Qualifying Examination in Topology
August 30, 1978

Instructions

"NEW RULES": Answer any four problems.

Please use a separate packet of paper for each problem. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

NOTE: Graders are, in general, more favorably impressed by your clear understanding of basic concepts than by your ability to quote high-powered theorems. "Compact" means that each open cover has a finite subcover. $\mathbb{R}^n$ denotes real Euclidean $n$-space; and $S^n$ is the standard $n$-sphere, i.e., the set of points in $\mathbb{R}^{n+1}$ at unit distance from the origin. $\mathbb{Z}$ denotes the additive group of integers.
Qualifying Examination in Topology

1. Let $X$ be a compact metric space, and let $Y$ be a Hausdorff space. Suppose that there is a continuous mapping $f$ of $X$ onto $Y$. Prove directly that $Y$ is regular and has a countable basis. (It then follows, of course, that $Y$ is metrizable, but you are not asked to prove this, and you should not use the metrizability of $Y$ in your proof.)

2. Let $X$ be a paracompact Hausdorff space, and let $Y$ be a compact Hausdorff space. Prove that $X \times Y$, with the usual product topology, is paracompact. Include a definition of "paracompact".

3. Let $X$ be a compact Hausdorff space with the property that each of its points is a $G_{\delta}$ set (≡ the intersection of countably many open sets). Prove that $X$ is "first countable", i.e., has a countable basis at each of its points. (Explain..."
7. Let
\[ 0 \rightarrow C' \xrightarrow{i} C \xrightarrow{j} C'' \rightarrow 0 \]
be a short exact sequence of free chain complexes. Define the homology modules
\[ H_\ast(C'), \ H_\ast(C), \ H_\ast(C''), \]
and the maps of the long exact sequence
\[ \cdots \rightarrow H_k(C') \xrightarrow{i_k} H_k(C) \xrightarrow{j_k} H_k(C'') \xrightarrow{\partial_k} H_{k-1}(C') \rightarrow \cdots. \]
Prove exactness at \( H_k(C'') \).

8. Define "orientability" for a topological manifold without boundary. Give examples of orientable and non-orientable manifolds. Show that each simply connected manifold is orientable.

9. Suppose that \( X \) is a connected simplicial complex and let \( \tilde{X} \) be its universal covering space. Prove that \( \tilde{X} \) is contractible if, and only if, \( H_i(\tilde{X}; \mathbb{Z}) = 0 \) for each \( i > 0 \).

10. Consider the torus \( T = S^1 \times S^1 \) where \( S^1 \) is the unit circle. Let \( A = S^1 \times x_0 \) and \( B = x_0 \times S^1 \). Also, let \( X \) denote the space obtained from \( T \) by attaching two 2-cells by means of maps
\[ f : S^1 \xrightarrow{} A, \ \text{degree} \ 2. \]
\[ g : S^1 \xrightarrow{} B, \ \text{degree} \ 3. \]

[Diagram: Figure 1]

Compute the (singular) homology groups \( H_k(X; \mathbb{Z}) \) of \( X \), for all \( k \) (See Fig. 1.).

11. Let \( \varphi : X \vee Y \xrightarrow{} X \vee Y \) denote a map from the wedge of two spaces, where
\[ X \vee Y = (X \times Y_0) \cup (x_0 \times Y) \subset X \times Y, \ x_0 \in X, \ y_0 \in Y, \]
and \( X \) and \( Y \) are finite connected polyhedra. Let \( L(\varphi) \) denote the Lefschetz number of \( \varphi \). Show how to express \( L(\varphi) \) in terms of Lefschetz numbers \( L(f) \) and \( L(g) \), where \( f : X \xrightarrow{} X \) and \( g : Y \xrightarrow{} Y \) are appropriately chosen self maps.
Define complex projective space $\mathbb{CP}^n$, $n \geq 2$. Compute the homotopy groups $\pi_k(\mathbb{CP}^n, x_0)$, $k \leq 2n + 1$. 
Qualifying Examination in Topology
January 17, 1979

Instructions

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Qualifying Examination in Topology

1. Let \( X \) be a compact Hausdorff space, and let \( X_1 \supset X_2 \supset \ldots \) be a nested sequence of closed, nonempty connected subsets of \( X \). Prove that \( \bigcap_{i=1}^{\infty} X_i \) is nonempty and connected.

2. Prove that the product \( \mathbb{R}^J \) of uncountably many copies of \( \mathbb{R} \), with the usual product topology, is not metrizable. (Thus, \( J \) denotes an uncountable set.)

3. Let \( U_1, U_2, \ldots \), be a countable, open covering of a metric space \( X \). Show that this covering has a countable, open refinement that covers \( X \) and is \textit{star-finite} (\( \equiv \) each of its members intersects only finitely many other members of the refinement).

4. Let \( M \) be a compact Hausdorff space, and \( n \) a fixed positive integer. Suppose that each point of \( M \) has an open neighborhood homeomorphic to \( \mathbb{R}^n \). Show that \( M \) embeds topologically in \( \mathbb{R}^q \), for some positive integer \( q \).

5. State the classification theorem for \textit{closed} (\( \equiv \) compact, with no boundary) 2-manifolds. List all closed 2-manifolds with Euler characteristic equal to (a) zero, and (b) minus one. If your list in either part (a) or part (b) contains more than one 2-manifold, prove that the fundamental groups of those having the same Euler characteristic are not isomorphic.

6. Define "covering space" and state the "covering homotopy theorem". If \((\tilde{X}, p)\) is a covering space of \( X \) and \( A \subseteq X \), prove (assuming that each of \( A, X \), and \( \tilde{X} \) is path-connected and locally path-connected) that if the inclusion-induced homomorphism \( \pi_1(A) \to \pi_1(X) \) is a surjection, then \( p^{-1}(A) \) is path-connected.
7. Compute the fundamental group and integral homology groups of the space $X$ obtained from the plane $\mathbb{R}^2$ by collapsing to a point * the subset $A$ consisting of three concentric disjoint circles.

8. Define the cup product in cohomology. Show that for an orientable closed $4k$-manifold this gives rise to a symmetric bilinear pairing. What is its relation to the homology intersection pairing?

9. Prove or disprove the following assertions concerning $H_k(X;\mathbb{Z})$, the singular homology groups of a topological space $X$ with integral coefficients.
   a) If $X$ is compact, $H_k(X;\mathbb{Z}) = 0$ for $k$ sufficiently large.
   b) If $X$ is 0-connected (path connected) and $H_k(X) = 0$ for $k \geq 1$, then $X$ is contractible.
   c) If $X$ and $Y$ have finitely generated homology groups in each dimension, so does $X \times Y$, their cartesian product.
   d) If $X$ is a finite polyhedron of dimension $n$, $H_k(X;\mathbb{Z}) = 0$ for $k > n$.

10. Let $K$ denote a finite polyhedron and $\sigma_n(K) =$ number of $n$-simplexes of $K$. Show that the number
    \[ \sum (-1)^n \sigma_n(K) \]
    is a topological invariant for $|K|$, the underlying space of $K$.

11. Compute the cohomology ring structure of the space $S\mathbb{CP}^3 \times \mathbb{CP}^2$, where $\mathbb{CP}^k$ is complex projective $k$-space and $S =$ suspension.

12. State the "Generalized Schoenflies Theorem". Use it to prove that if $X$ is a compact set in $S^n$ such that $S^n - X$ is homeomorphic to $\mathbb{R}^n$, then $X$ is a cellular subset of $S^n$. Include a definition of "cellular subset of $S^n$". 
Qualifying Examination in Topology
August 29, 1979

Instructions

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Policy on Misprints

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1. Suppose that $i: S^n \to S^{n+k}$ is any piecewise-linear embedding. Compute

$$H_\ast(S^{n+k} - i(S^n); \mathbb{Z}),$$

where $k \geq 1$. What can you say about

$$\pi_1(S^{n+k} - i(S^n); *)$$

for various values of $k$ ($k \geq 2$)?

2. Show that $S^m \times \mathbb{R}P^n$ and $\mathbb{R}P^m \times S^n$ (with $m \neq n$ and $m, n \geq 2$) have the same homotopy groups, but do not have the same homotopy type. Is there a map between these spaces inducing an isomorphism on homotopy groups?

3. Let $i: S^1 \times D^n \to S^1 \times S^n$ ($n \geq 3$) be an imbedding, where $D^n$ denotes the unit ball in $\mathbb{R}^n$. Consider the identification space

$$M = (S^1 \times S^n - i(S^1 \times \text{Int } D^n)) \cup_h (D^2 \times S^{n-1})$$

where $h$ is the identity map of

$$S^1 \times S^{n-1} = (\partial D^2) \times S^{n-1} = \partial (S^1 \times S^n - i(S^1 \times \text{Int } D^n)).$$

Show that $\pi_1(M; *)$ is cyclic.

4. Prove that the product of a countable number of sequentially compact spaces is sequentially compact. (A space is called sequentially compact if each sequence in the space has a subsequence which converges to a point in the space.)
5. Let $X$ and $Y$ be topological spaces, with $Y$ compact. Let $p : X \times Y \rightarrow X$ be the usual projection onto the first factor. Show that $p$ maps each closed set in $X \times Y$ to a closed set in $X$.

6. Let $X$ be an uncountable set well-ordered by "<" and such that each of its sections is countable. (For $x \in X$, the section $S_x$ determined by $x$ is: $\{y \in X \mid y < x\}$.) Show that $X$ with the order topology is not paracompact. Include a definition of "paracompact."

7. Prove that every regular, $\sigma$-compact Hausdorff space is paracompact. (A space is called $\sigma$-compact if it is a countable union of compact subspaces.) Include a definition of "paracompact".

8. Suppose that $X$ is a first countable, Hausdorff space and $A$ is a subset of $X$ which intersects each compact subspace of $X$ in a set closed in $X$. Prove that $A$ is closed in $X$.

9. Let $K$ be a finite polyhedron (= underlying space of a finite geometric complex in some $\mathbb{R}^n$) and $L$ a contractible subpolyhedron. Show that $|K|$ and $|K|/|L|$ have the same homotopy type. (The underlying topological space of the polyhedron $K$ is denoted by $|K|$. The quotient space obtained from $|K|$ by identifying $|L|$ to a point is denoted by $|K|/|L|$.)

10. Let $X$ denote the space obtained from $S^2$ by identifying the north and south poles to a single point. ($X$ is a "pinched 2-sphere".) Compute the integral homology groups of $X$, as well as the homotopy groups $\pi_1(X)$ and $\pi_2(X)$.
11. Let $X$ denote an arbitrary set. Let $X^{n+1}$ denote the set of $(n+1)$-tuples $(x_0, \cdots, x_n)$, $x_i \in X$. Let $C_n$, $n \geq 0$, denote the free abelian group on $X^{n+1}$ and define

$$\partial_n : C_n \to C_{n-1}, \quad n \geq 1$$

by the formula

$$\partial_n(x_0 \cdots x_n) = \sum_{i=0}^{n} (-1)^i (x_0 \cdots \hat{x_i} \cdots x_n)$$

where "\(^\hat{\cdot}\)" denotes deletion. Set $\partial_0 = 0$. If $C = \{C_n, \partial_n\}$ is the resulting chain complex, compute the homology groups $H_n(C)$, $n \geq 0$.

12. Prove that each finite polyhedron (= underlying space of a finite geometric complex in some $\mathbb{R}^n$) is an ENR. (An ENR, or "Euclidean neighborhood retract" is a space homeomorphic to a retract of an open set in some Euclidean space $\mathbb{R}^k$.)
Qualifying Examination in Topology
January 16, 1980

Instructions

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Policy on Misprints

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Qualifying Examination in Topology

1. Let $X$ and $Y$ denote finite polyhedra. Prove that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. ($\chi =$ Euler characteristic.)

2. Using covering space techniques, find all subgroups of index two in $\mathbb{F}_2$, the free (non-abelian) group of rank two.

3. Prove that whenever $S^3$ is expressed as the union of two connected, non-empty open subsets $U$ and $V$, it is always true that $U \cap V$ is connected.

4. Suppose that
   \[ f : \mathbb{R}P^n \to \mathbb{R}P^n \]
   is the map $[t_0, \ldots, t_n] \mapsto [t_0^k, \ldots, t_n^k]$. Determine the values of $n$ and $k$ for which the Lefschetz number $L(f)$ is not zero.

5. Let $X$ be a compact metric space and $Y$ a separable metric space. Let $C(X, Y)$ denote the space of continuous functions from $X$ into $Y$, with the compact-open topology. (This space's topology has as a subbasis all the sets
   \[ S(C, U) = \{ f \in C(X, Y) | f(C) \subseteq U \}, \]
   where $C$ is compact in $X$ and $U$ is open in $Y$.) Show that $C(X, Y)$ is separable metric.

6. Let $\{U_1, \ldots, U_n\}$ be a finite open covering of the normal Hausdorff space $X$. Prove that there exist continuous functions
   \[ f_i : X \to [0,1], \quad i = 1, \ldots, n, \]
   such that:
   1.Closure $f_i^{-1}(\mathbb{R}-\{0\}) \subseteq U_i$ for each $i$, and
   2.$\sum_{i=1}^n f_i(x) = 1$ for each $x$.

   State a non-trivial theorem in which such "partitions of unity" play an important role, and sketch its proof.

7. Let $S^3$ denote the unit quaternions with multiplication $\mu : S^3 \times S^3 \to S^3$. Show that if $f_1, f_2 : S^3 \to S^3$ are given maps, that the composition
   \[ S^3 \xrightarrow{f_1 \times f_2} S^3 \times S^3 \xrightarrow{\mu} S^3 \]
   has degree equal to $\deg(f_1 + f_2)$, i.e.
   \[ \deg(f_1 f_2) = \deg f_1 + \deg f_2. \]
Let $S_\Omega$ denote the minimal uncountable well-ordered set (it is uncountable, yet each of its sections is countable), with the order topology. Let $\overline{S}_\Omega$ denote $S_\Omega$ with a largest element added (again, with the order topology). Show that $S_\Omega \times \overline{S}_\Omega$ is not normal by showing that the "diagonal"

$$A = \{(x, x) | x \in S_\Omega\}$$

and the "top"

$$B = S_\Omega \times \{\Omega\}$$

are disjoint closed sets in $S_\Omega \times \overline{S}_\Omega$ that fail to have disjoint neighborhoods.

9. Let $X$ be pathwise connected, locally pathwise connected, and locally simply connected. Let $p : \tilde{X} \to X$ be a regular covering. Define the group $A = A(\tilde{X}, p)$ of covering (or "deck") transformations of $(\tilde{X}, p)$. State and prove a theorem relating $A$ to $\pi_1(X, *)$. (You may assume that $\tilde{X}$ is connected.)

10. Suppose that $p : E \to S^n (n \geq 1)$ is a Hurewicz fiber space, with pathwise connected fiber $F$. Prove that

$$i_* : \pi_j(F, *) \to \pi_j(E, *)$$

is injective for each $j$ if and only if there is a section $s : S^n \to E$. (Here, $i : F \to E$ is the natural embedding.) You may assume the basic theorems from homotopy theory.

11. Let $C$ be the cone over a compact metric space $X$. (That is, $C = X \times [0, 1]$ with $X \times \{0\}$ squashed to a point.) Suppose that $C$ embeds in $\mathbb{R}^n$ as a retract of an open set. Prove from first principles that each continuous mapping into $C$ defined on a closed subset of a normal space $Y$ extends continuously to all of $Y$.

12. Suppose that $f : S^n \to S^n$ is a mapping such that the vectors $\vec{0}x$ and $\vec{0}f(x)$ are perpendicular for each $x \in S^n$, where $0$ is the origin in $\mathbb{R}^{n+1}$. Prove that $f$ is homotopic to the identity mapping of $S^n$. Then show that such an $f$ exists if, and only if, $n$ is odd.
Qualifying Examination in Topology
August 26, 1981

Instructions

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1. Let $F = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$. For $f \in F$ and $\varepsilon > 0$ let $U_{f\varepsilon} = \{ g \in F \mid |f(x) - g(x)| < \varepsilon \text{ for all } x \in X \}$. For $f \in F$ and $h \in F$ with $h$ strictly positive, let $U_{fh} = \{ g \in F \mid |f(x) - g(x)| < h(x) \}$. Let $T_1$ and $T_2$ be the topologies on $F$ generated by the set of all $U_{f\varepsilon}$'s and the set of all $U_{fh}$'s respectively.

Which of the following properties are enjoyed by $(F, T_1)$? by $(F, T_2)$?

(1) locally compact
(2) locally connected
(3) 1st countable
(4) metric
(5) Lindelöf

[Proof that the spaces do or do not have the properties is not required]

2. Give an example of a space which is $T_3$ (regular and Hausdorff) but not $T_4$ (normal).

3. Prove that although a space can be the union of two metrizable subspaces without being metrizable, every union of two Lindelöf metrizable subspaces is metric.

4. Use Euler's formula to prove that the three utilities graph is not planar.

Can it be drawn on the torus? The graph has vertices $U_i, V_i$, $i = 1, 2, 3$ and for each pair $(U_i, V_j)$ there is an edge.
5. Identify the surface represented by
\[ aabcb^{-1}c^{-1}d^{e^{-1}}f^{-1}g^{-1}fhih^{-1} \].

6. Suppose \( M \) is a closed, orientable manifold and \( p : M \to M \) is a covering map of degree \( > 1 \). What can you conclude about \( \pi_1(M) \)? Give an example.

7. State and prove the simplicial approximation theorem.

8. Suppose \( p : E \to B \) has the homotopy extension property with respect to every space and it has unique path lifting. Show that if \( B \) is path connected then for \( b, b' \in B \) the spaces \( p^{-1}(b) \) and \( p^{-1}(b') \) are homeomorphic.

9. Let \( M \) be a polyhedral closed, orientable 3-manifold. Show that \( \pi_1(M) = 1 \) implies that \( M \) has the homotopy type of \( S^3 \).

10. Define an excisive triad \( (X; X_1, X_2) \) and derive its Mayer-Vietoris sequence.

11. Let \( p : E \to S^2 \) be a fiber bundle with fiber \( F \). Derive the exact sequence
\[
\cdots \to H_q(F) \xrightarrow{i_*} H_q(E) \to H_{q-2}(F) \to H_{q-1}(F) \xrightarrow{i_*} H_{q-1}(E) \to \cdots
\]
12. Let $A$ be a closed subset of $X$ and denote by $X/A$ the identification space obtained by smashing $A$ to a point. State sufficient conditions and prove using these conditions the isomorphism

$$H_k(X, A) \cong \tilde{H}_k(X/A)$$

for all $k$. 
Qualifying Examination in Topology
January 20, 1982

Instructions

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1. Show that the space of irrational numbers cannot be embedded as a closed subset of Euclidean $n$-space $\mathbb{R}^n$. Show that the irrationals can be embedded as a closed subset of the countably infinite product of real lines.

2. Suppose that $X$ is a compact Hausdorff space such that the diagonal

$\Delta = \{(x,x) \mid x \in X\}$

is a $G_δ$ subset of $X \times X$. Show that $X$ is metrizable. (A "$G_δ$") is a countable intersection of open sets.)

3. Suppose that the real line $\mathbb{R}$ is expressed as a product $X \times Y$. Show that one of $X,Y$ is a point.

4. Describe a cell decomposition of real projective space $\mathbb{RP}^n$. Calculate the Euler characteristic and the fundamental group of $\mathbb{RP}^n$.

5. Prove that the plane $\mathbb{R}^2$ is not the union of any family of disjoint simple closed curves. (A "simple closed curve" is a space homeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$.)

6. Show that every locally connected compact metric space is the continuous image of the unit interval $[0,1]$.
7. Let $X$ be a topological space with the following (integral) homology groups:

$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0, 3 \\ \mathbb{Z}_3 & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $H_q(X; G)$ with the following coefficients:

a) $G = \mathbb{Z}_2$

b) $G = \mathbb{Z}_3$

c) $G = \mathbb{Q}$, the rationals.

8. Let $X = S^1 \vee S^1$ be the figure eight space with wedge point $x_0$. Then $\pi_1(X, x_0)$ is a free group on two generators $a$ and $b$. Find the covering space $\tilde{X}$ of $X$ corresponding to the infinite cyclic group generated by $b$. Is $p : \tilde{X} \to X$ a regular covering?

9. Let $X$ be the twice pinched 2-sphere obtained from $S^2$ by choosing four distinct points $\{x_1, x_2, y_1, y_2\}$ and identifying $x_1$ with $x_2$ and $y_1$ with $y_2$. Find a CW (or simplicial) decomposition of $X$ and compute its homology groups.

10. A space $X$ has the fixed point property (f.p.p.) if every map $f : X \to X$ has a fixed point. Let $\mathbb{R}P^q$ and $\mathbb{C}P^k$ denote real and complex projective spaces. Discuss the f.p.p. for

$$\mathbb{R}P^q \times \mathbb{C}P^k, \quad \mathbb{R}P^q \times \mathbb{R}P^k,$$

proving your assertions.
11. Let $p : E \to B$ denote a Hurewicz fibration with fiber $F$ over a contractible base $B$. Show that there is a fiber homotopy equivalence

\[ E \longrightarrow B \times F \]

\[ \downarrow \quad \quad \quad \downarrow \]

\[ B \]

12. A space $X$ has category $n$ ($\text{cat } X = n$) if it can be covered by $n$ open sets $U_1, \ldots, U_n$ such that each $U_j$ is contractible in $X$ and $n$ is minimal with this property. Show that

\[ \text{cat (2-sphere)} = 2, \quad \text{cat (torus)} = 3. \]
Qualifying Examination

TOPOLOGY

August 25, 1982

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1. Suppose that $X$ is a compact metric space and that $Y$ is a separable metric space having metric $d$. Let $Y^X$ be the set of all continuous functions from $X$ to $Y$. For $f, g$ in $Y^X$ define

$$
\rho(f, g) = \begin{cases} 
\sup \{d(f(x), g(x)) | x \in X \}, & \text{if this number is less than 1}, \\
1 & \text{otherwise}.
\end{cases}
$$

(You may assume that $\rho$ is a metric on $Y^X$.) Prove that $Y^X$ is separable.

2. Prove: In a separable, normal space there is no discrete closed subset of cardinality that of the continuum.

3. Let $X = [0,1] \times [0,1]$ with the topology generated by this "dictionary" order: $(x, y) < (x', y')$ if either $x < x'$, or if $x = x'$ and $y < y'$. Prove that

   a) $X$ is not path connected, and that

   b) $T = [0,1] \times \{1\}$ is not the intersection of countably many open sets in $X$. 
4. Define "piecewise-linear map". Show that the composition of two piecewise linear maps is piecewise-linear.

5. Show that any compact, convex set in Euclidean space $\mathbb{E}^n$ which contains an open set is homeomorphic with the n-cell $B^n$.

6. Show that every continuous function $f : S^k \to S^n$ $(k < n)$ is homotopic to a constant. $S^m$ denotes the m-sphere.
   (Hint: Approximate f by a simplicial map.)

7. Let $T = S^1 \times S^1$ be the torus with base point p. Let X be the unreduced suspension of T:

   $X = T \times [0,1]/T \times \{0\} \sim (p,0), \ T \times \{1\} \sim (p,1)$.

   (a) Find a CW-decomposition of X.

   (b) Compute the cellular homology groups of X.
8. (a) Define "excisive couple".

(b) Suppose \( \{X_1, X_2\} \) is an excisive couple of subsets of \( X \).
    Derive the corresponding Mayer-Vietoris sequence.

9. Let \( \mathbb{CP}^2 \) denote the complex projective plane. Define a map
    \( \alpha : \mathbb{CP}^2 \to \mathbb{CP}^2 \) by
    \[ \alpha(z_0 : z_1 : z_2) = (\overline{z}_0 : \overline{z}_1 : \overline{z}_2), \]
    where \( \overline{z} \) denotes complex conjugation.

(a) Show that the Lefschetz number \( L(\alpha) \) of \( \alpha \), is not zero.

(b) Compute \( L(\alpha) \).

10. Let \( G \) be a finitely-generated abelian group and let \( p > 0 \) be an
    integer. Describe a construction which yields a topological space \( X \)
    with singular homology groups
    \[ \tilde{H}_i(X) = \begin{cases} 
    G & \text{if } i = p \\
    0 & \text{if } i \neq p 
    \end{cases} \]
11. Let \( p : E \to B \) be a Hurewicz fibration, with \( B \) path connected. Prove that the fibers \( p^{-1}(b) \) and \( p^{-1}(b') \) have the same homotopy type, where \( b, b' \) are arbitrary points in \( B \).

12. Give an example of two spaces \( X \) and \( Y \) such that

\[
H_*(X; \mathbb{Z}) \cong H_*(Y; \mathbb{Z})
\]

but \( H^*(X; \mathbb{Z}) \) and \( H^*(Y; \mathbb{Z}) \) are not isomorphic as algebras, with the cup-product being the product in \( H^*(\cdot; \mathbb{Z}) \). Include the computations of the cup-product.
Qualifying Exam

TOPOLOGY

January 19, 1983

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Policy on Misprints

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1. For each $x$ in the real line $\mathbb{R}$, let $\mathbb{R}_x$ be a copy of the real line with its usual topology. Consider the topological space

$$X = \coprod_{x \in \mathbb{R}} \mathbb{R}_x,$$

with the usual product topology. Does $X$ contain a countable subset that is dense? Does there exist a countable basis for the topology of $X$? Justify your answers.

2. Give (with proof) an example of a connected subset of the plane $\mathbb{R}^2$ that is the union of countably many (but more than one!) mutually disjoint subspaces each homeomorphic to $[0,1]$.

3. Suppose that $X$ is a normal topological space, and that $E$ and $F$ are subsets of $X$ such that each of $E, F$ is a countable union of closed sets, and $\overline{E} \cap F = \phi = E \cap \overline{F}$. (The bar denotes closure in $X$.) Prove that $E$ and $F$ have disjoint open neighborhoods in $X$.

4. (A) State the classification theorem for closed surfaces. (A closed surface is a compact, connected 2-manifold without boundary.)

(B) Describe all closed surfaces with these Euler characteristics: $-1, -2$. 

5. Define the (Euclidean) join $A \ast B$ of two subsets $A$ and $B$ of Euclidean $n$-space $\mathbb{R}^n$. Show that $A \ast B$ is compact if each of $A$ and $B$ is compact.

6. A subset $P \subset \mathbb{R}^n$ is defined to be a polyhedron if each point $a \in P$ has a cone neighborhood $N = \{a\} \ast L$ in $P$, where $L$ is compact. Show that each compact polyhedron is a finite union of simplices.

7. Let $P$ and $Q$ denote finite polyhedra with $p_0 \in P$, $q_0 \in Q$.

Define the "wedge" $P \vee Q$ by $P \vee Q = (P \times q_0) \cup (P_0 \times Q) \subset P \times Q$ and let $j : P \vee Q \subset P \times Q$ denote the inclusion map. Show that the homology sequence of the pair $(P \times Q, P \vee Q)$ "splits" so that

$$H_q(X \times Y) \cong H_q(X \vee Y) \oplus H_q(X \times Y, X \vee Y)$$

where $H$ denotes singular homology with coefficients in a P.I.D. $R$.

8. Let $X = S^1 \vee S^1$, the wedge of 2 circles, i.e. $X$ is a "figure eight."

If $\varphi : X \to S^3$ is an imbedding of $X$ in the 3-sphere $S^3$, compute $H_*(S^3 - \varphi(X))$, the (singular) homology of $S^3 - \varphi(X)$ with integer coefficients.

9. Let $K$ denote a finite polyhedron with the property that every map $f : S^n \to K$, $n > 2$ ($S^n$ an $n$-sphere) is homotopic to a constant map.

If $\tilde{K}$ is the universal cover of $K$, show that $\tilde{K}$ is contractible.
10. A compact \( n \)-manifold \( M \) is called spherical if the Hurewicz homomorphism \( h : \pi_n(M) \rightarrow H_n(M;\mathbb{Z}) \) is non-trivial. Prove that \( M \) is spherical if and only if there is a map \( f : S^n \rightarrow M \) such that the induced homomorphism \( f_* : H_n(S^n;\mathbb{Z}) \rightarrow H_n(M;\mathbb{Z}) \) is non-trivial. Give an example of a spherical manifold which is not a sphere.

11. Let \( M \) denote an orientable compact \( n \)-manifold and consider a map \( f : S^n \rightarrow M \). Define the degree of \( f \) (over \( \mathbb{Z} \)) and denote it by \( \text{deg} \ f \). Show that

\[
(\text{deg} \ f) \quad H_q(M,\mathbb{Z}) = 0 \quad \text{for} \quad 0 < q < n.
\]

\([S^n \text{ is the } n\text{-sphere}]

12. Let \( X \) denote a topological space and \( H^* \) the singular cohomology functor with rational coefficients \( \mathbb{Q} \). Define the cup-length of \( X \) as the maximum \( n \) such that

\[
x_1 \cup x_2 \cup \ldots \cup x_n \neq 0
\]

for elements \( x_i \in H^{k_i}(X;\mathbb{Q}), \ k_i \geq 1 \). Compute the cup-length of \( \mathbb{C}P^r \times \mathbb{C}P^s \), and justify your arguments. \( \mathbb{C}P^k \) is complex projective \( k \)-space.
Qualifying Exam

TOPOLOGY

August 24, 1983

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Policy on Misprints

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1. Construct a homeomorphism from the Klein bottle to the connected sum of two projective planes. Define carefully your standard model of each of these 2-manifolds. Also, give a careful definition of the "connected sum" of two oriented manifolds and explain how it relates to the definition used in this problem. Standard results from PL topology may be assumed but should be stated explicitly.

2. A topological space \( X \) is **zero-dimensional** if for every \( x \in X \) and neighborhood \( U \) of \( x \) there is a \( V \subset U \) with \( x \in V \) such that \( V \) is both open and closed. Prove that if \( X \) is separable, metric, and zero-dimensional and if \( H \) and \( K \) are disjoint closed subsets of \( X \), then \( X \) is the union of two disjoint open sets, one containing \( H \) and the other containing \( K \).

