Qualifying Examination in Algebra

August, 1978

Instructions:

1. Do four problems.

2. Write each solution in a separate booklet, putting your name and the question number on each booklet.
1. Let $G$ be a finite simple group and let $p$ be a prime dividing $|G|$. Suppose that $G$ has $p+1$ $p$-Sylow subgroups and that $P$ is a $p$-Sylow subgroup. Show that

(i) $G$ is isomorphic to a subgroup of the symmetric group of degree $p+1$. (3 points)

(ii) $p^2$ does not divide $|G|$. (2 points)

(iii) The centralizer of $P$ in $G$ is equal to $P$. (2 points)

(iv) $|G|$ divides $p(p^2 - 1)$. (3 points)

2. Let $G$ be a finite group and let $N_1, N_2, \ldots, N_k$ be normal subgroups. Suppose that $\bigcap_1^k N_i = \langle 1 \rangle$, and that $G/N_1 = S_1$ is simple. If all the groups $S_i$ are non-isomorphic prove that

$$G \cong S_1 \times S_2 \times \ldots \times S_k$$

(10 points)

3. Let $R$ be a ring with $1$ and let $V$ be a left $R$-module satisfying the ascending and descending chain conditions on submodules. If $T : V \to V$ is an $R$-endomorphism of $V$ prove that there is an integer $m$ such that

$$(\ker T^m) + (\text{im} T^m) = V$$

and

$$(\ker T^m) \cap (\text{im} T^m) = 0$$

(10 points)

4. Let $I_1$ and $I_2$ be minimal left ideals of a ring $R$ (with $1$).

(i) Show that if the product $I_1 I_2$ is nonzero then $I_1 \cong I_2$ as $R$-modules. (5 points)

(ii) Conversely, if $R$ is semisimple Artinian and $I_1 \cong I_2$ as $R$-modules show that $I_1 I_2$ is nonzero. (5 points)
Let $R = \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients. Let $I = (4, x)$ be the ideal of $R$ generated by 4 and $x$.

(i) Compute the radical of $I$. (3 points)
(ii) Prove that $I$ is primary. (4 points)
(iii) Prove that $I$ is not a power of its associated prime. (3 points)

6. Let $F$ be a field, $R$ a subring with 1 and let $\alpha$ be a nonzero element of $F$. Suppose that $R[\alpha] = F = R[1/\alpha]$.

(i) Show that $F$ is integral over $R$. (5 points)
(ii) Show that $R$ is a field. (5 points)

7. Let $\mathbb{Q}$ be the rational field and let $\zeta$ be a primitive fifth root of unity. The field $\mathbb{Q}(\zeta)$ is a Galois extension of $\mathbb{Q}$ of degree 4.

Prove that $\mathbb{Q}(\zeta)$ has a unique subfield $K$ of degree 2 over $\mathbb{Q}$ (4 points) and that $K = \mathbb{Q}(\sqrt{5})$ (6 points).

8. Let $K_i$ $(i = 1, 2)$ be Galois extensions of a field $k$ with Galois groups $G_i$. Assume that $K_1, K_2$ are included in some field and let $K_3 = K_1K_2$ be their composite.

(i) Show that $K_3$ is a Galois extension of $k$. (4 points)
(ii) Show that if $n_1 = [K_1:k]$ and $n_2 = [K_2:k]$ are relatively prime integers then the Galois group $G_3$ of $K_3/k$ is isomorphic to $G_1 \times G_2$. (6 points)
Qualifying Examination in Algebra
January, 1979

Instructions:

1. Do four problems.

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1. Let $G$ be an infinite group. Suppose that for any two subgroups $H$ and $K$ of $G$ either $H \subseteq K$ or $K \subseteq H$. Prove that there exists a prime $p$ such that $G$ is isomorphic to the multiplicative group of all complex numbers $z$ which satisfy $z^{p^n} = 1$ for some $n = 0, 1, 2, \ldots$.

2. Let $G$ be a group of order 360 and suppose $M$ is a maximal subgroup of $G$ which is isomorphic to the alternating group $A_5$. Show that $G$ is isomorphic to $A_6$.

3. Let $A$ be an $n \times n$ matrix over a field $F$ which can be partitioned as

$$A = \begin{bmatrix} B & 0 \\ D & C \end{bmatrix}$$

where $B$ and $C$ are square. Suppose that the characteristic polynomials $p(x)$ of $B$ and $q(x)$ of $C$ are relatively prime. Prove that $A$ is similar over $F$ to the matrix

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}.$$

Give an example to show that the relatively prime condition cannot be omitted.
4. If $R$ is a ring with identity let $J(R)$ denote its Jacobson radical.
Assume $R/J(R)$ is left Artinian and $J(R)$ is nilpotent. Let $S$ be
the ring of all matrices

$$
\begin{bmatrix}
  a & b \\
  0 & c \\
\end{bmatrix}
$$

where $a, b, c \in R$. Show that $S/J(S)$ is left Artinian and $J(S)$ is
nilpotent.

5. Let $A$ and $B$ be ideals of a commutative ring $R$ (with identity) and
suppose that $A + B = R$.
   a) (8 pts) Show that, as $R$-modules,

   $$
   A \oplus B \cong R \oplus AB
   $$

   Here $\oplus$ means external direct sum. (Hint. Define
   $\varphi: A \oplus B \to R$ by $\varphi(a, b) = a - b$.)
   b) (2 pts) Give an example of such ideals $A$ and $B$ such that
   neither is principal. (Hint. Take $R = \mathbb{Z}[x]$, the polynomial
   ring over the integers $\mathbb{Z}$ in the variable $x$.)

6. Let $K$ be a field and let $F$ be a finite Galoisian extension. Suppose
   that $g(x)$ is an irreducible polynomial in $K[x]$ and that $g(x)$ has an
   irreducible factor of degree 3 in $F[x]$. Prove that the degree of
   $g(x)$ is divisible by 3.
7. Let $K \subseteq F$ be finite fields with $F = K[\alpha]$ and $\alpha^{15} = 1$.

Show that the degree of the extension satisfies $|F : K| \leq 4$.

8. Suppose $K$ is a field and $F$ is a finite extension of $K$.

a) (5 pts) If there are only a finite number of intermediate fields, show that $F = K[\alpha]$ for some $\alpha \in F$.

b) (5 pts) Suppose $F = K[\alpha, \beta]$ where char $K = p$, $\alpha^p$ and $\beta^p \in K$ and $|F : K| = p^2$. Show that there are infinitely many intermediate fields.
QUALIFYING EXAM IN ALGEBRA
August 1979

Do four problems.
Write each solution in a separate booklet.
Write your name on each booklet.

