* is said to have an *essential singularity* at 0 if $z^n f(z)$ does not extend to a holomorphic function on Δ for any n.

Show that if f has an essential singularity at 0, then f assumes values arbitrarily close to every complex number in any neighborhood of 0—that is, for any $w \in \mathbb{C}$ and $\forall \epsilon$ and $\delta > 0$, there exists $z \in \Delta^*$ with

$$|z| < \delta \quad \text{and} \quad |f(z) - w| < \epsilon.$$

2. (AG) Let $S \subset \mathbb{P}^3$ be a smooth algebraic surface of degree d, and $S^* \subset \mathbb{P}^{3*}$ the *dual surface*, that is, the locus of tangent planes to S.

(a) Show that no plane $H \subset \mathbb{P}^3$ is tangent to S everywhere along a curve, and deduce that S^* is indeed a surface.

(b) Assuming that a general tangent plane to S is tangent at only one point (this is true in characteristic 0), find the degree of S^*.

3. (T) Let $X = S^1 \vee S^1$ be a figure 8, $p \in X$ the point of attachment, and let α and $\beta : [0,1] \to X$ be loops with base point p (that is, such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = p$) tracing out the two halves of X. Let Y be the CW complex formed by attaching two 2-discs to X, with attaching maps homotopic to

$$\alpha^2\beta \quad \text{and} \quad \alpha\beta^2.$$

(a) Find the homology groups $H_i(Y, \mathbb{Z})$.

(b) Find the homology groups $H_i(Y, \mathbb{Z}/3)$.

4. (DG) Let $f = f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be smooth, and let $S \subset \mathbb{R}^3$ be the graph of f, with the Riemannian metric ds^2 induced by the standard metric on $\mathbb{R}^3$. Denote the volume form on S by ω.

(a) Show that

$$\omega = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}.$$

(b) Find the curvature of the metric ds^2 on S

5. (RA) Suppose that $\mathcal{O} \subset \mathbb{R}^2$ is an open set with finite Lebesgue measure. Prove that the boundary of the closure of $\mathcal{O}$ has Lebesgue measure 0.

6. (A) Let R be the ring of integers in the field $\mathbb{Q}(\sqrt{-5})$, and S the ring of integers in the field $\mathbb{Q}(\sqrt{-19})$.

(a) Show that R is not a principal ideal domain

(b) Show that S is a principal ideal domain

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 3, 2009 (Day 3)

1. (A) Let $c \in \mathbb{Z}$ be an integer not divisible by 3.

(a) Show that the polynomial $f(x) = x^3 - x + c \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

(b) Show that the Galois group of f is the symmetric group $\mathfrak{S}_3$.

2. (CA) Let τ_1 and $\tau_2 \in \mathbb{C}$ be a pair of complex numbers, independent over $\mathbb{R}$, and $\Lambda = \mathbb{Z}\langle\tau_1, \tau_2\rangle \subset \mathbb{C}$ the lattice of integral linear combinations of τ_1 and τ_2. An entire meromorphic function f is said to be *doubly periodic* with respect to Λ if

$$f(z + \tau_1) = f(z + \tau_2) = f(z) \quad \forall z \in \mathbb{C}.$$

(a) Show that an entire holomorphic function doubly periodic with respect to Λ is constant.

(b) Suppose now that f is an entire meromorphic function doubly periodic with respect to Λ, and that f is either holomorphic or has one simple pole in the closed parallelogram

$$\{a\tau_1 + b\tau_2 : a, b \in [0, 1] \subset \mathbb{R}\}.$$

Show that f is constant.

3. (DG) Let M and N be smooth manifolds, and let $\pi : M \times N \to N$ be the projection; let α be a differential k-form on $M \times N$. Show that α has the form $\pi^*\omega$ for some k-form ω on N if and only if the contraction $\iota_X(\alpha) = 0$ and the derivative $\mathcal{L}_X(\alpha) = 0$ for any vector field X on $M \times N$ whose value at every point is in the kernel of the differential $d\pi$.

4. (RA) Show that the Banach space ℓ^p can be embedded as a summand in $L^p(0, 1)$—in other words, that $L^p(0, 1)$ is isomorphic as a Banach space to the direct sum of ℓ^p and another Banach space.

5. (T) Find the fundamental groups of the following spaces:

(a) $SL_2(\mathbb{R})$

(b) $SL_2(\mathbb{C})$

(c) $SO_3(\mathbb{C})$

6. (AG) Let $X \subset \mathbb{A}^n$ be an affine algebraic variety of pure dimension r over a field K of characteristic 0.

(a) Show that the locus $X_{\text{sing}} \subset X$ of singular points of X is a closed subvariety.

(b) Show that X_{sing} is a proper subvariety of X.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 4 February 2003 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1a. Let k be any field. Show that the ring $k[x]$ has infinitely many maximal ideals.

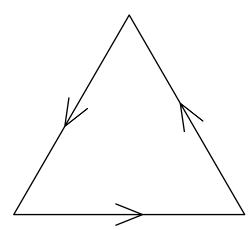
2a. Let f be an entire function. Suppose that f vanishes to even order at every zero of f. Prove there exists a holomorphic function g such that $g^2 = f$.

3a. Let k be a field. Let a, b be relatively prime positive integers. Is there an element in the field of fractions of the k-algebra $A = k[X, Y]/(Y^a - X^b)$ that generates the integral closure of A (i.e., generates it as k-algebra)? If so, find such an element; if not, prove not.

4a. Let $(f(v)\cos(u), f(v)\sin(u), g(v))$ be a parametrization of a surface of revolution $S \subset \mathbb{R}^3$ where $(u, v) \in (0, 2\pi) \times (a, b)$. If S is given the induced metric from $\mathbb{R}^3$, prove that the following map from S to $\mathbb{R}^2$ is locally conformal where $\mathbb{R}^2$ is given the standard Euclidean metric:

$$(u, v) \to \left(u, \int \frac{\sqrt{(f'(v))^2 + (g'(v))^2}}{f(v)} dv \right).$$

5a. Let X be the space obtained by identifying the three edges of a triangle using the same orientation on each edge, as shown below.



Compute $\pi_1(X)$, $H_*(X)$, and $H_*(X \times X)$.

6a. New Real Analysis Problem 1.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday 5 February 2003 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1b. Let $X \subset \mathbb{C}^2$ be the curve defined by $x^2(y^2-1) = 1$, and let $\overline{X} \subset \mathbb{P}^2$ be its closure.

(i) Find the singularities of $\overline{X}$ and classify them into nodes, cusps, and so on.

(ii) Find the genus of the smooth completion of X.

2b. Let $0 < s < 1$. Evaluate the integral

$$\int_0^\infty \frac{x^{s-1}}{1+x}\,dx.$$

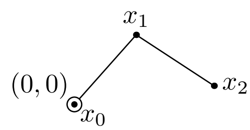
3b. (i) Consider $\mathbb{R}^n$ with the standard Euclidean metric and let $p \in \mathbb{R}^n$ be an arbitrary point. For any $x \in \mathbb{R}^n$ let $\rho_p(x)$ be the distance from p to x. Viewing $\rho_p(x)$ as a smooth function of x away from p, verify that $|\operatorname{grad}(\rho_p(x))|^2 = 1$ and that the integral curves of $\operatorname{grad}(\rho_p(x))$ are straight lines. (Here $\operatorname{grad}(\rho_p(x))$ refers to the usual gradient vector field of the function $\rho_p(x)$.)

(ii) More generally, given a smooth function f on a Riemannian manifold (M, g_{ij}), define $\operatorname{grad}(f)$ to be the vector field given locally by

$$\sum_{i,j} \left(g^{ij} \frac{df}{dx_i} \right) \frac{\partial}{\partial x_j}.$$

Show that if $|\operatorname{grad}(f)|^2 = 1$ then the integral curves of the vector field $\operatorname{grad}(f)$ are geodesics.