3. Let \( X = S^1 \times S^1 \), \( \tilde{X} = \mathbb{R} \times S^1 \), and \( p : \tilde{X} \to X \) be the covering projection defined by

\[
p(x,z) = (e^{2\pi i x}, z^2) .
\]

(Here, \( S^1 \) is regarded as the unit circle in the complex plane, and \( \mathbb{R} \) = reals.) For each of the following functions \( f_i : S^1 \to X \), decide whether or not \( f_i \) lifts to \( \tilde{X} \). Justify your answers.

(i) \( f_1(z) = (z, z) \);  
(ii) \( f_2(z) = (1, z^4) \).
1. Let \( X \) be compact and Hausdorff. Prove that the following are equivalent:

A. Each covering of \( X \) by \( G_\delta \) sets contains a countable subcovering;

B. Each nonempty subset \( Y \) of \( X \) contains a point that is isolated in \( Y \). (That is, \( X \) is scattered in its original topology.)

5. Suppose that \( X \) is a paracompact Hausdorff space each of whose closed subsets is a \( G_\delta \) set. Prove that each subspace of \( X \) is paracompact.

6. Let \( X \) be compact, Hausdorff, and first countable. Let \( Y \) be normal, Hausdorff, and countably compact. (There are two versions of "countably compact" and they may be assumed equivalent for a Hausdorff space \( Y \): (i) For each infinite subset of \( Y \) there is a limit point in \( Y \). (ii) Each countable open covering of \( Y \) contains a finite subcovering.) Prove that \( X \times Y \) is normal and Hausdorff.
7. Compute the cohomology of the p-sphere $S^p$ with integral coefficients and then compute the cohomology ring structure (with integer coefficients) of $S^2 \times S^3$. State carefully the theorems you employ to justify the computation.

8. Let $\mathbb{C}P^k$ denote complex projective space of complex dimension $k$. Express, for each $i \geq 1$, the homotopy groups $\pi_i(\mathbb{C}P^k,*)$ in terms of homotopy groups of spheres and justify your conclusions.

9. Let $K$ denote a finite polyhedron of dimension $n$ and $K^p$, $0 < p < n$, the $p$-skeleton of $K$. Show that if $K^p$ is contractible in $K$ to a point, then $H_i(K;\mathbb{Z}) = 0$ for $0 < i < p$.

10. Let $K$ denote a finite polyhedron (geometric complex) and $a_p$ the number of simplexes of $K$ of dimension $p$. Set $\chi(K) = \sum_{i=0}^{\infty} (-1)^pa_p$, the Euler characteristic of $K$. Show that $\chi(K)$ depends only upon the homotopy type of $|K|$, the underlying space of $K$. 
11. Let $X$ denote a compact, connected \textit{ENR} (Euclidean neighborhood retract). Suppose for every field $\mathbb{K}$, $H_p(X,\mathbb{K}) = 0$, \( p \geq 1 \). Show that $H_p(X,\mathbb{Z}) = 0$, \( p \geq 1 \).

12. Let $S$ denote a compact orientable surface of genus $p$ (p handles). Give a presentation of the fundamental group $\pi_1(S, \ast)$ and justify your answer.
Qualifying Exam

TOPOLOGY

January 18, 1984

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Policy on Misprints

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1. Prove that a Hausdorff space is metrizable if it is the continuous image of a compact, metrizable space.

2. Define what it means for a surjective mapping $f$ between topological spaces to be a **covering map**. Show that if the domain $X$ of $f$ is compact and the range $Y$ of $f$ is Hausdorff, then $f$ is a covering map if it is a **local homeomorphism**. (This means that each point in $X$ has an open neighborhood that is mapped homeomorphically by $f$ to an open set in $Y$.)

3. Let $A$ be a subset of the real line that is the intersection of countably many open sets. Show that $A$ has an equivalent metric that is **complete**.

4. A space is called a **Baire space** if, whenever it contains a countable collection of closed sets each having empty interior, it follows that the union of these closed sets also has empty interior. Is the real line with the topology generated by all half-open intervals $(a,b)$ a Baire space? Is the real line with its usual topology a Baire space? How about the rationals with the usual subspace topology from the real line? Justify your answers from the definition. (Feel free to use the equivalent statement about open sets, if you like.)
5. State and prove the Simplicial Approximation Theorem for mappings between finite simplicial complexes. Use it to prove that the n-sphere is simply connected for $n \geq 2$.

6. Prove: The Continuum Hypothesis implies that there is a regular Hausdorff space $X$ such that the spread of $X$ is $\omega_1$, while the spread of $X^2$ is $\omega_2$. (The spread of a space $Y$ is the smallest cardinal $k$ such that the cardinality of each discrete subspace of $Y$ is less than $k$.)

7. Let $\Sigma_n$ denote the closed, connected, orientable 2-manifold of genus $n$. (Equivalently, $\Sigma_n$ is the connected sum of $n$ copies of the torus $S^1 \times S^1$.) Prove that there is no continuous, degree-one mapping: $\Sigma_1 \to \Sigma_2$. 
8. Suppose a space $X$ has the property that every compact subset of $X$ is contractible in $X$ to a point. Show that for any coefficients $R$, $H_k(X; R) = 0$ for $k \geq 1$, where $H$ is the singular homology functor.

9. A space $X$ has (Ljusternik-Schnirelmann) category $\leq n$ ($\text{Cat}(X) \leq n$) if $X$ can be covered by $n$ open sets $U_1, \ldots, U_n$ such that each $U_i$ is contractible in $X$ to a point. Suppose $X$ is a finite cell complex with a single 0-cell, an arbitrary number of 1-cells and a single 2-cell. Prove that $\text{cat}(X) \leq 3$.

10. In what follows $\mathbb{RP}^2$ is real projective 2-space and $S^k$ is the $k$-sphere. Prove first that

$$H_k(S^1 \vee S^2; \mathbb{Z}_2) \cong H_k(\mathbb{RP}^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

then show that there is no map $f : \mathbb{RP}^2 \to S^1 \vee S^2$ which induces such an isomorphism.
11. Let $X$ and $Y$ denote connected finite complexes (either simplicial or cellular). Prove that for any coefficient domain $R$ and $k \geq 0$

$$H_k(X \times Y; R) \cong H_k(X \times Y, X \vee Y; R) \oplus H_k(X \vee Y; R)$$

where $H$ is the singular homology functor. State conditions on $X$ and $Y$ which insure the validity of this splitting for more general topological spaces.

12. Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration (or a locally trivial fiber space with fiber $F$ and paracompact $B$ --hence a fiber bundle). Show that if $F$ and $B$ are $n$-connected, $n \geq 0$ then $E$ is $n$-connected. (Here $X$ is $n$-connected, $n \geq 1$, if the homotopy groups $\pi_i(X) = 0$ for $i \leq n$).

13. Let $X$ denote a $K(\mathbb{Z}, 2)$, i.e., $\pi_i(X) = 0$ for $i \neq 2$ and $\pi_2(X) \cong \mathbb{Z}$. Compute the cohomology ring structure of $X$, with integer coefficients.
Qualifying Exam

TOPOLOGY

August 29, 1984

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Notation: $H_\ast$, $H^\ast$ denote homology and cohomology functors. $S^k$ = k-sphere, $D^k$ = k-ball = k-disk, $\mathbb{Z}$ = integers, $\mathbb{R}$ = real numbers.

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1. Suppose $X$ is a space with a complete metric $d$, and $f : X \to X$ is a function with the property that for all $x \neq y$ in $X$, 
\[ d(f(x), f(y)) \leq \frac{1}{2} d(x, y). \]
Prove that $f$ is continuous and that there is exactly one point $x \in X$ with $f(x) = x$.

2. Let $A$ be a subset of the real line that is the intersection of countably many open sets. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that $f$ is continuous at each point of $A$ but not continuous at any point of $\mathbb{R} - A$.

3. Prove that there is no continuous, monotone mapping of the 2-sphere onto the circle. (A mapping is monotone if the inverse image of each point in its range is connected.) Is there such a mapping of the 2-sphere onto the torus (=product of two circles)? You may assume that the 2-sphere is simply connected.

4. Let $X$ be a paracompact Hausdorff space and $f : X \to Y$ a continuous closed function onto $Y$ (it takes closed sets to closed sets). Prove that $Y$ is paracompact.
5. Show that any compact connected metric space $X$ containing at least two points contains distinct points $x$ and $y$ such that $X - \{x\}$ and $X - \{y\}$ are connected.

6. Prove that every product of countably many sequentially compact spaces is sequentially compact. (A space is called sequentially compact if each sequence has a convergent subsequence.)

7. Suppose that $S^1 \times D^{n-1} + S^n$ is an imbedding.

Compute $H_\ast( S^n/S^1 \times \{0\}; \mathbb{Z} )$. 
8. Let $X$ and $Y$ be two connected polyhedra with base points $x_0$ and $y_0$, respectively. Assume that $\tilde{H}_*(X; \mathbb{Z})$ consists of $p$-torsion and $\tilde{H}_*(Y, \mathbb{Z})$ consists of $q$-torsion with $(p,q) = 1$ ($p$ and $q$ are relatively prime). Prove that

$$H_*(X \times Y, X \vee Y; \mathbb{Z}) = 0$$

where $X \vee Y = X \times \{y_0\} \cup x_0 \times Y$ and $\tilde{H}_*$ is reduced homology.

9. Define $\mathbb{R}P^k$, real projective space, $k \geq 1$ and show that $\mathbb{R}P^k$ imbeds in $\mathbb{R}P^{k+1}$. Then show that $\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^{k+1}$.

10. Let $M$ be an orientable closed 2-manifold (compact surface without boundary). Let $M \# \mathbb{R}P^2$ denote the connected sum of $M$ and real projective 2-space. By considering the homology $H_*(M \# \mathbb{R}P^2; \mathbb{Z})$ show that $M \# \mathbb{R}P^2$ is not orientable.
11. Let \((M, \partial M)\) be a compact, \(\mathbb{Z}\)-orientable, connected manifold of dimension \(m\) with non-empty boundary \(\partial M\). Show that \(H_{m-1}(M; \mathbb{Z})\) and \(H_{m-1}(M, \partial M)\) are torsion free groups. Show that

\[ H_m(M; G) = 0, \quad H_m(M, \partial M; G) \cong G \]

for any abelian group \(G\).

12. Let \(X\) denote a triangulable space (e.g. a polyhedron).

Define the Euler characteristic \(\chi(X)\) of \(X\). Show that \(\chi(X)\) is a topological invariant and that \(\chi(X \times Y) = \chi(X) \chi(Y)\).

13. Let \(F \xrightarrow{i} E \xrightarrow{p} B\) denote a Hurewicz fibration. Show that if \(F\) is acyclic over \(R\) (a P.I.D.), i.e. \(H_*(F; R) = H_*\) (point; \(R\)), then \(p\) induces isomorphisms \(p_*: H_*(E; R) \to H_*(B; R)\), where \(H_*\) is singular homology.
Qualifying Exam

TOPOLOGY

January 16, 1985

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Notation: $H_*$, $H^*$ denote homology and cohomology functors. $S^k = k$-sphere, $D^k = k$-ball = $k$-disk, $\mathbb{Z} =$ integers, $\mathbb{R} =$ real numbers.

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1. Let \( f : X \to Y \) be a continuous surjection of topological spaces that is \textit{closed} (i.e., maps closed sets to closed sets) and has compact point-inverses. If \( X \) has a countable basis for its topology, prove that \( Y \) does also.

2. State the Baire Category Theorem for complete metric spaces. Prove directly that \( \mathbb{R}^\omega \) (the countably infinite product of real lines with the usual product topology) has a complete metric. Use this fact to prove that \( \mathbb{R}^\omega \) is not a countable union of compact subspaces.

3. Let \( f : S^n \to S^n \) be a continuous map. Prove that if \( f \) is fixed-point-free then \( f \) is homotopic to the antipodal map.
4. Let \( p : \tilde{X} + X \) be a covering of degree \( n \) and suppose \( X \) is a finite polyhedron. Show that \( \tilde{X} \) has a triangulation which makes \( p \) a simplicial map and use this to establish the Euler characteristic formula
\[
\chi(\tilde{X}) = n \cdot \chi(X).
\]

5. Let \( \mathbb{Z}_+ \) denote the positive integers with the discrete topology. Prove that \( \mathbb{Z}_+^\omega \) (the countably infinite product of copies of \( \mathbb{Z}_+ \) with the usual product topology) is homeomorphic to the space of irrationals in the real line.

6. Prove that the 2-sphere has the following "unicoherence" property: if a pair of points is separated in the 2-sphere by the union of two disjoint closed sets then the pair is separated by one of the closed sets.
7. Prove $X$ is hereditarily separable if and only if 
(*) if $Y$ is an uncountable subspace then there is no 
well-order $<$ on $Y$ such that $\{y \in Y : a \leq y\}$ is open in 
$Y$ for each $a \in Y$.

8. Let $(X,x_0)$ and $(Y,y_0)$ be pathwise connected spaces with 
base points. Show that for all $n$

$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \oplus \pi_n(Y, y_0).$$

9. Let $X$ be a finite cell complex. Define the cellular 
chain complex of $X$ and prove that the cellular homology 
groups of $X$ are isomorphic to the singular homology 
groups.
10. Consider the space

\[ X = S^n_1 \times \cdots \times S^n_k. \]

Suppose that \( U = \{U_i \mid i = 1, \ldots, m\} \) is a finite cover where each \( U_i \) is open in \( X \), and each \( U_i \) is contractible in \( X \) to a point. Show that

\[ m \geq k+1. \]

11. Suppose that \( \widetilde{p} : X \to X \) is a covering of degree \( n \).

If \( T : \Delta^k \to X \) is a map (\( \Delta^k \) denotes the \( k \)-simplex), denote by \( \{\tilde{T}_1, \ldots, \tilde{T}_n\} \) the set of its lifts. Consider the map

\[ \varphi_k : C_kX \to C_k\tilde{X} \]

which takes \( T \) to \( \sum_{i=1}^{n} \tilde{T}_i \), where \( C_k(X) \) is the group of singular chains. Prove that

(a) \( \varphi \) is a chain map;

(b) \( p^* \circ \varphi^* : H_*(X) \to H_*(X) \) is multiplication by \( n \).

What does this imply about lens spaces?
12. Let \( C = \{ C_n, d_n \mid n \geq 0 \} \) be a chain complex over the principal ideal domain \( R \). Assume that each \( C_n \) is finitely generated and free and that \( H_n(C) = \{0\} \) for all \( n \). Prove that there is a chain homotopy

\[
D : C \to C
\]

such that

\[
\delta D + D \delta = \text{id},
\]

where \( \text{id} : C \to C \) is the identity.

13. Let \((X, A)\) be a pair with the following property:

if \( f : K \to X \) is a map of the polyhedron \( K \) into \( X \),

then \( f \) is a homotopic to a map \( f' : K \to A \). Prove that the singular homology groups \( H_*(X, A) \) with arbitrary coefficients are all trivial.
14. Let $X = S^2 \times \mathbb{RP}^2$. Compute

$$H_q(X; \mathbb{Z}) \quad \text{and} \quad H_q(X; \mathbb{Z}/2\mathbb{Z}) \quad q \geq 0.$$ 

State the theorems used in the calculation.
Qualifying Exam

TOPOLOGY

August 28, 1985

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Notation: $H_*$, $H^*$ denote homology and cohomology functors. $S^k = k$-sphere, $D^k = k$-ball = $k$-disk, $\mathbb{Z} = \text{integers}$, $\mathbb{R} = \text{real numbers}$.

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1. Show that the product of a paracompact Hausdorff space and a compact Hausdorff space is paracompact. Include a definition of "paracompact".

2. Prove that a locally compact separable metric space is topologically complete (that is, has some complete metric).

3. Let \( C \) denote a collection of compact, connected subsets of the plane, no two of which intersect. Suppose that each element of \( C \) separates the plane into three complementary domains. Show that \( C \) has only countably many elements.

4. Let \( X \) be the space obtained from \( S^1 \times \mathbb{R} \) by removing the interiors of \( k \) disjoint 2-disks.

   (a) Compute (with proofs) the fundamental group, \( \pi_1(X) \).

   (b) What would your answer to part (a) be if \( S^1 \times \mathbb{R} \) were replaced by \( S^2 \times \mathbb{R} \) and the 2-disks were replaced by 3-disks (assume each is a 3-simplex)?

   (c) Let \( Y \) be the union of two copies of the projective plane having exactly one point \( y \) in common. Compute \( \pi_1(Y, y) \).
5. Given the torus \( T = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = 1\} \), define \( p : T \to T \) by \( \alpha(z_1, z_2) = (-z_1, z_2) \).

(a) Does \( p \) have any fixed points?

(b) Let \( T^* = T/(z_1, z_2) \sim p(z_1, z_2) \), with the quotient topology. Identify \( T^* \) as a topological space. (Prove your assertions.)
6. Let \( C(\mathbb{R}, \mathbb{R}) \) denote the set of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \). Then \( F \subseteq C(\mathbb{R}, \mathbb{R}) \) is pointwise-bounded if, for each \( x \in \mathbb{R} \),

\[
\{f(x) \mid f \in F\}
\]
is bounded. For compact \( C \subseteq \mathbb{R} \), \( \varepsilon > 0 \), and \( f \in C(\mathbb{R}, \mathbb{R}) \), define

\[
B_{\varepsilon}(f, C) = \{g \in C(\mathbb{R}, \mathbb{R}) \mid \text{l.u.b.} \{ |f(x) - g(x)| \mid x \in C \} < \varepsilon \}.
\]
The set of all \( B_{\varepsilon}(f, C) \) forms a basis for the topology of compact convergence on \( C(\mathbb{R}, \mathbb{R}) \). Which of the following subsets of \( C(\mathbb{R}, \mathbb{R}) \) are pointwise-bounded? Which are compact in the topology of compact convergence?

(a) \( \{f_n(x) \mid n \in \mathbb{Z}_+\} \), where \( f_n(x) = x + \sin nx \);

(b) \( \{g_n(x) \mid n \in \mathbb{Z}_+\} \), where \( g_n(x) = x^n \);

(c) \( \{h_n(x) \mid n \in \mathbb{Z}_+\} \), where \( h_n(x) = |x|^{1/n} \);

(d) the set of all polynomials of degree at most 4 all of whose coefficients have absolute value less than 1.
7. Let $G$ be a group which acts on the space $X$ in a properly discontinuous manner (i.e. given $x \in X$ there is a neighborhood $V$ of $x$ such that $gV \cap V = \varnothing$ for all $g \neq 1$ in $G$).

(a) Show that the natural projection $p : X \to X/G$ is a regular cover.

(b) Determine the group of covering transformations of $p : X \to X/G$.

8. Let $f,g : S^n \to S^n$ be maps with the property that $f(x) \neq -g(x)$ for all $x \in S^n$.

(a) Show that $f$ is homotopic to $g$.

(b) Using (a), show that if $S^n$ admits a nowhere-vanishing tangent vector field, then $n$ is odd.

(c) If $n$ is odd, show that $S^n$ admits a nowhere-vanishing tangent vector field.
9. Let $X$ be a path-connected space with base-point $x_0$, and consider the function

$$\varphi : \pi_1(X, x_0) \rightarrow H_1(X; \mathbb{Z})$$

which takes $[a]$ to the singular homology class $H_1(a) ([S^1])$ where $[S^1] \in H_1(S^1; \mathbb{Z})$ is a fixed generator. Show that

(a) $\varphi$ is a homomorphism,

(b) $\varphi$ is surjective, and

(c) $\ker \varphi$ is the commutator subgroup $[\pi_1, \pi_1]$ of $\pi_1(X, x_0)$.

10. Define the lens space $L(p, q)$ and show that it admits a CW-structure.

11. Consider the complex projective $n$-space $\mathbb{C}P^n$.

(a) Show that $\mathbb{C}P^n$ is an orientable closed manifold.

(b) Using (a) and induction on $n$, compute the cup-product in $H^*(\mathbb{C}P^n; \mathbb{Z})$. 
12. Let \( X = S^1 \cup D^2 \), where \( f : S^1 \to S^1 \) is the map \( z \mapsto z^3 \). Define, for any space \( Y \),

\[
h_n(Y) = H_n(Y \times X; \mathbb{Z})
\]

(a) Using the properties of singular homology, verify that \( h_n(Y) \) is also a homology theory (dimension axiom not required).

(b) Determine \( h_*(S^n) \), \( n \geq 0 \).
Qualifying Exam

TOPOLOGY

January 15, 1986

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

Notation: $H_*$, $H^*$ denote homology and cohomology functors. $S^k = k$-sphere, $D^k = k$-ball = $k$-disk, $\mathbb{Z} =$ integers, $\mathbb{R} =$ real numbers, $\mathbb{R}^n = $ Euclidean $n$-space, $\mathbb{C} = $ complex numbers.

Policy on Misprints

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1. Let $X$ be a compact, connected Hausdorff space. Prove that $X$ cannot be expressed as the union of a countable number (greater than one) of mutually exclusive, nonempty closed subsets.

2. Prove that paracompactness is hereditary with respect to $F_\sigma$ sets. That is: if $X$ is paracompact and

$$\{A_n \mid n \in \mathbb{Z}_+\}$$

is a family of closed subsets of $X$, then

$$A = \bigcup_{n \in \mathbb{Z}_+} A_n$$

is paracompact.

3. Let $P$ be the real line with the topology generated by the basis

$$\{(a,b) \mid a \text{ and } b \text{ rational}\}$$

Show that $P$ is homeomorphic to the set of irrationals in the real line with the usual topology.
4. Let $P, T, K$ denote the projective plane, torus and Klein bottle respectively. Let $\#$ denote the connected sum of manifolds. Prove that $T\#P$ and $K\#P$ are homeomorphic. Include a definition of the "connected sum" operation.

5. Let $X$ be the "Warsaw circle". That is, $X = A \cup B \cup C$ where

$$A = \{(x, y) | y = \sin \frac{1}{x}, \ 0 < x \leq 1\}$$

$$B = \{(0, y) | -1 \leq y \leq 1\}$$

and $C$ is an arc from $(0,0)$ to $(1, \sin 1)$ as indicated.

(This is a subspace of $\mathbb{R}^2$ with the usual topology.) Prove that each continuous mapping of a finite simplicial complex into $X$ is homotopic to a constant, but that the identity mapping $i : X \to X$ is not homotopic to a constant (i.e., $X$ is not contractible).
6. Let
\[ G \subset SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad-bc = 1, \ a,b,c,d \in \mathbb{Z} \right\} \]
and
\[ X = \{ z \in \mathbb{C} \mid \text{Im} \ z > 0 \} \]
and assume that $G$ acts on $X$ as follows:
\[ (g,z) \mapsto \frac{az + b}{cz + d}, \text{ where } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]

(a) Give an example where $X \to X/G$ is not a covering;

(b) Give an example where $X \to X/G$ is a covering;

(c) Characterize the groups $G$ for which $X \to X/G$ is a covering and $X/G$ is compact.

7. Let $A = S^2 \vee S^4 \vee S^6$ ($\vee$ = wedge) and $\varphi : A \to \mathbb{R}^8$ an imbedding. Compute $H_*(\mathbb{R}^8 - A; \mathbb{Z})$. 
8. Let $M$ denote the class of compact, 4-manifolds and 
$\chi : M \rightarrow \mathbb{Z}$, the Euler characteristic. Show that $\chi$ is 
surjective.

9. Let $E \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ denote the subspace of ordered pairs 
$(u,v)$ such that $u \cdot v = 0$ and $\|u\| = \|v\| = 1$.

Let $p : E \rightarrow S^n$ denote the projection. $p(u,v) = u$.

(a) Show that $p$ is a locally trivial fiber space.

(b) Show that $p$ admits a section $\sigma : S^n \rightarrow E$

$(p \sigma = \text{identity})$ if, and only if, $n$ is odd.

10. Define complex projective space $\mathbb{CP}^n$. Show that $\mathbb{CP}^n$

can be covered by $n+1$ open sets $U_i$ such that each $U_i$
is contractible in $\mathbb{CP}^n$ to a point in $\mathbb{CP}^n$. Show that

$n+1$ is minimal with this property.
11. Recall that $\mathbb{CP}^k$ is naturally imbedded in $\mathbb{CP}^n$, $k \leq n$. Show that $\mathbb{CP}^5 \times \mathbb{CP}^7$ is not a retract of $\mathbb{CP}^6 \times \mathbb{CP}^8$.

12. Let $M$ denote a connected, orientable $2n$-manifold such that $H_i(M; \mathbb{Z}) = 0$ for $0 < i < n$.

(a) Show that $H_n(M; \mathbb{Z})$ is free abelian and finitely generated.

(b) Define a bilinear form $\varphi : H_n(M; \mathbb{Z}) \otimes H_n(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ by $\varphi(a \otimes b) = (a \cup b) \cap \mu$, where $a, b \in H^n(M; \mathbb{Z})$ and $\mu$ is a fundamental class of $M$. Show that the matrix associated with $\varphi$ is symmetric or skew-symmetric.

(c) Construct an example where the associated matrix for $\varphi$ is

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
$$

(The associated matrix is given by $a_{ij} = \varphi(u_i \otimes u_j)$, where $u_i$ form a basis.)
Qualifying Exam

TOPOLOGY

August 27, 1986

Instructions

Answer any four problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems.

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1. Let $\tilde{\mathbb{R}}$ denote the real line with the topology having the collection of half-open intervals

$$\{ [a, b) \mid a, b \in \mathbb{R}, \quad a < b \}$$

as a basis. Prove that the product space $\tilde{\mathbb{R}} \times \tilde{\mathbb{R}}$ contains a countable, dense subset but has no countable basis for its topology. Deduce that $\tilde{\mathbb{R}}$ has no countable basis for its topology.

2. Let $Q$ be the countably infinite product of copies of the unit interval:

$$Q = \prod_{n \in \mathbb{Z}_+} [0,1]_n,$$

with the usual product topology. Let $z$ be the point of $Q$ having all its coordinates zero. Prove that there is a homeomorphism $h$ of $Q$ onto $Q$ such that each coordinate of $h(z)$ lies in the open interval $(0,1)$.
3. Suppose \( X \) is an uncountable metric space. Prove that there is a continuous function from \( X \) into \( \mathbb{R} \) whose range is uncountable.

4. Let \( G \) be a path-connected topological group. Prove that \( G \) is \( n \)-simple, for all \( n \). (Include a definition of "\( n \)-simple".)

5. Let \((K,L)\) be a pair of finite CW-complexes, and \( B \) a space. Denote by

\[ p : B^K \to B^L \]

the map that is induced by restriction. (Notation. \( B^X \) is the space of maps of \( X \to B \), with the compact-open topology.) Prove that \( p \) is a Hurewicz fibration. (Include a definition of "Hurewicz fibration".)
6. Let $M$ be a compact (possibly nonorientable) 3-manifold. Show that

$$\chi(\partial M) = 2 \cdot \chi(M),$$

where $\chi$ denotes Euler characteristic. Deduce that $P^2$ (real projective 2-space) bounds no compact 3-manifold. Find a compact 3-manifold whose boundary is the connected, nonorientable 2-manifold with $\chi = 0$. (Include a definition of "Euler characteristic" for a finite CW-complex.)

7. Let $X$ denote a compact triangulable space and let $H_q(X;\mathbb{Z})$ and $H^q(X,\mathbb{Z}), \ q \geq 0,$ denote the $q$-th singular homology and cohomology groups of $X$. Show that each is a finitely generated abelian group.
8. Let $M$ denote a compact connected manifold (without boundary) and $f : M \to M$. Define the degree of $f$ under appropriate orientability conditions. Using this notion show that when degree $f \neq 0$, $f^{-1}(p_0) \neq \emptyset$ for any $p_0 \in M$.

9. Show that $s^p \times s^q$ cannot be imbedded in $s^{p+q}$, where $s^n$ denotes the $n$-sphere.

10. Let $K$ denote a finite polyhedron and $|K|$ its underlying topological space. Show that $|K|$ may be imbedded in a finite dimensional Euclidean space $\mathbb{R}^N$. If $\varphi : |K| \to \mathbb{R}^N$ is such an imbedding show that $\varphi(|K|)$ is a neighborhood retract in $\mathbb{R}^N$. 
11. Let $G = \mathbb{Z}_2 = \{1, -1\}$. Then Euclidean $n$-space $\mathbb{R}^n$ is a $\mathbb{Z}_2$-space under the action $(-1)x = -x$. The unit sphere $S^{n-1}$ is a $\mathbb{Z}_2$-set (invariant set) under this action. If $X$ is any $\mathbb{Z}_2$-set in $\mathbb{R}^n$ define

$$\text{genus } X = \{\min p| \exists \text{ a } \mathbb{Z}_2\text{-map } f : X \to \mathbb{R}^p - 0\}.$$

Show that genus $S^{n-1} = n$.

12. Let $p : E \to S^n$ be a fiber space in the sense of Hurewicz (universal covering homotopy property), $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$. Let $F = p^{-1}(x), \ x \in S^n$, denote a typical fiber. Prove that the homomorphism

$$\pi_k(F, y) \xrightarrow{\pi_k(i)} \pi_k(E, y), \quad y \in F$$

induced by the inclusion map $i : F \to E$, is injective for all $k$ if, and only if, the map $p$ admits a section, i.e., there is a map $\sigma : S^n \to E$ such that $p\sigma =$ identity.
Qualifying Exam

TOPOLOGY

January 14, 1987

Instructions: Answer any four problems.
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1. Suppose that \( X \) is a complete metric space and that \( G \) is a \( G_\delta \) subset of \( X \) (i.e., \( G \) is the countable intersection of open subsets of \( X \)). Prove that \( G \) with the subspace topology has a complete metric.

2. Prove that the product of a paracompact Hausdorff space and a compact Hausdorff space is paracompact. Include a definition of "paracompact".

3. Let \( X \) be separable metric and let \( Y \) be Hausdorff. Let \( f : X \rightarrow Y \) be a closed, continuous surjection with each point-inverse \( f^{-1}(y) \) compact. Prove that \( Y \) is separable and metrizable. (\( f \) is closed means that \( f \) maps closed sets in \( X \) to closed sets in \( Y \).)
4) Let $G$ be a finite group, and denote by $G^*G$ the join of $G$ with itself. Suppose that

$$\varphi : (G^*G) \times G \to G^*G$$

is the map defined by the rule $((g_1, t, g_2), g) \mapsto (g_1 g, t, g_2 g)$.