1. Let \( p \) be a prime and let \( G \) be a finite group with the property \((*)\) that every
element of order a power of \( p \) is contained in a conjugacy class of size a power
of \( p \).
   1) If \( p \mid |G| \), show that \( G \) has a central subgroup \( Z \) of order \( p \).
      (4 points)
   2) If \( Z \) is as above, show that \( G/Z \) has property \((*)\).
      (4 points)
   3) Deduce that a Sylow \( p \)-subgroup of \( G \) is normal.
      (2 points)

2. Let \( G \) be a finite group with the property that all of its proper subgroups are
abelian. Let \( N \) be a proper normal subgroup of \( G \). Show that either \( N \)
is contained in the center of \( G \) or else \( G \) has a normal abelian subgroup of
prime index.

3. Recall that an \( R \)-module \( X \) is said to be completely reducible if every submodule is a
direct summand or, equivalently, if \( X \) is a (possibly infinite) direct sum of simple
submodules. Let \( M \) be an \( R \)-module and let \( K \) be a submodule such that
\( M/K \) is completely reducible. Suppose \( N \subseteq M \) and \( N \cap K = 0 \). Prove that
   1) \( N \) is a direct summand of \( M \).
      (5 points)
   2) \( N \) is Artinian iff it is Noetherian. (HINT: Show that \( N \) is completely
      reducible.)
      (5 points)

4. In this problem we consider commutative rings with \( 1 \).
   1) Suppose every ideal \( I \neq R \) of a ring \( R \) is prime. Prove that \( R \)
is a field. (HINT: Consider the ideal \((x^2)\) for \( x \in R \)).
      (4 points)
   2) Suppose that every nonzero proper ideal of a ring is maximal. Prove that
      there are at most two such ideals.
      (4 points)
   3) Suppose that \( I \) is an ideal of a ring \( R \) which is maximal with respect to
      the property that it is proper and not prime. Deduce that \( I \) is contained in at
      most two other proper ideals of \( R \). (HINT: Apply parts (1) and (ii) in turn
      to suitable factor rings of \( R \)).
      (2 points)

(MORE ON BACK)
Let $V$ be an infinite dimensional vectorspace over a field $F$ and let $R$ be the ring of all $F$-linear transformations from $V$ to $V$. Note that this makes $V$ into a simple right $R$-module. Let $S$ be the set of all $r \in R$ with finite dimensional range, and let $I$ be any nonzero 2-sided ideal of $R$.

1. Show that $S$ is a 2-sided ideal of $R$. (3 points)
2. If $v \in V$ with $v \neq 0$, show that $v \cdot I = V$. (3 points)
3. If $r \in R$ has 1-dimensional range, prove that $r \in rI \subseteq I$. (2 points)
4. Deduce that $S \subseteq I$. (2 points)

6. Let $F$ be a field and suppose $\alpha \in F$. Let $m$ and $n$ be relatively prime positive integers.

1. Let $K \supseteq F$ be a field. Show that $K$ contains an $(mn)$-th root of $\alpha$ iff it contains both an $m$-th root and an $n$-th of $\alpha$. (5 points)

2. Prove that the polynomial $x^{mn} - \alpha$ is irreducible over $F$ iff both $x^m - \alpha$ and $x^n - \alpha$ are irreducible over $F$. (HINT: Use field extension degrees.) (5 points)

7. Let $E \supseteq F$ be a finite degree, Galois field extension. Let $f(x) \in F[x]$ be an irreducible polynomial of prime degree. Suppose $f(x)$ reduces over $E$. Show that $f(x)$ splits completely in $E[x]$.

8. Let $E \supseteq F$ be a finite degree, Galois field extension. Assume that all intermediate fields, properly between $E$ and $F$, have equal degrees over $F$. Prove that all of these intermediate fields are Galois extensions of $F$. 
QUALIFYING EXAM IN ALGEBRA

January 1980

Do four problems.
Write each solution in a separate booklet.
Write your name on each booklet.

1. Let $G$ be a finite group.
   a) Suppose $N, H, K \subseteq G$ are subgroups such that $NH = G$ and $(N \cap H)K = G$. Show that $N(K \cap H) = G$. (3 points)
   b) Suppose that the intersection of all of the maximal subgroups of $G$ is trivial. Show that if $A \triangleleft G$ is abelian, then there exists $U \subseteq G$ such that $AU = G$ and $A \cap U = 1$.
   (HINT: Use induction on $|A|$.) (7 points)

2. Let $G$ be a finite group and let $p$ be a prime. Suppose that every proper subgroup of $G$ of order divisible by $p$ is a $p$-group. Show that every two distinct Sylow $p$-subgroups of $G$ intersect trivially.
   (HINT: Consider the normalizer of a maximal intersection of Sylow $p$-subgroups.)

3. Let $R$ be a commutative ring with $1$ and let $I$ be an ideal of $R$, maximal with respect to the property that $R/I$ is not Noetherian. Prove that $I$ is a prime ideal. (HINT: If $AB \subseteq I$, study the $R$-module $(A+I)/I$.)
4. Let $R$ be a ring with 1 (not necessarily commutative) and let $V$ be a faithful, simple right $R$-module. Note that a module is said to be faithful if it is not annihilated by any nonzero element of the ring.

a) If $A$ and $B$ are nonzero ideals of $R$, prove that $AB \neq 0$. (5 points)

b) If $M$ is a minimal right ideal of $R$, prove that $M$ and $V$ are isomorphic right $R$-modules. (5 points)

5. Let $F$ be a field of characteristic $p > 0$.

a) If $F \subseteq F[\alpha]$ is an extension of fields and $\alpha^p \in F$, show that $|F[\alpha] : F| = 1$ or $p$. (3 points)

b) Let $L = F(x, y)$ be the field of rational functions in two variables over $F$. Let $K = F(x^p, y^p) \subseteq L$. Show that $|L : K| = p^2$. (4 points)

c) Show that there exist infinitely many fields $E$ such that $K \subseteq E \subseteq L$. (3 points)

6. Let $F \subseteq E$ be a finite degree field extension and let $G = \{ \sigma \in \text{Aut}(E) \mid \sigma^\sigma = \sigma \text{ for all } \sigma \in F \}$. Suppose $\theta \in E$ and that the distinct elements of the form $\theta^\sigma$ for $\sigma \in G$ form a basis for $E$ over $F$.

a) Show that $E = F[\theta]$. (5 points)

b) Show that if $\alpha \in E$ and $\sigma^\sigma = \sigma$ for all $\sigma \in G$, then $\alpha \in F$. (5 points)
7. Let \( K[x, y] \) be the polynomial ring over the field \( K \) in the two variables \( x \) and \( y \). For a fixed integer \( n \geq 1 \), let \( I \) be the ideal of \( K[x, y] \) generated by the \( n+1 \) monomials \( x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n \) each of total degree \( n \).