4b. A *mechanical linkage* is a collection of points (some fixed, some not) in the plane connected by rigid struts, each with a fixed length. Its *configuration space* is the set of all solutions to the constraints that the struts have a fixed length, with the topology induced from the product of the plane with itself. For instance, this mechanical linkage

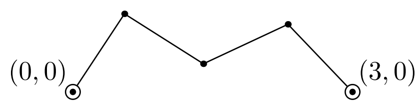


(in which ⊙ denotes a fixed vertex) can be described by the equations

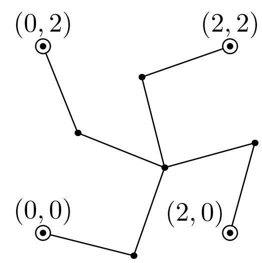
$$\{x_0, x_1, x_2 \in \mathbb{R}^2 | x_0 = (0,0), |x_0 - x_1| = 1, |x_2 - x_1| = 1\}$$

The configuration space of this linkage is the torus $S^1 \times S^1$.

Identify topologically the configuration space of the linkages



and



All edges have length 1, and the fixed vertices are at the indicated locations.

Hint: Consider the position of the central point, and compute the Euler characteristic.

5b. Let H_d be the space of degree d curves in $\mathbb{P}^2$, where $d > 1$. We identify H_d with the projectivization of the vector space of degree d homogeneous polynomials in three variables, so $H_d = \mathbb{P}^N$ for some N.

(i) Find N, the dimension of H_d.

(ii) For a fixed point $p \in \mathbb{P}^2$ find the dimension of the set $\Sigma_p \subset H_d$ of curves that have a singularity at p.

(iii) Find the dimension of the set $\Sigma \subset H_d$ of singular curves.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 6 February 2003 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let $Z \subset \mathbb{P}^n$ be a variety of degree d. Choose a point $P \notin Z$ and let PZ be the union of lines containing the points P and Q, where the union is taken over all points $Q \in Z$. Prove that the degree of PZ is at most d. (Hint: Intersect with a suitable hyperplane and use induction on dimension.)

2c. Let p, q, and r be non-constant non-vanishing entire holomorphic functions that satisfy the equation

$$p + q + r = 0.$$

Prove there exists an entire function h such that p, q and r are constant multiples of h.

3c. Let M be a smooth manifold with a connection ∇ on the tangent bundle. Recall the following definitions of the torsion tensor T and curvature tensor R: For arbitrary vector fields X, Y and Z on M we have

$$T(X,Y) := \nabla_Y X - \nabla_X Y - [X,Y]$$

and

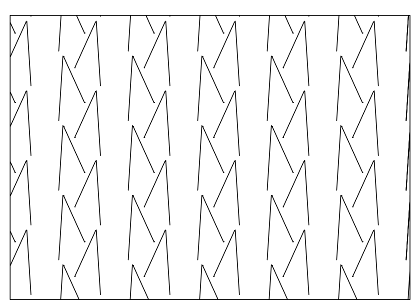
$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z.$$

Assuming we have a torsion-free connection ($T = 0$), verify the following identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

(Hint: Begin by assuming that X, Y, Z are coordinate vector fields, then justify that there is no loss in generality in doing this.)

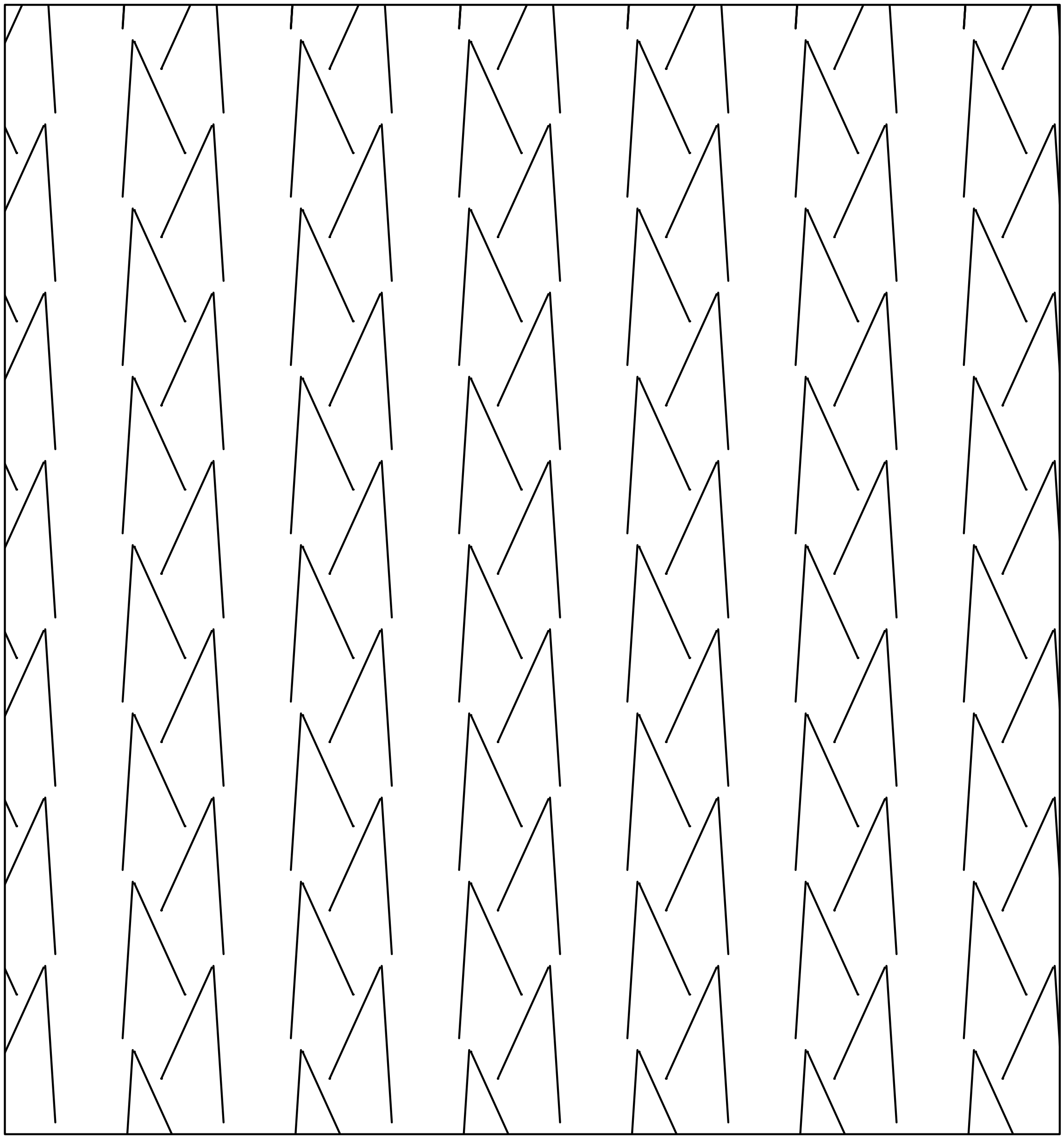
4c. (i) What is the symmetry group G of the following pattern? What is the topological space $\mathbb{R}^2$ modulo G?



(ii) What is the commutator subgroup of G? Draw generators for the commutator subgroup on a copy of the pattern (see Page 6).

5c. Let ρ be a two-dimensional (complex) representation of a finite group G such that $\rho(g)$ has 1 as an eigenvalue for every $g \in G$. Prove that ρ is the sum of two one-dimensional representations.

6c. Let k be a field. Let f, g be polynomials in $k[x, y]$ with no common factor. Show that the quotient ring $k[x, y]/(f, g)$ is a finite dimensional vector space over k.



QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday 10 February 2004 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1a. Prove the following theorem of Banach and Saks:

Theorem. *Given in L^2 a sequence $\{f_n\}$ which weakly converges to* 0*, we can select a subsequence $\{f_{n_k}\}$ such that the sequence of arithmetic means*

$$\frac{f_{n_1} + f_{n_2} + \cdots + f_{n_k}}{k}$$

strongly converges to 0.

(Recall: We say that the sequence $\{f_n\}$ *strongly converges* to f when $||f - f_n|| \to 0$. We say that the sequence $\{f_n\}$ *weakly converges* to f if for every $g \in L^2$, the sequence (f_n, g) converges to (f, g).)

2a. Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. Let $\overline{\mathbf{Z}}$ denote the subset of all elements of $\overline{\mathbf{Q}}$ that satisfy a monic polynomial with coefficients in the ring $\mathbf{Z}$ of integers. You may assume that $\overline{\mathbf{Z}}$ is a ring.

(i) Show that the ideals (2) and $(\sqrt{2})$ in $\overline{\mathbf{Z}}$ are distinct.

(ii) Prove that $\overline{\mathbf{Z}}$ is not Noetherian.

3a. Let B denote the open unit disk in the complex plane $\mathbf{C}$.

(i) Does there exist a surjective, complex-analytic map $f : \mathbf{C} \to B$?

(ii) Does there exist a surjective, complex-analytic map $f : B \to \mathbf{C}$?

4a. (i) Draw a picture of a compact, orientable 2-manifold S (without boundary) of genus 2. On your picture, draw a base-point x and a simple closed curve γ on S that represents a non-trivial element of $\pi_1(S, x)$ but represents the zero element of $H_1(S)$. Justify your answer.

(ii) Let $p : \tilde{S} \to S$ be a two-to-one covering space. Let $\tilde{x}$ be one of the two points in $p^{-1}(x)$. Show that there is a closed path $\tilde{\gamma}$ based at $\tilde{x}$ in $\tilde{S}$ such that $\gamma = p \circ \tilde{\gamma}$.

5a. Given $0 < b < a$, define

$$g(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u),$$

for $(u, v) \in \mathbf{R} \times \mathbf{R}$. The image is a torus. Compute the Gaussian curvature of this torus at points $g(0, v)$.

6a. (i) Let X by a smooth hypersurface of degree d in $\mathbf{P}^n$. What is the degree of the projection of X from one of its points onto a general hyperplane in $\mathbf{P}^n$?

(ii) Prove that every smooth quadric hypersurface in $\mathbf{P}^n$ is rational. (A variety X is rational if it admits a birational map to some projective space.)

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday 11 February 2004 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1b. Let H be a Hilbert space, and let P be a subset of H (not necessarily a subspace). By the orthogonal complement of P we mean the set

$$P^{\perp} = \{y : y \perp x \text{ for all } x \in P\}.$$

(i) (4 points) Show that $P^{\perp}$ is always a closed vector subspace of H.

(ii) (6 points) Show that $P^{\perp\perp}$ is the smallest closed vector subspace that contains P.

2b. Prove that each of the following rings contains infinitely many prime ideals:

(i) (2 points) The ring $\mathbf{Z}$ of integers.

(ii) (2 points) The ring $\mathbf{Q}[x]$ of polynomials over $\mathbf{Q}$.

(iii) (3 points) The ring of regular functions on an affine algebraic surface over $\mathbf{C}$. (You may assume standard results from algebraic geometry.)

(iv) (3 points) The countable direct product of copies of R, for any nonzero commutative ring R with unity.

3b. Show that if $f(z)$ is a polynomial of degree at least 2, then the sum of the residues of $1/f(z)$ at all the zeros of $f(z)$ must be 0.

4b. Let X be a smooth, compact, oriented manifold.

(i) (4 points) Give a clear statement of the Poincaré duality theorem as it applies to the singular homology of X. Deduce from the duality theorem that, if X is connected, there is an isomorphism $\epsilon : H^n(X) \to \mathbf{Z}$.

(ii) (6 points) Use the universal coefficient theorem and the Poincaré duality theorem to show that, if $a \in H^i(X)$ is not a torsion element, then there exists $b \in H^{n-i}(X)$ such that the cup product $a \smile b$ is nonzero.

5b. Let $M^2 \subset \mathbf{R}^3$ be an embedded oriented surface and let S^2 be the unit sphere. The Gauss map $G : M \to S^2$ is defined to be $G(x) = \vec{N}(x)$ for any $x \in M$, where $\vec{N}(x)$ is the unit normal vector of M at x. Let h and g denote the induced Riemannian metric on M and S^2 from $\mathbf{R}^3$ respectively. Prove that if the mean curvature of M is zero everywhere, then the Gauss map G is a conformal map from (M, h) to (S^2, g).

(Recall: If (Σ_1, g_1), (Σ_2, g_2) are two Riemannian manifolds, a map $\varphi : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$ is called *conformal* if $g_1 = \lambda \varphi^* g_2$ for some scalar function λ on Σ_1.)

6b. Consider $X = \mathbf{P}^1 \times \mathbf{P}^1$ sitting in $\mathbf{P}^3$ via the Segre embedding. Prove that the Zariski topology on X is different from the product topology induced by the Zariski topology on both $\mathbf{P}^1$ factors.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 12 February 2004 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1c. Let f be a continuous, real-valued, increasing function on an interval $[a, b]$ such that $f(a) > a$ and $f(b) < b$. Let $x_1 \in [a, b]$, and define a sequence via $x_n = f(x_{n-1})$. Show that $\lim_{n\to\infty} x_n$ exists. If we call this number x^*, show that $f(x^*) = x^*$.

2c. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).

3c. Suppose f is a biholomorphism between two closed annuli in $\mathbf{C}$

$$A(R) = \{z \in \mathbf{C} \mid 1 \le |z| \le R\} \qquad \text{and} \qquad A(S) = \{z \in \mathbf{C} \mid 1 \le |z| \le S\},$$

with $R, S > 1$.

(i) Show that f can be extended to a biholomorphic map from $\mathbf{C} \setminus \{0\}$ to $\mathbf{C} \setminus \{0\}$.

(ii) Prove that $R = S$.

4c. Use homotopy groups to show that there is no retraction $r : \mathbf{RP}^n \to \mathbf{RP}^k$ if $n > k > 0$. (Here $\mathbf{RP}^n$ is real projective space of dimension n.)

5c. Let $\alpha : I \to \mathbf{R}^3$ be a regular curve with nonzero curvature everywhere. Show that if the torsion $\tau(t) = 0$ for all $t \in I$, then $\alpha(t)$ is a plane curve (i.e., the image of α lies entirely in a plane).

6c. Let X be a k-dimensional irreducible subvariety of $\mathbf{P}^n$. In the Grassmannian $\mathbf{G}(1, n)$ of lines in $\mathbf{P}^n$, let $S(X)$ be the set of lines which are secant to X, i.e., which meet X in at least two distinct points. Consider also the union $C(X)$ of all these secant lines, which is a subset of $\mathbf{P}^n$.

(i) Prove that if X is not a linear subspace of $\mathbf{P}^n$, then the closure of $S(X)$ is an irreducible subvariety of $\mathbf{G}(1, n)$ of dimension $2k$.

(ii) Prove that the closure of $C(X)$ is an irreducible subvariety of $\mathbf{P}^n$ of dimension at most $2k + 1$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 25 January 2005 (Day 1)

1. (a) Show that, up to isomorphism, there is a unique group of order 15.

(b) Show that, up to isomorphism, there are exactly two groups of order 10.

2. Let m and n be positive integers, and k another positive integer less than m and n. Let $N = mn - 1$, and realize the complex projective space $\mathbb{P}^N$ as the space of nonzero $m \times n$ complex matrices modulo scalars. Let $X_k \subset \mathbb{P}^N$ be the subset of matrices of rank k or less. Show that X_k is an irreducible closed algebraic subset of $\mathbb{P}^N$, and compute its dimension.