Prove that

(a) the natural projection

$$p : G^*G \to (G^*G)/G$$

is a regular cover, where $(G^*G)/G$ is the orbit space;

(b) $(G^*G)/G$ is of the homotopy type of a bouquet of $n-1$ circles, where $n = |G|$; and

(c) $G^*G$ is of the homotopy type of a bouquet of circles, and determine their number.
5) Let $F$ be a free group and $R$ a normal subgroup. Assume that $F$ and $R$ are both finitely generated, and put $\pi = F/R$.

(a) Construct a finite 2-dimensional complex $X$ such that $\pi_1(X,x_0) \cong \pi$.

(b) If $Y$ is a space such that $\pi_1(Y,y_0) \cong \pi$, show that there is a map $f : (X,x_0) \to (Y,y_0)$ such that $\pi_1(f) : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is an isomorphism.

6) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a map such that $f(-x) = -f(x)$, for all $x \in \mathbb{R}^3$. Prove that $f^{-1}(0) \cap S^2 \neq \emptyset$, where $S^2 \subset \mathbb{R}^3$ is the unit sphere centered at the origin.

7) Compute the singular homology groups $H_*(S^n/S^k;\mathbb{Z})$, where $S^k = (\mathbb{R}^{k+1} \times \{0\}) \cap S^n$, for all $k \leq n$. 
8) Let $X$ be a space such that $\tilde{H}^k(X;G) = \{0\}$ for $k \neq n$ and $\tilde{H}^n(X;G) \cong G$ is any abelian group. Assume that $G$ and $\tilde{H}_k(X;\mathbb{Z})$ are finitely generated for all $k$. Prove that $\tilde{H}_k(X;\mathbb{Z})$ is a torsion group for all $k \neq n$, and if $G$ is moreover torsion free, then $\tilde{H}_k(X;\mathbb{Z}) = \{0\}$ for $k \neq n$ and $\tilde{H}_n(X;\mathbb{Z}) \cong \mathbb{Z}$.

9) Let $X$ be a CW-complex, and assume that for every finite CW-complex $K$ of dimension $\leq n$, all maps $f : K \to X$ are null-homotopic. Prove that $\tilde{H}_k(X;G) = \{0\}$ for all $k \leq n$.

10) Let $X$ be $\mathbb{S}^6 \vee \mathbb{P}_2(\mathbb{C})$, and $Y = \mathbb{P}_3(\mathbb{C})$, where $\mathbb{P}_n(\mathbb{C})$ is complex projective $n$-space. Prove that $H_*(X;G) \cong H_*(Y;G)$ for all coefficient groups $G$, but there is no map $f : X \to Y$ which induces an isomorphism $H_*(f) : H_*(X;G) \to H_*(Y;G)$ on the homology groups.
11) Suppose that $X$ is a finite CW-complex such that
\[ \tilde{H}_n(X \times \mathbb{P}_{2n}(\mathbb{R}); \mathbb{Z}) = \{0\} \], where $\mathbb{P}_{2n}(\mathbb{R})$ is real projective 2n-space. Prove that $\tilde{H}_n(X; \mathbb{Z})$ is a torsion-group of odd-order.

12) Let $M^n$ be a compact $n$-manifold (without boundary) imbedded in Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$. Show that $M^n$ is orientable. Include a definition of orientability.
Qualifying Exam

TOPOLOGY

August 26, 1987

Instructions: Answer four problems.

DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PAGE ONE!

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proctor and indicate your interpretation in your solution. In such cases
do not interpret the problem in such a way that it becomes trivial.
1. Prove that a product
\[ \prod_{\alpha \in A} X_{\alpha} \]
of nondegenerate Hausdorff spaces (with the usual product topology) has a countable basis for its topology if and only if:

(a) each \( X_{\alpha} \) has a countable basis; and
(b) the index set \( A \) is countable.

2. Prove that a compact Hausdorff space is metrizable if and only if each of its points has a neighborhood that is metrizable (in the subspace topology).

3. Let \( B \) denote the countably infinite box product of copies of \( \mathbb{R} \), each factor with its usual topology. Let \( b_0 \in B \) denote the point \((0,0,0,...)\). Describe the (connected) component of \( B \) containing \( b_0 \), and prove that it is such.
4. Prove that the fundamental group of a topological group is abelian. Include a definition of "topological group".

5. Prove that \( \chi(K) \leq 2 \) for any combinatorial surface \( K \).

(A combinatorial surface is a compact, connected 2-manifold triangulated as a finite simplicial complex. You may assume that each of its maximal simplices has dimension 2, that each 1-simplex belongs to exactly two 2-simplices, and that the 2-simplices containing a given vertex fit together to form a cone with apex the given vertex and base a polygonal simple closed curve. Euler characteristic \( \chi \) is defined as \( n_0 - n_1 + n_2 \), where \( n_i \) = number of simplices of dimension \( i \). Give a direct proof.) That is, do not assume the standard machinery of algebraic topology such as duality, alternate ways to compute \( \chi \), etc.

6. Prove: If \( G \) acts as a group of homeomorphisms on a simply connected space \( X \), and if each point \( x \in X \) has a neighborhood \( U \) so that \( U \cap g(U) = \emptyset \) for all \( g \in G - \{e\} \), then \( \pi_1(X/G) \) is isomorphic to \( G \). (\( e \) is the identity element of \( G \).) You may assume that \( X \) is path-connected, locally path connected, and locally simply connected.
7. Let \( X \) be a space and \( A_1, A_2 \) closed subspaces. Assume that \( X = A_1 \cup A_2 \) and that \( A_1, A_2 \) and \( A_1 \cap A_2 \) are neighborhood deformation retracts in \( X \). Prove that the morphism

\[
H_\ast(i) : H_\ast(A_1, A_1 \cap A_2) \to H_\ast(X, A_2)
\]

is an isomorphism, where \( i : (A_1, A_1 \cap A_2) \to (X, A_2) \) is the natural imbedding and \( H_\ast \) is singular homology with arbitrary coefficients.

8. Let \( X = \bigcup_n A_n \), where each \( A_n \) is compact, \( A_n \subset A_{n+1} \), where \( X \) is Hausdorff and has the weak topology with respect to the subspaces \( A_n \). That is, a set \( F \subset X \) is closed if and only if \( F \cap A_n \) is closed for each \( n \). Suppose that for all \( n \), the natural imbedding \( i_n : A_n \to X \) is null-homotopic. Prove that \( \tilde{H}_\ast(X) = \{0\} \), where \( \tilde{H}_\ast \) is singular homology with arbitrary coefficients.
9. Show that the following two statements are equivalent.

(i) Any map \( f : D^n \rightarrow D^n \) has a fixed point.

(ii) For any map \( f : D^n \rightarrow \mathbb{R}^n \) either \( f^{-1}(0) \neq \emptyset \) or there is an element \( x \in \partial D^n \) and a real number \( \lambda > 0 \) such that \( f(x) = \lambda x \).

Here \( D^n \) is the unit disk centered at the origin \( 0 \in \mathbb{R}^n \).

10. Suppose that \( X \) and \( Y \) are finite cell-complexes, and assume that for each prime \( p > 0 \) either \( \tilde{H}_\ast(X; \mathbb{Z}) = \{0\} \) or \( \tilde{H}_\ast(Y; \mathbb{Z}_p) = \{0\} \). Prove that \( H_\ast(X \times Y, X \vee Y; \mathbb{Z}) = \{0\} \) where \( X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y \), and \( \tilde{H}_\ast \) is singular homology.
11. State the Poincaré Duality Theorem and use it to compute the cohomology ring $H^\ast(\mathbb{P}_n(\mathbb{C}); \mathbb{Z})$, where $\mathbb{P}_n(\mathbb{C})$ is complex projective n-space.

12. Prove that $\mathbb{P}_2(\mathbb{R})$ cannot be imbedded in $\mathbb{R}^3$, where $\mathbb{P}_2(\mathbb{R})$ is real projective 2-space. State a general result of which the statement above is a special case.
Qualifying Exam

TOPOLOGY

January 13, 1988

Instructions: Answer four problems.

HOWEVER: DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PAGE ONE!

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1. Let $X$ be a compact, connected Hausdorff space. Suppose that $p \in X$. Define

$$Q = \cap \{U \subseteq X | p \in U, \text{ and } U \text{ is clopen in } X\}.$$ 

Prove that $Q$ is connected. ("clopen" means both open and closed.)

2. Suppose that $X$ is a compact Hausdorff space such that the diagonal

$$\Delta = \{(x,x) | x \in X\}$$

is a $G_\delta$ subset of $X \times X$. Prove that $X$ is metrizable. (A $G_\delta$ is a countable intersection of open sets.)

3. Let

$$X = \prod_{S \in \mathcal{P}(\mathbb{Z}_+)} \{0,1\}_S,$$

with the usual product topology. ($\{0,1\}$ is a two-point discrete space; $\mathcal{P}(\mathbb{Z}_+)$ is the set of all subsets of the positive integers $\mathbb{Z}_+$; thus, $X$ is the product of continuumwise-many copies of a two-point discrete space.) Prove that there is a sequence of points in $X$ that has no convergent subsequence. State a theorem that answers the question: "Is $X$ compact?"
4. a) State the classification theorem for closed surfaces. (A closed surface is a compact, connected 2-manifold without boundary.)

b) Describe all closed surfaces with these Euler characteristics: -1, -2.

c) There is exactly one orientable closed surface that can double-cover a closed surface of Euler characteristic -1. What is this covering surface?

5. Suppose that \( p: \tilde{X} \rightarrow X \) with \( p(\tilde{x}_0) = x_0 \) is a covering map with \( p_*\pi_1(\tilde{X}, \tilde{x}_0) \) a normal subgroup of \( \pi_1(X, x_0) \). Prove that if \( f: I \rightarrow X \) is a closed path (i.e., \( f(0) = f(1) \)), then either every lifting of \( f \) is closed, or none is. (Assume that \( \tilde{X} \) and \( X \) are path connected and locally path connected.) I denotes the closed unit interval \([0,1]\).

6. Let \( X \) be the torus \( S^1 \times S^1 \) and let \( A \) be a finite subset of \( X \).

a) If the cardinality of \( A \) is 1, what is the fundamental group of \( X - A \)? Give reasons.

b) If the cardinality of \( A \) is 2, what is the fundamental group of \( X - A \)? Give reasons.

c) If \( x_0 \) is a point of \( S^1 \), collapse \( \{x_0\} \times S^1 \) to a point in \( X \) and let \( Y \) be the quotient space. What is the fundamental group of \( Y \)? Give reasons.

7. Let \( X \) be a path-connected space, \( A \subset X \) a closed subspace, \( SA \) the suspension of \( A \), and \( X \cup CA \) the space obtained by attaching the cone on \( A \). Denote the natural projection \( X \cup CA \rightarrow (X \cup CA)/X \cong SA \) by \( p \). Prove that

\[
H_*(p): H_*(X \cup CA, X) \rightarrow H_*(SA, \{\tilde{a}_0\})
\]

is an isomorphism, where \( \tilde{a}_0 = [A \times \{0\}] \in SA \), and \( H_* \) is singular homology with arbitrary coefficients.
8. Let $K$ be a finite simplicial complex, and define $\chi(K)$ to be $\sum (-1)^i a_i$, where $a_i$ is the number of the simplices of dimension $i$. Prove that $\chi(K)$ is a homotopy-type invariant.

9. Compute $H_*(\mathbb{P}_2(\mathbb{R}) \# \mathbb{P}_2(\mathbb{R}); \mathbb{Z})$ where $\mathbb{P}_2(\mathbb{R})$ is the real projective plane and $\#$ the connected sum.

10. Define $X_p$ to be $S^2 \cup_f D^3$ where $f_p : \partial D^3 \to S^2$ is a map of degree $p$. Compute $H_*(\bigcup_{p \in \mathbb{P}} X_p; \mathbb{Z})$, where $\mathbb{P}$ is an infinite set of distinct primes.

11. Let $M$ be a compact manifold ($\partial M = \emptyset$), and $\mathcal{U} = \{U_i : i = 1, \ldots, n\}$ an open cover of $M$. If each $U_i$ is contractible to a point in $M$, and if $M$ is not of the homotopy type of a sphere, prove that $n \geq 3$ ($n$ is the cardinality of $\mathcal{U}$).

12. Consider the Hurewicz fibration $p : E \to B$ with a section $s : B \to E$. Denote the fiber by $F$. Show that for any space $X$ and any map $X \to F$, $f$ is null-homotopic if, and only if, the composite $X \xrightarrow{f} F \xrightarrow{i} E$ is null-homotopic, where $i : F \to E$ is the natural imbedding.
Qualifying Exam

TOPOLOGY

August 31, 1988

Instructions: Answer four problems.

However: Do not choose more than two problems from page one!

Please use different sheets of paper for each problem since not all of
your answers will be graded by the same person. Indicate clearly which
problems you are doing. Hand in no more than the requested number of
problems. To facilitate impartial grading, do not place your name or
initials on your answer sheets. Instead, the proctor will assign you a
number to be placed on your cover sheet and all answer sheets.

Notation: $H_*, H^*$ denote homology and cohomology functors.

$S^k = k$-sphere, $D^k = k$-ball = $k$-disk, $\mathbb{Z} = \text{integers}$, $\mathbb{R} = \text{real numbers}$.

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as possible. Nevertheless, the exam may contain misprints. If you are
convinced a problem has been stated incorrectly, mention this to the
proctor and indicate your interpretation in your solution. In such cases
do not interpret the problem in such a way that it becomes trivial.
1. Let $X$ be a compact Hausdorff space, and let $A$ and $B$ be nonempty closed subspaces of $X$ such that no connected set in $X$ intersects both $A$ and $B$. Prove that

$$X = H \cup K,$$

where $H$ and $K$ are disjoint, closed, and $A \subseteq H$, $B \subseteq K$.

2. Prove directly, rather than by just quoting a theorem, that each open subset of $\mathbb{R}^n$ (with the usual Euclidean metric topology) is paracompact. Include a definition of "paracompact".

3. Let $\mathbb{R}^n$ have the usual Euclidean metric topology, and let $X$ be a closed subspace of $\mathbb{R}^n$. Let $Q$ be the quotient space obtained from $\mathbb{R}^n$ by identifying $X$ to a point. (If

$$q: \mathbb{R}^n \rightarrow Q$$

is the identification mapping then $U \subseteq Q$ is open if and only if $q^{-1}(U)$ is open in $\mathbb{R}^n$.)

Prove that $Q$ is metrizable if and only if $X$ is compact.

DO NOT DO MORE THAN TWO PROBLEMS ON THIS PAGE
4. Let \( \Sigma \) denote a compact connected surface (no boundary) other than the 2-sphere \( S^2 \). If \( f:S^2 \to \Sigma \) is any map, determine the homomorphism \( f_*: H_q(S^2 ; \mathbb{Z}) \to H_q(\Sigma ; \mathbb{Z}) \) for each \( q \geq 0 \). Verify your results.

5. Let \( K \subset S^n \) be a subset homeomorphic to the unit interval \( I \). Give a direct proof (do not merely invoke the "duality" theorem) that the reduced homology

\[
\tilde{H}_*(S^n - K; \mathbb{Z}) = \{0\}.
\]

6. Let \( X \) be the "Hawaiian Earring". That is, \( X \) is the union of the circles

\[
(x - 1/n)^2 + y^2 = 1/n^2, \quad n = 1, 2, 3, \ldots,
\]

with the subspace topology from \( \mathbb{R}^2 \). Let \( Y \) denote the identification space obtained from \( \mathbb{R} \) by identifying all the integers to a single point.

(a) Decide with proof whether or not each of \( X \) and \( Y \) is compact,

(b) Decide with proof whether or not each of \( X \) and \( Y \) has a countable fundamental group.
7. Show that a covering space \( p: \tilde{X} \to X \) is a regular cover if, and only if, the group of covering transformations acts transitively on \( p^{-1}(x) \) for some \( x \in X \). Here, \( X \) is path connected and locally path connected.

8. Compute \( H^*(\mathbb{RP}^5;A) \) where \( A = \mathbb{Z}, \mathbb{Z}_4 \) and \( \mathbb{Z}_3 \), as abelian groups.

9. Let \( X = (D^2 \times S^1) \cup_f (D^2 \times S^1) \), where \( f: S^1 \times S^1 \to S^1 \times S^1 \) is given by \( f(x,y) = (x,xy) \). Prove that \( H_*(X;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z}) \).

10. Let \( K \) denote a compact surface (no boundary) whose fundamental group is finitely presented by \( \{a,b| abab = 1\} \). Denote by \( \tilde{K} \) the covering space corresponding to the subgroup of the fundamental group generated by \( a^2 \) and \( b \). Identify \( K \) and \( \tilde{K} \) and then show that every map \( f: K \to K \) admits a "lift" \( \tilde{f}: \tilde{K} \to \tilde{K} \).
11. Let \( f: (S^n, K_1, S^n - K_1) \to (S^n, K_2, S^n - K_2) \) denote a map of triples where \( K_1, K_2 \) are compact subsets of \( S^n \). Assume that \( f: S^n \to S^n \) has degree 1 and that \( (f|K_1)_*: H_*(K_1; \mathbb{Z}) \to H_*(K_2; \mathbb{Z}) \) is an isomorphism. Prove that

\[
(f|S^n - K_1)_*: H^*(S^n - K_2; \mathbb{Z}) \to H^*(S^n - K_1)
\]

is an isomorphism.

12. Compute the cohomology rings of complex projective space \( \mathbb{C}P^3 \) and \( S^2 \times S^4 \). Show that these spaces are not of the same homotopy type.
Qualifying Exam

TOPOLOGY

January 19, 1989

Instructions: Answer four problems.

However: Do not choose more than two problems from page one!

Please use different sheets of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems. To facilitate impartial grading, do not place your name or initials on your answer sheets. Instead, the proctor will assign you a number to be placed on your cover sheet and all answer sheets.

Notation: \( \text{H}_*, \text{H}^* \) denote homology and cohomology functors.
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1. Prove that every closed subset of the real line either is countable or has the cardinality of the continuum.

2. Let $X$ be the set of all non-decreasing functions on the closed unit interval $I$ with values in $I$. That is, $X = \{ f : I \rightarrow I \mid x < y \text{ in } I \text{ implies } f(x) \leq f(y) \}$. Give $X$ the topology it inherits as a subspace of the product space $I^I$. Show that $X$ is compact, Hausdorff, separable, and every point has a countable local basis (that is, $X$ is 1st countable). Comment: $X$ is not metrizable.

3. Let $\mathbb{N}$ denote the positive integers with the discrete topology, and $\beta\mathbb{N}$ the Stone–Čech compactification of $\mathbb{N}$. Show that if the continuum hypothesis holds, then there is point $p$ of $\beta\mathbb{N} - \mathbb{N}$ such that The intersection of every countable family of open sets in $\beta\mathbb{N} - \mathbb{N}$ containing $p$ is open.

DO NOT DO MORE THAN TWO PROBLEMS ON THIS PAGE.
4. Let \( p: \mathring{X} \to X \) be a map such that \( p^{-1}(\{x_0\}) \) is discrete in \( \mathring{X} \). Assume that \( p : \mathring{X} \to X \) satisfies the Covering Homotopy Property. Prove that if \( \mathring{X} \) is path-connected, then \( p \) is a homeomorphism if, and only if, there is a section \( s : X \to \mathring{X} \), i.e. \( p \circ s = \text{id}_X \).

5. Compute \( \pi_1(S^1 \times S^1 - F_n \times x_0) \), where \( F_n \subset S^1 \times S^1 \) has cardinality \( n \) (i.e. \( F_n \) is a set of \( n \) distinct elements), with \( 0 \leq n \leq 2 \). Here \( S^1 \) is the unit circle. (You may assume that the result is independent of the imbedding \( F_n \subset S^1 \times S^1 \). Also you may assume the van Kampen Theorem.)

6. Let \( S^2_{2p} \to \mathbb{R}P^2 \) be the standard 2-fold cover of the real projective plane \( \mathbb{R}P^2 \), and let \( M \) be a 2-dimensional closed manifold. Prove that \( M \) is orientable if and only if, for every map \( f: M \to \mathbb{R}P^2 \) there is a map \( \tilde{f}: M \to S^2 \) such that \( p \circ \tilde{f} = f \).
7. Let $T$ denote the torus and $D$ a 2-dimensional disk in $T$. Then $\partial D$ is homeomorphic to a circle $C$. Let $X$ denote the space obtained from $T - \text{int } D$ by identifying $x \in \partial D$ with its antipode. Compute $H_\ast(X; \mathbb{Z})$ and $H_\ast(X; \mathbb{Z}_2)$.

8. Let $S^n$ denote the $n$-sphere, and $A$ a closed subset of $S^n$ homeomorphic to the wedge $S^p \vee S^q$ of a $p$-sphere and a $q$-sphere. Assume that $1 \leq p, q < n$. Compute the homology groups $H_\ast(S^n - A; \mathbb{Z})$.

9. Let $f$ and $g$ denote the homotopic maps from $S^{n-1}$ to $Y$ where $Y$ is a Hausdorff space and $n \geq 1$. Let $D^n$ denote the $n$-ball with $\partial D^n = S^{n-1}$. Prove that $D^n \cup_f Y$ and $D^n \cup_g Y$ have the same homotopy type.
10. Let \( f: \mathbb{RP}^k \to \mathbb{RP}^n \) denote an imbedding where \( n > k \). (\( \mathbb{RP}^k \) is the real projective \( k \)-space.) Let \( A = f(\mathbb{RP}^k) \). Show that there is no retraction of \( \mathbb{RP}^n \) onto \( A \).

11. Let \( M \) denote a compact connected \( n \)-manifold imbedded in \( \mathbb{R}^{n+1} \). Show that \( M \) is orientable. Include an appropriate discussion of your definition of orientability.

12. Let \( X = S^2 \vee S^3 \vee S^5 \) and \( Y = S^2 \times S^3 \). Show that the cohomology groups \( H^*(X;\mathbb{Z}) \) and \( H^*(Y;\mathbb{Z}) \) are isomorphic but there is no map \( f: X \to Y \) inducing the isomorphism.
Qualifying Exam

TOPOLOGY

August 30, 1989

Instructions: Answer four problems.

However: Do not choose more than two problems from page one!

Please use different sheets of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems. To facilitate impartial grading, do not place your name or initials on your answer sheets. Instead, the proctor will assign you a letter code to be placed on your cover sheet and all answer sheets.

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$\mathbb{Z} = \text{integers.}$

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1. Let $X$ be the real line with the half-open-interval topology: that is $X = \mathbb{R}$ with the topology that has

$$\{[a,b) \mid a < b \text{ in } \mathbb{R}\}$$

as an open basis. Prove that $X$ is paracompact but $X^2$ is not. Include a definition of "paracompact".

2. State the Tietze Extension Theorem, which characterizes normal topological spaces in terms of real-valued mappings. Assuming this theorem, let $C$ be this subspace of $\mathbb{R}^2$:

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup Y,$$

where $Y$ is an arbitrary, but fixed, subset of the unit circle $S^1$. Prove that if $X$ is any metrizable space, $A$ is any closed subspace of $X$ and $f$ is any continuous mapping of $A$ into $C$, then $f$ extends to a continuous mapping of $X$ into $C$.

3. A space $X$ is called Tychonoff if one-point sets are closed, and for each point $x \in X$ and closed set $K$ with $x \in K$, there is a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(K) = 1$. Prove that a space $X$ is Tychonoff if and only if $X$ can be embedded in some product of (many) copies of $[0,1]$.

DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM THIS PAGE.
4. State the Classification Theorem for surfaces. (A surface is a compact, connected 2–manifold without boundary.) Use it to show that there is a unique surface with Euler characteristic $-1$. Describe this surface. Find all surface(s) that are 2–sheeted covering spaces of the surface with Euler characteristic $-1$.

5. Let $X$ be a compact subspace of $\mathbb{R}^n$. Give a self–contained, elementary proof that each continuous mapping of $X$ into $S^n$ is homotopic to a constant.

6. If $K$ is a geometric simplicial complex, prove that the underlying topological space $|K|$ is a "Euclidean neighborhood retract". That is, show that $|K|$ can be embedded in some $\mathbb{R}^n$ as a retract of an open set in $\mathbb{R}^n$. 
7. Let $M$ denote a connected $n$–manifold. Give a precise definition of $M$ being orientable (over $\mathbb{Z}$). Show that $M$ always admits a 2–fold cover $\tilde{M}$ which is orientable. Identify $\tilde{M}$ where $M$ is the projective plane and Klein Bottle.

8. Let $M$ denote a topological space and $\Lambda(M) = M^{S^1}$, the space of continuous maps from $S^1$ to $M$ with the compact–open topology. If $s_0 \in S^1$ and $x_0 \in M$, let

$$\Omega(M,x_0) = \{ \omega \in \Lambda(M): \omega(s_0) = x_0 \}$$

and assume $M$ is path connected and simply connected. Show that

$$\pi_n(\Lambda(M),*) = \pi_n(\Omega(M,x_0),*) \oplus \pi_n(M,x_0), \quad n \geq 1.$$ 

where $* = \text{constant map of } x_0$.

9. Let $M$ denote a fixed space and $X \subset M$ a closed subset with inclusion map $i_X: X \to M$. Set

$$f(X) = \ker i_X^*: H^*(M) \to H^*(X),$$

where $H^*$ is singular cohomology with coefficients in a field $\mathbb{F}$. If $(X;X_1,X_2)$ is an excisive triad of closed subsets of $M$ show that $x_1 \in f(X_1)$, $x_2 \in f(X_2)$ implies the cup product $x_1 x_2 \in f(X)$, where $X = X_1 \cup X_2$. 
10. Is there a finite CW–complex $X$ for which $\pi_2(X, x_0)$ is not finitely generated as an abelian group? Justify your assertions.

11. Let $\mu: S^n \times S^n \rightarrow S^n$ be a map such that $\mu(x, e) = x = \mu(e, x)$ for all $x \in S^n$, where $e$ is a basepoint in $S^n$. Prove that $n$ is odd.

12. Define $X$ to be the space $(S^n \times S^n)/\sim$ obtained from $S^n \times S^n$ by identifying $(x, e)$ and $(e, x)$ to $x$ for all $x \in S^n$, where $e \in S^n$ is a basepoint. Compute all cup–products in $H^*(X, \mathbb{Z})$. 
Qualifying Exam

TOPOLOGY

January 18, 1990

Instructions: Answer four problems.

HOWEVER: Do not choose more than two problems from page one!

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1. A space $X$ is **completely regular** if for every $x \in X$ and open neighborhood $U$ of $x$, there is a continuous map $f: X \to [0,1]$ with $f(x) = 0$ and $f(X \setminus U) = 1$. A space $X$ is **Baire** if for each countable family $\mathcal{K}$ of closed sets whose union contains $X$, there is at least one $K \in \mathcal{K}$ which contains a nonempty open set. Prove that if $X$ is a nonempty completely regular space and every continuous real valued function on $X$ is bounded, then $X$ is Baire.

2. Suppose $X$ is a compact Hausdorff space and $U$ is a proper open subset of $X$ (i.e. both $X \setminus U$ and $U$ are nonempty.) Show that, if $x \in U$, then the component of $U$ containing $x$ has a limit point on the boundary of $U$.

3. Let $X$ denote the union of two circles meeting at a single point, with the subspace topology from $\mathbb{R}^2$:

   ![Diagram of two circles meeting at a single point](image)

   Find a suitable connected, three-fold (i.e., $p^{-1}(x_0)$ is 3 points) covering space of $X$, and use that same three-fold covering space to prove that the fundamental group of $X$ is not abelian. Give complete statements of any results from covering space theory that you use.

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**DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM THIS PAGE.**
4. State the Simplicial Approximation Theorem. Use it to show that if \( k < m \) then each continuous mapping \( S^k \to S^m \) is homotopic to a constant.

5. Let \( X \subset \mathbb{R}^n \) be compact, and \( f: X \to S^k \) a continuous mapping that is not homotopic to a constant. Prove from first principles that there is a compact set \( Y \subset \mathbb{R}^n \) so that \( f \) does not extend to a continuous map of \( X \cup Y \) into \( S^k \), but if \( Y_0 \) is any proper closed subset of \( Y \) then \( f \) extends to a continuous map of \( X \cup Y_0 \) into \( S^k \).

6. State the Classification Theorem for surfaces. (A surface is a compact, connected 2–manifold without boundary.) Show by direct construction that each non–orientable surface with even Euler characteristic embeds in the product \( S^2 \times S^1 \).

7. Exhibit a space \( X \) which is 1–connected such that:
   (a) \( H_q(X; \mathbb{Z}) \) is non–trivial for all \( q > 1 \); and
   (b) For each prime \( p \), \( H_q(X; \mathbb{Z}_p) = 0 \) for \( q \) sufficiently large. Justify the steps in your argument.

8. Define the 3–manifold \( L(n,k) \) for relatively prime \( (n,k) \), \( 0 \leq k < n \). Compute the homology groups of \( L(n,k) \). (\( L(n,k) \) is called a lens space.)

9. Given a short exact sequence of chain complexes

\[
0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0,
\]

define the connecting homomorphism \( H_q(C'') \xrightarrow{\partial} H_{q-1}(C') \) and show that the sequence

\[
\ldots \xrightarrow{i_*} H_q(C') \xrightarrow{j_*} H_q(C) \xrightarrow{\partial} H_q(C') \xrightarrow{\partial} H_{q-1}(C') \to \ldots
\]

is exact at \( H_q(C'') \).
10. If $X$ is a space, let $\chi(X)$ denote the alternating sum of the ranks $H_q(X; \mathbb{Q})$, where $\mathbb{Q}$ is the field of rationals. Let $M$ denote complex projective space of (complex) dimension $n$, let $N = M \times M$, and let $D$ be the diagonal in $N$. Compute $\chi(N-D)$. State carefully the results you employ during your argument.

11. Let $X$ be the Cartesian product of the circle $S^1$ and the Klein bottle.
   (a) Compute $\pi_1(X)$;
   (b) If $\tilde{X}$ is the universal covering of $X$, compute $\pi_2(\tilde{X})$;
   (c) Compute $H_*(X; \mathbb{Z}_2)$ and $H_*(X; \mathbb{Z}_3)$.