a) If \( f(x, y) = \sum_{i, j} k_{ij} x^i y^j \) belongs to \( I \), show that \( f(x, y) \) has no terms of total degree \( < n \), that is show that \( k_{ij} = 0 \) if \( i + j < n \). (3 points)

b) Prove that \( I \) cannot be generated as an ideal by fewer than \( n+1 \) elements. (HINT: If \( I \) is generated by \( f_1, f_2, \ldots, f_m \), write each \( x^i y^{n-1} \) as a \( K[x, y] \) -linear combination of the \( f_j \)'s and consider the terms of total degree \( n \).) (7 points)

8. Let \( F \) be a field of characteristic \( p > 0 \) and let \( V \) be a vector space over \( F \) of finite dimension \( n \geq 1 \). Let \( g \in \text{GL}(V) \), the multiplicative group of all nonsingular linear transformations on \( V \).

a) Let \( g \) have order \( p^m \) for some integer \( m \geq 0 \). Show that the minimal polynomial of \( g \) on \( V \) is \( (x-1)^k \) for some integer \( k \geq 1 \). (3 points)

b) Let \( g \) and \( k \) be as above. If \( W = \{v \in V \mid v = vg\} \), show that \( \dim_F W \geq n/k \). (4 points)

c) If the minimal polynomial of \( g \) is \( (x-1)^2 \), show that \( g \) has order \( p \). Furthermore if \( n \) is odd and if \( h \) is conjugate to \( g \) in \( \text{GL}(V) \), prove that there exists a nonzero vector \( v \in V \) with \( vg = v = vh \). (3 points)
1. a) If a finite $p$-group has more than one maximal subgroup, prove that it has at least $p+1$ maximal subgroups. (5 points)

b) Suppose that the finite group $G$ has a total of no more than five subgroups. Prove that $G$ is abelian. (5 points)

2. Let $\epsilon$ be a primitive $n^{th}$ root of unity in the complex numbers.
Let $\mathbb{Q}$ denote the rationals, and let $\alpha \in \mathbb{Q}[\epsilon]$ have the property that $\alpha^n = 2$ for some positive integer $m$. Show that $m = 1$ or 2.

3. Let $R$ be a commutative Noetherian ring with 1.

a) Let $I, J$ be ideals of $R$ and suppose $R/I$ is an Artinian ring. Prove that $JJI$ is an Artinian $R$-module. (5 points)

b) If all prime ideals of $R$ are maximal, prove that $R$ is an Artinian ring. (5 points)
4. Let \( V \) be a finite dimensional vector space over some field \( F \).

Let \((\ ,\ )\) be an \( F \)-bilinear form on \( V \). For subspaces \( U \subseteq V \), write \( r(U) = \{v \in V \mid (U, v) = 0\} \) and \( l(U) = \{v \in V \mid (v, U) = 0\} \).

a) Show that \( \dim l(V) = \dim r(V) \). (5 points)

b) If \( l(V) = 0 \) and \( U \subseteq V \), show that \( \dim l(U) = \dim r(U) \). (5 points)

5. Let \( G \) be a finite group and let \( p \) be a prime.

a) Let \( H \) be a subgroup of \( G \) which contains \( N_G(P) \) for some Sylow \( p \)-subgroup \( P \) of \( G \). Suppose \( P \subseteq H^g \) for some \( g \in G \). Show that \( g \in H \). (4 points)

b) Let \( K \) be a subgroup of \( G \) which contains \( N_G(Q) \) for every nonidentity \( p \)-subgroup \( Q \) of \( K \). Show that \( |K \cap K^g| \) is not divisible by \( p \) if \( g \not\in K \). (6 points)
6. Let \( k \) be a field and let \( F = k(x_1, x_2, x_3, x_4) \) where the \( x_i \) are algebraically independent over \( k \). Let \( s_1, s_2, s_3, s_4 \) be the elementary symmetric functions in the \( x_i \) and put \( K = k(s_1, s_2, s_3, s_4) \). Finally, let \( L = K(\xi_1, \xi_2, \xi_3) \) where

\[
\xi_1 = x_1x_2 + x_3x_4 \\
\xi_2 = x_1x_3 + x_2x_4 \\
\xi_3 = x_1x_4 + x_2x_3
\]

Note that we have the tower of fields \( K \subseteq L \subseteq F \).

a) Prove that \( F/L \) is a Galois extension and determine the Galois group (up to isomorphism). (6 points)

b) Prove that \( L/K \) is a Galois extension and determine the Galois group (up to isomorphism). (4 points)

7. Let \( R \) be a (not necessarily commutative) ring with \( 1 \). Let \( E \subseteq A \) be right \( R \)-modules. We say that \( E \) is essential in \( A \) (and write \( E \text{ ess } A \)) if for all nonzero submodules \( X \subseteq A, \) we have \( E \cap X \neq 0 \).

a) If \( E \text{ ess } A \) and \( F \text{ ess } A \), show that \( (E \cap F) \text{ ess } A \). (2 points)

b) If \( E_1 \text{ ess } A_1 \) and \( E_2 \text{ ess } A_2 \), show that \( (E_1 \oplus E_2) \text{ ess } (A_1 \oplus A_2) \). (3 points)

c) If \( E \text{ ess } A \) and \( a \in A \), let \( I = \{ r \in R \mid ar \in E \} \).

Show that \( I \) is a right submodule (i.e. right ideal) of \( R \) and that \( I \text{ ess } R \). (5 points)
8. Let $A$ be a finite dimensional algebra over a field $F$ of characteristic zero. Suppose that $A$ has a basis $G$ which forms a group with respect to the multiplication in $A$. Let $S$ be the $F$-subspace of $A$ spanned by the nonidentity elements of $G$.

a) If $s \in S$, show that the linear transformation of $A$ defined by right multiplication by $s$ has trace zero. (3 points)

b) Show that if $a \in A$ is nilpotent, then $a \in S$. (3 points)

c) If $A$ is commutative, show that $A$ has no nonzero nilpotent elements. (4 points)

Policy on Misprints

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Qualifying Exam in Algebra
January 1982

Do FOUR problems.

1. Let $P$ be a nonabelian group of order $p^4$, where $p$ is a prime, and let $A$ be a subgroup of $P$ maximal with the property of being normal and abelian. Show that $|A| = p^3$. HINT: Consider the centralizer of $A$.

2. Let $K = \mathbb{Q}([\zeta]$ where $\mathbb{Q}$ is the field of rational numbers and $\zeta$ is the complex number given by $\zeta = e^{2\pi i/63}$.
   
   a) How many subfields of $K$ are extensions of degree 2 or 3 over $\mathbb{Q}$? (5 points)

   b) How many of the subfields counted in part (a) are contained in the real numbers? (5 points)

   Justify your answers!