3. (a) Consider $f(x) \in L^1(\mathbb{R}^n)$, $\mathbb{R}^n$ equipped with Lebesgue measure. Show that the function

$$\Phi_f : \mathbb{R}^n \to L^1(\mathbb{R}^n) \text{ given by } \Phi_f(y)(x) = f(x+y)$$

is continuous.

(b) Let $f(x) \in L^1(\mathbb{R})$ and also absolutely continuous. Prove that

$$\lim_{h \to 0} \int \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| dx = 0.$$

4. Let $-1 < a < 1$ and $0 < \beta < \pi$. Compute

$$\int_0^\infty \frac{x^a \, dx}{1 + 2x \cos\beta + x^2}$$

and express the answer as a rational function of π, $\sin a\beta$, $\sin a\pi$, $\sin\beta$ with rational coefficients.

5. Let $f : [a,b] \to \mathbb{R}_+$ be smooth, and let $X \subset \mathbb{R}^3$ be the surface of revolution formed by revolving $z = f(x)$ about the x-axis.

(a) For which functions f is the Gaussian curvature of X

- always positive?
- always negative?
- identically zero?

(b) Characterise by a differential equation those functions f such that X is a minimal surface.

6. Illustrate how a Klein bottle, K, may be decomposed as the union of two Möbius bands, joined along their common boundary. Using this decomposition and van Kampen's theorem, obtain a presentation of $\pi_1(K)$, and hence show that there is surjection from $\pi_1(K)$ to the dihedral group of order $2n$, for all n.

Prove that, for all $n \geq 3$, there exists a connected n-sheeted covering space, $\tilde{K}_n \to K$, that is *not* a normal covering. What is the topology of the covering space $\tilde{K}_5$ in your example?

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday 26 January 2005 (Day 2)

1. (a) Show that every group of order p^n, p prime, has a nontrivial center.

(b) Let G be a group of order p^n, let k be a (possibly infinite) field of characteristic p, and let M be a finite-dimensional k-vector space on which G acts. Show that if $\sigma \in G$ satisfies $\sigma^p = 1$, then σ fixes (pointwise) a nonzero subspace of M.

(c) Under the same assumptions as in the previous part, show that there is a nonzero vector of M fixed by G.

2. Let H_1 and H_2 be two distinct hyperplanes in $\mathbb{P}^n$. Show that any regular function on $\mathbb{P}^n - (H_1 \cap H_2)$ is constant.

3. (a) A (formal) sum $\sum_{n=1}^{\infty} a_n$ of complex numbers $a_n \in \mathbb{C}$ is *Cesaro summable* provided the limit

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} S_i$$

exists, where $S_i := \sum_{j=1}^{i} a_j$. Given a sequence of complex numbers $\{a_n\}_{n\geq 1}$, assume $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$, and $\sum_{n=1}^{\infty} a_n$ is Cesaro summable. Show that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let $f(\theta) \in C^0(S^1)$ (a continuous function on the unit circle) with the property $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 |n| < \infty$, where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta.$$

Show that

$$S_n f \to f \text{ uniformly}$$

as $n \to \infty$ where $S_n f$ are the partial Fourier sums of f, i.e.

$$S_n f(\theta) = \sum_{k=-n}^{n} \hat{f}(k) e^{ik\theta}.$$

4. Let $f(z)$ be a holomorphic function on $\{|z| < 1\}$ and continuous up to $\{|z| \leq 1\}$. Let M be the supremum of $|f|$ on $\{|z| \leq 1\}$. Let L be the intersection of $\{|z| \leq 1\}$ and $\left\{\operatorname{Re} z = \frac{1}{2}\right\}$. Let m be the supremum of $|f|$ on L. Show that $|f(0)|^3 \leq m\, M^2$. (*Hint:* Consider the product of some functions obtained from f.)

5. Let C_1 and C_2 be smooth curves in $\mathbb{R}^3$. Suppose that there exist unit-speed parametrizations $\rho_1 : \mathbb{R} \to \mathbb{R}^3$, $\rho_2 : \mathbb{R} \to \mathbb{R}^3$ of C_1, C_2 such that:

- the curvatures κ_1, κ_2 of ρ_1, ρ_2 coincide and are never zero;
- the torsions τ_1, τ_2 of ρ_1, ρ_2 coincide.

Show that there exists an isometry I of $\mathbb{R}^3$ taking C_1 to C_2.

6. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$, and let $f : S^m \to S^n$ be a map satisfying $f(-x) = -f(x)$ for all x. Assuming that m and n are both at least 1, show that the resulting map $\bar{f} : \mathbb{RP}^m \to \mathbb{RP}^n$ induces a non-zero map $\bar{f}_* : \pi_1(\mathbb{RP}^m) \to \pi_1(\mathbb{RP}^n)$. Use $\mathbb{Z}/2$ cohomology to show that $m \leq n$.

In the case that n is even and $m = n$, show that $\bar{f}$ must have a fixed point.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 27 January 2005 (Day 3)

1. Let p be a prime, and let K be a field of characteristic not equal to p that contains the pth roots of unity. Show that every cyclic extension L of K of degree p can be obtained by adjoining a root of the polynomial $x^p - a$ for some $a \in K$.

2. Let X be the Veronese surface, i.e. the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. If $C \subset X$ is a closed irreducible curve, show that there exists a hypersurface $H \subset \mathbb{P}^5$ such that $H \cap X = C$, where the intersection is considered set-theoretically.

3. Show that

$$\|f * g\|^2_{L^2(\mathbb{R})} \le \|f * f\|_{L^2(\mathbb{R})} \|g * g\|_{L^2(\mathbb{R})}$$

for all $f, g \in L^2(\mathbb{R})$ (with the understanding that either side may be infinite). Can there be such an inequality with $L^1(\mathbb{R})$ instead of $L^2(\mathbb{R})$? (Hint: use delta functions.)

4. By applying the Argument Principle to the domain which is the intersection of a quadrant and the disk centered at the origin of radius R with $R \to \infty$, find out how many roots of the following equation lie in each of the four quadrants:

$$z^4 + z^3 + 4z^2 + 2z + 3 = 0.$$

(*Hint:* First observe that there are no zeroes on the nonnegative real axis and the imaginary axis. Then verify that there are no zeroes on the negative real axis by separately grouping certain terms together in the case $\operatorname{Re} x \le -1$ and in the case $\operatorname{Re} x > -1$. Then consider the change of arguments along the positive imaginary axis and a large quarter-circle.)

5. Let ω be a closed non-degenerate 2-form on a compact smooth manifold M, and let $f : M \to \mathbb{R}$ be smooth.

(a) Show that there is a unique vector field X on M such that $\iota_X \omega = df$.

(b) For each $t \in \mathbb{R}$, let $\rho_t : M \to M$ be the time-t flow of the vector field X. Show that $\rho_t^\star \omega = \omega$ and $\rho_t^\star f = f$ for all $t \in \mathbb{R}$.

(c) Let M be the unit 2-sphere in $\mathbb{R}^3$ and let ω be the standard volume form on M. Find a function $f : M \to \mathbb{R}$ so that the corresponding map ρ_t is rotation about the z-axis by the angle t.

6. Give an example of a pair of CW complexes, (X, A), satisfying all of the following three conditions: (i) there exists an n such that X is obtained from A by adding a single n-cell,

$$X = A \cup_\phi e^n;$$

(ii) the attaching map $\phi : S^{n-1} \to A$ for this n-cell induces the zero map $H_{n-1}(S^{n-1}) \to H_{n-1}(A)$; and (iii) the space A is not a retract of X. Justify your answer.

Qualifying exam, Spring 2006, Day 1

(1) Let $\phi : A \to B$ be a homomorphism of commutative rings, and let $\mathfrak{p}_B \subset B$ be a maximal ideal. Set $A \supset \mathfrak{p}_A := \phi^{-1}(\mathfrak{p}_B)$.