12. Let $X$ be a compact connected 4-dimensional manifold with $\partial X = \emptyset$ and $\pi_1(X) = 0$.
   (a) Prove that the cup product
       \[ \beta: H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \to H^4(X; \mathbb{Z}) \]
       \[ \beta(x, y) = x \cup y \]
       can be represented by a symmetric matrix with integer coefficients, call it $M$, such that $\det M = \pm 1$ and the inverse of $M$ also has integer coefficients. Quote the full statements of any theorems that you use.
   (b) Find the corresponding matrices for $X_1 = \mathbb{C}P^2 \# \mathbb{C}P^2$ and $X_2 = S^2 \times S^2$ and use them to prove that $X_1$ and $X_2$ are not homeomorphic.
Qualifying Exam

TOPOLOGY

August 29, 1990

Instructions: Answer four problems.

However: Do not choose more than two problems from page one!

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1. A space $X$ is called limit point compact if for each infinite set $A \subset X$, there exists at least one point $p$ in $X$ such that $p$ is a limit point of $A$. Prove that the product of a compact space and a limit point compact space is limit point compact. Include a definition of "$p$ is a limit point of $A$". (Caution: the product of two limit point compact spaces need not be limit point compact.)

2. Let $X$ be a compact, connected Hausdorff space. Prove that $X$ cannot be expressed as a countably infinite union of pairwise disjoint, nonempty closed sets. (You may assume the truth of the "to the boundary" theorem: If $Y$ is a compact, connected Hausdorff space and $U$ is a nonempty, proper open subset of $Y$ then each component of $U$ has a limit point in the boundary $\overline{U} - U$ of $U$.)

3. Suppose that $X$ is a compact metrizable space and that $Y$ is a space having metric $d$. Assume that $Y$ has a countable basis for the $d$-metric topology. Let $Y^X$ be the set of all continuous functions from $X$ to $Y$. For $f, g$ in $Y^X$ define

$$
\rho(f, g) = \begin{cases} 
\sup \{ d(f(x), g(x)) | x \in X \}, & \text{if this number is less than 1,} \\
1, & \text{otherwise.}
\end{cases}
$$

(You may assume that $\rho$ is a metric on $Y^X$.) Prove that $Y^X$ has a countable basis for the $\rho$-metric topology.

DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM THIS PAGE!
4. Let \( X \) be the underlying space of a finite simplicial (or "geometric") complex, and let \( f : X \to X \) be a continuous (not necessarily simplicial) mapping that is homotopic to a constant. Prove that \( f \) has a fixed point. (You may assume that each continuous mapping of each \( n \)-disk \( D^n \) into itself has a fixed point and that \( X \) is a retract of an open set in some \( \mathbb{R}^N \). Recall that \( x \) is a fixed point for \( f : X \to X \) if \( f(x) = x \).

5. A surface is a compact, connected 2-manifold without boundary. Assume that each surface is homeomorphic to the underlying space of a finite simplicial (or "geometric") complex, and that its Euler characteristic can be consistently calculated in the usual way from the number of simplices in such a complex. An involution of a space \( X \) is a homeomorphism \( h \) of \( X \) onto \( X \) such that \( h(h(x)) = x \) for each \( x \) in \( X \).

(a) Prove that if a surface \( M \) admits an involution with no fixed points then its Euler characteristic \( \chi(M) \) is an even integer.

(b) Prove that every orientable surface admits an orientation-reversing involution with no fixed points.

6. A space \( X \) is called 1-connected if it is pathwise-connected and each continuous mapping \( f : S^1 \to X \) extends to a continuous mapping of \( D^2 \to X \). (\( S^1 = \partial D^2 \), where \( D^2 \) is the 2-disk.) Suppose that a Hausdorff space \( X \) is the union of three closed subspaces \( X_0, X_1, \) and \( A \), where \( X_0 \cap X_1 = \phi \), each \( A \cap X_i \) is a point \( p_i \), \( A \) is homeomorphic to \( [0,1] \) with \( p_i \) corresponding to \( i \), and each \( X_i \) is 1-connected. (That is, \( X \) consists of 1-connected spaces \( X_0 \) and \( X_1 \) joined by an arc.) Prove that \( X \) is 1-connected. (Caution: Two 1-connected spaces with a single point in common need not have a 1-connected union.)
7. Compute the cohomology ring structure of $\mathbb{C}P^n$.

8. Let $G$ be a finitely-generated abelian group and let $n > 0$. Construct a finite cell-complex $X$ such that

$$\tilde{H}_k(X) = \begin{cases} G & k = n \\ 0 & \text{otherwise} \end{cases}$$

9. Let $(X, A)$ be a pair of compact metric spaces, $A \subset X$. Let $X/A$ be the space obtained by collapsing $A$ to the point $* \in X/A$. Give conditions on $(X, A)$ such that

$$H_q(X, A) \to H_q(X/A, *)$$

is an isomorphism for all $q$.

10. Let $P_n(F)$ be the $n$–projective space in $F^{n+1}$, where $F = \mathbb{R}$ or $\mathbb{C}$. Compute $\pi_i(P_n(F), *)$ for $i \leq dn + d - 2$, $d = \dim_{\mathbb{R}} F$. 


11. Assume that $X''$ and $X$ are $H$-spaces, and let $p : X'' \to X$ be a covering which is an $H$-map. Prove that any covering transformation $\phi : X'' \to X''$ is homotopic to the identity map of $X''$. Assume that $X''$ and $X$ are pathwise and locally pathwise connected.

12. Let $p : E \to B$ be a fiber space in the sense of Hurewicz, and assume that $B$ is pathwise-connected. Prove that the fibers $p^{-1}(b)$ are all of the same homotopy type. When are they of the same homeomorphism type? You may assume that the spaces in question are compactly generated.
Qualifying Exam

TOPOLOGY

January 16, 1991

Instructions: Answer four problems.

HOWEVER: DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PAGE ONE!

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Notation: $H_*, H^*$ denote homology and cohomology functors.

$S^k = k$-sphere, $D^k = k$-ball = $k$-disk, $\mathbb{Z} = \text{integers}$, $\mathbb{R} = \text{real numbers}$.

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1. Let $X$ be a closed subspace of $\mathbb{R}^n$ (with the usual metric topology) with the property that $X$ contains no non-empty open set of $\mathbb{R}^n$. Let $Q$ be the quotient space of $\mathbb{R}^n$ in which $X$ (and only $X$) is identified to a point. (If $q: \mathbb{R}^n \to Q$ is the quotient map, then $U \subset Q$ is defined to be open if and only if $q^{-1}(U)$ is open in $\mathbb{R}^n$.) Prove that $Q$ is metrizable if and only if $X$ is compact. State clearly any metrization theorems that you use.

2. Define what it means for a topological space $Z$ to be locally connected at a point $p$ in $Z$. Let $Y$ be a subspace of $X$, where the topology of $X$ is generated by the metric $d$. Suppose that $Y$ has the following property with respect to $d$: For each $\varepsilon > 0$, $Y$ can be expressed as the union of finitely many connected subspaces, each of diameter less than $\varepsilon$. Prove that $Y$, the closure of $Y$ in $X$, is locally connected at each of its points.

3. Let $X$ be this compact subspace of $\mathbb{R}^2$:

$$X = \{(0,y)\mid -1 \leq y \leq 1\} \cup \{(x, \sin(1/x))\mid 0 < |x| \leq \pi\}.$$ 

Let $S$ denote the suspension of $X$. $S$ can be considered as a subspace of $\mathbb{R}^3$ consisting of the union of two geometric cones over $X$ that intersect only along $X$. For example, if

$$X \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$$

then we can take

$$S = \{t \cdot (0,0,\pm) + (1-t) \cdot x \mid x \in X, 0 \leq t \leq 1, \pm = \pm 1\}.$$

Show that $S$ is not simply connected. That is, find a continuous mapping of the unit circle $S^1$ into $S$ that does not extend to a continuous mapping of the unit 2–disk $D^2$ into $S$. You may assume that $X$ is connected and has the obvious three path components.
4. A surface is a compact, connected 2-manifold without boundary.
   
   (a) State the classification theorem for surfaces.
   
   (b) Decide (with proof) exactly which surfaces embed in $\mathbb{R}^3$. (You may restrict yourself to smooth or piecewise-linear embeddings.)
   
   (c) Show that each punctured surface (i.e., surface with an open 2-disk removed) embeds in $\mathbb{R}^3$.
   
   (d) Show that every surface embeds in $\mathbb{R}^4$.

5. Let $f: S^n \to \mathbb{R}^{n+1}$ (n≥1) denote a continuous map such that $f(x) = f(-x)$ for each $x \in S^n$. Show there exists an $x_0 \in S^n$ such that $f(x_0)$ and $x_0$ are perpendicular vectors, i.e. $(f(x_0), x_0) = 0$ where $(\ , \ )$ is dot product.

6. Let $O(n,2)$ be the Stiefel manifold of orthonormal 2-frames $(x, v)$ in $\mathbb{R}^n$ (n≥2). Denote by $p: O(n,2) \to S^{n-1}$ the map $p(x, v) = x$. Prove that $p$ has a section if, and only if, $n$ is even. Do not merely state an equivalent theorem for the proof.
7. Let \( f: S^n \times S^n \to S^n \) be a map of bidegree \((1,p)\). Let \( A = S^n \times S^n \) and \( X = D^{n+1} \times S^n \). Let \( g: A \to A \) be defined by \( g(x,y) = (x,f(x,y)) \) and let \( Y = X \cup gX \).

Compute \( H_*(Y; \mathbb{Z}) \). The bidegree of \( f \) is \((d_1,d_2)\) if \( f_*(u+v) = d_1u + d_2v \), where \( u \) and \( v \) are generators of

\[
H_n(S^n) \oplus H_n(S^n) \cong H_n(S^n \times S^n).
\]

8. Let \([A,B]\) denote the set of homotopy classes of maps from space \( A \) to space \( B \). Classify the elements in the set \([\mathbb{R}P^n; T^2]\), where \( T^2 = S^1 \times S^1 \), the 2–dimensional torus and \( \mathbb{R}P^n \) is real projective \( n \)–space. Justify your conclusions.

9. Let \( M \) denote a compact \( n \)–manifold, \( n \geq 2 \) and \( h: \pi_n(M) \to H_n(M; \mathbb{Z}) \) the Hurewicz homomorphism. \( M \) is called spherical if \( h \) is non–trivial.

a) Prove that \( M \) is spherical if there is a map \( f: S^n \to M \) such that the induced homomorphism on homology (over \( \mathbb{Z} \)) is non–trivial.

b) Prove that if \( M \) is spherical, \( M \) is orientable (over \( \mathbb{Z} \)).

c) Give an example of a spherical manifold which is not a sphere.
10. In each of the following, find examples of two connected spaces $X$ and $Y$ such that

(a) $\pi_1(X) \cong \pi_1(Y) \neq \{1\}$ but $H_*(X;\mathbb{Z}) \neq H_*(Y;\mathbb{Z})$

(b) $H_*(X;\mathbb{Z}) \cong H_*(Y;\mathbb{Z})$, but $\pi_1(X) \neq \pi_1(Y)$

(c) $H_*(X;\mathbb{Z}) \cong H_*(Y;\mathbb{Z})$, $\pi_1(X) \cong \pi_1(Y) \neq \{1\}$, but $H^*(X;\mathbb{Z}) \neq H^*(Y;\mathbb{Z})$ as rings.

Justify your answers.

11. Let $X$ denote a fixed topological space and $A \subset X$ any closed subset. Let $i_A^*: A \to X$ denote the inclusion map, $K$ a field and

$$i_A^*: H^*(X;K) \to H^*(A;K)$$

the induced homomorphism in cohomology. Let $p(A) = \text{kernel } i_A^*$, an ideal in the cohomology ring $H^*(X;K)$. Show that if $X$ is a finite polyhedron and $A_1$, $A_2$ are finite subpolyhedra, then

$$a_1 \in p(A_1), a_2 \in p(A_2) \implies a_1 a_2 \in p(A_1 \cup A_2)$$

where $a_1 a_2$ is cup product. (Hint: Use relative cup products.)

12. Let $M$ denote a compact, connected, orientable (over $\mathbb{Z}$) $n$–manifold. Define "orientable over a field $K"$ and show that $M$ is orientable over $K$. Prove that $H_n(M-x;K) = 0$ for any $x \in M$. 
Qualifying Exam

TOPOLOGY

August 28, 1991

Instructions: Answer four problems,

HOWEVER, DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PAGE ONE!

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1. Let $X$ and $Y$ be Hausdorff spaces, with $X$ compact and $Y$ paracompact. Prove that the product space $X \times Y$ is paracompact. Include a definition of "paracompact".

2. Let $f : X \rightarrow Y$ be a continuous surjection of topological spaces that is closed (i.e., maps closed sets to closed sets) and has each of its point-inverses compact. If $X$ has a countable basis for its topology, prove that $Y$ does also.

3. Let $B(I)$ denote all bounded real-valued functions defined on $I = [0, 1]$, and let $C(I)$ be the subset consisting of all the continuous functions. We use the uniform topology on $B(I)$ and on $C(I)$. That is, define the metric $\rho$ on $B(I)$ by:

\[
\rho(f, g) = \text{l.u.b.}_{x \in I} |f(x) - g(x)|
\]

and use the metric topology determined by $\rho$.

(a) Show that $C(I)$ contains a countable, dense subset. Give the complete proof. Don't just quote a theorem.

(b) Show that $B(I)$ does not contain a countable, dense subset.
4. Let $\mathcal{U}$ be a convex, bounded open set in $\mathbb{R}^n$. Prove that $\overline{\mathcal{U}}$ is homeomorphic to the closed unit $n$-ball $D^n$.

5. (1) Show that if the real projective plane $\mathbb{P}^2$ or the Klein bottle $K$ covers a manifold $N$, then $N$ is homeomorphic to $\mathbb{P}^2$ or $K$, respectively. You may assume that $N$ has a triangulation, i.e. that $N$ is homeomorphic to the underlying space of a simplicial complex.

(2) Is it possible for either $\mathbb{P}^2$ or $K$ to cover itself non-trivially? Justify each answer with a proof or construction.

6. Let $K$ be a compact Hausdorff space obtained from the torus $S^1 \times S^1$ by removing a round open disk and then attaching a closed disk along a curve of the form $\{\text{point}\} \times S^1$ that is disjoint from the deleted disk. (See figure. $K$ could be taken to be a finite simplicial complex.)

(a) Construct explicitly the universal covering space $\tilde{K}$ of $K$ and describe the covering map;

(b) Calculate the fundamental group of $K$.

[Torus $S^1 \times S^1$]

Add to torus

Remove from torus
7. Let $C = \{C_n, \partial\}$ denote a chain complex of free abelian groups. Assume that 
$$H_\ast(C) = \bigoplus_n H_n(C)$$
is a finite group. Prove that $H_\ast(C;G) = \{0\}$ and $H^\ast(C;G) = \{0\}$ for all abelian groups $G$ such that $|H_\ast(C)|, G) = 1$, i.e., $H_\ast(C)$ and $G$ have relatively prime orders.

8. Let $M$ denote a compact, connected $n$-manifold, $n \geq 1$. Show that $M$ is orientable if $H_1(M;\mathbb{Z}_2) = 0$. Include the definition of orientable that you employ.

9. Let $M_1$ and $M_2$ denote two compact, connected $n$-manifolds, $n \geq 2$. Define their connected sum $M_1 \# M_2$. Then

a) Show that $M_1 \# M_2$ is orientable if $M_1$ and $M_2$ are orientable.

b) Compute the cohomology ring (over $\mathbb{Z}$) of $\mathbb{C}P^2 \# \mathbb{C}P^2$, where $\mathbb{C}P^2$ is complex projective 2-space.

10. Let $\mu : S^n \times S^n \to S^n$ denote a map such that for some $e \in S^n$, $\mu(x, e) = \mu(e, x) = x$, for all $x \in S^n$. Show that $n$ is odd.
11. Let $A$ denote a disjoint union of imbedded circles in $S^n$, $n \geq 2$. Assume that each circle is imbedded as a neighborhood deformation retract. Compute $H_*(S^n/A; \mathbb{Z})$.

12. Let $\gamma \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \alpha$ denote an exact sequence of abelian groups and $X$ a topological space. Give an outline for the existence of an associated exact sequence

$$
\gamma \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow H^n(X; G_1) \rightarrow H^n(X; G_2) \rightarrow H^n(X; G_3) \rightarrow H^{n+1}(X; G_1) \rightarrow \cdots
$$

a) Prove exactness at $H^n(X; G_2)$

b) Compute $\beta_*$ when $X = \mathbb{RP}^k$ and the given sequence of abelian groups is

$$
\gamma \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \alpha
$$

where $\gamma(u) = 2u$ and $\alpha$ is induced by inclusion.
Qualifying Exam

TOPOLOGY

January 15, 1992

Instructions: Answer four problems.

HOWEVER: DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PAGE ONE!

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1. A topological space $X$ is called *locally compact at a point* $x \in X$ if for some open set $U$ in $X$ and some compact set $K$, we have $x \in U \subset K \subset X$. If $X$ is locally compact at each of its points, $X$ is called *locally compact*. Suppose that $X$ is a locally compact subspace of $\mathbb{R}^n$ with the usual Euclidean topology. Show that:

(a) there is an open set $W \subset \mathbb{R}^n$ such that $X \subset W$ and $X$ is closed relative to $W$; and

(b) $X$ can be embedded as a closed subspace of $\mathbb{R}^{n+1}$.

2. Let $X$ be a topological space whose topology is generated by a fixed metric $d$. Suppose that for each $\epsilon > 0$, $X$ can be expressed as a union of a finite number of its connected subspaces each of which has $d$-diameter less than $\epsilon$. Prove that $X$ has a basis for its topology that consists of open, *connected* sets.

3. Let $X$ be this subspace of $\mathbb{R}^2$ with the usual Euclidean topology:

$$X = \bigcup_{n=1}^{\infty} \{(x,y) | [x - (1/n)]^2 + y^2 = (1/n)^2 \}.$$ 

($X$ is sometimes called the *Hawaiian earring.*) Prove that the fundamental group $\pi_1(X, x_0)$ is uncountable, where $x_0 = (0,0) \in \mathbb{R}^2$.
4. Let $X = S^1 \lor S^1$ be the wedge of two circles. That is, $X$ is the union of two circles intersecting in exactly one point, $x_0$. We take $X$ to be a subspace of $\mathbb{R}^2$ with the usual Euclidean topology. Describe a three-to-one covering $p : \tilde{X} \to X$, where $\tilde{X}$ is connected but for some pair of points $\tilde{x}_1, \tilde{x}_2$ with $p(\tilde{x}_1) = p(\tilde{x}_2) = x_0$, there is no homeomorphism $h : \tilde{X} \to \tilde{X}$ with $p \circ h = p$ and $h(\tilde{x}_1) = \tilde{x}_2$. Prove the nonexistence of $h$ from first principles, not just by quoting a theorem. (Note that with correctly-chosen notation, $p^{-1}(x_0) = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$.)

5. A closed surface is a compact, connected Hausdorff space each of whose points has a neighborhood homeomorphic to $\mathbb{R}^2$. State the Classification Theorem for closed surfaces. Prove that the fundamental group of the Klein Bottle is not abelian. (The Klein bottle is the unique nonorientable closed surface with Euler characteristic equal to 0.)

6. A polyhedron is the underlying space of a finite simplicial complex. State the Simplicial Approximation Theorem for mappings from one polyhedron to another. (Include a definition of "simplicial approximation", of course.) Use the Simplicial Approximation Theorem to prove that the set of homotopy classes of maps from one polyhedron to another is always countable.
7. Let $X$ denote a topological space and $X_1$, $X_2$ subsets of $X$ such that $X = X_1 \cup X_2$. Let $S(X) = S(X; R)$ denote the singular chain complex of $X$ with coefficients in a P.I.D.$R$. Let $S(X_1) + S(X_2)$ denote the subcomplex of $S(X)$ generated by $S(X_1)$ and $S(X_2)$. Consider the inclusion induced maps.

$$i_1 : S(X_1, X_1 \cap X_2) \rightarrow S(X, X_2)$$

$$i_2 : S(X_2, X_1 \cap X_2) \rightarrow S(X, X_1)$$

$$j : S(X_1) + S(X_2) \rightarrow S(X)$$

Show that if any of $i_1, i_2, j$ induce isomorphisms on homology, so do the remaining two.

8. Let $S^n, n \geq 1$, denote the unit sphere in $\mathbb{R}^{n+1}$. Show that for all $k \in \mathbb{Z}$, $S^n$ admits a self map $f : S^n \rightarrow S^n$ of degree $k$. Also show that except for one value of $k$ (depending on $n$) that a map of degree $k$ always has fixed points.

9. Let $X$ denote the wedge of 2 circles, i.e., $X = S^1 \vee S^1$ (the union of two $S^1$'s with a point in common). Let $\varphi : X \rightarrow S^3$ denote an imbedding. Compute $H_q(S^3 - \varphi(X); \mathbb{Z})$ for all $q \geq 0$. Hint: Use $H_*(S^3 - \psi(I))$ where $\psi : I = [0, 1] \rightarrow \mathbb{R}^3$ is an imbedding.

10. Let $X$ denote a connected, finite CW-complex and $x_0$ a 0-call (base point) in $X$. Define

$$W_n(X) = \left\{(x_1, \ldots, x_n) : x_i \in X \text{ and for some } j, 1 \leq j \leq n, \quad x_j = x_0\right\}$$

Compute $H_*(W_3(X); \mathbb{Z})$ under the assumption that $H_q(X; \mathbb{Z})$ is torsion free (as a $\mathbb{Z}$-module) for $q \geq 0$.

11. Let $X$ denote a compact connected ENR (Euclidean neighborhood retract). Assume that for every prime $p$, $H_n(X; \mathbb{Z}_p) = 0$ for $n > 0$. Prove that $H_n(X; \mathbb{Z}) = 0$ for $n > 0$.

12. Let $M$ denote a compact, orientable $n$-manifold. Let $f : S^n \rightarrow M$ denote a map. Define (up to sign) the degree of $f$. If the degree of $f$ is $d(f)$, and $0 < q < n$, show that for every $u \in H_q(M; \mathbb{Z})$, $d(f)u = 0$. 

3.
Qualifying Exam

TOPOLOGY

August 26, 1992

Instructions: Answer four problems.

HOWEVER: DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PAGE ONE!

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QUALIFYING EXAM – TOPOLOGY

DO NOT CHOOSE MORE THAN 2 PROBLEMS FROM THIS PAGE!

1. Let $U$ be a nonempty, open connected subspace of $\mathbb{R}^2$ with the usual Euclidean topology. Prove that for each pair $p, q$ of points of $U$ there can be found a continuous function $f : [0, 1] \to U$ with $f(0) = p$ and $f(1) = q$. ($f$ is called a path in $U$ from $p$ to $q$.)

2. Let $X$ and $Y$ be topological spaces, with $Y$ compact. Prove from first principles that the product projection onto the first factor

$$\pi : X \times Y \to X$$

is closed. That is, $\pi$ maps closed sets to closed sets.

3. Describe a quotient (or “identification”) space $X$ of the 2-disk $D^2$ whose fundamental group $\pi_1(X, x_0)$ is cyclic of given order $n \geq 2$. Describe the universal covering space $\tilde{X}$ of $X$ and the covering map $p : \tilde{X} \to X$. Give a careful statement of a result implying that $X$ has the correct fundamental group.

4. Let $X$ denote the countably infinite product of circles

$$X = S^1 \times S^1 \times S^1 \times \ldots,$$

with the usual product topology. Compute the fundamental group $\pi_1(X, \bar{1})$, where $\bar{1} = (1, 1, \ldots) \in X$. You may assume $\pi_1(S^1, 1) \cong \mathbb{Z}$.

DO NOT CHOOSE MORE THAN 2 PROBLEMS FROM THIS PAGE!
5. Let \( f : X \to Y \) denote a continuous map between 0-connected topological spaces \( X \) and \( Y \). Suppose that for all \( q \geq 0 \), \( f_* : H_q(X; \mathbb{Z}) \to H_q(Y; \mathbb{Z}) \) is an isomorphism. Prove that for all \( q \geq 0 \) and abelian groups \( G \):

\[
f_* : H_q(X; G) \to H_q(Y; G)
\]

is an isomorphism. Define all homomorphisms used in your proof.

6. Let \( X \) denote a topological space.
   a) Define the suspension \( \Sigma X \) of \( X \) and show that \( \Sigma \) induces a functor from \( \text{Top} \to \text{Top} \), where \( \text{Top} \) is the category of topological spaces and (continuous) maps.
   b) Let \( X \) denote a compact, connected orientable manifold of dimension \( n \) and \( f : X \to X \) a map of degree \( k \). Show that \( \Sigma f \) induces a homomorphism \( (\Sigma f)_* : H_{n+1}(\Sigma X; \mathbb{Z}) \to H_{n+1}(\Sigma X; \mathbb{Z}) \) which is multiplied by \( k \).

7. Let \( X \) and \( Y \) denote path-connected spaces, \( x_0 \in X \), \( y_0 \in Y \). Then the wedge \( X \vee Y \) is defined by

\[
X \vee Y = X \times y_0 \cup x_0 \times Y \subset X \times Y.
\]

Also, the cone \( CX \) is defined by

\[
CX = X \times I / X \times \{1\}
\]

and \( i : X \to CX \) is defined by \( i(x) = [x, 0] \). Identify \( X \) and \( i(X) \). Now let

\[
J = (CX \times Y) \cup (X \times CY) \subset CX \times CY.
\]

Show that for any coefficients \( R \)

\[
H_n(J; R) \approx H_{n-1}(X \times Y, X \vee Y; R), \quad n \geq 1.
\]

8. Let \( K = \{K_n, \partial^K\} \) and \( L = \{L_n, \partial^L\} \) denote chain complexes of \( R \)-modules, \( R \) a commutative ring with unit and let \( f : K \to L \) denote a chain map. Define a new chain complex \( T = \{T_n, \partial^T\} \) by

\[
T_n = L_n \oplus K_{n-1}, \quad \partial^T(y, x) = (\partial^L y + f x, -\partial^K x).
\]

   a) Verify that \( \partial^T \partial^T = 0 \).
   b) If \( K = L \) and \( f = \) identity, verify that \( T \) is acyclic, i.e. \( H_*(T) = 0 \).

9. Let \( X \) denote the connected sum \((S^1 \times S^1) \# (S^1 \times S^1)\), i.e., \( X \) is a “sphere with two handles.” Calculate the cohomology ring \( H^*(X; \mathbb{Z}) \). You may use the cohomology ring of the torus \( S^1 \times S^1 \).
Problems/with Solutions

1. A topological space is called locally connected if each component of each of its open subsets is open. Let

\[ f : X \rightarrow Y \]

be a continuous, surjective mapping of topological spaces that is closed. That is, \( f \) always maps closed sets to closed sets. Assume that \( X \) is locally connected.

Prove:
(a) (9 points) For each \( y \in Y \), and each neighborhood \( U \) of \( y \), the point \( y \) is an interior point of the component of \( U \) that contains \( y \); and in fact:

(b) (1 point) \( Y \) is locally connected.

Solution Suppose \( U \) is an open neighborhood of \( y \in Y \). Let \( V \) be the union of all components of \( f^{-1}(U) \) that intersect \( f^{-1}(y) \). Then \( V \) is open,

\[ f^{-1}(y) \subseteq V \subseteq f^{-1}(U), \]

and \( f(V) \) is connected, since it is the union of a family of connected sets (images of components of \( f^{-1}(U) \)) each of which contains \( y \). Further, if \( C \) is the component of \( U \) that contains \( y \), then:

\[ y \in Y \setminus f(X \setminus V) \subseteq f(V) \subseteq C \subseteq U, \]

where \( Y \setminus f(X \setminus V) \) is open since \( f \) is closed. This completes part (a).

Part (a) reveals that if \( U \) is any open set in \( Y \), and \( y \in U \), then \( y \) is an interior point of the component \( C \) of \( U \) that contains \( y \). Since this is true for each \( y \in C \), it follows that \( C \) is open, as desired.
2. Let $X$ be a non-compact Hausdorff space whose topology has a countable basis. Prove from first principles (i.e., don't just quote theorems) that:

(a) (6 points) There is an infinite subset $A$ of $X$ such that $A$ has no limit point in $X$; and

(b) (4 points) The product projection onto the second coordinate

$$p_2 : X \times [0,1] \to [0,1]$$

is not a closed mapping. (A closed mapping is one that always maps closed sets to closed sets.)

Solution. Let $B$ be a countable basis for the topology of $X$. Let $V$ be an open cover of $X$ with no finite subcover, and let

$$B' = \{ B \in B | \text{some member of } V \text{ contains } B \}.$$  

Then $B'$ is a countable covering of $X$ with no finite subcover. Choose an enumeration

$$B' = \{ W_i | i \in \mathbb{Z}_+ \}.$$

Let $x_1 \notin W_1$, and in general $x_n \notin W_1 \cup \ldots \cup W_n$. In fact, since $X \setminus (W_1 \cup \ldots \cup W_n)$ is an infinite set, we can choose $x_n \notin \{ x_1, \ldots, x_{n-1} \}$. This ensures that $A = \{ x_n | n \in \mathbb{Z}_+ \}$ is an infinite set. Further, $A$ has no limit point in $X$ since $X$ is Hausdorff, and $B'$ is an open cover of $X$ such that each $W_n \cap A$ is finite.

Let

$$F = \{ (x_n, 1/n) | n \in \mathbb{Z}_+ \},$$

a closed set in $X \times [0,1]$. Then $p_2(F)$ is not closed in $[0,1]$.

3. (a) (5 points) Describe all path-connected covering spaces $p : E \to S^1$ of the circle. Explain how you know that your list is complete, giving careful statements of any results from covering space theory that you use.
(b) (5 points) Let $X$ be the wedge of two circles:

$$X = S^1 \times \{y_0\} \cup \{x_0\} \times S^1 \subset S^1 \times S^1.$$ 

Describe all path-connected 2-sheeted covering spaces $p : E \to X$.