3. Let $F$ be a field and let $R = F[x,y,z]$ be the polynomial ring in the three indeterminates $x,y,z$ over $F$. Let $I$ be the ideal of $R$ generated by $xy$, $xz$ and $yz$. Show that $I$ is an intersection of three prime ideals of $R$. 
4. Let \( V \) be a vector space over the reals with a positive definite symmetric bilinear form \((\ , \ )\). Let \( v \in V \) be some fixed vector. Suppose \( u_1, u_2, \ldots, u_n \in V \) are vectors such that

i) \( (u_i, v) > 0 \) for all \( i \)

ii) \( (u_i, u_j) \leq 0 \) for all \( i \neq j \).

Show that \( u_1, u_2, \ldots, u_n \) are linearly independent.

5. A group \( G \) is said to be **complete** if

i) all its automorphisms are inner, and

ii) its center is trivial.

a) Show that \( \text{Sym}(3) \), the symmetric group on three symbols, is complete. (5 points)

b) Suppose a complete group \( G \) is contained as a normal subgroup in some group \( X \). Show that \( G \) is a direct factor of \( X \). (5 points)

6. Let \( K \) be a field of characteristic zero which contains a primitive \( n \)th root of unity \( \zeta \). Suppose that \( a \in K \) and that the polynomial \( f(x) = x^{2n} + ax^n + 1 \) is irreducible in \( K[x] \). Let \( L \) be a splitting field for \( f(x) \) over \( K \).

a) If \( \alpha \in L \) is a root of \( f(x) \), find the remaining roots. (3 points)

b) Determine the degree \( |L:K| \). (3 points)

c) Find the number of elements of order 2 in the Galois group \( \text{Gal}(L/K) \). (4 points)

**HINT** for (c): Consider separately the cases \( n \) odd and \( n \) even.
7. Let $R$ be a (not necessarily commutative) ring with 1, and let $e$ be a nonzero idempotent in $R$. Let $J = J(R)$ be the Jacobson radical of $R$.

a) Show that the ideal $eJ e$ of the ring $eRe$ is quasi-regular. (3 points)

b) If $V$ is a simple right $R$-module, show that either $Ve = 0$ or $Ve$ is a simple right $eRe$-module. (4 points)

c) Conclude that $eJe$ is the Jacobson radical of $eRe$. (3 points)

8. Let $G$ be the group defined by the presentation

$$G = \langle a, b, x, t \mid a^2 = b^2 = 1, [a, b] = 1, x^3 = 1, a^x = b, b^x = ab, t^2 = 1, x^t = x^{-1}, a^t = a, b^t = ab \rangle.$$ 

Prove that $G \cong \text{Sym}(4)$, the symmetric group on four symbols.

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QUALIFYING EXAM IN ALGEBRA

August, 1982

Do FOUR problems.

1) Let $G$ be a finite group and let $P$ be a subgroup of prime order $p$.

Assume that $P = C_G(P)$, that is $P$ is self-centralizing.

   a) Show that $p^2$ does not divide $|G|$.

   b) If $N < G$ and $P \not\subseteq N$, show that $p$ divides $|N| - 1$.

   c) If $N < G$ and $P \subseteq N$, show that the index $|G:N|$ divides $p-1$.

2) Let $G$ be a finite group and let $H$ be a solvable subgroup of $G$.

Let $N = N_G(H)$ be the normalizer of $H$ in $G$ and assume that $N/H$

is a nonabelian simple group. Prove that $N = N_G(N)$.
3) Let $R$ be a commutative Noetherian ring with $1$ and let $M$ be a nonzero right $R$-module. Let $S$ be the set of all ideals $I$ of $R$ such that $mI = 0$ for some nonzero $m \in M$.

a) Show that the maximal members of $S$ (with respect to inclusion) are prime ideals. (4 pts)

b) For each maximal member $P$ of $S$ let $U(P) = \text{ann}_M(P) = \{m \in M \mid mP = 0\}$. Show that the sum of these submodules $U(P)$ is direct. (4 pts)

c) If $M$ is a finitely generated $R$-module, deduce that $S$ contains only finitely many maximal members. (2 pts)

4) Let $R$ be a ring with $1$ which has only finitely many units.

a) Let $J$ be the Jacobson radical of $R$. Prove that $J$ is finite and that $R/J$ has only finitely many units. (5 pts)

b) If $R$ is commutative and infinite, prove that $R$ has infinitely many maximal ideals. (5 pts)
5) Let $F = \mathbb{Z}/7\mathbb{Z}$ be the field of integers modulo 7. Let $I = (x^2 - x - 1)$ and $J = (x^2 + 1)$ be the ideals of the polynomial ring $F[x]$ generated by the irreducible polynomials $x^2 - x - 1$ and $x^2 + 1$. Since $F[x]/I$ and $F[x]/J$ are finite fields with the same number of elements, they are isomorphic. Find an explicit isomorphism $F[x]/J \rightarrow F[x]/I$ and prove it is an isomorphism.

6) Let $\theta = \sqrt{5} + \sqrt{5}$. Show that $\mathbb{Q}[\theta]$ is a Galois extension of the rationals $\mathbb{Q}$ and determine the Galois group $\text{Gal}(\mathbb{Q}[\theta]/\mathbb{Q})$ as an explicit group of permutations of the roots of the minimal polynomial of $\theta$ over $\mathbb{Q}$. 
7) Let $A$ be an $n \times n$ complex matrix. Prove that the following statements are equivalent.
   
   a) $\text{rank } A^2 = \text{rank } A$
   
   b) The multiplicity of $0$ as a root of the minimum polynomial of $A$ is at most one.
   
   c) There is an $n \times n$ matrix $X$ such that
      
      \[ AXA = A, \quad XAX = X, \quad AX =XA. \]

8) Let $G = \text{GL}(6,2)$, the group of invertible $6 \times 6$ matrices over the field \( \mathbb{Z}/2\mathbb{Z} \). Let $E$ be the set of all matrices in $G$ of the form
   
   \[
   \begin{bmatrix}
   I & 0 \\
   X & I
   \end{bmatrix}
   \]
   
   where $I$ is the $3 \times 3$ identity matrix, $0$ is the $3 \times 3$ zero matrix and $X$ is any $3 \times 3$ matrix over $\mathbb{Z}/2\mathbb{Z}$.
   
   a) Prove that $E$ is an elementary abelian $2$-subgroup of $G$. (3 pts)
   
   b) Show that $C_G(E) = E$, that is $E$ is self-centralizing in $G$. (3 pts)
   
   c) If $N = N_G(E)$ is the normalizer in $G$ of $E$, prove that $N/E \cong \text{GL}(3,2) \times \text{GL}(3,2)$. (4 pts)
Do FOUR problems.