(a) Show that $\mathfrak{p}_A$ is prime but in general non maximal.

(b) Assume that A, B are finitely generated algebras over a field k and ϕ is a morphism of k-algebras. Show that in this case $\mathfrak{p}_A$ is maximal.

(2) Let V be a 4-dimensional vector space over k, and let $Gr^2(V)$ denote the set of 2-dimensional vector subspaces of V. Set $W = \Lambda^2(V)$, and let $\mathbb{P}^5$ be the 5-dimensional projective space, thought of as the set of lines in W.

Define a map of sets $Gr^2(V) \to \mathbb{P}^5$ that sends a 2-dimensional subspace $U \subset V$ to the line $\Lambda^2(U) \subset \Lambda^2(V) = W$.

(a) Show that the above map is injective and identifies $Gr^2(V)$ with the set of points of a projective subvariety of $\mathbb{P}^5$.

(b) Find the dimension of the above projective variety, and its degree.

(3) Are there any non-constant bounded holomorphic functions defined on the complement $\mathbb{C} \setminus I$ of the unit interval

$$I = \{a \in \mathbb{R} \mid 0 \leq a \leq 1\} \subset \mathbb{C}$$

in the complex plane $\mathbb{C}$?

(4) Let X be the topological space obtained by removing one point from a Riemann surface of genus $g \geq 1$. Compute the homotopy groups $\pi_n(X)$.

(5) Let γ be a geodesic curve on a regular surface of revolution $S \subset \mathbb{R}^3$. Let $\theta(p)$ denote the angle the curve forms with the parallel at a point $p \in \gamma$ and $r(p)$ be the distance to the axes of revolution. Prove Clairaut's relation: $r \cos\theta = const$.

(6) Define the function f on the interval $[0, 1]$ as follows. If $x = 0.x_1x_2x_3...$ is the unique non-terminating decimal expansion of $x \in (0, 1]$, define $f(x) = \max_n \{x_n\}$. Prove that f is measurable.

Qualifying exam, Spring 2006, Day 2

(1) Describe irreducible representations of the finite group A_4.

(2) Show that every morphism of projective varieties $\mathbb{P}^2 \to \mathbb{P}^1$ is constant.

(3) Let $g(z)$ be an entire holomorphic function. Define the function $F(z)$ on $\mathbb{C} \setminus [-1, 1]$ by

$$F(z) = \int_{-1}^{1} \frac{g(x)}{x - z} \, dx.$$

(a) Show that $F(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and can be analytically continued across the open interval $(-1, 1)$.
(b) Call $F_-(z)$ and $F_+(z)$ the analytic continuations from below and from above $(-1, 1)$ respectively. Calculate $F_+(z) - F_-(z)$ on $(-1, 1)$.

(4) Let X be the blow-up of $\mathbb{CP}^2$ at one point. Compute the groups $H^i(X, \mathbb{Z})$.

(5) Let $F(x, y, z)$ be a smooth homogenous function of degree n, i.e. $F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x, y, z)$. Prove that away from the origin the induced metric on the conical surface

$$\Sigma = \{(x, y, z) \mid F(x, y, z) = 0\}$$

has Gaussian curvature equal to 0.

(6) Let $p > 0$. Let l^p denote sequences $\underline{x} = \{x_n\} \in \mathbb{R}^{\mathbb{N}}$ (here $\mathbb{N}$ denotes the set of natural numbers), such that $\sum |x_n|^p$ converges. We define a topology on l^p with the basis

$$B^r(\underline{x}) = \{\underline{y} \mid \sum |y_n - x_n|^p < r\}.$$

For which p does this topology arise from a norm?

Qualifying exam, Spring 2006, Day 3

(1) Let $\zeta = e^{2\pi i/37}$ and let $\alpha = \zeta + \zeta^{10} + \zeta^{26}$. Find (with proof) the degree of $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$.

(2) Let $X \subset \mathbb{A}^n$ be an algebraic subvariety, defined by a non-trivial *homegeneous* ideal $I \subset k[t_1, ..., t_n]$.
(a) Show that the point 0 is contained in X.
(b) Assume that X is non-singular at 0. Show that $X = W$ for some linear subspace $W \subset \mathbb{A}^n$.

(3) Let f be a holomorphic function on $\mathbf{C}$ whose image lands in the upper half plane. Prove that f is constant without using Picard's theorem.

(4) Let f be a continuous map $\mathbf{CP}^n \to \mathbf{CP}^n$.
(a) Prove that if n is even, then f necessarily has a fixed point.
(b) Verify that the map $f : \mathbf{C}^4 \to \mathbf{C}^4$ defined by $f(z_1, z_2, z_3, z_4) = (\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3)$ induces a map $\mathbf{CP}^3 \to \mathbf{CP}^3$ with no fixed points.

(5)
(a) Let f, g be two C^∞-maps between manifolds $X \to Y$. Let ω^k be a closed differential form on Y, and consider $f^*(\omega), g^*(\omega) \in \Omega^k(X)$. Assume that there exists a homotopy between f and g, i.e., a smooth map $h : \mathbb{R} \times X \to Y$ such that $h|_{0\times X} = f$ and $h|_{1\times X} = g$. Show that $f^*(\omega) - g^*(\omega)$ is exact.
(b) Deduce that every closed k-form ($k \geq 1$) on $\mathbb{R}^n$ is exact.

(6) Does there exist a continuous function on the interval $[0, 1]$ such that

$$\int_0^1 x^n f(x)dx = \begin{cases} 1, & n = 1 \\ 0, & n = 2, 3, ... \end{cases}$$

Qualifying exam, Spring 2007, Day 1

All problems are worth 10 points.

Problem 1.

(a) Let T be a linear map $V \to V$, where V is a finite-dimensional vector space over an algebraically closed k. Show that V admits a non-zero eigenvector.

(b) For T as above, show that V can be decomposed as a direct sum $V \simeq \bigoplus_i V_i$ of T-stable subspaces V_i, such that for every i the restriction of T to V_i equals the sum of a scalar operator T_i', and a *nilpotent* operator T_i'' such that $(T_i'')^{\dim(V_i)-1} \neq 0$.

(c) Deduce that an $n \times n$ matrix over k can be conjugated to one in the Jordan canonical form.

Problem 2. Let P_1, P_2, P_3, P_4 be a collection of 4 distinct points in $\mathbb{P}^2$. Show that this set is a complete intersection if and only if either all 4 points lie on the same line, or no 3 points lie on the same line.

Problem 3. Let $f \in \mathbb{C}(z)$ be a rational function. Assume that f has no poles on the closed right half-plane $Re(z) \geq 0$ and is bounded on the imaginary axis. Show that f is bounded on the right half-plane.

Problem 4. Show that S^3 minus two linked circles is homotopy equivalent to $S^1 \times S^1$.

Problem 5.

(a) Define orientability of a smooth manifold.

(b) Let X be a n-dimensional submanifold of $\mathbb{R}^{n+1}$. Show that X is orientable if and only if there exists a nowhere-vanishing normal vector field along X.

Problem 6. Let V be a separable Banach space, and let V^* be its continuous dual. Consider the closed unit ball $B(0,1) \subset V^*$ endowed with the *weak* topology. Construct a norm $||\cdot||_{new}$ on V^* such that the above topology is equivalent to one corresponding to this norm.

Qualifying exam, Spring 2007, Day 2

All problems are worth 10 points.

Problem 1.
(a) Show that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
(b) Let M/K be a Galois field extension, and let $K \subset L_1, L_2 \subset M$ be two intermediate fields. Show that $L_1 \simeq L_2$ as K-algebras if and only if $Gal(M/L_1)$ is conjugate to $Gal(M/L_2)$ as subgroups of $Gal(M/K)$.

Problem 2. Construct an isomorphism from a smooth conic in $\mathbb{CP}^2$ to $\mathbb{CP}^1$.