**Note.** $p : E \to X$ and $p' : E' \to X$ are equivalent if there is a homeomorphism $h : E' \to E$ such that $p \cdot h = p'$.

**Solution.** There is a Classification Theorem:

Let $X$ be path-connected and locally path-connected. Let $p : E \to X$ and $p' : E' \to X$ be path-connected covering spaces of $X$. Let $p(e_0) = p'(e'_0) = b_0$. Then $p$ and $p'$ are equivalent covering maps if and only if $p_*(\pi_1(E, e_0))$ and $p'_*(\pi_1(E', e'_0))$ are conjugate subgroups of $\pi_1(X, b_0)$.

(a) It follows that there is a unique path-connected covering space of $S^1$ for each subgroup of $\pi_1(S^1) \cong \mathbb{Z}$. We list these covering spaces. First, corresponding to the subgroup $n \cdot \mathbb{Z}, n \geq 1$, we have the covering $p : S^1 \to S^1$ described in complex-number rotation by $p(z) = z^n$. Last, corresponding to the trivial subgroup, we have the "universal" covering $p : \mathbb{R} \to S^1$ described by $p(x) = e^{ix}$.

(b) If $p : E \to X$ is a 2-sheeted covering of the wedge $X$, then the restriction of the covering to the pre-image of each of the wedge-circles is a 2-sheeted covering (possibly disconnected) of that circle. Thus, there are two possible coverings for each of the wedge-circles, giving four 2-sheeted coverings of $X$. Since one of these is the trivial (disconnected) 2-sheeted covering of $X$, we have three
4. A closed surface is a compact, connected 2-manifold without boundary.

(a) (2 points) State the Classification Theorem for closed surfaces; and

(b) (8 points) Describe all closed surfaces with these Euler characteristics:
    $-1, -2$. If you use the connected sum operation in your description, explain what it means.

Solution. (a) We have the

Theorem 1. Each closed surface $M^2$ is uniquely expressible as the connected sum of $S^2$ with $n$ tori and $h$ projective planes, where $n \geq 0$ and $0 \leq h \leq 2$. ($M^2$ is orientable if and only if $h = 0$.)

Alternately:

Theorem 1'. Each orientable closed surface is uniquely expressible as the connected sum of $S^2$ and $m$ tori, $m \geq 0$. Each non-orientable closed surface is uniquely expressible as the connected sum of $S^2$ and $k$ projective planes, $k \geq 0$. 
In the notation of Theorem 1, we calculate the Euler characteristic as

$$\chi(M^2) = 2 - 2n - h, \ h \leq 2,$$

(In Theorem 1', $\chi = 2 - 2m$ or $2 - k$.)

(b) If $\chi(M^2) = -1$, then $M^2$ must be the connected sum

$$T^2 \# P^2 (= P^2 \# P^2 \# P^2).$$

If $\chi(M^2) = -2$, then $M^2$ must be one of 2 possibilities:

$$T^2 \# T^2$$

or

$$T^2 \# P^2 \# P^2 (= P^2 \# P^2 \# P^2 \# P^2).$$
5. Let $X$ denote a finite simplicial complex. $X$ is called unicoherent if whenever $X = K \cup L$, where $K$ and $L$ are subcomplexes with $K$ and $L$ connected, then $K \cap L$ is connected. Prove that for $n > 1$, $S^n$ and $\mathbb{R}P^n$ are unicoherent. Exhibit an example of finite simplicial complex which is not unicoherent.

Solution Consider the tail end of the Mayer-Vietoris sequence (over $\mathbb{Z}$)

$$H_1(K \cup L) \xrightarrow{\Delta} H_0(K \cap L) \xrightarrow{\varphi} H_0(K) \oplus H_0(L) \xrightarrow{\psi} H_0(K \cup L) \to 0$$

where $\varphi(u) = (i_*(u), j_*(u))$ and $\psi(x, y) = k_*(x) - \ell_*(y)$ and $i : K \cap L \to K$, $j : K \cap L \to L$, $k : K \to K \cup L$, $\ell : L \to K \cup L$ are inclusions. Identifying $\mathbb{Z} = H_0(K) = H_0(L) = H_0(K \cup L)$, kernel $\psi = \{(a, a) \in \mathbb{Z} \oplus \mathbb{Z}\} = \text{image } \varphi$.

a) if $X = K \cup L = S^n$, $n > 1$, $H_1(K \cup L) = 0$ and $\varphi$ injects. $\therefore H_0(K \cap L) \cong \mathbb{Z}$ and $K \cap L$ is 0-connected.

b) if $X = K \cup L = \mathbb{R}P^n$, $n > 1$, $H_1(K \cup L) = \mathbb{Z}_2$ and since $H_0(K \cap L)$ is free, image $\Delta = \text{kernel } \varphi = 0$. Thus, again we have $H_0(K \cap L) \cong \mathbb{Z}$.

Finally, $S^1$ is not unicoherent.

6. Let $D^n$ denote the unit ball in $\mathbb{R}^n$ $n \geq 2$, and for $x \in \mathbb{R}^n - D^n$, set $f(x) = x/(1 - ||x||)$. Show that $f : \mathbb{R}^n - D^n \to \mathbb{R}^n - D^n$ and then discuss completely when $f \sim$ identity as maps from $\mathbb{R}^n - D^n$ to $\mathbb{R}^n - D^n$.

Solution

(a) Let $f(x) = \frac{x}{1 - ||x||}$, $||x|| > 1$

(a) To show $||f(x)|| > 1$, we proceed as follows:
\[ \frac{\|x\|}{1 - \|x\|} > 1 \]
\[ \iff \frac{\|x\|}{\|x\| - 1} > 1 \]
\[ \iff \|x\| > \|x\| - 1 \]
\[ \iff 0 > -1 . \]

(b) \[ f(x) = -\left( \frac{1}{\|x\| - 1} \right) x \quad \therefore x \text{ and } f(x) \text{ are opposites.} \]
\[ = -\varphi(x)x \quad \varphi(x) > 0 \]
\[ H(x, t) = -[(1 - t)\varphi(x) + t]x \]

shows \( f \sim \alpha \), where \( \alpha(x) = -x \).

(c) Let \( \overline{f} : S^{n-1} \rightarrow S^{n-1} \) denote the obvious map on \( S^{n-1} \mapsto S^{n-1} \). Then, \( f \sim id \iff f \sim id \). Then, \( f \sim id \iff n \) is even.

7. Let \( X \) denote a finite complex and \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \). Prove that \( H_1(X; \mathbb{Z}_p) = 0 \) for all primes \( p \) if, and only if, \( H_1(X; \mathbb{Z}) = 0 \). Is the same statement true for cohomology, i.e. \( H^1(X; \mathbb{Z}_p) = 0 \) for all primes \( p \) if, and only if, \( H^1(X; \mathbb{Z}) = 0 \).

**Solution**

By the universal coefficient theorem for homology

\[ H_1(X; \mathbb{Z}_p) = H_1(X; \mathbb{Z}) \otimes \mathbb{Z}_p \oplus H_0(X; \mathbb{Z}) \ast \mathbb{Z}_p \]

since \( H_0(X; \mathbb{Z}) \) is f.g. free, \( H_0(X; \mathbb{Z}) \ast \mathbb{Z}_2 = 0 \)

\[ H_1(X; \mathbb{Z}_p) = H_1(X; \mathbb{Z}) \otimes \mathbb{Z}_p \]

\[ \therefore H_1(X; \mathbb{Z}) = 0 \text{ implies } H_1(X; \mathbb{Z}_p) = 0 \text{ for all primes } p. \] On the other hand, suppose \( H_1(X; \mathbb{Z}) \neq 0 \). Since \( H_1(X; \mathbb{Z}) \) is finitely generated as a \( \mathbb{Z} \)-module \( H_1(X; \mathbb{Z}) = F \oplus T \) where \( F \) is free and \( T \) is a direct sum of finite cyclic groups. If \( F \) is not trivial, \( F \otimes \mathbb{Z}_p \) contains \( \mathbb{Z}_p \) and \( H_1(X; \mathbb{Z}_p) \neq 0 \). Thus, if \( T \) is non-trivial, choose \( p \) so that
$T$ contains $p$-torsion in which case $H_1(X; \mathbb{Z}_p)$ contains $\mathbb{Z}_p$ and $H_1(X; \mathbb{Z}_p) \neq 0$ thus, $H_1(X; \mathbb{Z}_p) = 0$ for all $p$ implies $H_1(X; \mathbb{Z}) = 0$.

As far as cohomology is concerned, $H^1(\mathbb{R}P^n; \mathbb{Z}) = 0$ but $H^1(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2$ so in this case the result is false.

8. Show that $S^p \times S^q$ cannot be imbedded in $S^{p+q}, p \geq 1, q \geq 1$. Also, show that $S^p \times S^q$ can be imbedded in $S^{p+q+1}$.

**Solution** First observe that $H_k(S^p \times S^q) \neq H_k(S^{p+q})$ for $k = p$ or $k = q$ and hence if $\varphi : S^p \times S^q \to S^{p+q}$ is an imbedding, $A = \varphi(S^p \times S^q)$ is a proper subset of $S^{p+q}$ and hence $A \subset \mathbb{R}^{p+q} \subset S^{p+q}$ and $A$ being homeomorphic to $S^p \times S^q$, $H^{p+q}(S^p \times S^q) = H^p(S^p) \otimes H^q(S^q) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$. We may proceed in two different ways.

a) The exact sequence

$$\to H_0(\mathbb{R}^{p+q}) \to H_0(\mathbb{R}^{p+q}, \mathbb{R}^{p+q} - A) \to 0$$

tell us that

$$H_0(\mathbb{R}^{p+q}, \mathbb{R}^{p+q} - A) = 0.$$

On the other hand, by Alexander Duality ($A$ is an ENR)

$$H_k(\mathbb{R}^n, \mathbb{R}^n - A) \simeq H^{n-k}(A)$$

and hence

$$0 = H_0(\mathbb{R}^{p+q}, \mathbb{R}^{p+q} - A) \simeq H^{p+q}(A) = \mathbb{Z}$$

which is a contradiction.

b) Alternatively, since $A$ is an ENR, there is an open set $U \subset \mathbb{R}^{p+q}, U \supset A$. 

Qualifying Exam
TOPOLOGY
August 30, 1993

Instructions: Answer four problems, 10 points per problem. /

HOWEVER: DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PART I!

Please use different sheets of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems. To facilitate impartial grading, do not place your name or initials on your answer sheets. Instead, the proctor will assign you a letter code to be placed on your cover sheet and all answer sheets.

Notation: $H_*, H^*$ denote homology and cohomology functors.

$S^k = k$-sphere, $D^k = k$-ball = $k$-disk, $\mathbb{Z} =$ integers,

$\mathbb{Q} =$ rationals, $\mathbb{R} =$ real numbers.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
Qualifying Exam: TOPOLOGY
August 30, 1993

1. Let \( f : S^{2n} \to S^{2n} \) be a continuous map.
   
   (a) (8 pts) Prove that there exists an \( x \in S^{2n} \) with \( f(x) = x \) or \( f(x) = -x \).

   (b) (2 pts) Is statement (a) true for all self-maps of odd-dimensional spheres? Prove it or provide a counterexample.

**SOLUTION:** We need:

**Lemma:** Let \( f, g : S^k \to S^k \) be continuous maps with \( f(x) \neq -g(x) \) for each \( x \). Then \( f \) is homotopic to \( g \).

**Proof.** The homotopy \( H : S^k \times I \to S^k \) is:

\[
H(x, t) = \frac{t \cdot g(x) + (1 - t) \cdot f(x)}{\|t \cdot g(x) + (1 - t) \cdot f(x)\|}.
\]

(The denominator can be zero only when \( t = 1/2 \) and \( f(x) = -g(x) \), since each of \( f(x) \) and \( g(x) \) has norm 1.)

Now suppose \( f : S^{2n} \to S^{2n} \) is given. Let \( g_1 \) be the identity map of \( S^{2n} \). The lemma implies that if \( f(x) \neq -x \) for each \( x \), then \( f \) is homotopic to \( g_1 \). Let \( g_2 \) be the antipodal map of \( S^{2n} \). The lemma implies that if \( f(x) \neq x \) for each \( x \), then \( f \) is homotopic to \( g_2 \).

Hence, if \( f(x) \neq -x \) and \( f(x) \neq x \) for each \( x \), it follows that \( g_1 \) is homotopic to \( g_2 \). But \( g_1 \) and \( g_2 \) cannot be homotopic, since \( g_1 \) has degree 1 and \( g_2 \) has degree \((-1)^{2n+1} = -1 \). This last claim about the degree of \( g_2 \) follows from the fact that a reflection in one of the coordinate hyperplanes of \( \mathbb{R}^{2n+1} \) has degree \(-1 \), and that \( g_2 \) is the composition of \( 2n + 1 \) such reflections.

The statement of part (a) fails for self-maps of odd-dimensional spheres. To see this, let \( (z_1, \ldots, z_n) \) be an \( n \)-tuple of complex numbers with \( |z_1|^2 + \cdots + |z_n|^2 = 1 \). (This represents a typical point of \( S^{2n-1} \).) If \( \theta \) is not a multiple of \( \pi \), then this rotation of \( S^{2n-1} \):

\[
(z_1, \ldots, z_n) \to (z_1 e^{i\theta}, \ldots, z_n e^{i\theta})
\]

sends no point to itself or to its antipode.
2. Let $W$ be the one-point union of two circles. ($W$ is sometimes called a "figure-8"). Construct explicitly a connected, 3-sheeted covering space of $W$ whose only covering transformation is the identity. Justify all your claims from first principles.

**SOLUTION:** The covering is most easily described by a picture:

Any covering transformation $h$ is a homeomorphism that maps the set $\{v_1, v_2, v_3\}$ onto itself. But $v_3$ must map to itself, since $\bar{W} - \{v_3\}$ is disconnected, while each of $\bar{W} - \{v_1\}$ and $\bar{W} - \{v_2\}$ is connected. Thus, the set

$$F = \{x \in \bar{W} \mid h(x) = x\}$$

is nonempty. An easy argument shows that $F$ is both open and closed in $\bar{W}$. Hence, $h$ is the identity.
3. (a) \(7\) pts) Construct a connected space \(X\) such that

\[ H_\ast(X; \mathbb{Z}_3) \cong H_\ast(S^3 \vee S^4; \mathbb{Z}_3), \]

and

\[ H_\ast(X; \mathbb{Z}_2) \cong H_\ast(S^4 \vee S^5; \mathbb{Z}_2). \]

(b) \(3\) pts) Also compute for your example \(X, H_\ast(X; \mathbb{Q})\). \((S^p \vee S^q\) denotes the wedge, or 1-point union, of \(S^p\) and \(S^q\).\)

**SOLUTION:** Construct a finite CW-complex \(X\) as follows. Start with the one-point-union, or “wedge”, \(W = S^3 \vee S^4\), of a 3-sphere and a 4-sphere. To obtain \(X\), attach a 4-cell to \(S^3 \subset W\) by a map of degree 3, and a 5-cell to \(S^4 \subset W\) by a map of degree 2.

We obtain, using \(\mathbb{Z}\) coefficients:

\[
H_i(X) = \begin{cases} 
\mathbb{Z}, & i = 0 \\
\mathbb{Z}_3, & i = 3 \\
\mathbb{Z}_2, & i = 4 \\
0, & \text{otherwise.}
\end{cases}
\]

By the Universal Coefficient Theorem:

\[ H_i(X; \mathbb{Z}_n) = H_i(X) \otimes \mathbb{Z}_n + \text{Tor}(H_{i-1}(X), \mathbb{Z}_n). \]

In order for the right hand side to be non-zero, \(i\) must be 0, or \(H_i(X)\) must be \(\mathbb{Z}_n\), or \(H_{i-1}(X)\) must be \(\mathbb{Z}_n\). For \(n = 3\), this happens when \(i = 0, 3, \text{or } 4\), giving

\[
H_i(X; \mathbb{Z}_3) = \begin{cases} 
\mathbb{Z}_3, & i = 0 \\
\mathbb{Z}_3 \otimes \mathbb{Z}_3 = \mathbb{Z}_3, & i = 3 \\
\mathbb{Z}_2 \otimes \mathbb{Z}_3 + \text{Tor}(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3, & i = 4 \\
0, & \text{otherwise.}
\end{cases}
\]

For \(n = 2\), we must have \(i = 0, 4, \text{or } 5\), giving

\[
H_i(X; \mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2, & i = 0 \\
\mathbb{Z}_2 \otimes \mathbb{Z}_2 + \text{Tor}(\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2, & i = 4 \\
\text{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2, & i = 5 \\
0, & \text{otherwise.}
\end{cases}
\]

For rational coefficients \(\mathbb{Q}\), the homology is zero in all positive dimensions, since the integral homology is finite in those dimensions.
4. (a) (8 pts) Show that if \( M \) is a compact, connected and orientable 3-manifold (without boundary) with \( H_1(M; \mathbb{Z}) = 0 \), then \( M \) has the homology of a 3-sphere.

(b) (2 pts) Do these conditions force \( M \) to be the same homotopy type as a 3-sphere? Explain your answer briefly.

**SOLUTION:**

**Proof.** We employ Poincaré duality first to conclude that \( H_q(M; \mathbb{Z}) \cong H^{3-q}(M; \mathbb{Z}) \) and hence \( H_3(M; \mathbb{Z}) \cong \mathbb{Z} \), \( H_q(M; \mathbb{Z}) = 0 \) for \( q > 3 \) and

\[
H_2(M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}).
\]

But,

\[
H^1(M; \mathbb{Z}) = \text{Hom}[H_1(M; \mathbb{Z}), \mathbb{Z}] + \text{Ext}[\mathbb{Z}, \mathbb{Z}]
\]

\[
= 0.
\]

Therefore the first part follows:

The answer to the last part is no. There are orientable 3-manifolds whose fundamental group is non-trivial but when made abelian is trivial. For example, the orbit space of \( S^3 \) modulo the free action of the dodecahedral group.
5. Let $X$ denote a finite, connected CW-complex.

(a) (7 pts) Prove that $H_1(X; \mathbb{Z}_p) = 0$ for all primes $p \geq 2$, if, and only if, $H_1(X; \mathbb{Z}) = 0$.

(b) (3 pts) Is the same statement true for cohomology, i.e. when $H_1$ is replaced by $H^1$?

**SOLUTION:** (a.1) Suppose $H_1(X; \mathbb{Z}) = 0$. Then applying UCT we have

$$H_1(X; \mathbb{Z}_p) \cong H_1(X; \mathbb{Z}) \otimes \mathbb{Z}_p \oplus \text{Tor}(H_0(X, \mathbb{Z}), \mathbb{Z}_p)$$

$$\cong 0 \otimes \mathbb{Z}_p \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_p)$$

$$\cong 0 \oplus 0 = 0.$$  

(a.2) First the key point: $H_q(X; \mathbb{Z})$ is a finitely generated abelian group and hence $H_1(X; \mathbb{Z}) = F \oplus T$ where $F$ is a free abelian group and $T$ is a finite abelian group. Suppose, $H_1(X; \mathbb{Z}_p) = 0$ for every prime $p \geq 2$. If $H_1(X; \mathbb{Z}) \neq 0$, either $F$ or $T$ is non-zero. If $T = 0$, $F \neq 0$

$$H_1(X; \mathbb{Z}_p) = H_1(X; \mathbb{Z}) \otimes \mathbb{Z}_p \oplus \text{Tor}[\mathbb{Z}, \mathbb{Z}_p]$$

$$= H_1(X; \mathbb{Z}) \otimes \mathbb{Z}_p$$

and $H_1(X; \mathbb{Z}_p) = F \otimes \mathbb{Z}_p$ is a non-trivial direct sum of $\mathbb{Z}_p$'s, which contradicts, $H_1(X; \mathbb{Z}_p) = 0$.

If $T \neq 0$, let $p$ denote a prime where $T$ contains $p$-torsion. Then,

$$H_1(X; \mathbb{Z}_p) = F \otimes \mathbb{Z}_p \oplus T \otimes \mathbb{Z}_p$$

and $T \otimes \mathbb{Z}_p \neq 0$ which contradicts $H_1(X; \mathbb{Z}_p) = 0$.

(b) The corresponding statement is false for cohomology. Let $X = \mathbb{R}P^2$. Then, $H^1(X; \mathbb{Z}) = 0$ but $H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2$. 


6. (a) (3 pts) Define $\mathbb{RP}^n$, $n$-dimensional real projective space and give its structure as a CW-complex.

(b) (4 pts) Assuming the homology $H_\ast(\mathbb{RP}^n; \mathbb{Z})$, compute the cohomology ring $H^\ast(\mathbb{RP}^n; \mathbb{Z}_2)$.

(c) (3 pts) Then, use the long exact sequence in cohomology associated to the coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \to 0$$

where $\alpha$ is multiplication by 2, to compute the cohomology ring $H^\ast(\mathbb{RP}^n; \mathbb{Z})$.

**SOLUTION:**

**Proof.** (a) Represent $\mathbb{RP}^n$ as $\mathbb{RP}^{n-1} \cup_p D^n$ where $p : S^{n-1} \to \mathbb{RP}^n$ is the canonical 2-fold cover. This yields the homology $H(\mathbb{RP}^n; \mathbb{Z})$ by induction as:

$$H_q(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \text{ or } q = n \text{ odd} \\ \mathbb{Z}_2 & q \text{ odd } q < n \\ 0 & 0 < q \text{ even, or } q > n. \end{cases}$$

Then, using the UCT (explicit argument here!)

$$H^q(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \leq q \leq n \\ 0 & q > n \end{cases}$$

and the inclusion $\mathbb{RP}^{n-1} \to \mathbb{RP}^n$ induces isomorphisms in dimensions $\leq n - 1$.

(b) Let $x$ denote the generator of $H_1(\mathbb{RP}^n; \mathbb{Z}_2)$. We assert that $H^\ast(\mathbb{RP}^n; \mathbb{Z}_2)$ is the truncated polynomial ring $\{\mathbb{Z}_2[x], x^{n+1} = 0\}$. We proceed by induction and need only show $x^{n-1} \cup x$ is non-zero. But, using Poincaré duality (over $\mathbb{Z}_2$) with $\mu$ the generator of $H_\mu(\mathbb{RP}^n; \mathbb{Z}_2)$,

$$(x^{n-1} \cup x) \cap \mu = x^{n-1} \cap (x \cap \mu)$$

$x \cap \mu$ is the generator of $H_{n-1}(\mathbb{RP}^n; \mathbb{Z}_2)$ and

$$x^{n-1} \cap (x \cap \mu) = 1$$

is given by evaluation. Thus $x^{n-1} \cup x \neq 0$ by Poincaré duality.

(c) For the cohomology ring $H^\ast(\mathbb{RP}^n; \mathbb{Z})$ we use the UCT for cohomology to obtain the additive structure:

$$H^q(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, \text{ or } q = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } q \leq n, q \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$
Let $y$ denote the generator of $H^2(RP^n; \mathbb{Z})$ and $z$ the generator of $H^n(RP^n; \mathbb{Z})$ when $n$ is odd. Assert

$$H^*(RP^n; \mathbb{Z}) = \begin{cases} \{ \mathbb{Z}_2[y] : y^{k+1} = 0 \}, & n = 2k \\ \{ \mathbb{Z}_2[y] : y^{k+1} = 0 \} \oplus E(z), & n = 2k + 1. \end{cases}$$

To verify this consider the coefficient sequence

$$\ldots \to H^q(RP^n; \mathbb{Z}) \xrightarrow{\alpha^*} H^q(RP^n; \mathbb{Z}) \xrightarrow{\beta^*} H^q(RP^n; \mathbb{Z}_2) \to H^{q+1}(RP^n; \mathbb{Z}) \to$$

whence $\alpha^*$ is multiplication by 2 and $\beta^*$ is a ring homomorphism. If $q$ is even $q < n$ and $y_q$ is the generator of $H^q(RP^n; \mathbb{Z})$ $\alpha^*(y_q) = 0$ and $\beta^*$ is an isomorphism. Thus

$$y_q = y^q$$

and the result follows.
7. (10 pts) Let \( f : [0,1] \to S^n, n \geq 2 \), denote an imbedding and let \( K = f([0,1]) \).

Use a direct argument (not Alexander Duality) to compute \( H_*(S - K; \mathbb{Z}) \).

**SOLUTION:** We assert the reduced homology \( \tilde{H}_q(S^n - K; \mathbb{Z}) = 0 \) for every \( q \).

Consider a \( q \)-cycle \( z_q \) with support in \( S^n - K \). Subdivide \( I \) into 2 equal parts \( I = I_1 \cup I_2 \) and let \( A_i = f(I_i) \) so that

\[
K = A_1 \cup A_2 \quad \text{and} \quad A_1 \cap A_2 = \text{a point } x_0.
\]

Then, using the Mayer-Vietoris sequence for the open cover \( S^n - A_1, S^n - A_2 \) we have

\[
0 \to \tilde{H}^q(S^n - K) \xrightarrow{\cong} \tilde{H}^q(S^n - A_1) \oplus \tilde{H}^q(S^n - A_2) \to \tilde{H}^q(S^n - x_0) = 0.
\]

Thus, if \( z_q \) does not bound in \( S^n - K \), it does not bound in either \( S^n - A_1 \) or \( S^n - A_2 \). Let \( K_2 = \) whatever \( A_i \) has the property that \( z_q \) does not bound in \( S^n - A_i \). Thus \( K \supset K_2 \). Do the same for \( K_2 \) as \( K \) and continue to obtain \( K \supset K_2 \supset K_3 \supset \cdots \) a sequence of compact sets with \( \bigcap_j K_j \), being a point \( y \), diam \( K_j \to 0 \), and \( z_q \) does not bound in every \( S^n - K_j \). But \( z_q \) bounds in \( S^n - y \) hence in \( S^n - W, W \) a neighborhood of \( y \) (because of compact supports). For \( j \) sufficiently large \( K_j \subset W \) and hence \( z_q \) bounds in \( S^n - K_j \) a contradiction. Thus, \( \tilde{H}_q(S^n - K; 0) = 0 \).
Qualifying Exam
TOPOLOGY
August 29, 1994

Instructions: Answer four problems, 10 points per problem.

HOWEVER: DO NOT CHOOSE MORE THAN TWO PROBLEMS FROM PART I!

Please use different sheets of paper for each problem since not all of your answers will be graded by the same person. Indicate clearly which problems you are doing. Hand in no more than the requested number of problems. To facilitate impartial grading, do not place your name or initials on your answer sheets. Instead, the proctor will assign you a letter code to be placed on your cover sheet and all answer sheets.

Notation: $H_*$, $H^*$ denote homology and cohomology functors.

$S^k =$ k-sphere, $D^k =$ k-ball = k-disk, $\mathbb{Z} =$ integers,

$\mathbb{Q} =$ rationals, $\mathbb{R} =$ real numbers, $\mathbb{C} =$ complex numbers.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
QUALIFYING EXAM: TOPOLOGY
Part I
DO NOT CHOOSE MORE THAN 2 PROBLEMS FROM THIS PAGE.

1. A **closed surface** is a compact, connected metrizable 2-manifold without boundary.
   (a) (3 pts) State the Classification Theorem for closed surfaces. (You may wish to include a definition of the **connected sum** of two closed surfaces.)
   (b) (2 pts) Which orientable closed surface(s) can be 2-sheeted covering spaces of the connected sum $P^2 \# P^2 \# P^2$ of three real projective planes?
   (c) (5 pts) Find all compact, connected metrizable 2-manifolds, possibly with boundary, having Euler characteristic $-1$.

2. Let $U$ be a connected, bounded open subset of $\mathbb{R}^3$. Suppose that $U$ can be expressed as $\bigcup_{n=1}^{\infty} U_n$, where $U_n$ is an open set in $\mathbb{R}^3$ with $\overline{U}_n \subseteq U_{n+1}$, $U_n$ is homeomorphic to $S^1 \times \mathbb{R}^2$, and $\overline{U}_n$ is homeomorphic to $S^1 \times D^2$. We choose basepoint $\ast$ in $U_1$, and assume it is known that each group $\pi_1(U_n, \ast)$, $n \geq 1$, can be identified with $\mathbb{Z}$ in such a way that the generator of $\pi_1(U_n, \ast)$ maps to $(n + 1)$-times the generator of $\pi_1(U_{n+1}, \ast)$ under inclusion.
   (a) (8 pts) Prove from first principles that $\pi_1(U, \ast)$ is isomorphic to $\mathbb{Q}$, the additive group of rationals.
   (b) (2 pts) Calculate the homotopy groups $\pi_k(U, \ast)$, $k \geq 2$.

3. (a) (4 pts) Let $X$ be a regular, Hausdorff topological space, and $p : \tilde{X} \rightarrow X$ a covering space of $X$. Prove that for any compact $C \subseteq \tilde{X}$, the set of covering transformations $\varphi$ such that $\varphi(C) \cap C \neq \emptyset$, is finite. Include a definition, of "covering space". You may assume that $X$, $\tilde{X}$ are path connected and locally path connected.
   (b) (6 pts) Let $X$ be the 1-point union, or wedge, of two circles. ($X$ is sometimes called a "figure-8".) Let $G = \pi_1(X, x_0)$, where $x_0$ is the point common to the two circles. Prove the following proposition using the theory of covering spaces, and without quoting results from the theory of free groups: If $N$ is a finitely-generated, normal subgroup of $G$ such that $N$ has infinite index in $G$, then $N$ is the trivial group. State explicitly any results from covering space theory that you use.

DO NOT CHOOSE MORE THAN 2 PROBLEMS FROM THIS PAGE.
Part II

4. (a) (5 pts) Let \( X = \mathbb{R}P^m \times \mathbb{C}P^k \) where \( m \) and \( k \) are even and \( f : X \to X \) a map. Define
\[
L(f) = \Sigma(-1)^q \text{Tr}(f^*_q)
\]
where \( \text{Tr} = \text{Trace} \) and \( f^*_q : H^q(X; \mathbb{Q}) \to H^q(X, \mathbb{Q}) \) is the endomorphism induced by \( f \). Show that \( L(f) \neq 0 \) for any \( f \).