1) Let $T$ be a linear transformation on a finite dimensional complex vector space, $V$. Suppose that the subspace of $V$ spanned by all eigenvalues of $T$ is 1-dimensional. Prove that the minimal polynomial of $T$ is equal to $(x-\alpha)^n$, where $n = \dim V$ and $\alpha$ is some complex number.

2) Let $R$ be a right Artinian ring with 1. Suppose that $R$ has only finitely many units (i.e. elements with 2-sided inverses). Show that $R$ itself has only finitely many elements.

3) Let $S$ be the set of all complex numbers which are algebraic over the rationals $\mathbb{Q}$ and which have a minimal polynomial over $\mathbb{Q}$ that is solvable by radicals.

   a) (5 pts.) Show that $S$ is a field.

   b) (5 pts.) Show that, in the complex numbers, $S$ is closed under taking $n$th roots (for arbitrary positive integers $n$). Show also that $S$ is minimal with this property.
4) The socle of a finite group \( G \) is, by definition, the subgroup generated by all nonidentity minimal normal subgroups of \( G \). This characteristic subgroup is denoted \( \text{soc}(G) \).

a) (4 pts) Suppose \( N \) is a normal subgroup of \( G \) with \( \text{soc}(G) \subseteq N \subset G \). Show that \( \text{soc}(G) \subseteq \text{soc}(N) \).

b) (3 pts) Suppose \( G \) has no nontrivial abelian normal subgroup and show that then the centralizer \( C_G(\text{soc}(G)) = 1 \).

c) (3 pts) Suppose \( \text{soc}(G) \subseteq N \subset G \) and that \( G \) is as in part b). Show that then \( \text{soc}(G) = \text{soc}(N) \).

5) Let \( R \) be an integral domain with quotient field \( K \) and suppose that \( R \) is integrally closed in \( K \) (that is, all elements of \( K \) which are integral over \( R \) are in \( R \)). Let \( F \) be an algebraic field extension of \( K \).

a) (5 pts) Suppose that \( \alpha \in F \) is a root of a monic polynomial in \( R[x] \). Show that the minimal polynomial for \( \alpha \) over \( K \) has coefficients in \( R \).

b) (5 pts) Show that there is a subring \( S \) of \( F \) such that \( F \) is the field of quotients of \( S \) and \( S \cap K = R \).

6) Let \( G \) be a finite group which has exactly eight Sylow 7-subgroups. Show that there exists a normal subgroup \( N \) of \( G \) such that the index \( |G:N| \) is divisible by 56 but not by 49.
7) a) (3 pts) Determine the minimal polynomial, \( p(x) \), of \( \sqrt{2} + \sqrt{2} \) over the rationals, \( \mathbb{Q} \).

b) (3 pts) Determine the degree of the splitting field \( F \) for \( p(x) \) over \( \mathbb{Q} \) in \( \mathbb{C} \), the complex numbers.

c) (4 pts) Determine the Galois group of \( F \) over \( \mathbb{Q} \).

8) Let \( R \subseteq S \) be commutative rings with the same \( \mathfrak{l} \). Let \( P \) be a minimal prime ideal of \( R \) (i.e. \( P \) contains no properly smaller prime ideal of \( R \), in particular, \( P = (0) \) if \( (0) \) is a prime ideal). Show that there exists a prime ideal \( Q \) of \( S \) with \( Q \cap R = P \).

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**POLICY ON MISPRINTS**

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Qualifying Exam

ALGEBRA

August 22, 1983

Instructions: Do four problems.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $G$ be a simple group of order 168.

   a) Show that $G$ has precisely 8 Sylow 7-subgroups. (2 pts)

   b) Show that $G$ is isomorphic to a subgroup $\tilde{G}$ of the
      alternating group $A_8$ and that no element of order 2 in
      $\tilde{G}$ has a fixed point. (3 pts)

   c) Show that $G$ has no element of order 6. (2 pts).

   d) Find the number of Sylow 3-subgroups of $G$. Prove your
      answer. (3 pts)

2. Let $\Lambda = \mathbb{Z} / 4\mathbb{Z}$ be the ring of integers mod 4 and let

   $$ R = \begin{pmatrix}
   \Lambda & \Lambda & \Lambda \\
   0 & \Lambda & \Lambda \\
   0 & 0 & \Lambda
   \end{pmatrix} $$

   be the ring of 3×3 upper triangular matrices over $\Lambda$.

   a) Explicitly describe the Jacobson radical of $R$. Prove
      your answer. (5 pts)

   b) Find the number of different (up to isomorphism) simple
      right $R$-modules and explicitly construct one such module
      in each isomorphism class. Prove your answer. (5 pts)
3. Let $\mathbb{Q}$ be the field of rational numbers and let $\omega$ and $\varepsilon$ be the complex numbers

$$\omega = e^{2\pi i / 3}, \quad \varepsilon = e^{2\pi i / 5}$$

a) Show that $\omega$ is not an element of $\mathbb{Q}[\varepsilon]$. (4 pts)

b) Find the degree of the extension $\mathbb{Q}[\varepsilon + \omega] \supset \mathbb{Q}$. Prove your answer. (6 pts)

4. Let $V$ be a finite dimensional complex vector space and let $S$ and $T$ be linear transformations on $V$ satisfying

$$ST - TS = S^2 - S.$$ 

For each complex number $\mu$ let $V_\mu$ be the subspace of $V$ defined by

$$V_\mu = \{ v \in V \mid (S-\mu I)^m v = 0 \ \text{for some integer} \ m > 1 \}$$

where $I$ is the identity transformation.

a) Show that each subspace $V_\mu$ is invariant under $T$, that is $TV_\mu \subseteq V_\mu$. (5 pts)

b) If $V_\mu \neq 0$ show that $\mu = 0$ or $1$. (Hint. Consider the trace of $S^2 - S$ in its action on $V_\mu$.) (5 pts)
5. a) Let \( A \) be an abelian group and let \( \sigma \) be an automorphism of \( A \) with \( \sigma^2 = 1 \). Assume that no nonidentity element of \( A \) is fixed by \( \sigma \). Show that \( a^\sigma = a^{-1} \) for all \( a \in A \). (2 pts)

b) Let \( G \) be a group of order \( 2 \cdot p^n \) where \( p \) is an odd prime. If \( N \) is a minimal normal subgroup of \( G \), prove that \( N \) is cyclic. (8 pts)

6. Let \( R \) be a ring with \( 1 \) and let \( V \) be a right \( R \)-module. Assume that \( V \) is the direct sum \( V = \alpha R \oplus W \) where \( W \) is a submodule, \( \alpha \in V \) and the annihilator in \( R \) of \( \alpha \) is zero.

a) Let \( E \) be the ring of \( R \)-endomorphisms of \( V \) with \( E \) acting on the right. Show that \( \alpha E = V \) and that there exists \( e \in E \) with \( Ve \subseteq \alpha R \) and \( \alpha e = \alpha \). (3 pts)

b) Let \( S \) be the ring of \( E \)-endomorphisms of \( V \) with \( S \) acting on the right. Prove that for each \( s \in S \) there exists an element \( r \in R \) such that \( vs = vr \) for all \( v \in V \). (Hint. First use the element \( e \) of part (a) to show that \( \alpha s \in \alpha R \).) (4 pts)

c) Let \( A \) be a finitely generated infinite abelian group written additively. If an element \( \mu \) is in the center of the endomorphism ring of \( A \), show that there exists an integer \( n \in \mathbb{Z} \) with \( a\mu = an \) for all \( a \in A \). (3 pts)
7. Let $K$ be a finite degree extension field of the rationals $Q$. Show that there exists a finite degree extension $E \supset K$ such that $E$ is not normal over $K$.