Problem 3.
(a) Prove the *Schwarz Lemma*: Let $f(z)$ be analytic for $|z| < 1$ and satisfy the conditions $f(0) = 0$ and $|f(z)| \leq 1$; then $|f(z)| \leq |z|$.
(b) Let $f(z)$ be analytic on the closed unit disk and satisfy $|f(z)| \leq c < 1$. Show that there is a unique solution in the unit disk to the equation $f(z) = z$.

Problem 4. Consider the topological space, obtained from an annulus by identifying antipodal points on the outer circle and antipodal points on the inner circle (separately). Compute its fundamental group.

Problem 5. Let S be a compact surface contained in a closed ball in $\mathbb{R}^3$ of radius r. Show that there exists at least one point $p \in S$ where the Gauss curvature and the absolute value of mean curvature are bounded below by $\frac{1}{r^2}$ and $\frac{1}{r}$ respectively.

Problem 6. Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x_1, ..., x_n) = \exp(-\sum a_{ij} x_i x_j).$$

Prove that $f \in L^1(\mathbb{R}^n)$ if and only if the matrix A is positive definite. Compute $\|f\|_1$ if this is the case.

Qualifying exam, Spring 2007, Day 3

*All problems are worth 10 points. Problems marked with * give extra bonus.*

Problem 1. Let G be a group and M a G-module (i.e., an abelian group, endowed with an action of G).

(a) Define the cohomology group $H^1(G, M)$.

(b) Show that $H^1(G, M)$ is in bijection with the set of isomorphism classes of short exact sequences of G-modules

$$0 \to M \to \tilde{M} \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ is endowed with the trivial G-action. (We say that two such short exact sequences are isomorphic if there exists an isomorphism of G-modules $\tilde{M}_1 \to \tilde{M}_2$, which maps $M \subset \tilde{M}_1$ *identically* to $M \subset \tilde{M}_2$, and such that induced map $\mathbb{Z} \simeq \tilde{M}_1/M \to \tilde{M}_2/M \simeq \mathbb{Z}$ is also the identity map.)

Problem 2. Let $X \subset \mathbb{A}^{2n^2}$ be the variety of pairs (A, B), where A and B are $n \times n$ matrices such that $A \cdot B = 0$ and $B \cdot A = 0$.

(a) Show that each irreducible component of X has dimension n^2.

(b*) Show that each irreducible component of X is smooth away from the locus where it meets other irreducible components.

Problem 3. Prove that the cosecant function can be written as

$$\csc z = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - z^2},$$

for $z \in \mathbb{C},\ z \neq k\pi\ (k \in \mathbb{Z})$.

Problem 4. Calculate the cohomology $H^i(X, \mathbb{Z})$ (just as abelian groups), where X is the Grassmannian of 2-planes in $\mathbb{C}^4$.

Problem 5.

(a) Let G, H be Lie groups. Let $\phi : G \to H$ be a homomorphism and let K be the kernel of ϕ. Assume that G is connected and K is discrete. Show that K lies in the center of G.

(b) Deduce that the fundamental group of a Lie group is abelian.

Problem 6. Prove Picard's theorem: Let v be a continuously differentiable time-dependent vector field defined on a domain of $\mathbb{R} \times \mathbb{R}^n$, containing a point (t_0, x_0). Show that there exists an open interval $t_0 \in I \subset \mathbb{R}$ and a unique differentiable function $\gamma : I \to \mathbb{R}^n$ with $\gamma(t_0) = x_0$ and $d\gamma|_t = v(t, \gamma(t))$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday January 29 2008 (Day 1)

1. Let $K = \mathbb{C}(x)$ be the field of rational functions in one variable over $\mathbb{C}$, and consider the polynomial

$$f(y) \;=\; y^4 + x \cdot y^2 + x \;\in\; K[y].$$

(a) Show that f is irreducible in $K[y]$.

(b) Let $L = K[y]/(f)$. Is L a Galois extension of K?

(c) Let L' be the splitting field of f over K. Find the Galois group of L'/K

2. Let f be a holomorphic function on the unit disc $\Delta = \{z : |z| < 1\}$. Suppose $|f(z)| < 1$ for all $z \in \Delta$, and that

$$f(\frac{1}{2}) = f(-\frac{1}{2}) = 0.$$

Show that $|f(0)| \leq \frac{1}{3}$.

3. Let $\mathbb{CP}^n$ be complex projective n-space.

(a) Describe the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$.

(b) Let $i : \mathbb{CP}^n \to \mathbb{CP}^{n+1}$ be the inclusion of $\mathbb{CP}^n$ as a hyperplane in $\mathbb{CP}^{n+1}$. Show that there does not exist a map $f : \mathbb{CP}^{n+1} \to \mathbb{CP}^n$ such that the composition $f \circ i$ is the identity on $\mathbb{CP}^n$.

4. Let f be the function on $\mathbb{R}$ defined by

$$f(t) = t, \quad -\pi < t \leq \pi$$

and

$$f(t + 2\pi) = f(t) \quad \forall t.$$

Find the Fourier expansion of f.

5. Let X, Y, Z and W be homogeneous coordinates on projective space $\mathbb{P}^3$ over a field K, and $Q \subset \mathbb{P}^3$ be the surface defined by the equation $XY - ZW = 0$.

(a) Show that Q is smooth and irreducible.

(b) Show that Q is birational to $\mathbb{P}^2$, that is, the function field of Q is isomorphic to $K(s, t)$.

(c) Show that Q is *not* isomorphic to $\mathbb{P}^2$.

6. (a) Define the *curvature* and *torsion* of a differentiable arc in $\mathbb{R}^3$.

(b) Let $\Delta \subset \mathbb{R}^3$ be an arc given parametrically by the $\mathcal{C}^\infty$ vector-valued function $t \mapsto v(t) \in \mathbb{R}^3$ for t in the interval $I = (-1, 1) \subset \mathbb{R}$. Under what conditions is the map

$$\phi : (-\epsilon, \epsilon) \times (0, \eta) \;\rightarrow\; \mathbb{R}^3$$

given by

$$\phi(t, s) \;\mapsto\; v(t) + s \cdot v'(t)$$

an immersion for some positive ϵ and η?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 30 2008 (Day 2)

1. Let $X \subset \mathbb{R}^3$ be the cone $x^2 = y^2 + z^2$, and let Y be the torus $(\sqrt{x^2+y^2}-2)^2 + z^2 = 1$, that is, the torus obtained by rotating the circle $(x-2)^2+z^2-1 = y = 0$ around the z-axis.

(a) Show that for any point $p \in X$ other than the vertex $(0,0,0)$, there is a neighborhood of p in X isometric to an open subset of the Euclidean plane $\mathbb{R}^2$.

(b) Show that no open subset of Y is isometric to any open subset of the Euclidean plane.

2. Let V be an n-dimensional vector space over a field K, and $Q : V \times V \to K$ a symmetric bilinear form. By the *kernel* of Q we mean the subspace of V of vectors v such that $Q(v,w) = 0$ for all $w \in V$, and by the *rank* of Q we mean n minus the dimension of the kernel of Q.

Let $W \subset V$ be a subspace of dimension $n-k$, and let Q' be the restriction of Q to W. Show that

$$\operatorname{rank}(Q) - 2k \;\leq\; \operatorname{rank}(Q') \;\leq\; \operatorname{rank}(Q).$$

3. Find the solution of the differential equation

$$y''' - y'' - y' + y \;=\; 0$$

satisfying the conditions

$$y(0) = y'(0) = 0 \quad \text{and} \quad y''(0) = 1.$$

4. Let S be a compact orientable 2-manifold of genus g, and let S_2 be its *symmetric square*, that is, the quotient of the ordinary product $S \times S$ by the involution exchanging factors.

(a) Show that S_2 is a manifold.

(b) Find the Euler characteristic $\chi(S_2)$.