(b) (5 pts) Let \( M \) denote a compact, connected, orientable \( n \)-manifold and \( f : M \to M \) a map. Let \( \mu \) denote a generator of \( H_n(M; \mathbb{Z}) \). \( f \) is called orientation reversing if \( f_*(\mu) = -\mu \). Show that \( \mathbb{C}P^k \), \( k \geq 1 \), admits an orientation reversing homeomorphism if, and only if, \( k \) is odd.

5. Let \( M^n \) be an \( n \)-dimensional compact manifold such that
\[
H_i(M^n; \mathbb{Z}/2\mathbb{Z}) = H_i(S^n; \mathbb{Z}/2\mathbb{Z}).
\]
Prove that \( M \) is orientable.

6. Let \( G \) be a finitely presented group.
   (a) Show that there exists a compact topological space \( X \) such that \( \pi_1(X) = G \). Justify your computations or arguments by quoting the relevant theorems.
   (b) Suppose \( G \) is a free group on \( n \) generators. Construct a compact connected topological manifold without boundary \( X \) such that \( \pi_1(X) = G \). Justify your computation of \( \pi_1(X) \) by quoting the relevant theorems.

7. Let \( X \) be a simply-connected compact manifold without boundary of dimension four. Suppose that the Euler characteristic of \( X \) is 4.
   (a) Determine all possibilities for the cohomology ring structure of \( X \) over \( \mathbb{R} \).
   (b) Give examples to illustrate each possibility in (a).
1. A closed surface is a compact, connected metrizable 2-manifold without boundary.

(a) State the Classification Theorem for closed surfaces. Include a definition of “connected sum”.

(b) Three surfaces with boundary are shown below. If 2-disks are attached to each component of their boundaries, closed surfaces result. Identify these 3 closed surfaces among the model surfaces given in your statement of the Classification Theorem.

\[ \begin{align*}
\mathcal{M}_1 & \quad \mathcal{M}_2 & \quad \mathcal{M}_3 \\
\end{align*} \]

Solution:

(a) If \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are closed surfaces, then the connected sum \( \mathcal{M}_1 \# \mathcal{M}_2 \) is obtained by choosing a closed 2-disk \( D_1 \subset \mathcal{M}_1 \) and identifying \( \mathcal{M}_1 - \text{Int} \, D_1 \) with \( \mathcal{M}_2 - \text{Int} \, D_2 \) via a homeomorphism of \( \partial D_1 \) with \( \partial D_2 \). The result is a closed surface whose definition does not depend upon the choices made in this construction.

The Classification Theorem says that if \( M \) is an orientable closed surface then \( M \) is homeomorphic to the connected sum of \( S^2 \) and a finite number \( k \geq 0 \) of tori, \( T^2 = S^1 \times S^1 \). (The Euler characteristic of \( M \) is \( 2 - 2k \).) If \( M \) is a nonorientable closed surface then \( M \) is homeomorphic to the connected sum of \( S^2 \) and a finite number \( k \geq 0 \) of real projective planes, \( P^2 \). (The Euler characteristic of \( M \) is \( 2 - k \) in this case.)

An alternate statement of the Classification Theorem is that each closed surface \( M \) is homeomorphic to the connected sum of \( S^2 \) with \( m \geq 0 \) tori and \( n \) projective planes, where \( 0 \leq n \leq 2 \). The integers \( m \) and \( n \) are unique, and \( M \) is orientable if and only if \( n = 0 \). We have that \( \chi(M) = 2 - 2m - n \).
(b) We have:

<table>
<thead>
<tr>
<th>$\chi(M_i)$</th>
<th>Components of $\partial M_i$</th>
<th>Orientation Status</th>
<th>$\chi(\tilde{M}_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>-2</td>
<td>2</td>
<td>Orientable</td>
</tr>
<tr>
<td>$M_2$</td>
<td>-3</td>
<td>1</td>
<td>Orientable</td>
</tr>
<tr>
<td>$M_3$</td>
<td>-2</td>
<td>1</td>
<td>Non-Orientable</td>
</tr>
</tbody>
</table>

We conclude that:

$\tilde{M}_1$ is a torus, $T^2$

$\tilde{M}_2$ is $T^2 \# T^2$

$\tilde{M}_3$ is $T^2 \# P^2 = P^2 \# P^2 \# P^2$

2. (a) Define the meaning of "$p : E \to B$ is a covering map".

(b) Let $X$ be the familiar "figure-8" subspace of $\mathbb{R}^2$ : $X$ is the union of two circles with a single point $x_0$ in common. Give a covering space argument to show that $\pi_1(X, x_0)$ contains elements $a$ and $b$ such that $(ab)^2 \neq a^2 b^2$. Give a careful statement of any theorems from covering space theory that you use. Do not use results from the theory of free groups.

Solution:

(a) A continuous surjective map $p : E \to B$ is called a covering map if each point $b$ of $B$ has a neighborhood $U$ such that $p^{-1}(U)$ can be expressed as the union of disjoint open sets $V_\alpha$ in $E$ such that for each $\alpha, p$ restricts to a homeomorphism of $V_\alpha$ onto $U$. 
(b) We must choose a suitable covering map \( p : \tilde{X} \to X \). Many choices are possible, including the “universal” covering space where \( \tilde{X} \) is simply connected. Here is a three-to-one covering that works:

Let \( I = [0,1] \), and \( f : I \to X \) a map with \( f(I) = A \), \( f(0) = x_0 = f(1) \), and \( f|_{(0,1)} \) a homeomorphism onto \( A \setminus \{x_0\} \). Similarly, let \( g : I \to X \) be a map with \( g(I) = B \), \( g(0) = x_0 = g(1) \), and \( g|_{(0,1)} \) a homeomorphism onto \( B \setminus \{x_0\} \). Then \( f \) and \( g \) have liftings \( \tilde{f}, \tilde{g} : I \to \tilde{X} \) with \( \tilde{f}(0) = \tilde{x}_0 = \tilde{g}(0) \). \((\tilde{f}(I)) \) is the lower half of the circle \( A_1 \) and \( \tilde{g}(I) \) is the upper half of the circle \( B_1 \).

We let \( a \) be the path homotopy class \([f]\) of the loop \( f \) based at \( x_0 \). Let \( b \) be the path homotopy class \([g]\) of the loop \( g \) based at \( x_0 \). We claim that in \( \pi_1(X,x_0),(ab)^2 \neq a^2b^2 \). If suffices to show that \( f \ast g \ast f \ast g \) is not path homotopic to \( f \ast f \ast g \ast g \), where “\( \ast \)” denotes path product. But the final point of \( f \ast g \ast f \ast g \) is \( \tilde{x}_1 \), while the final point of \( f \ast f \ast g \ast g \) is \( \tilde{x}_0 \).

The desired conclusion then follows from a standard result about lifting homotopies.

**Proposition.** Let \( p : E \to B \) be a covering map with \( p(e_0) = b_0 \). Let \( F : I \times I \to B \) be continuous, with \( F(0,0) = b_0 \). Then \( F \) lifts to a continuous map

\[ \tilde{F} : I \times I \to E \]

with \( \tilde{F}(0,0) = e_0 \). If \( F \) is a path homotopy, then \( \tilde{F} \) is a path homotopy. In particular if two paths in \( B \) have initial point \( b_0 \) and are path-homotopic, then their lifts starting at \( e_0 \) must have the same final point.

3. Let \( A, B, C \) be chain complexes.
(a) Explain the meaning of the statement that there is a short exact sequence of chain complexes

\[ 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0. \]

(b) State the homology result which is a consequence of the short exact sequence in (a).

(c) Define the connecting homomorphism in (b).

Solution:

(a) For each \( q \) we have chain homomorphisms

\[ 0 \to A_q \xrightarrow{\alpha} B_q \xrightarrow{\beta} C_q \to 0 \]

with \( \ker \alpha_q = 0, \im \alpha_q = \ker \beta_q, \im \beta_q = C_q \).

(b) **Theorem** There is an induced long exact sequence in homology

\[ \cdots H_q(A) \xrightarrow{\alpha} H_q(B) \xrightarrow{\beta} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \to \cdots \]

where \( \partial \) is the connecting homomorphism.

(c) \[ 0 \to A_q \xrightarrow{\alpha} B_q \xrightarrow{\beta} C_q \to 0 \]

\[ \partial \downarrow \quad \downarrow \partial \quad \downarrow \partial \]

\[ 0 \to A_{q-1} \xrightarrow{\alpha_{q-1}} B_{q-1} \xrightarrow{\beta_{q-1}} C_{q-1} \to 0 \]

Suppose \([z] \in H_q(C)\) is a homology class represented by the cycle \( z \in C_q, \partial z = 0 \).

Since \( \beta_q \) is surjective, \( \exists b \in B_q \) with \( \beta_q b = z \). Now \( \beta_{q-1} \partial b = \partial \beta_q b = \partial z = 0 \). Since the sequence is exact, \( \exists a \in A_{q-1} \) with \( \alpha_q(a) = \partial b \).

The chain \( a \in A_{q-1} \) is a cycle:

\[ \alpha_{q-2}(\partial a) = \partial(\alpha_{q-1} a) = \partial(\partial b) = 0. \]

Since \( \alpha_{q-2} \) is injective, \( \partial a = 0 \).

Definition: \( \partial_*[z] = [a] \).

Independent of choice: of \( b \): Suppose \( \beta_q b' = \beta_q b = z \) and \( \alpha_{q-1}(a') = \partial b' \). Then \( \beta_q(b - b') = 0 \) so \( \exists m \in A_q \) with \( \alpha_q(m) = b - b' \). We get

\[ \partial \alpha_q(m) = \alpha_{q-1} \partial(m) \]
\[ \partial(b - b') = \alpha_{q-1}(\partial m) \]
\[ \alpha_{q-1}(a - a') = \alpha_{q-1}(\partial m) \]
Since \( \alpha_{q-1} \) is a monomorphism \( a - a' = \partial m \) so \( [a] = [a'] \).

4. Let \( S^2 \) denote the 2-sphere and let \( p, q, r \) be distinct points on \( S^2 \). Let \( X \) be the identification space obtained by identifying \( p, q, r \) to a single point \( x \in X \).

(a) Find a cell structure for \( X \).

(b) Compute the integer homology groups of \( X \).

Solution:

(a) We may assume that \( p, q, r \) are on the equator.

A cell decomposition of \( S^2 \) is obtained by taking
3 0-cells: \( p, q, r \)
3 1-cells: \( e_1, e_2, e_3 \)
2 2-cells: \( f_1, f_2 \)

A cell decomposition of \( X \) has
1 0-cell: \( x \)
3 1-cells: \( e_1, e_2, e_3 \)
2 2-cells: \( f_1, f_2 \)

(b)

\[
\begin{align*}
C_0(X) &= \mathbb{Z}x \\
C_1(X) &= \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \\
C_2(X) &= \mathbb{Z}f_1 \oplus \mathbb{Z}f_2
\end{align*}
\]

We describe the boundary maps:

\[
\begin{align*}
\partial e_1 &= x - x = 0 & \partial e_2 &= x - x = 0 & \partial e_3 &= x - x = 0 \\
\partial f_1 &= e_1 + e_2 + e_3 & \partial f_2 &= e_1 + e_2 + e_3
\end{align*}
\]

Thus \( \partial(f_1 - f_2) = 0 \).

\[
\begin{align*}
Z_0(X) &= C_0(X) \\
B_0(X) &= 0
\end{align*}
\]

\( \Rightarrow H_0(X) \simeq \mathbb{Z} \)
\[ Z_1(X) \cong C_1(X) \]
\[ B_1(X) \cong \mathbb{Z}_{(e_1 + e_2 + e_3)} \]
\[ \Rightarrow H_1(X) \cong \mathbb{Z} + \mathbb{Z} \]
\[ Z_2(X) \cong \mathbb{Z}(f_1 - f_2) \]
\[ B_2(X) = 0 \]
\[ \Rightarrow H_2(X) \cong \mathbb{Z} \]

5. Let \( R \) be a commutative ring with 1. Define an \( R \)-homology sphere to be a Hausdorff topological space \( X \) such that \( H_i(X; R) \cong H_i(S^n; R) \) for some sphere \( S^n \).

(a) Let \( W \) be a compact connected \((n+1)\)-manifold and let \( X = \partial W \). Suppose that \( H_i(W; \mathbb{Z}_2) = 0 \) for all \( i > 0 \). Prove that \( X \) is a \( \mathbb{Z}_2 \)-homology sphere.

(b) Give an example of \( R \) and an \( R \)-homology sphere which is a non-orientable connected compact manifold. Justify your answer by quoting appropriate theorems. You may quote standard computations without proof.

Solution: (a) By Poincaré-Lefschetz duality, \( H_i(W, \partial W; \mathbb{Z}_2) \cong H^{n+1-i}(W; \mathbb{Z}_2) \).
From Universal Coefficients Formula, \( H^r(W; \mathbb{Z}_2) = \text{Hom}_{\mathbb{Z}_2}(H_r(W; \mathbb{Z}_2), \mathbb{Z}_2) = 0 \) for \( r > 0 \) (using the hypothesis). Hence \( H_i(W, \partial W; \mathbb{Z}_2) = 0 \) for \( i < n+1 \).
Since \( W \) is a connected manifold, \( H_{n+1}(W, \partial W; \mathbb{Z}_2) \cong \mathbb{Z}_2 \). Apply the homology exact sequence for the pair \((W, \partial W)\) with \( \mathbb{Z}_2 \)-coefficients:

\[ \cdots \rightarrow H_{i+1}(W, \partial W) \xrightarrow{\delta} H_i(\partial W) \rightarrow H_i(W) \rightarrow H_i(W, \partial W) \rightarrow \cdots \]

It follows that \( H_i(\partial W; \mathbb{Z}_2) = 0 \) unless \( i = 0 \) or \( n \) and \( H_n(\partial W; \mathbb{Z}_2) = H_0(\partial W; \mathbb{Z}_2) = \mathbb{Z}_2 \).

(b) Let \( R = \mathbb{Z}_p \) for \( p \neq 2 \). First, consider \( \mathbb{R}P^k \). By the Universal Coefficients Theorem, \( H_i(\mathbb{R}P^n; \mathbb{Z}_p) = H_i(\mathbb{R}P^n) \otimes \mathbb{Z}_p \oplus \text{Tor}(H_{i-1}(\mathbb{R}P^n), \mathbb{Z}_p) \).
\( H_{2i}(\mathbb{R}P^k) = 0 \) for \( i > 0 \) and \( H_{2i+1}(\mathbb{R}P^k) = \mathbb{Z}_2 \). Hence, for \( i > 0 \), \( H_i(\mathbb{R}P^n; \mathbb{Z}_p) = \mathbb{Z}_2 \otimes \mathbb{Z}_p = 0 \) since \((2, p) = 1\). Now consider \( X = \mathbb{R}P^k \times S^m \) for any \( m > 0 \) which is a connected compact topological manifold without boundary. By Künneth formula,

\[ H_{2k+m}(X) = H_{2k}(\mathbb{R}P^k) \otimes H_m(S^m) \oplus \text{Tor}(H_i(\mathbb{R}P^k), H_{2k+m-1-i}(S^m)) = 0. \]

Therefore, \( X \) is non-orientable. But,

\[ H_i(X; \mathbb{Z}_p) = \oplus H_i(\mathbb{R}P^k; \mathbb{Z}_p) \otimes H_{i-r}(S^m; \mathbb{Z}_p) \oplus \oplus \text{Tor}(H_s(\mathbb{R}P^k; \mathbb{Z}_p), H_{i-1-s}(S^m; \mathbb{Z}_p)) = 0 \]
for \( i \neq 0, m \),
\[ H_0(X; \mathbb{Z}_p) = \mathbb{Z}_p = H_m(X; \mathbb{Z}_p). \]

Hence \( X \) is a \( \mathbb{Z}_p \)-homology sphere. As another example, the Klein bottle is an \( R \)-homology circle if \( R = \mathbb{Z}_p \) and \( p \) is an odd prime (with similar calculations).

6. Let \( \{X_i : i = 1, 2, 3, \cdots \} \) be an infinite sequence of topological spaces and \( f_i : X_i \to X_{i+1} \) be maps. The mapping telescope of this sequence is defined to be the identification space \( T = \bigsqcup_{i=1}^{\infty} X_i \times [0,1]/\sim \), where \( (x, 1) \sim (y, 0) \) if \( x \in X_i, y \in X_{i+1} \) and \( y = f_i(x) \) describes the identification. Consider the case where all \( X_i = S^m \) \( (m > 0) \) and \( f_i : X_i \to X_{i+1} \) is a map of degree \( d \).

(a) Compute \( H_*(T) \) when \( d = 0 \).

(b) Compute \( H_*(T; \mathbb{Q}) \) when \( d \neq 0 \).

Solution: Let
\[
T_n = \left( \left( \prod_{i=1}^{n-1} X_i \times [0,1] \right) \prod X_n \times [0, \frac{1}{2}) \right)/\sim
\]
with a similar identification. Then \( \{T_n : n = 2, 3, \cdots \} \) form an increasing sequence of open subspaces of \( T \), and \( T = \bigcup_{n=2}^{\infty} T_n \). (Therefore, \( H_*(T) = \lim_{n \to \infty} H_*(T_n) \) in general.) For the case \( X_i = S^m \) and deg \( (f_i) = d \), \( H_*(T_n) \cong H_*(S^m) \) and the inclusion \( T_n \subseteq T_{n+1} \) induces multiplication by \( d \) in \( H_*(T_n) \to H_*(T_{n+1}) \). To see this, consider the following commutative diagram in which the vertical arrows are homotopy equivalences, since \( X_n \) is a deformation retract of \( T_n \):

\[
\begin{array}{ccc}
X_n & \xrightarrow{f} & X_{n+1} \\
\downarrow \text{inclusion} & & \downarrow \text{inclusion} \\
T_n & \subseteq & T_{n+1}
\end{array}
\]

(a) If \( d = 0 \), then \( f \) is null-homotopic. Hence \( T_n \to T_{n+1} \) is null-homotopic. Passing to the increasing union: \( \cup_n T_n \to \cup_n T_{n+1} \) which is identity: \( T \to T \) is null-homotopic. Hence \( T \) is contractible and \( H_i(T) = 0 \) for \( i > 0 \), \( H_0(T) = \mathbb{Z} \).

(b) If \( d \neq 0 \), then \( T_n \to T_{n+1} \) induces an isomorphism \( H_*(T_n; \mathbb{Q}) \cong H_*(T_{n+1}; \mathbb{Q}) \). Hence, the non-zero homology groups are obtained as direct limits of isomorphisms \( \mathbb{Q} \to \mathbb{Q} \). This shows \( H_0(T; \mathbb{Q}) = \mathbb{Q}, H_m(T; \mathbb{Q}) = \mathbb{Q} \), and all other homology groups vanish.
TOPOLOGY QUALIFYING EXAM
August 30, 1995

DO FOUR PROBLEMS, ALL OF THEM CARRY EQUAL MARKS (CREDIT).

1. Identify the vector space of all \( n \times n \) real matrices with \( \mathbb{R}^n \). Let \( Y_n = SO(n) \) denote the topological subspace of all \( n \times n \) orthogonal matrices of determinant one. Using the natural embedding

\[
A \mapsto \begin{pmatrix} 0 & & \vdots & & \end{pmatrix}
\begin{pmatrix} A & & \end{pmatrix}
\begin{pmatrix} 0 & \end{pmatrix}
\begin{pmatrix} 0 & \end{pmatrix}
\begin{pmatrix} 0 & \end{pmatrix}
\begin{pmatrix} 1 & \end{pmatrix}
\]
we obtain an inclusion \( Y_n \subset Y_{n+1} \).

(a) Prove that \( Y_n \) is a compact topological space for all \( n \geq 1 \).
(b) Show that \( Y_{n-1} \) is a closed subgroup of \( Y_n \), and that for \( n \geq 2 \), the space of cosets \( Y_n / Y_{n-1} \) is homeomorphic to \( S^{n-1} \).
(c) Using induction and part (b), show that \( Y_n \) is connected for all values of \( n \geq 1 \).

2. Let \( \mathbb{R}P^n \) denote the space of unoriented lines through the origin in \( \mathbb{R}^{n+1} \). Does there exist a continuous map \( f : \mathbb{R}P^n \to \mathbb{R}^{n+1} - 0 \) such that for all \( x \in \mathbb{R}P^n \), \( f(x) \) is orthogonal to \( x \)? Justify your answer.

3. Let \( \tilde{X} \to X \) be a connected \( n \)-sheeted covering space of a compact, connected surface without boundary. Show that \( \tilde{X} \) is also a compact surface without boundary and that \( \chi(\tilde{X}) = n \cdot \chi(X) \). Use this to explicitly determine all connected surfaces (up to homeomorphism) which are double covers of the Klein bottle.

4. Suppose that \( n \) and \( m \) are integers with \( n > m > 0 \). Let \( M^n \) and \( N^n \) denote compact, connected manifolds without boundary of dimensions \( m \) and \( n \) respectively. Prove that they are not homotopy equivalent.

5. Let \( X = \mathbb{C}P^n \) denote complex projective space of dimension \( 2n \).
(a) Prove that \( X \) is a compact, connected \( 2n \)-dimensional manifold.
(b) Show in detail that \( X \) can be given a CW–complex structure with one cell in every even dimension \( i = 0, 2, 4, \ldots, 2n \).
(c) Calculate the cohomology ring \( H^*(X, \mathbb{Z}) \).

6. Let \( (C_*, \partial_*) \) denote a chain complex of free \( \mathbb{Z} \)-modules such that \( C_i = 0 \) for \( i < 0 \). Show that the homology groups of \( C_* \) are all zero if, and only if, there exist homomorphisms \( D_i : C_i \to C_{i+1} \) for \( i \geq -1 \), such that \( \partial_{i+1} D_i + D_{i-1} \partial_i = id \) for all \( i \geq 0 \).

7. Let \( M \) denote a connected, non-orientable, compact 3–manifold without boundary. Prove that its fundamental group must be an infinite group.
1. Let $X \subset \mathbb{R}^n$ be a compact subset and assume there is an open set $U \subset \mathbb{R}^n$ such that $X \subset U$ and $X$ is a retract of $U$ (i.e. $X$ is a neighborhood retract in $\mathbb{R}^n$). Show that there exists an $\epsilon > 0$ such that given any topological space $Y$ and two maps $f, g : Y \to X$, with

$$d(f, g) = \inf_{y \in Y} \|f(y) - g(y)\| < \epsilon,$$

then they are homotopic, i.e. $f \simeq g : Y \to X$.

2. Let $f : X \to Y$ be a continuous function. Define the mapping cylinder of $f$ to be the quotient space

$$Z_f = X \times [0,1] \cup Y / S$$

where $S$ is the relation which identifies the points $(x, 1)$ and $f(x)$. Let $i : X \to Z_f$ be given by $i(x) = [(x, 0)]$ and $j : Y \to Z_f$ be given by $j(y) = [y]$. Identify $X$ and $Y$ with their images under $i$ and $j$ respectively, so that $i$ and $j$ become inclusions.

(a) Show that there is a retraction $r : Z_f \to Y$ such that $r \circ i = f$.

(b) Show that in fact $Y$ is a strong deformation retract of $Z_f$.

(c) Use this to show that $f$ induces an integral homology equivalence between $X$ and $Y$ (i.e. an isomorphism between all the homology groups with integer coefficients) if and only if $H_*(Z_f, X; \mathbb{Z})$ is identically zero.

(d) Use part (c) to show that if a continuous map induces an integral homology equivalence, then it induces one with coefficients in any abelian group.

3. Let $A \subset \mathbb{R}^3$ denote the union of the 3 coordinate axes in $\mathbb{R}^3$, i.e.

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0 \text{ or } y = z = 0 \text{ or } x = z = 0\}.$$

Let $X = \mathbb{R}^3 - A$. Compute $H_*(X; \mathbb{Z})$.

4. Let $f : S^{2n-1} \to S^n$ denote a continuous map, and $H_f = D^{2n} \cup_f S^n$ the space obtained by attaching a 2n-cell to $S^n$ using $f$.

(i) Calculate the integral homology of $H_f$.

(ii) Find an $n$ and an $f$ such that $H^*(H_f, \mathbb{F}_2)$ has a non-trivial product. Justify your answer.

5. Let $X$ and $Y$ be finite CW–complexes such that $H_*(X \times Y, \mathbb{Q}) \cong H_*(X, \mathbb{Q})$. What can you say about $H_*(Y, \mathbb{Q})$? Justify your answer in detail.

6. (a) Prove that if $M$ is a compact connected orientable odd-dimensional manifold without boundary, then its Euler characteristic $\chi(M)$ must be zero.

(b) Is this result valid when $M$ is non–orientable? Explain.
TOPOLOGY QUALIFYING EXAMINATION
August 28, 1996

Do four problems only. Each carries the same credit.

Problem 1 Let \( p : \tilde{X} \to X \) be an \( n \)-sheeted covering space of a finite CW-complex. Prove that \( \chi(\tilde{X}) = n \chi(X) \). Use this result to determine, up to isomorphism, all two-sheeted covering spaces of the Klein bottle.

Problem 2 Denote by \( M_1 \) and \( M_2 \), respectively, the mapping cylinders of the natural projections

\[
p_1 : S^m \times S^n \to S^m, \quad p_2 : S^m \times S^n \to S^n, \quad m, n > 0
\]

Compute \( H_*(M_1 \cup M_2; \mathbb{Z}) \) in terms of the homology of \( X \). Here \( M_1 \cup M_2 \) is the union of the mapping cylinders along \( S^m \times S^n \). (Recall that the mapping cylinder of a map \( f : X \to Y \) is the space \( X \times I \cup_f Y \) obtained from the disjoint union \( X \times I \sqcup Y \) by identifying \( (x, 1) \in X \times I \) with \( f(x) \in Y \).)

Problem 3 Consider the space

\[
O_{n+1,2} = \{(z_1, z_2) \mid z_1, z_2 = 0\} \subset S^n \times S^n
\]

where \( S^n \) is the unit sphere in the Euclidean space \( \mathbb{R}^{n+1} \) with the standard inner product. Denote by \( p : O_{n+1,2} \to S^n \) the projection on the first factor. Prove that there is a section \( s : S^n \to O_{n+1,2} \) if, and only if, \( n \) is odd.

Problem 4 Let \( \{C, \partial\} \) and \( \{C', \partial'\} \) be two chain complexes of free abelian groups, such that \( C_n = \{0\} = C'_n \) for \( n < 0 \). Assume that \( H_*(C) \) and \( H_*(C') \) are finitely generated groups. Prove that \( H_*(C \otimes C') = \{0\} \) if, and only if, the orders of \( H_*(C) \) and \( H_*(C') \) are finite and relatively prime.

Problem 5 Let \( \mathbb{P}_2(\mathbb{R}) \) and \( T \) denote, respectively, the real projective plane and the torus \( S^1 \times S^1 \). Prove that any map

\[
f : \mathbb{P}_2(\mathbb{R}) \to T,
\]

is homotopic to a constant map.

Problem 6 Define \( X \) to be the space obtained from \( S^2 \times S^2 \) by identifying \( (x, e) \) with \( (e, x) \), where \( e \in S^2 \) is a basepoint. Determine the cohomology ring \( H^*(X; \mathbb{Z}) \). Show that if \( X = \bigcup_{i=1}^n U_i \), where \( U_i \) is contractible in \( X \), then \( n \geq 3 \).
Problem 1 Find topological spaces $X$ and $Y$ so that $\pi_k(X) \simeq \pi_k(Y)$ for all $k$, but $X$ and $Y$ are not homotopy equivalent.

Solution Here is one example. Let $X = S^2 \times P^3$ and $Y = P^2 \times S^3$. Then $\pi_1(X) \simeq \pi_1(Y) \simeq \mathbb{Z}_2$ and for $k \geq 2$, $\pi_k(X) \simeq \pi_k(Y) \simeq \pi_k(S^2 \times S^3)$. $X$ is an orientable 5-manifold, so $H_5(X) \simeq \mathbb{Z}$. $Y$ is a nonorientable 5-manifold, so $H_5(Y) = 0$. Thus they are not homotopy equivalent.

Problem 2 Let $X$ denote the space obtained from $S^2$ by identifying $p$ distinct points on the equator. Describe a cell structure for $X$ and use it to compute the homology groups of $X$.

Solution Consider the cell decomposition of $S^2$ indicated below. There are $p$ 0-cells $E^0_i$ corresponding to the points identified in $X$. There are $p$ 1-cells $E^1_i$ corresponding to the edges of the equator between the points identified in $X$. We may assume that these are oriented so that $\partial E^1_k = E^0_k - E^0_{k-1}$. There are two 2-cells $E^2_1$ and $E^2_2$ corresponding to the upper and lower hemispheres. These may be oriented so that

$$\partial E^1_1 = \partial E^1_2 = E^0_1 + E^0_2 + \ldots + E^0_p.$$

It follows that $X$ has one 0-cell, $e^0$, $p$ 1-cells $e^1_1$, and two 2-cells $e^2_1$, $e^2_2$. Moreover $\partial e^1_1 = 0$ and $\partial e^1_2 = \partial e^2_1 = e^0_1 + e^0_2 + \ldots + e^0_p$. Thus the homology of $X$ is torsion free with Betti numbers $b_0 = 1$, $b_1 = p - 1$, $b_2 = 1$, and $b_k = 0$ for $k > 2$.

Problem 3 Show that there is no one-to-one continuous map $\mathbb{R}^n \rightarrow \mathbb{R}^2$ for $n > 2$. 

1
Solution Argue by contradiction. Suppose \( f : \mathbb{R}^n \to \mathbb{R}^2 \) is a one-to-one continuous map and \( n > 2 \). We may assume that \( f(0) = 0 \) so \( f : \mathbb{R}^n - \{0\} \to \mathbb{R}^2 - \{0\} \) is a one-to-one continuous map. Consider the unit circle \( S \subset \mathbb{R}^2 - \{0\} \). It represents a generator of \( \pi_1(\mathbb{R}^2 - \{0\}) \). Since \( f \) is one-to-one, \( f^{-1}(S) \) is a loop in \( \mathbb{R}^n - \{0\} \) whose image under \( f_\# : \pi_1(\mathbb{R}^n - \{0\}) \to \pi_1(\mathbb{R}^2 - \{0\}) \) is non-trivial. This contradicts \( \pi_1(\mathbb{R}^n - \{0\}) = 1 \) for \( n > 2 \).