8. Let $F = GF(q)$ be a finite field of order $q$ and let $R = F[x]$ be the polynomial ring over $F$. Suppose $f(x) \in R$ is irreducible of degree 6 and let $M$ be the ideal of $R$ generated by $f(x)$. Find the number of ideals $N$ of $R$ such that the rings $R/M$ and $R/N$ are isomorphic. Prove your answer.
Qualifying Exam

ALGEBRA

January 16, 1984

Instructions: Do four problems.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. The socle \( \text{soc}(G) \) of a finite group \( G \) is the subgroup generated by all minimal normal subgroups of \( G \).

   a) Let \( S = \text{soc}(G) \). Prove that \( S = \text{soc}(S) \). \( (5 \text{ points}) \)

   b) If \( G = \text{soc}(G) \) and \( M \) is minimal normal in \( G \), prove that \( M \) is simple. \( (5 \text{ points}) \)

2. Let \( F \subseteq E \) be a finite degree field extension. Suppose for each intermediate field \( K \) such that \( E \) is primitive (i.e. singly generated) over \( K \), that \( E \) is separable over \( K \). Show that \( E \) is separable over \( F \).
3. Let $R$ be a commutative ring with $1$ and let $A_1, A_2, \ldots, A_n$ be ideals of $R$. Set $B = \bigcap_{i=1}^{n} A_i$.

a) Suppose that $A_i + A_j = R$ for all $i \neq j$. Prove that $R/B$ is isomorphic to the (external) direct sum of the rings $R/A_i$. (7 points)

b) Show by example that the above isomorphism does not hold in general if the assumption $A_i + A_j = R$ (for all $i \neq j$) is replaced by the weaker condition $A_1 + A_2 + \ldots + A_n = R$. (3 points)

4. Let $p$ be a prime, let $q$ be a power of $p$ and set $\omega = e^{2\pi i/q}$. Suppose $c_0, c_1, \ldots, c_{q-1} \in \mathbb{Z}$ (the integers) and that

$$\sum_{i=0}^{q-1} c_i \omega^i \in \mathbb{Z}.$$ 

Prove that

$$\sum_{i=0}^{q-1} c_i \omega^i \equiv \sum_{i=0}^{q-1} c_i \mod p.$$
5. A group $X$ is said to be **perfect** if $X = X'$, where $X'$ is the commutator subgroup of $X$. Let $\varphi : G \rightarrow H$ be a homomorphism from the finite group $G$ onto the perfect group $H$.

a) Show that there exists a normal, perfect subgroup $N$ of $G$ with $\varphi(N) = H$. (5 points)

b) Assume that the kernel of $\varphi$ is solvable. Show that the subgroup $N$ of part (a) is unique. (5 points)

6. Let $f(x) = x^5 - 6x + 2$ be a polynomial over the rationals $\mathbb{Q}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_5$ be its roots in the complex numbers and set $K = \mathbb{Q}[\alpha_1, \alpha_2, \ldots, \alpha_5]$.

a) Show that the degree $|K: \mathbb{Q}| = 120$. (3 points)

b) Let $g(x) = \prod_{1 \leq i < j \leq 5} (x - \alpha_i \alpha_j)$. Show that $g(x) \in \mathbb{Q}[x]$ and that $g(x)$ is irreducible over $\mathbb{Q}$. (5 points)

c) Determine the splitting field of $g(x)$ in the complex numbers. (2 points)
7. Let $R$ be a right Artinian ring with $1$ and let $J = J(R)$ be its Jacobson radical. Let $I$ be a 2-sided ideal of $R$.

a) Suppose (for this part only) that $I \cap J = 0$.
Prove that there exists a 2-sided ideal $K$ of $R$ with $R = I \oplus K$. 

b) Show that every unit of $R/I$ is of the form $u + I$ where $u$ is a unit of $R$. (Hint. Consider the cases $I \cap J = 0$ and $I \subseteq J$). 

8. An abelian group $D$ (written additively) is said to be divisible if, for every integer $n$ and element $d \in D$, there exists $x \in D$ with $nx = d$. Let $D \subseteq A$ be additive abelian groups with $D$ divisible.

a) If $D \neq A$, show that there exists a subgroup $B \subseteq A$ with $B \neq 0$ and $B \cap D = 0$. 

b) Show that $D$ is a direct summand of $A$. 

Qualifying Exam

ALGEBRA

August 27, 1984

Instructions: Do four problems.

Policy on Misprints

The Doctoral Exam Committee strives to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $G$ be a finite group and let $P$ be a subgroup of prime order $p$. Suppose $P$ is equal to its centralizer $C_G(P)$ in $G$.

   a. Show that $P$ is a Sylow $p$-subgroup of $G$. (3 points)

   b. If $P$ is normal in $G$, show that $G$ is solvable. (3 points)

   c. Let $q \neq p$ be a prime divisor of $|G|$. If $P$ is normal in $G$, find the number of elements of $G$ of order $q$. Prove your answer. (4 points)

2. **Definition.** Let $S$ be a (not necessarily commutative) ring with $1$. A proper ideal $P$ of $S$ is **prime** if whenever $P$ contains the product $AB$ of two ideals of $S$ then $P \supset A$ or $P \supset B$.