(c) Find the Betti numbers of S_2.

5. Prove the identity

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$

6. Let $\mathbb{P} \cong \mathbb{P}^n$ be the space of nonzero homogeneous polynomials of degree n in two variables, mod scalars, and let $\Delta \subset \mathbb{P}$ be the locus of polynomials with a repeated factor.

(a) Show that Δ is an irreducible subvariety of $\mathbb{P}$.

(b) Show that $\dim \Delta = n - 1$.

(c) What is the degree of Δ?

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 31 2008 (Day 3)

1. For c a nonzero real number, evaluate the integral

$$\int_0^\infty \frac{\log z}{z^2+c^2}\,dz$$

2. Let $\mathbb{G}(1,4)$ be the Grassmannian parametrizing lines in $\mathbb{P}^4$, and let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface. Let $F \subset \mathbb{G}(1,4)$ be the set of lines contained in Q.

(a) Show that F is an algebraic subvariety of $\mathbb{G}(1,4)$.

(b) Show that F is irreducible.

(c) What is the dimension of F?

3. Let S be a compact orientable 2-manifold of genus 2 (that is, a 2-holed torus), and let $f : S \to S$ be any orientation-preserving homeomorphism of finite order.

(a) Show that f must have a fixed point.

(b) Is this statement still true if we drop the hypothesis that f is orientation-preserving? Prove or give a counterexample.

(c) Is this statement still true if we replace S by a compact orientable 2-manifold of genus 3? Again, prove or give a counterexample

4. (a) State Fermat's Little Theorem on powers in the field $\mathbb{F}_{37}$ with 37 elements.

(b) Let k be any natural number not divisible by 2 or 3, and let $a \in \mathbb{F}_{37}$ be any element. Show that there exists a unique solution to the equation

$$x^k = a$$

in $\mathbb{F}_{37}$.

(c) Solve the equation

$$x^5 = 2$$

in $\mathbb{F}_{37}$.

5. Let X be a Banach space.

(a) Define the *weak topology* on X by describing a basis for the topology.

(b) Let $A : X \to Y$ be a linear operator between Banach spaces that is continuous from the weak topology on X to the norm topology on Y. Show that the image $A(X) \subset Y$ is finite-dimensional.

6. Let $V \cong \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$.

(a) Show that the n^{th} symmetric power $V_n = \mathrm{Sym}^n V$ is irreducible.

(b) Which V_n appear in the decomposition of the tensor product $V_2 \otimes V_3$ into irreducible representations?

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday January 27, 2009 (Day 1)

1. Let $\mathbb{P}^{n^2-1}$ be the space of nonzero $n \times n$ matrices mod scalars, and consider the subset

$$\Sigma \;=\; \{(A,B) : AB = 0\} \;\subset\; \mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}.$$

(a) Prove that Σ is a Zariski closed subset of $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}$.

(b) Is Σ irreducible?

(c) What is the dimension of Σ?

2. Consider the integral

$$\int_0^\infty \sin x \cdot x^{a-1} dx.$$

(a) For which real values of a does the integral converge absolutely? For which does it converge conditionally?

(b) Evaluate the integral for those values of a for which it does converge.

3. (a) Let p be a prime number. Show that a group G of order p^n $(n > 1)$ has a nontrivial normal subgroup, that is, G is not a simple group.

(b) Let p and q be primes, $p > q$. Show that a group G of order pq has a normal Sylow p-subgroup. If G has also a normal Sylow q-subgroup, show that G is cyclic.

(c) Give a necessary and sufficient condition on p and q for the existence of a non-abelian group of order pq. Justify your answer.

4. Let $X = S^1 \vee S^1$ be a figure 8.

(a) Exhibit two three-sheeted covering spaces $f : Y \to X$ and $g : Z \to X$ such that Y and Z are not homeomorphic.

(b) Exhibit two three-sheeted covering spaces $f : Y \to X$ and $g : Z \to X$ such that Y and Z are homeomorphic, but not as covering spaces of X (i.e., there is no homeomorphism $\phi : Y \to Z$ such that $g \circ \phi = f$).

(c) Exhibit a normal (that is, Galois) three-sheeted covering space of X.

(d) Exhibit a non-normal three-sheeted covering space of X.

(e) Which of the above would still be possible if we were considering two-sheeted covering spaces instead of three-sheeted?.

5. Suppose T is a bounded operator in a Hilbert space V and there exist a basis $\{e_k\}$ for V such that $Te_k = \lambda_k e_k$. Prove that T is compact if $\lambda_k \to 0$ as $k \to \infty$.

6. Let $\Sigma \subset \mathbb{R}^3$ be a smooth 2-dimensional submanifold, and $n : \Sigma \to \mathbb{R}^3$ a smooth map such that $n(p)$ is a unit length normal to Σ at p. Identify the tangent bundle $T\Sigma$ as the subspace of pairs $(p, v) \in \Sigma \times \mathbb{R}^3$ such that $v \cdot n(p) = 0$, where $\cdot$ designates the Euclidean inner product. Suppose now that $t \to p(t)$ is a smoothly parametrized curve in $\mathbb{R}^3$ that lies on Σ. Prove that this curve is a geodesic if and only if

$$p''(t) \cdot (n(p(t)) \times p'(t)) = 0 \quad \forall t$$

Here, p' is the derivative of the map $t \to p(t)$ and p'' is the second derivative.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday January 28, 2009 (Day 2)

1. Let $C \subset \mathbb{P}^n$ be a smooth algebraic curve.

(a) Let $\Lambda \subset \mathbb{P}^n$ be a general $(n-4)$-plane. Show that the projection map $\pi_\Lambda : C \to \mathbb{P}^3$ is an embedding.

(b) Now let $\Lambda \subset \mathbb{P}^n$ be a general $(n-3)$-plane. Show that the projection map $\pi_\Lambda : C \to \mathbb{P}^2$ is birational onto its image, and that the image curve has only nodes (ordinary double points) as singularities.

2. Show that the function defined by

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

is analytic in the open disc $|z| < 1$, but has no analytic continuation to any larger domain.

3. (a) Let K be the splitting field of the polynomial $f(x) = x^3 - 2$ over $\mathbb{Q}$. Find the Galois group G of $K/\mathbb{Q}$ and describe its action on the roots of f.

(b) Let K be the splitting field of the polynomial $X^4 + aX^2 + b$ (where a, $b \in \mathbb{Q}$) over the rationals. Assuming that the polynomial is irreducible, prove that the Galois group G of the extension $K/\mathbb{Q}$ is either C_4, or $C_2 \times C_2$, or the dihedral group D_8.

4. Let $\{f_n\}$ be a sequence of functions on the interval $X = (0, 1) \subset \mathbb{R}$, and suppose $f_n \to f$ in $L_p(X)$ for all $p : 1 \leq p < \infty$. Does it imply that $f_n \to f$ almost everywhere? Does it imply that there is a subsequence of f_n converging to f almost everywhere? Prove your answer or give a counterexample.

5. View S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, and in particular S^1 as the unit circle in $\mathbb{C}$. Define an action of S^1 on S^{2n+1} by

$$\lambda : (z_1, \ldots, z_{n+1} \mapsto (\lambda z_1, \ldots, \lambda z_{n+1})$$

The quotient is the space $\mathbb{CP}^n$. View the projection map $\pi : S^{2n+1} \to \mathbb{CP}^n$ as a principal S^1-bundle.

(a) Explain why the restriction to S^{2n+1} of the 1-form

$$A = \frac{1}{2} \sum_{1 \leq k \leq n=1} (\overline{z}_k dz_k - z_k d\overline{z}_k)$$

defines a connection on this bundle.

(b) What is the pullback to S^{2n+1} of the curvature 2-form of this connection?

6. Let $X = S^2 \times \mathbb{RP}^3$ and $Y = S^3 \times \mathbb{RP}^2$

(a) Find the homology groups $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ for all n.