Problem 4 Show that for any map \( f : S^4 \to S^2 \times S^2 \), the induced map in homology \( f_* : H_4(S^4) \to H_4(S^2 \times S^2) \) is trivial.

Solution Let \( a \) be a generator of \( H_2(S^2) \). Since the homology of \( S^2 \) is free, UCT shows that \( H^2(S^2) \) is dual to \( H_2(S^2) \). Let \( \alpha \in H^2(S^2) \) be a dual generator. Then

\[
(\alpha \times \alpha)(a \times a) = (-1)^{2^2}\alpha(a)\alpha(a) = 1.
\]

Thus \( \alpha \times \alpha \) and \( a \times a \) are generators of \( H^4(S^2 \times S^2) \) and \( H_4(S^2 \times S^2) \). Let \( u = p_1^*(\alpha) \) and \( v = p_2^*(\alpha) \) where \( p_i \) are the projections. Then

\[
u \cup v = p_1^*(\alpha) \times p_2^*(\alpha) = \alpha \times \alpha.
\]

Suppose \( f : S^4 \to S^2 \times S^2 \). Then \( f^*(\alpha \times 1) = f^*(1 \times \alpha) = 0 \) because \( H^2(S^4) = 0 \). Let \( b \in H_4(S^4) \) be a generator. Then \( f_*(b) = t(a \times a) \) for some integer \( t \). We get:

\[
t = \langle \alpha \times \alpha, t(a \times a) \rangle \\
= \langle \alpha \times \alpha, f_*(b) \rangle \\
= \langle f^*(\alpha \times \alpha), b \rangle \\
= \langle f^*(\alpha \times 1) \cup (1 \times \alpha), b \rangle \\
= \langle f^*(\alpha \times 1) \cup f^*(1 \times \alpha), b \rangle \\
= 0.
\]

Problem 5 Let \( M \) be a closed orientable manifold of dimension \( 4n + 2 \). Show that its Euler characteristic, \( \chi(M) \) is even.

Solution Use real coefficients and write \( b_i \) for the Betti numbers. By Poincaré duality, \( b_i = b_{4n+2-i} \). Thus

\[
\chi(M) = \sum_{i=0}^{4n+2} b_i = 2 \sum_{i=0}^{2n} b_i - b_{2n+1}.
\]
It remains to show that $b_{2n+1}$ is even. Choose a basis for $H^{2n+1}(M)$ and let $A$ be the matrix of the cup product pairing with this basis. Since $H^{2n+1}(M)$ supports a dual pairing, the pairing is nonsingular, $\det(A) \neq 0$. Since the dimension is odd, the pairing is skew-symmetric and $A^t = -A$. Thus

$$\det(A) = \det(A^t) = \det(-A) = (-1)^{b_{2n+1}} \det(A)$$

so $b_{2n+1}$ is even.

**Problem 6** Let $X$ and $Y$ be topological spaces and let $X \ast Y$ be their join. Show that there are split short exact sequences for all $p \geq 0$:

$$0 \to \tilde{H}_{p+1}(X \ast Y) \to \tilde{H}_p(X \times Y) \to \tilde{H}_p(X) \oplus \tilde{H}_p(Y) \to 0.$$ 

**Solution** Consider $X \ast Y$ as an identification space of $X \times Y \times [0,1]$. Let $A$ be the image of $X \times Y \times [0,1]$, and let $B$ be the image of $X \times Y \times (0,1)$. Then $A \cup B = X \ast Y$ and $A \cap B = X \times Y \times (0,1)$. Note that $A$ is homotopy equivalent to $X$, $B$ is homotopy equivalent to $Y$, and $A \cap B$ is homotopy equivalent to $X \times Y$. Thus in the reduced Mayer-Vietoris sequence of $A, B$, we get

$$\ldots \to \tilde{H}_{p+1}(X \ast Y) \to \tilde{H}_p(X \times Y) \to \tilde{H}_p(X) \oplus \tilde{H}_p(Y) \to \ldots$$

Finally, the maps induced by injections of the factors split the sequence.
TOPOLOGY QUALIFYING EXAMINATION  
January, 1998

Do four problems only. Each carries the same credit.

**Problem 1** Let $p : \bar{X} \to X$ be a cover. Suppose that $f, g : Y \to \bar{X}$ are maps such that $p \circ f$ and $p \circ g$ are equal and assume that $f, g$ agree at $y_0 \in Y$. Show that if $Y$ is connected, then $f = g$.

**Problem 2** For $n \geq 1$, let $f : D^{n+1} \to \mathbb{R}^2$ be a map such that $f(-x) = -f(x)$ for all $x \in \partial D^{n+1}$. Prove that $f^{-1}(0) \cap \partial D^{n+1} \neq \emptyset$.

**Problem 3** Let $K$ be a CW-complex with 2-skeleton $K^{(2)}$. Let $\phi : K^{(2)} \to K$ denote the natural inclusion. Prove that the induced homomorphism $H_n(\phi) : H_n(K^{(2)}) \to H_n(K)$ is an isomorphism for $n = 1$ and surjective for $n = 2$.

**Problem 4** Let $\pi$ be a finitely generated abelian group. Let $n$ be a positive integer. Construct a CW-complex $X$ such that $H_n(X) \cong \pi$, $H_0(X) \cong \mathbb{Z}$ and $H_m(X) = \{0\}$ for $m \neq n$.

**Problem 5** If $M$ is a manifold with boundary, then the *double* of $M$ is defined by identifying two copies of $M$ along their boundaries by the identity map. Let $M = D^2 - \bigcup_i D_\varepsilon(x_i)$ where $\{D_\varepsilon(x_i)\}$ are $n$ mutually disjoint open disks of radius $\varepsilon$ in the interior of $D^2$ centered at $\{x_i\}$. Let $W$ be the double of $M$. Determine the fundamental group and Euler characteristic of $W$.

**Problem 6** Let $T$ be the 2-torus and $K$ the Klein bottle. Compute the cohomology ring $H^*(T \times K; \mathbb{Z}_2)$.

**Problem 7** Let $M^{2n+1}$ be a compact, connected $(2n+1)$-manifold which is possibly nonorientable. Show that the Euler characteristic of $M$ is zero.
1. Identify the vector space of all $n \times n$ real matrices with $\mathbb{R}^{n^2}$. Let $Y_n = SO(n)$ denote the topological subspace of all $n \times n$ orthogonal matrices of determinant one. Using the natural embedding

$$
A \mapsto \begin{pmatrix}
A & 0 \\
0 & 1
\end{pmatrix}
$$

we obtain an inclusion $Y_n \subset Y_{n+1}$.

(a) Prove that $Y_n$ is a compact topological space for all $n \geq 1$.

(b) Show that $Y_{n-1}$ is a closed subgroup of $Y_n$, and that for $n \geq 2$, the space of cosets $Y_n/Y_{n-1}$ is homeomorphic to $S^{n-1}$.

(c) Using induction and part (b), show that $Y_n$ is connected for all values of $n \geq 1$.

2. Let $\mathbb{P} P^n$ denote the space of unoriented lines through the origin in $\mathbb{R}^{n+1}$. Given a continuous map $f : \mathbb{P} P^n \to \mathbb{R}^{n+1} - 0$ prove that there is an $x \in \mathbb{P} P^n$, such that $f(x)$ is orthogonal to $x$.

3. Prove that if $M$ is an odd–dimensional, compact, connected manifold without boundary then its Euler characteristic $\chi(M)$ vanishes. Use this to deduce that a connected, non–orientable, compact 3–manifold without boundary must have an infinite fundamental group.

4. Let $A \subset \mathbb{R}^3$ denote the union of the 3 coordinate axes in $\mathbb{R}^3$, i.e.

$$
A = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0 \text{ or } y = z = 0 \text{ or } x = z = 0\}.
$$

Let $X = \mathbb{R}^3 - A$. Compute $H_*(X; \mathbb{Z})$.

5. Let $S^n$ denote the $n$–dimensional sphere and fix a basepoint $x_0 \in S^n$. Let $X$ denote the space obtained from $S^n \times S^n$ by identifying the pairs of points $(x, x_0) \sim (x_0, x)$ for all $x \in S^n$. Calculate the integral cohomology ring $H^*(X; \mathbb{Z})$.

6. Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there exists an $x \in S^{2n}$ with $f(x) = x$ or $f(x) = -x$. Is this statement true for odd–dimensional spheres? (prove or provide a counterexample).
TOPOMETRY QUALIFYING EXAM

January 1999

Do four problems, all of them carry equal marks.

1. Let $Y$ denote a compact, Hausdorff topological space and $f : Y \to Y$ a continuous function. Prove that there exists a non-empty closed subspace $A \subseteq Y$ with $A = f(A)$.

2. Let $X = \mathbb{CP}^n$ denote complex projective space of dimension $2n$.
   (a) Prove that $X$ is a compact, connected $2n$-dimensional manifold.
   (b) Show in detail that $X$ can be given a $CW$–complex structure with one cell in every even dimension $i = 0, 2, 4, \ldots, 2n$.
   (c) Calculate the cohomology ring $H^*(X, \mathbb{Z})$.
   (d) Show that there is no orientation reversing homotopy equivalence $\phi : \mathbb{CP}^k \to \mathbb{CP}^k$ if $k$ is even.

3. Let $f : X \to Y$ be a continuous function. Define the mapping cylinder of $f$ to be the quotient space
   $$Z_f = X \times [0,1] \cup Y / S$$
   where $S$ is the relation which identifies the points $(x, 1)$ and $f(x)$. Let $i : X \to Z_f$ be given by $i(x) = [(x, 0)]$ and $j : Y \to Z_f$ be given by $j(y) = [y]$. Identify $X$ and $Y$ with their images under $i$ and $j$ respectively, so that $i$ and $j$ become inclusions.
   (a) Show that there is a retraction $r : Z_f \to Y$ such that $r \circ i = f$.
   (b) Show that in fact $Y$ is a strong deformation retract of $Z_f$.
   (c) Use this to show that $f$ induces an integral homology equivalence between $X$ and $Y$ (i.e. an isomorphism between all the homology groups with integer coefficients) if and only if $H_*(Z_f, \mathbb{Z})$ is identically zero.
   (d) Use part (c) to show that if a continuous map induces an integral homology equivalence, then it induces one with coefficients in any abelian group.

4. Let $X$ be a finite $CW$–complex. Prove that $H_1(X, \mathbb{F}_p) = 0$ for all primes $p$ if and only if $H_1(X, \mathbb{Z}) = 0$. Is the same statement true for cohomology?

5. Let $X$ be a simply connected CW-complex, and let $f : X \to T^n$, where $T^n$ is the $n$-dimensional torus $S^1 \times \cdots \times S^1$. Prove that $f$ is homotopic to the constant map.

6. Let $W$ be a compact connected oriented manifold of dimension $m$, and assume that its boundary $\partial W$ is the disjoint union of two closed $(m-1)$–dimensional submanifolds $\partial_1 W$ and $\partial_{-1} W$. Prove that $\forall k \geq 0$,
   $$H_{m-k}(W, \partial_1 W; \mathbb{Z}) \cong H^k(W, \partial_2 W; \mathbb{Z})$$
   and use this to show that if $H_k(W, \partial_1 W; \mathbb{Z}) = 0 \forall k \geq 0$, then $H_*(\partial_1 W; \mathbb{Z}) \cong H_*(\partial_2 W; \mathbb{Z})$. 

TOPOLOGY QUALIFYING EXAM

August 1999

Do four problems, all of them carry equal marks.

1. Let $X$ be a topological space; $A \subseteq X$ a closed subspace, and $U \subseteq A$ a subset open in $A$. Let $V$ be any set open in $X$ with $U \subseteq V$. Prove that $U \cup (V - A)$ is open in $X$.

2. Prove that $\mathbb{C}P^{2k}$ is not the boundary of a compact $(4k+1)$-manifold.

3. Calculate the cohomology ring of the n-dimensional real projective space $H^*(\mathbb{R}P^n, \mathbb{Z}_2)$ and use this to show that there is no continuous map $f : \mathbb{S}^n \to \mathbb{S}^m$, $n > m$ such that $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$.

4. Let $A$ denote a finitely generated abelian group, and $n > 0$ an integer. Construct a path-connected finite CW-complex $X$ such that $H_n(X, \mathbb{Z}) = A$, and $H_i(X, \mathbb{Z}) = 0$ for all $i \neq 0, n$.

5. For this question, all spaces are path-connected, locally path connected, Hausdorff, and possess a universal cover. Let $X_1, X_2$ be path-connected spaces with $\pi_1(X_i) = G_i$ for $i = 1, 2$. Find a group theoretic condition on the group $G_1 \times G_2$ which is equivalent to the condition that every cover of $X_1 \times X_2$ is a product of a cover of $X_1$ with a cover of $X_2$. If $\pi_1(X_1) = \pi_1(X_2) = \mathbb{Z}/2\mathbb{Z}$, is every cover of $X_1 \times X_2$ a product cover?

6. Let $M$ be a closed orientable surface, and suppose that

$$M = \bigcup_{i=1}^{k} A_i,$$

where each $A_i$ is open in $M$. Assume that for each $i$ the subspace $A_i$ is contractible. Show that $k \geq 2$ always holds, and that in fact $k \geq 3$ if $M \not\cong S^2$. 
1. Let $\Delta \subset S^n \times S^n$ be the diagonal subspace. Prove that the projection map

$$p : S^n \times S^n - \Delta \to S^n$$

given by $(x, y) \mapsto x$ is a homotopy equivalence.

2. Denote

$$O_{n+1,2} = \{(x, y) \in S^n \times S^n | x \perp y\} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

and let $p : O_{n+1,2} \to S^n$ be the projection on the first factor. Prove that there is a continuous map $\sigma : S^n \to O_{n+1,2}$ such that $p \circ \sigma = \text{id}_{S^n}$ if and only if $n$ is odd.

3. Let $M$ be a compact, connected, orientable surface. Let $P = \{x_0, x_1, x_2\}$ denote a collection of three distinct points on $M$. Determine $\pi_1(M - \{x_0, x_1, x_2\}, b)$, where $b \notin P$, providing a careful justification of every step. (Hint: your answer should depend on the genus of the surface).

4. Show that $\mathbb{C}P^{2n}$ admits no orientation reversing homotopy equivalence.

5. Let $(X, x_0), (Y, y_0)$ denote two pointed spaces with non-trivial rational homology in positive dimensions. Prove that there is no retraction $X \times Y \to X \setminus Y$, where

$$X \setminus Y = \{(x, y) \in X \times Y | x = x_0 \text{ or } y = y_0\}.$$

6. Let $f : S^{2n-1} \to S^n$ denote a continuous map, and $H_f = D^{2n} \cup_f S^n$ the space obtained by attaching a $2n$-cell to $S^n$ using $f$.

(i) Calculate the integral homology of $H_f$.

(ii) Find an $n$ and an $f$ such that $H^* (H_f, \mathbb{F}_2)$ has a non-trivial product. Justify your answer.
1. Suppose that $\omega, \omega' \in \Omega(X, x_0)$ are freely homotopic loops. Prove that $[\omega]$ and $[\omega']$ are conjugate elements in $\pi_1(X, x_0)$.

2. Suppose that $f : S^n \to \mathbb{R}^{n+1}$ is a continuous map such that $f(x) \neq \lambda x$, for all $x \in S^n$ and real numbers $\lambda$. Prove that $n$ is odd.

3. Let $K \subset S^n$ be a subset homeomorphic to the unit interval $[0, 1]$. Compute the homology of the complement of $K$, $H_*(S^n - K; \mathbb{Z})$.

4. Let $\Delta^2$ be the standard 2-simplex. Put $X = \Delta^2/\sim$, where $(v_0, v_1) \sim (v_1, v_2) \sim (v_0, v_2)$, with $v_0, v_1$ and $v_2$ being the vertices of $\Delta^2$. The edges are identified by the induced linear homeomorphisms (the space $X$ is called the *dunce cap*). Compute the integral homology $H_*(X; \mathbb{Z})$.

5. Let $M$ denote a closed, non-orientable 3-manifold. Prove that its fundamental group $\pi_1(M, x_0)$ must be an infinite group.

6. Prove that if $M$ is a closed, oriented manifold of dimension $4k+2$, then $\chi(M)$ is even.
TOPOLOGY QUALIFYING EXAM

January 2002

Do four problems; all carry equal marks.

In answering the questions below, either give a proof or else provide a counterexample. Write down enough details so that your argument can be followed and understood; justify the steps involved in your proof.

Problem 1. Let $\gamma$ be a smooth curve in the plane $\mathbb{R}^2$. Let $A$ be the set of all values $r \in \mathbb{R}$ such that the circle of radius $r$ about the origin is tangent to the curve $\gamma$ at some point. Show that the interior of $A$ (as a subset of $\mathbb{R}$) is empty.

Problem 2. Let $X$ be the complement of two circles $\{x^2 + y^2 = 1, z = 1\}$ and $\{x^2 + y^2 = 1, z = -1\}$ in $\mathbb{R}^3$. Show that $X$ is path-connected and determine the fundamental group $\pi_1(X)$ and the homology $H_\ast(X, \mathbb{Z})$ of the space $X$.

Problem 3. Consider $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $Y = \mathbb{CP}^2 \vee \mathbb{CP}^1$.
(a) For any (abelian) coefficient group $G$ compute the homology $H_\ast(X, G)$ and $H_\ast(Y, G)$;
(b) show that $X$ and $Y$ are not homotopy equivalent.

Problem 4.
(a) Construct a connected, 2-dimensional CW–complex $X$ such that $\pi_1(X, x_0) = \mathbb{Z}/3$.
(b) Can you construct such an example where in addition $X$ is a closed surface?
(c) Construct a 6–dimensional closed connected manifold $M$ with $\pi_1(M, m_0) = \mathbb{Z}/3$.

Problem 5. Let $U(2)$ denote the group of $2 \times 2$ unitary matrices over $\mathbb{C}$, i.e. those matrices with $A \cdot \overline{A}^T = I$. Calculate $H_\ast(U(2), \mathbb{Z})$.

Problem 6. Let $M$ be a compact, connected, orientable $n$–manifold and $f : M \to M$ a continuous map such that $f_* : H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{Z})$ is an isomorphism. Prove that the induced homomorphisms $f_* : H_q(M, G) \to H_q(M, G)$ are isomorphisms for all $q \geq 0$ and any (abelian) coefficient group $G$. 
DO four problems; all carry equal marks.

In answering the questions below, either give a proof or else provide a
counterexample. Write down enough details so that your argument can be
followed and understood; justify the steps involved in your proof.

Problem 1. Let $M$ be a compact smooth manifold of dimension $n$, and let $f : M \to \mathbb{R}^n$
be a smooth map. Show that $f$ cannot be everywhere nonsingular.

Problem 2. Consider the space $X = GL_2^+(\mathbb{R})$ of $2 \times 2$ matrices with positive determinant,
with the topology induced as an open subspace of $\mathbb{R}^4$. Prove that $X$ is path connected
and compute its fundamental group.

Problem 3. Construct two finite, connected $CW$–complexes $X$ and $Y$ such that $H_*(X,\mathbb{Z}) \cong H_*(Y,\mathbb{Z})$ but where the isomorphism cannot be induced by any continuous function
$X \to Y$ or $Y \to X$.

Problem 4. Compute the ring structure of $H^*(T^n,\mathbb{Z})$, where $T^n$ is the $n$-dimensional
torus (a product of $n$ circles). Do the same for $H^*(T^n - \{x\},\mathbb{Z})$, where $x \in T^n$ is any
point.

Problem 5. Let $M^n$, $N^n$ be two closed, connected and oriented manifolds. Prove that if
$f : M \to N$ has degree one, then the induced map $f_* : \pi_1(M) \to \pi_1(N)$ is surjective.

Problem 6. Let $A$ be the union of two once linked circles in the 3–dimensional sphere
$S^3$; let $B$ be the union of two unlinked circles in $S^3$. Show that the cohomology groups of
$S^3 - A$ and $S^3 - B$ are isomorphic, but the cohomology rings are not.
TOPOLOGY QUALIFYING EXAM

January 2003

Do four problems; all carry equal marks.

In answering the questions below, either give a proof or else provide a counterexample. Write down enough details so that your argument can be followed and understood; justify the steps involved in your proof.

Problem 1. Let \( X = \mathbb{CP}^n \) denote complex projective space of dimension \( 2n \).
(a) Prove that \( X \) is a compact, connected 2n-dimensional manifold.
(b) Show in detail that \( X \) can be given a CW–complex structure with one cell in every even dimension \( i = 0, 2, 4, \ldots, 2n \).
(c) Calculate the cohomology ring \( H^*(X, \mathbb{Z}) \).

Problem 2.
(a) Prove that if \( M \) is a compact, odd-dimensional manifold without boundary, then its Euler characteristic \( \chi(M) \) must be zero.
(b) Prove that if \( M \) is a compact, oriented manifold without boundary of dimension \( 4k + 2 \), then \( \chi(M) \) is even.

Problem 3. Consider \( M \) a compact connected smooth manifold without boundary. Show that there is no submersion \( f : M \to \mathbb{R}^n \) for any \( n \geq 1 \).

Problem 4. Consider the topological space \( X \) obtained as the quotient of the standard torus \( T^2 = S^1 \times S^1 \) by the equivalence relation \( (e^{i\theta_1}, e^{i\theta_2}) \sim (e^{-i\theta_1}, e^{-i\theta_2}) \) where \( e^{i\theta} \) is the standard coordinate on \( S^1 \subset \mathbb{C} \).
(a) Is \( X \), with the induced quotient structure, a smooth manifold? Is it a topological manifold?
(b) Find a cell decomposition of \( X \) and compute its homology.

Problem 5. Let \( f : X \to X \) be a continuous function such that \( f \circ f \circ f = Id_X \) but \( f(x) \neq x \) for all \( x \in X \).
(a) Give an example of such a function \( f \) when \( X \) is a 2 dimensional torus \( T^2 \).
(b) Show that such a function \( f \) does not exist when \( X \) is the 2 dimensional sphere \( S^2 \).

Problem 6. Let \( M \) denote a connected, non–orientable, compact 3–manifold without boundary. Prove that its fundamental group must be an infinite group.
TOPOLOGY QUALIFYING EXAM

AUGUST 2003

Do four problems; all carry equal marks.

In answering the questions below, either give a proof or else provide a counterexample. Write down enough details so that your argument can be followed and understood; justify the steps involved in your proof.

1. Consider the space \( X = GL_2^+ (\mathbb{R}) \) of \( 2 \times 2 \) matrices with positive determinant, with the topology induced as an open subspace of \( \mathbb{R}^4 \). Prove that \( X \) is path connected and compute its fundamental group.

2. Let \( T = S^1 \times S^1 \) denote the torus and define an action of \( G = \mathbb{Z}/2 \) on it by \( (z, w) \mapsto (z^{-1}, -w) \). Show that this is a fixed-point-free action and prove that the orbit space \( T/G \) is a non-orientable surface. Identify it.

3. (a) Prove that a \( 1 \)-form \( \omega \) on \( S^1 \) is the differential of a function if and only if \( \int_{S^1} \omega = 0 \).
(b) Let \( \nu \) be any \( 1 \)-form on \( S^1 \) with non-zero integral. Prove that if \( \omega \) is any other \( 1 \)-form, then there exists a constant \( c \) such that \( c \nu - \omega = df \) for some function \( f \) on \( S^1 \).
(c) Show that the \( 1 \)-form defined on \( \mathbb{R}^2 - \{0\} \),

\[
\omega = \left( \frac{-y}{x^2 + y^2} \right) dx + \left( \frac{x}{x^2 + y^2} \right) dy,
\]

is closed but not exact. Use this together with (a) and (b) to prove that \( H_1^1(S^1) \cong \mathbb{R} \).

4. Let \( M \) denote a compact, connected \( n \)-manifold. Show that if \( M \) is orientable, then \( H_{n-1}(M, \mathbb{Z}) \) is torsion-free. If \( M \) is nonorientable, show that \( H_n(M, \mathbb{Z}/k\mathbb{Z}) = 0 \) if \( k \) is odd, that the torsion subgroup of \( H_{n-1}(M, \mathbb{Z}) \) is cyclic of order 2 and that \( H_1(M, \mathbb{Z}/2\mathbb{Z}) \neq 0 \).

5. Let \( A \) be the union of two once linked circles in \( S^3 \); let \( B \) be the union of two unlinked circles. Show that the cohomology groups of \( S^3 - A \) and \( S^3 - B \) are isomorphic, but the cohomology rings are not.

6. Let \( X \) be the space obtained by gluing two solid tori \( D^2 \times S^1 \) along their boundaries via the map \( f : \partial D^2 \times S^1 \to \partial D^2 \times S^1 \) given by \( f(x, y) = (p^x x, y) \) (where \( p \) is a fixed positive integer).

(a) Can \( X \) be given the structure of a topological manifold?
(b) Compute \( \pi_1(X) \).
(c) Compute the cohomology ring \( H^*(X, \mathbb{Z}) \).
In answering the questions below, either give a proof or else provide a counterexample. Write down enough details so that your argument can be followed and understood; justify the steps involved in your proof.

1. Let \( \Delta \subset \mathbb{S}^n \times \mathbb{S}^n \) denote the diagonal subspace. Show that the projection

\[
\mathbb{S}^n \times \mathbb{S}^n - \Delta \to \mathbb{S}^n
\]

given by \((x, y) \mapsto x\) is a homotopy equivalence.

2. Let \( M \) and \( N \) denote smooth manifolds.
   (a) If \( f : M \to N \) is a submersion and \( U \) is an open subset of \( M \), show that \( f(U) \) is an open subset of \( N \).
   (b) If \( M \) is compact and \( N \) is connected, show that every submersion \( f : M \to N \) is surjective.
   (c) Show that there exist no submersions of compact manifolds into Euclidean space.

3. Let \( X = \mathbb{C}P^n \) denote complex projective space of dimension \( 2n \).
   (a) Prove that \( X \) is a compact, connected \( 2n \)-dimensional manifold.
   (b) Show in detail that \( X \) can be given a \( CW \)-complex structure with one cell in every even dimension \( i = 0, 2, 4, \ldots, 2n \).
   (c) Calculate the cohomology ring \( H^*(X, \mathbb{Z}) \).
   (d) Show that if \( n \) is even, \( \mathbb{C}P^n \) admits no orientation reversing homotopy equivalence.

4. Let \( r_d : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \) be the mod \( d \) reduction homomorphism, and \( r_{d*} : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}/d\mathbb{Z}) \) be the induced homomorphism on cohomology. For \( c \in H^2(X, \mathbb{Z}/2\mathbb{Z}) \) define \( \beta(c) \in H^4(X, \mathbb{Z}/4\mathbb{Z}) \) as follows:
   1. Find \( \tilde{c} \in H^2(X, \mathbb{Z}) \) such that \( r_{2*}(\tilde{c}) = c \).
   2. Define \( \beta(c) = r_{4*}(\tilde{c} \cup \tilde{c}) \).

Show that if \( X \) is a compact, simply connected 4–manifold without boundary, then \( \beta(c) \) is well defined, i.e. that \( \tilde{c} \) exists and that \( \beta(c) \) is independent of the choice of \( \tilde{c} \).

5. Show that a product of two spheres can be embedded in Euclidean space of one dimension higher.

6. Let \( \mathbb{D} \) be a standard 2 dimensional disk with the boundary \( \mathbb{S}^1 \) and let \( x \neq y \) be two points in the interior of \( \mathbb{D} \). Show that there is no homeomorphism \( f : \mathbb{D} \to \mathbb{D} \) that satisfies the following:
   1. \( f \circ f = Id. \) (i.e. \( f \) is an involution on \( \mathbb{D} \))
   2. \( f|_{\mathbb{S}^1} = Id. \)
   3. \( f(x) = y. \)

   [Hint: consider the induced map \( f_* : \pi_1(\mathbb{D} - \{x, y\}) \to \pi_1(\mathbb{D} - \{x, y\}) \).]
TOPOLOGY QUALIFYING EXAM
AUGUST 2004

Do four problems; all carry equal marks.

In answering the questions below, either give a proof or else provide a counterexample. Write down enough details so that your argument can be followed and understood; justify the steps involved in your proof.

1. Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ denote a continuous involution on the standard closed disk (i.e. $f \cdot f = I$, the identity map on $\mathbb{D}^2$). Show that if $f$ is the identity on its boundary, then it is the identity on the whole disk.

2. Let $\mathbb{T}^2 \subset S^3$ be the standardly embedded torus in $S^3$. Let $K_{p,q} \subset \mathbb{T}^2$ be the $(p, q)$-torus knot given by the image of $\gamma : \mathbb{R} \to \mathbb{T}^2$, $\gamma(t) = (e^{ipt}, e^{iqt})$. Assuming that $1 < p < q$ with $p$ and $q$ relatively prime integers, compute:
   (a) $\pi_1(S^3 \setminus K_{p,q})$.
   (b) $H_*(S^3 \setminus K_{p,q}, \mathbb{Z})$.

3. Let $P_n = \mathbb{RP}^n$ denote the real projective space of dimension $n$ and let $d > 1$ denote an integer.
   (a) Write down the additive structure for $H_*(P_n, \mathbb{Z})$ (you are not asked to prove it).
   (b) Show that if $d = 4k + 2$, then the mod 2 reduction map $\mathbb{Z}/d \to \mathbb{Z}/2$ induces a cohomology isomorphism $H^*(P_n, \mathbb{Z}/d) \cong H^*(P_n, \mathbb{Z}/2)$.
   (c) Show that when $d = 4k$ and $n \geq 2$, there exist non-zero classes $\alpha \in H^1(P_n, \mathbb{Z}/d)$ and $\beta \in H^2(P_n, \mathbb{Z}/d)$ such that $\alpha^2 = 0$, and $\beta^i \neq 0$ for $1 \leq i \leq \lfloor n/2 \rfloor$ (the integer part of $n/2$).