Let $R$ be a ring and $e$ a nonzero idempotent element of $R$. Note that $eRe$ is a ring with identity $e$.

   a. If $I$ is an ideal of $R$, show that $eIe$ is an ideal of the ring $eRe$. Furthermore if $I$ is prime in $R$ and $e \not\in I$ show that $eIe$ is prime in $eRe$. (5 points)

   b. If $J$ is an ideal of the ring $eRe$, show that there exists a unique largest ideal $I$ of $R$ such that $eIe = J$. Furthermore if $J$ is prime in $eRe$, show that $I$ is prime in $R$. (5 points)
3. Let $F$ be a field with algebraic closure $\bar{F}$. Suppose that for each integer $n$ there exists at most one intermediate field $L$ such that the degree $(L:F) = n$. Let $F \subseteq K \subseteq \bar{F}$ with $(K:F) < \infty$.

a. Prove that $K$ is the splitting field of an irreducible polynomial over $F$. (5 points)

b. If $G = \text{Gal}(K/F)$, show that $G$ is cyclic. (5 points)

4. Let $A$ be the (complex) matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$$

a. Find the minimum polynomial of $A$. (3 points)

b. Find the Jordan canonical form $J$ of $A$. (3 points)

c. Find an invertible matrix $P$ such that $P^{-1}AP = J$. (4 points)

Justify your answers.
5. Let $G$ be a finite solvable group and let $x_1, x_2, \ldots, x_n$ be elements of $G$ of pairwise relatively prime orders. If the product $x_1 x_2 \cdots x_n = 1$, show that each $x_i = 1$.

6. Let $R$ be a commutative Noetherian ring with $1$ and let $R[x]$ be the polynomial ring in the indeterminate $x$. Let $I$ be an ideal of $R$ and define

$$S = R + Ix + I^2x^2 + \cdots + I^nx^n + \cdots \subseteq R[x].$$

a. Show that $S$ is a subring of $R[x]$ and that $S$ is a homomorphic image of a polynomial ring over $R$ in finitely many indeterminates. (5 points)

b. Let $J$ be an ideal of $R$. For each integer $n > 0$ let

$$J_n = J + (I \cap J)x + (I^2 \cap J)x^2 + \cdots + (I^n \cap J)x^n \subseteq S$$

and set

$$\widetilde{J} = \bigcup_{n=0}^{\infty} J_n.$$

Show that $\widetilde{J}$ is an ideal of $S$ and that $\widetilde{J} = J_nS$ for some $n$. (5 points)
7. Let $\mathbb{Q}$ be the field of rational numbers, let $\mathbb{C}$ be the complex field and let $\alpha \in \mathbb{C}$.

a. Let $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$ with $E$ a (finite) Galois extension of $\mathbb{Q}$. Suppose $U$ is a field with $\mathbb{Q}(\alpha) \subseteq U \subseteq E(\alpha)$. Show that $U = K(\alpha)$ where $K = U \cap E$. (Hint. Compute the degrees $(E(\alpha):U)$ and $(E(\alpha):K(\alpha))$. ) (5 points)

b. Assume $\alpha$ is transcendental over $\mathbb{Q}$ and let
$$\beta = \sqrt{2} + \sqrt{3} \alpha + \sqrt{5} \alpha^2 .$$
Show that $\sqrt{2}, \sqrt{3}, \sqrt{5} \in \mathbb{Q}(\alpha, \beta)$. (Hint. In part (a) choose $E$ to contain $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and deduce that $\beta \in K(\alpha)$. Now write $\beta$ as $p(\alpha)/q(\alpha)$ where $p(x), q(x) \in K[x]$.) (5 points)

8. Let $A$ and $B$ be $m \times n$ matrices over a field $F$.

a. Prove that:
$$\text{rank} \ (A+B) \leq \text{rank}(A) + \text{rank}(B) .$$ (4 points)

b. If equality occurs in part (a), show that every row vector $v$ in $F^m$ can be written as a sum $v = u + w$ of row vectors with $uA = 0$ and $wB = 0$. (6 points)
Qualifying Exam

ALGEBRA

January 14, 1985

Instructions: Do four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

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1. Let $G$ be a transitive permutation group of prime degree $p$. (Note that this implies that every nonidentity normal subgroup of $G$ is also transitive.) Assume that $G$ is solvable.

   a) Show that $G = PH$ where $P \triangleleft G$ has order $p$, $H$ is abelian and $P \cap H = 1$. (5 points)

   b) If $G$ is doubly transitive, find the order of $H$. (3 points)

   c) If $G$ is triply transitive, show that $p = 3$. (2 points)

2. Let $R$ be a commutative ring with $1$ and suppose $Q$ is a primary ideal of $R$. If the radical of $Q$ is a principal ideal, show that $Q$ is also a principal ideal.
3. Let \( f(x) \) be a nonconstant polynomial over the rationals \( \mathbb{Q} \) and let \( G \) be the Galois group of \( f \) over \( \mathbb{Q} \). For each integer \( k \geq 0 \) set \( f_k(x) = f(x^{2^k}) \) and let \( G_k \) be the Galois group of \( f_k(x) \) over \( \mathbb{Q} \).

a) Show that there is a homomorphism from \( G_1 \) onto 
\( G = G_0 \) with kernel which is an elementary abelian 2-group. (6 points)

b) Show that for some \( k \), there is a homomorphism of \( G_k \) onto \( G \) with kernel a nontrivial elementary abelian 2-group. (4 points)

4. Let \( F \) be the field of 3 elements and let \( V \) be a 2-dimensional vector space over \( F \). Let \( G \) be the group of all invertible linear transformations of \( V \) and let \( Z \) be its center. Show that \( G/Z \cong \text{Sym}(4) \), the symmetric group on 4 letters.
5. Let $G$ be a finite group and let $X$ and $L$ be normal subgroups of $G$. If $P$ is a Sylow $p$-subgroup of $L$, show that the normalizer of $PX$ satisfies

$$\mathcal{N}_G(PX) = \mathcal{N}_G(P) \cdot (X \cap L).$$

6. Let $R$ be a right Artinian ring with $1$ and let $J$ be its Jacobson radical. Let $e \neq 0$ be an idempotent of $R$ and assume that $eR$ is a simple right $R$-module.

a) Show that $Re/Je$ is a simple left $R$-module. (7 points)

b) Give an example to show that $Re$ need not be a simple left $R$-module. (Hint. Consider a ring of $2 \times 2$ upper triangular matrices.) (3 points)
7. Let $F$ be a field of prime characteristic $p$ and let $K$ be an algebraic extension of $F$. Let $S$ be the subfield of $K$ consisting of all elements $\alpha \in K$ such that $\alpha^{p^n} \in F$ for some integer $n \geq 0$ (depending on $\alpha$).

a) Suppose both $\beta, \gamma \in K$ are roots of the irreducible polynomial $f(x)$ over $F$. Show that they are both roots of the same irreducible polynomial $g(x)$ over $S$. (5 points)

b) If $K$ is normal over $F$, show that $K$ is separable over $S$. (5 points)

8. Let $V$ be an $n$-dimensional vector space over a finite field of $q$ elements and let $1 \leq k \leq n$.

a) Find the number of subspaces of dimension exactly $k$. (5 points)

b) Find the number of linear transformations $T : V + V$ of rank exactly $k$. (5 points)