(b) Find the homology groups $H_n(X, \mathbb{Z}/2)$ and $H_n(Y, \mathbb{Z}/2)$ for all n.

(c) Find the homotopy groups $\pi_1(X)$ and $\pi_1(Y)$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 29, 2009 (Day 3)

1. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an algebraic curve of bidegree (a, b) (that is, the zero locus of a bihomogeneous polynomial of bidegree (a, b)), and let $C' \subset \mathbb{P}^3$ be the image of C under the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.

(a) What is the degree of C'?

(b) Assume now that $\max(a, b) \geq 3$. Show that C' lies on one and only one quadric surface $Q \subset \mathbb{P}^3$ (namely, the quadric surface $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$).

2. Find the Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

around 0 of the function

$$f(z) = \frac{1}{z^2 + z + 1}$$

(a) valid in the open unit disc $\{z : |z| < 1\}$, and

(b) valid in the complement $\{z : |z| > 1\}$ of the closed unit disc in $\mathbb{C}$.

3. Let A be a commutative ring. Show that an element $a \in A$ belongs to the intersection of all prime ideals in A if and only if it's nilpotent.

4. Let f be a given real-valued function on $X = (0, 1) \subset \mathbb{R}$, and define a function $\phi : [1, \infty) \to \mathbb{R}$ by

$$\phi(p) = \|f\|^p_{L^p(X)}.$$

Prove that ϕ is convex.

5. Let $X \subset \mathbb{R}^2$ be a connected one-dimensional real analytic submanifold, not contained in a line. Prove that not every tangent line to X is bitangent—that is, it is not the case that for all $p \in X$ there exists $q \neq p \in X$ such that the tangent line to X at p equals the tangent line to X at q as lines in $\mathbb{R}^2$.

6. Let X and Y be two CW complexes.

(a) Show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(b) Let A and B be two subcomplexes of X such that $X = A \cup B$. Show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 19, 2010 (Day 1)

1. Let (X, μ) be a measure space with $\mu(X) < \infty$. For $q > 0$, let $L^q = L^q(X, \mu)$ denote the Banach space completion of the space of bounded functions on X with the norm

$$||f||_q = \left(\int_X |f|^q \mu \right)^{\frac{1}{q}}.$$

Now suppose that $0 < p \leq q$. Prove that all functions in L^q are in L^p, and that the inclusion map $L^q \hookrightarrow L^p$ is continuous.

2. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k, $\mathbb{G}(\ell, n)$ the Grassmannian of ℓ-planes in $\mathbb{P}^n$ for some $\ell < n - k$, and $C(X) \subset \mathbb{G}(\ell, n)$ the variety of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

3. Let λ be real number greater than 1. Show that the equation $ze^{\lambda - z} = 1$ has exactly one solution z with $|z| < 1$, and that this solution z is real. (Hint: use Rouché's theorem.)

4. Let k be a finite field, with algebraic closure $\overline{k}$.

(a) For each integer $n \geq 1$, show that there is a unique subfield $k_n \subset \overline{k}$ containing k and having degree n over k.

(b) Show that k_n is a Galois extension of k, with cyclic Galois group.

(c) Show that the norm map $k_n^\times \to k^\times$ (sending a nonzero element of k_n to the product of its Galois conjugates) is a surjective homomorphism.

5. Suppose ω is a closed 2-form on a manifold M. For every point $p \in M$, let

$$R_p(\omega) = \{v \in T_pM \ : \ \omega_p(v, u) = 0 \text{ for all } u \in T_pM\}.$$

Suppose that the dimension of R_p is the same for all p. Show that the assignment $p \mapsto R_p$ as p varies in M defines an integrable subbundle of the tangent bundle TM.

6. Let X be a topological space. We say that two covering spaces $f : Y \to X$ and $g : Z \to X$ are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that $g \circ h = f$. If X is a compact oriented surface of genus g (that is, a g-holed torus), how many connected 2-sheeted covering spaces does X have, up to isomorphism?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 20, 2010 (Day 2)

1. Let a be an arbitrary real number and b a positive real number. Evaluate the integral

$$\int_0^\infty \frac{\cos(ax)}{\cosh(bx)} dx$$

(Recall that $\cosh(x) = \cos(ix) = \frac{1}{2}(e^x + e^{-x})$ is the hyperbolic cosine.)

2. For any irreducible plane curve $C \subset \mathbb{P}^2$ of degree $d > 1$, we define the *Gauss map* $g : C \to \mathbb{P}^{2^*}$ to be the rational map sending a smooth point $p \in C$ to its tangent line; we define the *dual curve* $C^* \subset \mathbb{P}^{2^*}$ of C to be the image of g.

(a) Show that the dual of the dual of C is C itself.

(b) Show that two irreducible conic curves $C, C' \subset \mathbb{P}^2$ are tangent if and only if their duals are.

3. Let Λ_1 and $\Lambda_2 \subset \mathbb{R}^4$ be complementary 2-planes, and let $X = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2)$ be the complement of their union. Find the homology and cohomology groups of X with integer coefficients.

4. Let $X = \{(x, y, z) : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ be a cylinder. Show that the geodesics on X are *helices*, that is, curves such that the angle between their tangent lines and the vertical is constant.

5. (a) Show that if every closed and bounded subspace of a Hilbert space E is compact, then E is finite dimensional.

(b) Show that any decreasing sequence of nonempty, closed, convex, and bounded subsets of a Hilbert space has a nonempty intersection.

(c) Show that any closed, convex, and bounded subset of a Hilbert space is the intersection of the closed balls that contain it.

(d) Deduce that any closed, convex, and bounded subset of a Hilbert space is compact in the weak topology.

6. Let p be a prime, and let G be the group $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$.

(a) How many subgroups of order p does G have?

(b) How many subgroups of order p^2 does G have? How many of these are cyclic?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 21, 2010 (Day 3)

1. Consider the ring

$$A = \mathbb{Z}[x]/(f) \quad \text{where} \quad f = x^4 - x^3 + x^2 - 2x + 4.$$

Find all prime ideals of A that contain the ideal (3).

2. Let f be a holomorphic function on a domain containing the closed disc $\{z : |z| \leq 3\}$, and suppose that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$$

and find all such functions for which equality holds in this inequality.

3. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a differentiable, positive real function. Find the Gaussian curvature and mean curvature of the surface of revolution

$$S = \{(x, y, z) : y^2 + z^2 = f(x)\}.$$

4. Show that for any given natural number n, there exists a finite Borel measure on the interval $[0, 1] \subset \mathbb{R}$ such that for any real polynomial of degree at most n, we have

$$\int_0^1 p \, d\mu = p'(0).$$

Show, on the other hand, that there does *not* exist a finite Borel measure on the interval $[0, 1] \subset \mathbb{R}$ such that for any real polynomial we have

$$\int_0^1 p \, d\mu = p'(0).$$

5. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^4$.

(a) Find the homology groups $H_*(X, \mathbb{Z}/2)$

(b) Find the homology groups $H_*(X, \mathbb{Z})$

(c) Find the cohomology groups $H^*(X, \mathbb{Z})$

6. By a *twisted cubic curve* we mean the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$[X, Y] \mapsto \big[F_0(X, Y), F_1(X, Y), F_2(X, Y), F_3(X, Y)\big]$$

where $F_0, \ldots, F_3$ form a basis for the space of homogeneous cubic polynomials in X and Y.

(a) Show that if $C \subset \mathbb{P}^3$ is a twisted cubic curve, then there is a 3-dimensional vector space of homogeneous quadratic polynomials on $\mathbb{P}^3$ vanishing on C.

(b) Show that C is the common zero locus of the homogeneous quadratic polynomials vanishing on it.

(c) Suppose now that $Q, Q' \subset \mathbb{P}^3$ are two distinct quadric surfaces containing C. Describe the intersection $Q \cap Q'$.