4. Let $M$ be a path-connected $n$-manifold. Show that if $\pi_1(M, x)$ is a finite group of odd order, then $M$ is orientable.

5. Let $S^n(r)$ be a sphere of radius $r$, centered at the origin in $\mathbb{R}^{n+1}$ and let $r^2 = x_0^2 + x_1^2 + \cdots + x_n^2$. Let us put
   \[\omega = \frac{1}{r} \sum_{i=0}^{n} (-1)^{i+1} x_i \cdot dx_0 \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \cdots \wedge dx_n.\]
   a) Show that $dr \wedge \omega$ is a volume form.
   b) Show that $\omega$ represents a non-trivial element in the DeRham cohomology $H^n_{DR}(S^n(r))$.
   c) Modify the form $\omega$ to find a form representing a non-trivial element of the DeRham cohomology $H^n_{DR}(\mathbb{R}^{n+1} \setminus \{0\})$.

6. Prove that there exists no non-zero vector field on $S^2$, i.e. show that there is no continuous $f : S^2 \to \mathbb{R}^3 \setminus \{0\}$ such that $\forall x \in S^2$, $x$ is orthogonal to $f(x)$ (as vectors in $\mathbb{R}^3$).
Topology Qualifying Exam

JANUARY, 2005

Do four problems; all carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial.

Standard notation: \( \mathbb{R} \) denotes the real numbers, \( \mathbb{C} \) denotes the complex numbers, \( \mathbb{D}^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \} \) denotes the closed unit disk, \( S^{n-1} = \partial \mathbb{D}^n = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \) denotes the unit \( n \)-sphere, \( \mathbb{P}^n = S^n/(\{\pm 1\}) \) denotes real projective space of dimension \( n \), and \( \mathbb{C}P^n \) denotes complex projective space of complex dimension \( n \), i.e. real dimension \( 2n \).

1. Let \( f : S^2 \to S^2 \) be a continuous map such that \( \|f(p) - p\| < 1 \) for all \( p \in S^2 \). Must \( f \) be onto? (Prove your answer.)

2. Is there a map \( f : S^6 \to \mathbb{C}P^3 \) is a map of non-zero degree?

3. Assume \( n > 0 \) and that \( f : S^n \to S^n \) has degree two. Show that the identification space \( X = S^n \cup_f \mathbb{D}^{n+1} \) is a manifold only if \( n = 1 \).

4. Show that the spaces \( S^2 \times \mathbb{P}^4 \) and \( S^4 \times \mathbb{P}^2 \) have the same fundamental groups, i.e. that \( \pi_1(S^2 \times \mathbb{P}^4) \) and \( \pi_1(S^4 \times \mathbb{P}^2) \) are isomorphic. Are \( S^2 \times \mathbb{P}^4 \) and \( S^4 \times \mathbb{P}^2 \) homotopy equivalent? (Prove your answer.)

5. Consider the map \( f : A \times B \to S^2 \) where
   \[
   A = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, \; z = 0 \},
   \]
   \[
   B = \{ (x, y, z) \in \mathbb{R}^3 : (y - 1)^2 + z^2 = 1, \; x = 0 \},
   \]
   and
   \[
   f(p, q) = \frac{p - q}{|p - q|}
   \]
   for \( p \in A \) and \( q \in B \). Is the map \( f \) homotopic to a constant map? (Prove your answer.)

6. Let \( A \) and \( B \) be as in the previous problem. Find smooth functions
   \[
   a = a(x, y, z), \quad b = b(x, y, z), \quad c = c(x, y, z)
   \]
   defined on \( \mathbb{R}^3 \setminus B \) such that the one form
   \[
   \omega = a \, dx + b \, dy + c \, dz
   \]
   is closed (i.e. \( d\omega = 0 \)) and the integral \( \int_A \omega \) is nonzero.
Topology Qualifying Exam for Fall 2005

(1) Let $\mathbb{RP}^n$ denote the n-dimensional real projective space ($n > 0$), and let $f: \mathbb{RP}^n \to \mathbb{RP}^n$ be a continuous function.

   (a) If $n$ is even, show that $f$ has a fixed point (i.e., there is some $x \in \mathbb{RP}^n$ with $f(x) = x$).
   (b) Is the same true for odd $n$? Prove your answer.

(2) Let $M$ be an oriented, compact, connected, simply connected 3-dimensional manifold with connected (non-empty) boundary $\partial M$. Prove that $\partial M$ is homeomorphic to the sphere $S^2$. Hint: Consider the double.

(3) Let $Y_n$ be the space obtained by removing $n$ points from the torus. How many isomorphism classes of regular, connected, 2-fold covers of $Y_n$ are there?

(4) Let $M$ and $N$ be compact, connected, smooth manifolds of the same dimension, and assume that $N$ is simply connected. Show that an immersion $f: M \to N$ is a diffeomorphism.

(5) Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus and $S \subset M$ be the submanifold defined by
   $$S = \{(x,y) + \mathbb{Z}^2 : 2x = 3y\}.$$ 
   Find a differential form $\eta_S$ representing the Poincaré dual of $S$.

(6) Define embeddings $\phi, \psi: S^1 \to \mathbb{R}^3$ by
   $$\phi(\alpha) = (\cos \alpha, 1 + \sin \alpha, 0), \quad \psi(\beta) = (0, \cos \beta, \sin \beta).$$
   (Here $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ so $\phi$ parameterizes the circle $x^2 + (y-1)^2 - 1 = z = 0$ and $\psi$ parameterizes the circle $y^2 + z^2 - 1 = x = 0$.) Then define a map
   $$f: S^1 \times S^1 \to S^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$
   by
   $$f(\alpha, \beta) = \frac{\psi(\beta) - \phi(\alpha)}{||\psi(\beta) - \phi(\alpha)||}.$$ 
   (a) Calculate the degree of $f$.
   (b) Is there a map $g: S^2 \to S^1 \times S^1$ of nonzero degree? (Give either an example of such a map or a proof that there is no such map.)
1. Let $n \in \mathbb{Z}$ and $X = S^2 \cup_f S^1$ where the attaching map $f: \partial \mathbb{D}^2 \to S^1$ is $f(\cos \theta, \sin \theta) = (\cos \theta, \sin \theta).$ Find the fundamental group $\pi_1(X)$ and the homology groups $H_k(X, \mathbb{Z})$ of $X.$ For which values of $n$ is $X$ a manifold?

2. Let $M = \mathbb{R}^2 / \mathbb{Z}^2$ be the two-dimensional torus and $S \subset M$ be the submanifold defined by $s = \{(x, y) + n \cdot \mathbb{Z} : 3x = 7y\}.$ Find a differential form $\omega$ representing the Poincaré dual of $S.$

3. Calculate the cohomology ring $H^*(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$.

4. Let $P$ be a polygon with an even number of sides. Suppose that the sides are identified in pairs in any way whatsoever. Prove that the quotient space is a manifold. (That the sides are identified in pairs means the following. There is an enumeration $a_1, a_2, \ldots, a_{2n}$ of the edges of $P$ (not necessarily in cyclic order but without repetitions) and for each $k = 1, 2, \ldots, n$ a homeomorphism $a_k : a_k \to P_k$ SO that desired identification space $S$ is obtained from $P$ by identifying $x \in a_k$ with $y \in P_k.$)

5. Calculate the fundamental group $\pi_1(Y)$ of the space $Y$ which results from the cube $[0,1]^3$ by identifying opposite faces with a 90° right hand twist.

TURN PAGE.
6. Consider the three embedded circles
\[ \alpha := S^1 \times \{(0,0)\}, \quad \beta := S^1 \times \{(0, \frac{1}{2})\}, \quad \gamma := \{(1,0)\} \times \{(x,y) \in \mathbb{D}^2 : x^2 + y^2 = \frac{1}{2}\}. \]
in the solid torus
\[ M := S^1 \times \mathbb{D}^2 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \times \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}. \]

Let \( T_\alpha, T_\beta, T_\gamma \) be small open tubular neighborhoods of \( \alpha, \beta, \gamma \) respectively, so small that their closures are disjoint from each other and from the boundary of \( M \). Let
\[ X := M \setminus T_\alpha, \quad X_1 := X \setminus T_\gamma, \quad X_2 := X \setminus T_\beta. \]

(Each of \( T_\alpha, T_\beta, T_\gamma \) is the interior of a solid torus. In the picture the dotted curves represent the tubular neighborhoods that are removed.) Prove or disprove that \( X_1 \) and \( X_2 \) homeomorphic. Hint: \( S^3 \).
Topography Qualifying Exam

August, 2006

Do four problems; all carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial. Of course you must give proofs.

Standard notation: \( \mathbb{Z} \) denotes the integers, \( \mathbb{R} \) denotes the real numbers, \( \mathbb{C} \) denotes the complex numbers, \( I = [0, 1] \) denotes the unit interval,

\[
D^n := \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}
\]
denotes the closed unit disk,

\[
S^{n-1} := \partial D^n = \{ x \in \mathbb{R}^n : \|x\| = 1 \}
\]
denotes the unit \( n \)-sphere, and \( \mathbb{P}^n(\mathbb{C}) \) denotes complex projective space of complex dimension \( n \), i.e. real dimension \( 2n \).

1. Let \( X \subset S^2 \subset \mathbb{R}^3 \) denote the subset

\[
X = \{(x, y, z) \in S^2 : y^2 z = x^3 - xz^2 \}.
\]

Calculate the homology groups of \( X \). Is \( X \) a smooth submanifold of \( \mathbb{R}^3 \)?

2. Let \( M \) be a compact, connected, oriented 2-manifold. Is there a map \( f : M \to S^2 \) of degree two with exactly one critical point?

3. Let \( X \) denote the figure eight and \( a \) and \( b \) be the generators of its fundamental group. Find the covering space \( Y \) of \( X \) corresponding to the subgroup generated by \( a \).

4. Let \( X = S^2 \cup J \) where \( J = \{(0,0,z) \in \mathbb{R}^3 : -1 \leq z \leq 1 \} \) is the interval on the \( z \)-axis joining the north and south poles. Compute the homology groups of \( X \).

5. Show that \( \mathbb{P}^2(\mathbb{C}) \) and \( S^4 \vee S^2 \) have the same homology and cohomology groups but are not homotopy equivalent.

6. Show that \( \pi_2(S^3) \neq 0 \).
Do four problems; all carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial.

**Standard notation:** $\mathbb{Z}$ denotes the integers, $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ denotes the complex numbers, and

$$S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$$

denotes the unit $n$-sphere.

1. Show that any continuous map $f : S^3 \to S^1 \times S^1 \times S^1$ has degree zero.

2. Construct a compact topological space whose fundamental group is the group $S_3$ of permutations of $\{1, 2, 3\}$.

3. Compute the integer homology groups $H_k(T^2, \mathbb{Z})$ (for $k = 0, 1, 2$) of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

and compute the map $f_* : H_k(T^2, \mathbb{Z}) \to H_k(T^2, \mathbb{Z})$ in homology induced by the homeomorphism

$$f((x, y) + \mathbb{Z}^2) = (x + 2y, x + y) + \mathbb{Z}^2, \quad (x, y) \in \mathbb{R}^2.$$

4. Compute the cohomology ring of $CP^2$ (complex projective space) with real coefficients.

5. Compute the cohomology ring structure of the Klein bottle with $\mathbb{Z}/2$ coefficients.

6. Compute the integer cohomology groups of the product $S^2 \times RP^3$ of the two sphere with real projective three space.

7. Consider the distribution $\mathcal{D}$ on $\mathbb{R}^3$ defined by

$$\mathcal{D}_{(x,y,z)} = \{(u, v, w) \in \mathbb{R}^3 \mid T_{(x,y,z)}\mathbb{R}^3 : w - xv = 0\}, \quad (x, y, z) \in \mathbb{R}^3.$$

Show that $\mathcal{D}$ is not involutive. (A distribution on $M$ is sometimes called a subbundle of the tangent bundle $TM \to M$ and an involutive distribution is sometimes called an integrable subbundle.)

More problems on the next page.
8. Let \((M, g)\) be a compact Riemannian manifold without boundary, and \(f : M \rightarrow \mathbb{R}\) a smooth function.

(a) Give a coordinate-free proof that

\[
\int_M |\nabla f|^2 dV = \int_M f \Delta f,
\]

where \(\Delta = d\delta + \delta d\) is the Hodge Laplacian, \(\delta = (-1)^{n(p+1)+1} \ast d\ast\) is the divergence, \(\nabla f\) is the gradient, and \(dV\) is the Riemannian volume element.

(b) Let \(u : M \rightarrow \mathbb{R}\) be a smooth solution to

\[
\Delta u = c(x) \cdot u,
\]

where \(c(x) < 0\) is a smooth function on \(M\). Show that \(u\) is identically zero.

9. Let \(M\) be a compact Riemannian manifold, \(X\) a smooth vector field on \(M\), and \(u : M \rightarrow \mathbb{R}\) be a smooth function.

(a) Show that in coordinates for a function \(u\):

\[
\Delta u = -\sum_{i,j} \frac{1}{\sqrt{\det(g)}} \partial_i \left( g^{ij} u_j \sqrt{\det(g)} \right).
\]

(b) Suppose that \(u\) satisfies the equation

\[
\Delta u + g(X, \nabla u) + c(x) \cdot u \leq 0,
\]

where \(c(x) > 0\) is a smooth function on \(M\). Show that \(u \leq 0\) on \(M\).
If you selected the algebraic topology option, do two problems from part I and two from part II. If you selected the differential topology option, do two problems from part I and two from part III. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial.

**Part I: Basic Topology**

1. Let $X$ be the space obtained by identifying 3 points on the torus $T^2 = S^1 \times S^1$. Compute the homology and fundamental groups of $X$.

2. Prove that any continuous map $f : \mathbb{R}P^2 \rightarrow T^2$ must be null-homotopic, i.e. homotopic to some constant map.

3. Let $X$ be the space obtained by attaching two discs to $S^1$, where the first disk $D_1$ is attached via the map $f : \partial D_1 = S^1 \rightarrow S^1, z \rightarrow z^3$, and the second disk $D_2$ is attached via the map $g : \partial D_2 = S^1 \rightarrow S^1, z \rightarrow z^5$.
   
   (a) Compute the homology groups of $X$.
   
   (b) Compute the relative homology groups $H_i(X, S^1)$.
   
   (c) Is $X$ homeomorphic to $S^2$? Why?

**Part II: Algebraic Topology**

4. Compute the ring structure of $H^*(T^n; \mathbb{Z})$, where $T^n = S^1 \times \cdots \times S^1$ ($n$ factors) is the $n$-torus. Do the same for $H^*(T^n \setminus \{x\}; \mathbb{Z})$ for any $x \in T^n$.

5. If $n$ is even, show that every continuous map $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ has a fixed point. Is the same true for $n$ odd? Either prove it is or give a counterexample.

6. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

**Part III: Differential topology**

7. Let $M^n$ and $N^n$ be closed (i.e. compact and without boundary) manifolds of dimension $n$. Prove that an immersion $f : M^n \rightarrow N^n$ is a covering map.
8. Prove that a smooth distribution $\Delta$ is involutive if and only if the corresponding ideal of forms is a differential ideal.

9. Let $(M, g)$ be a compact Riemannian manifold, and let $\Delta$ be the Laplacian defined by $\Delta f = \delta df = -\ast d \ast df$.

(a) Let $u : M \to \mathbb{R}$ be a smooth, non-trivial solution to

$$\Delta u = c \cdot u,$$

where $c$ is a non-zero constant. Show that $c > 0$.

(b) Let $X$ be a smooth vector field, and $u : M \to \mathbb{R}$ be a smooth solution to

$$\Delta u + g(X, \nabla u) + f(x) \cdot u \geq 0,$$

where $f(x) > 0$ is a smooth function on $M$. Show that $u \geq 0$ on $M$. 
Topology/Geometry Qualifying Exam

(Both Options)

JANUARY, 2008

Do four problems; all carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial. All students should answer two questions from Part I. Students who have chosen the Algebraic Topology Option should answer two questions from Part II. Students who have chosen the Differential Topology Option should answer two questions from Part III.

Part I

1. By definition, a topological group is a set $G$ with both a topology and a group structure, such that the map $G \to G$ sending $x$ to $x^{-1}$ and the map $G \times G \to G$ sending $(x, y)$ to $xy$ are both continuous. Let $1 \in G$ denote the identity of this topological group $G$. Show that $\pi_1(G, 1)$ is abelian.

2. Let $D^n$ be the $n$-dimensional ball and $S^{n-1}$ the $(n-1)$-dimensional sphere, realized as the boundary of $D^n$. Prove that the following are equivalent:
   1. There is no retraction $D^n \to S^{n-1}$.
   2. Every continuous map $D^n \to D^n$ has a fixed point.

3. Let $X$ be a connected topological space, and $\pi : Y \to X$ a surjective covering space map. Suppose that the group of deck transformations of $\pi$ contains a subgroup $\mathbb{Z}_p$, where $p$ is a prime number, such that $\mathbb{Z}_p$ acts freely and transitively on fibers of $\pi$. Prove that either $Y$ is homeomorphic to a disjoint union of $p$ copies of $X$ or is connected.

Part II

4. Let $M$ denote a compact, orientable $n$-manifold. Let $f : S^n \to M$ be a continuous map of degree $d$. For $0 < q < n$, show that every $x \in H_q(M; \mathbb{Z})$ satisfies $d \cdot x = 0$.

5. Prove that there is no self-homotopy equivalence $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ which reverses orientation.
6. Let $X = A \cup B$ be a union of two contractible open subsets $A$ and $B$, and fix a commutative ring $R$. Show that the cup product of any two positive dimensional elements of $H^*(X; R)$ is zero.

Part III

7. Assume that $M$ and $N$ are smooth manifolds, that $M$ is compact, and that $N$ is non compact. Show that every smooth map $f : M \rightarrow N$ has at least one singular point.

8. Let $(M, g)$ be a compact oriented Riemannian manifold without boundary and let $\Omega_g$ be its volume form. Define the Laplace Beltrami operator $\Delta_g$ of $(M, g)$ and show that if $u$ and $v$ are eigenfunctions of $\Delta_g$ corresponding to distinct eigenvalues, then $\int_M uv \Omega_g = 0$.

9. Let $M$ be any differential manifold and $X, Y$ are two smooth vector fields on $M$ which are pointwise linearly independent and satisfy $[X, Y] = 2X - 3Y$. (i) Show that for every point $p \in M$, there is a submanifold passing through $p$ whose tangent space is everywhere spanned by $\{X, Y\}$. (ii) Does the conclusion remain true if the hypothesis that $X$ and $Y$ are pointwise linearly independent is dropped? (Proof or counter example.)
Summer 2008 Qualifying Exam

751 - 761

July 17, 2008

Do two 751 problems and two 761 problems. All carry equal marks.
If you think a problem is badly stated, state precisely what you think
is intended, but do not interpret a question so that it becomes trivial.

751 Problems

1. Let \( X = A \cup B \) be a topological space, which is a union of two subspaces \( A, B \subset X \).

   Assume that \( A \) and \( B \) are closed, that the pairs \( (X, A), (X, B), (A, A \cap B), (B, A \cap B) \) have the homotopy extension property, and that \( A, B, A \cap B \) are all contractible. Prove that \( X \) is contractible.

2. Give an example of \( X = A \cup B \) with \( A \) open and \( B \) closed such that \( A, B, A \cap B \) are contractible, but \( X \) is not. Justify your answer.

3. Let \( G \) be a finitely generated abelian group. Find a finite-dimensional path-connected topological space \( X_G \) with \( \pi_1(X_G) = G \).

761 Problems

4. Let \( g \) be the metric on the unit sphere \( S^2 \subset \mathbb{R}^3 \) it inherits as a submanifold of \( \mathbb{R}^3 \). (This is sometimes called the round metric on \( S^2 \).) The point \((x, y) \in \mathbb{R}^2\) is the stereographic projection of the point \((\xi, \eta, \zeta) \in S^2\) iff the three points
(0, 0, 1), (x, y, 0), and (ξ, η, ζ) are collinear; this defines a map \( \sigma : \mathbb{R}^2 \to S^2 \), \( \sigma(x, y) = (\xi, \eta, \zeta) \). Show that \( \sigma \) maps \( \mathbb{R}^2 \) diffeomorphically onto the complement of a point in \( S^2 \) and calculate the pull back metric \( \sigma^* g \). Use this formula to compute the area of \( S^2 \) by evaluating an integral over \( \mathbb{R}^2 \).

5 Let \( S \subset \mathbb{R}^3 \) be an oriented two-manifold and \( \nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x)) \) be the outward unit normal to \( S \) at \( x \). Show that \( \sigma \) maps \( \mathbb{R}^2 \) diffeomorphically onto the complement of a point in \( S^2 \) and calculate the pull back metric \( \sigma^* g \). Use this formula to compute the area of \( S^2 \) by evaluating an integral over \( \mathbb{R}^2 \).

6 Let \( S \subset \mathbb{R}^3 \) be an oriented two-manifold and \( \nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x)) \) be the outward unit normal to \( S \) at \( x \). Show that for any 2-form

\[
\omega = f_1 \, dx_2 \wedge dx_3 + f_2 \, dx_3 \wedge dx_1 + f_3 \, dx_1 \wedge dx_2
\]

the restriction \( \omega|_S \) of \( \omega \) to \( S \) is

\[
\omega|_S = (\nu \cdot f) \, dA
\]

where

\[
dA = \nu_1 \, dx_2 \wedge dx_3 + \nu_2 \, dx_3 \wedge dx_1 + \nu_3 \, dx_1 \wedge dx_2
\]

and

\[
\nu \cdot f = \nu_1 f_2 + \nu_2 f_2 + \nu_3 f_3.
\]

Use this formula and Stokes’ Theorem to prove the following form of the Divergence Theorem:

Assume that \( D \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary \( \partial D \) and that \( X \) is a smooth vector field defined on \( D \). Then

\[
\int_D \delta X^\flat \, dx_1 \wedge dx_2 \wedge dx_3 = \int_{\partial D} (X \cdot \nu) d\sigma,
\]

where \( d\sigma \) is the induced area element on \( \partial D \), \( X^\flat \) is the duality map from vector fields to one-forms, and \( \delta \) is the formal adjoint of the exterior derivative. (These are all defined with respect to the constant Riemannian metric on \( \mathbb{R}^3 \).)

In your proof you should relate \( d\sigma \), \( X^\flat \), and \( \delta \) to the Riemannian metric.

6 Show that the map \( \phi : S^1 \times S^1 \to \mathbb{R}^3 \) defined by

\[
\phi(u, v) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 + \cos \beta \\
0 \\
\sin \beta
\end{pmatrix}
\]

for \( u = (\cos \alpha, \sin \alpha) \) and \( v = (\cos \alpha, \sin \alpha) \) is an embedding and determine the degree of the map \( f : S^1 \times S^1 \to S^2 \) defined by

\[
f(u, v) = \frac{\phi(u, v) - p}{|\phi(u, v) - p|}
\]

where \( p \in \mathbb{R}^3 \setminus \phi(S^1 \times S^1) \). Note: The answer may depend on \( p \). Indicate the choice of orientations on \( S^1 \times S^1 \) and \( S^2 \) that you use to define the degree.
Answer two questions for Part I and two from Part II

Part I

(1) Let $X$ be a path-connected topological space. For $n > 1$ an integer, denote by $S_n$ the symmetric group on $n$ letters. State and prove a bijective correspondence between degree $n$ covering spaces of $X$ and group homomorphisms $\pi_1(X) \to S_n$. (Note that finding the accurate statement is part of the problem.)

(2) Let $\{H_k\}_{k\geq 0}$ be a sequence of finitely generated abelian groups with $H_0$ free. Show that there is a compact simplicial complex $X$ such that $H_k(X) \cong H_k$.

(3) Prove that $\tilde{H}_k(X) \cong \tilde{H}_{k+1}(X * S^0)$, where $*$ is the join operation.

Part II

(4) Let $X, Y$ be two closed topological manifolds. Compute the homology groups of the connected sum $X \# Y$ in terms of the homology groups of $X$ and $Y$.

(5) Let $X$ and $Y$ be two compact connected oriented manifolds of dimension 3. Suppose that 

$$\pi_1(X) \cong \mathbb{Z}_7 \oplus \mathbb{Z}^{\oplus 3}, \quad \pi_1(Y) \cong \mathbb{Z}_{14} \oplus \mathbb{Z}^{\oplus 4}.$$ 

Compute $H_*(X, \mathbb{Z}), H_*(Y, \mathbb{Z}),$ and $H_*(X \times Y, \mathbb{Q})$.

(6) Is $S^2 \vee S^4 \vee S^6$ homotopy equivalent to a compact manifold? Justify your answer.
Do four problems, two from each part. All carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial. Answer two questions from Part I and two from Part II.

**Part I**

1. The graph $K$ has six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and nine edges $a_ib_j (i, j = 1, 2, 3)$. The space $X$ is obtained from $K$ by attaching a 2-cell along each loop formed by a cycle of four edges in $K$. Find $\pi_1 (X)$.

2. Show that the reduced homology groups of the join $X \ast Y$ are given by $\tilde{H}_n(X \ast Y) \cong H_{n-1}(X \times Y, X \vee Y)$.

3. The transfer homomorphism $\tau$ of a double cover $p : \tilde{X} \to X$ assigns to each singular chain in $X$ the sum of its two lifts to $\tilde{X}$.

   (i) Show that the sequence
   
   $\begin{array}{cccc}
   0 & \longrightarrow & C_*(X; \mathbb{Z}_2) & \quad \longrightarrow \quad C_*(\tilde{X}; \mathbb{Z}_2) \\
   & & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \tau \\
   & & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad p \circ \quad \\
   & & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \# \\
   & & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad C_*(X; \mathbb{Z}_2) \\
   & \longrightarrow & \quad \longrightarrow & \quad \longrightarrow & \quad \longrightarrow & \quad \longrightarrow & \quad \longrightarrow & \quad \longrightarrow & 0
   \end{array}$

   is an exact sequence of chain complexes.

   (ii) Use the corresponding long exact sequence in homology to prove that an odd map from $S^n$ to itself has odd degree. (A map $f : S^n \to S^n$ is odd iff $f(-x) = -f(x)$.)

**Part II**

4. Compute the cup product structure of the cohomology ring $H^*(S^1 \times S^1; \mathbb{Z})$ of the torus $S^1 \times S^1$. Use this and the quotient map from $M_g$ to the wedge sum of $g$ tori to compute the cup product structure in $H^*(M_g; \mathbb{Z})$ for $M_g$ the closed orientable surface of genus $g \geq 2$.

5. Show that if $M$ is a compact contractible $n$-manifold for $n \geq 1$, then $\partial M$ is a homology $(n - 1)$-sphere, that is, $H_i(\partial M; \mathbb{Z}) \cong H_i(S^{n-1}; \mathbb{Z})$ for all $i$. 

   **Turn Page.**
6. Let $Y$ be an arbitrary topological space and $S^1$ the unit circle. Let $s_0 \in S^1$ and consider the pair $(S^1 \times Y, \{s_0\} \times Y)$. Prove that the connecting homomorphism

$$
\delta : H^k(\{s_0\} \times Y; R) \to H^{k+1}(S^1 \times Y, \{s_0\} \times Y; R)
$$

is trivial.
January 2010 Qualifying Exam

Algebraic Topology: 751 - 752

January 14, 2010

Do two 751 problems and two 752 problems. All carry equal marks. If you think a problem is badly stated, state precisely what you think is intended, but do not interpret a question so that it becomes trivial.

751 Problems

1. Let \( X \) be the space obtained from \( \mathbb{R}^3 \) by removing the three coordinate axes. Calculate \( \pi_1(X) \) and \( H_*(X) \).

2. True or False: A continuous map \( f: X \to Y \) which induces trivial maps \( f_* \) in the (reduced) \( \mathbb{Z} \)-homology is nullhomotopic. Explain your answer.

3. Show that if \( f: \mathbb{RP}^2 \to Y \) is a covering map of a CW-complex \( Y \), then \( f \) is a homeomorphism.

752 Problems

4. Is \( S^2 \times S^4 \) homotopy equivalent to \( \mathbb{CP}^3 \)? Explain.

5. Show that \( H^*(\mathbb{RP}^3; \mathbb{Z}) \cong H^*(\mathbb{RP}^2 \vee S^3; \mathbb{Z}) \) is a ring isomorphism. Is \( \mathbb{RP}^3 \) homotopy equivalent to \( \mathbb{RP}^2 \vee S^3 \)? Explain.

6. Let \( M \) be a closed, connected, oriented \( n \)-manifold and let \( f: S^n \to M \) be a continuous map of non-zero degree, i.e., the morphism \( f_*: H_n(S^n; \mathbb{Z}) \to H_n(M; \mathbb{Z}) \) is non-trivial. Show that \( M \) and \( S^n \) have the same \( \mathbb{Q} \)-homology.
January 2010 Qualifying Exam

Differential Topology: 751 - 761

January 14, 2010

Do two 751 problems and two 761 problems. All carry equal marks.
If you think a problem is badly stated, state precisely what you think
is intended, but do not interpret a question so that it becomes trivial.

751 Problems

1 Let $X$ be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.

2 True or False: A continuous map $f : X \to Y$ which induces trivial maps $f_*$ in the (reduced) $\mathbb{Z}$-homology is nullhomotopic. Explain your answer.

3 Show that if $f : \mathbb{RP}^{2n} \to Y$ is a covering map of a $CW$-complex $Y$, then $f$ is a homeomorphism.

761 Problems

4 Let $f : S^n \to S^n$ be $p \mapsto -p$.

1. Determine the values of $n$ for which $f$ preserves or reverses the orientation.

2. Use (1) to determine the values of $n$ for which $\mathbb{RP}^n$ is, or is not, orientable.

5 Let $\Delta$ be the distribution on $\mathbb{R}^3$ defined by the form $\omega = dx + zdy + ydz$, that is, $\omega$ generates the ideal of forms vanishing on $\Delta$.

1. Use $\omega$ to decide whether or not $\Delta$ is integrable.

2. Find the vector fields generating $\Delta$ and use them to find an alternate proof of (1).

6 Prove that a compact, oriented manifold without boundary is not contractible. (Do not simply quote a theorem, give as many details of the full argument as you can.)