Prove your answers in both parts.
Qualifying Exam

ALGEBRA

August 26, 1985

Instructions: Do four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

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1. Let $G$ be a finite group, $p$ a prime and $P$ a $p$-subgroup of $G$. Let $n$ be the number of Sylow $p$-subgroups of $G$ which contain $P$.

a) Prove that $n \equiv 1 \mod p$. (4 points)

b) Show that $n \geq |\text{Syl}_p(\mathcal{N}_G(P))|$, where the latter is the number of Sylow $p$-subgroups of the normalizer of $P$. (3 points)

c) Show that $G = S_4$, the symmetric group on 4 letters, provides an example of strict inequality in (b). (3 points)

2. Let $R$ be a commutative Noetherian ring with 1.

a) If $\theta : R \to R$ is a surjective (i.e. onto) ring homomorphism, prove that $\theta$ is one-to-one. (4 points)

b) Show that $R$ and the polynomial ring $R[x]$ are not isomorphic rings. (4 points)

c) Show by example that (b) can fail if $R$ is not Noetherian. (2 points)
3. Let \( Q \subseteq E \subseteq C \) be fields where \( Q \) is the rational numbers and \( C \) is the complex numbers. Assume that \( E \) is Galois over \( Q \) with abelian Galois group. Let \( p \) be an odd prime and let \( \alpha \in E \) be such that \( \alpha^p \in Q \).

a) If \( \alpha \notin Q \) show that \( Q[\alpha] = Q[e^{2\pi i/p}] \). 

b) Prove that \( \alpha = q\zeta \) where \( q \in Q \) and \( \zeta^p = 1 \).

(5 points) (5 points)

4. Let \( A, B \) and \( C \) denote linear transformations on a finite dimensional complex vector space \( V \).

a) Show that there exists a nonzero linear transformation \( X \) on \( V \) satisfying \( AX = XB \) if and only if \( A \) and \( B \) have a common eigenvalue. (Hint. Viewing the actions on the right, show that \( A \) acts on \( V/\ker(X) \) and \( B \) acts on \( \text{Im}(X) \).)

(7 points)

b) Show that the equation \( AX - XB = C \) has a unique solution if and only if \( A \) and \( B \) have no common eigenvalue. Here \( X \) is again a linear transformation.

(3 points)
5. Let \( G = N_1 \times N_2 \) be the direct product of the normal subgroups \( N_1 \) and \( N_2 \) of \( G \). A subgroup \( D \) of \( G \) is said to be diagonal if \( D \cap N_i = 1 \) and \( DN_i = G \) for \( i = 1,2 \).

a) Show that \( G \) has a diagonal subgroup if and only if \( N_1 \cong N_2 \). (4 points)

b) If \( N_1 \cong N_2 \) show that the number of diagonal subgroups of \( G \) is equal to \( |\text{Aut}(N_1)| \). (4 points)

c) If \( G \) has a normal diagonal subgroup, prove that \( G \) is abelian. (2 points)
6. Let \( \mathbb{Z} \) be the ring of integers, \( \mathbb{Q} \) the rational numbers and let \( R \) be the subring of the ring of \( 2 \times 2 \) matrices over \( \mathbb{Q} \) described by

\[
R = \begin{pmatrix}
\mathbb{Z} & \mathbb{Q} \\
0 & \mathbb{Q}
\end{pmatrix}.
\]

That is, \( R \) is the set of all matrices \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) with \( a \in \mathbb{Z} \) and \( b, c \in \mathbb{Q} \). Set

\[
N = \begin{pmatrix}
0 & \mathbb{Q} \\
0 & 0
\end{pmatrix} \subseteq R.
\]

a) Show that \( N \) is a 2-sided ideal of \( R \) which is minimal as a right ideal but not as a left ideal. (3 points)

b) Prove that \( R \) is right Noetherian but not left Noetherian. (5 points)

c) Show that every nonzero 2-sided ideal of \( R \) contains \( N \). (2 points)
7. Let \( F \subseteq E \) be an extension of fields of characteristic zero. Prove that the following are equivalent.

a) Every nonconstant polynomial \( f(x) \in F[x] \) has a root in \( E \).

b) Every finite degree field extension of \( F \) is \( F \)-isomorphic to a subfield of \( E \).

c) \( E \) contains an algebraic closure of \( F \).

8. Let \( D \) be a division algebra of finite dimension \( n \) over its center \( K \) and let \( T \) be the ring of all \( K \)-linear transformations on \( D \) (i.e. \( T = \text{End}_K(D) \)). Let \( \{d_1, d_2, \ldots, d_n\} \) be a \( K \)-basis for \( D \) and for each \( i \) let \( \ell_i : D \rightarrow D \) and \( r_i : D \rightarrow D \) denote the elements of \( T \) given by left and right multiplication by \( d_i \) respectively.

a) Show that \( S = \sum_{i=1}^{n} \sum_{j=1}^{n} Kr_i \ell_j \) is a subring of \( T \). (2 points)

b) Note that the additive group of \( D \) is a faithful module for the ring \( S \). Show that it is a simple \( S \)-module and that \( \text{End}_S(D) \) (i.e. the commuting ring) is equal to scalar multiplications by \( K \). (4 points)

c) Deduce that \( S = T \) and that the \( n^2 \) elements \( r_i \ell_j \) for \( i, j = 1, 2, \ldots, n \) form a basis for \( T \). (4 points)
Qualifying Exam

ALGEBRA

January 13, 1986

Instructions: Do four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let \( G \) be a finite group. A subgroup \( H \) of \( G \) is said to be subnormal in \( G \) if there exists a chain of subgroups

\[
H = H_0 < H_1 < \ldots < H_n = G
\]

where each is normal in the next.

(i) If \( H \) is a subnormal subgroup of \( G \) and 
\( (|H|,|G:H|) = 1 \), show that \( H \) is normal in \( G \). (5 points)

(ii) Let \( G = PS \) where \( P \) is a normal abelian Sylow \( p \)-subgroup of \( G \) and \( S \) is a subnormal subgroup of \( G \). Prove that \( S \) is normal in \( G \). (5 points)

2. Let \( K \) be an extension field of the rational numbers \( \mathbb{Q} \).
Suppose that there exist elements \( \alpha, \beta \in K \) with
\[
K = \mathbb{Q}(\alpha, \beta) \quad \text{and} \quad \beta^2 = \alpha^3.
\]

(i) If \( \beta \in \mathbb{Q}(\alpha) \) prove that \( (K: \mathbb{Q}) < \infty \). (5 points)

(ii) If \( (K: \mathbb{Q}) = \infty \) prove that \( K \) has two automorphisms which map \( \alpha \) to \( \alpha + 1 \). (5 points)
3. If $R$ is a ring with 1, let $J(R)$ denote its Jacobson radical.

(i) Let $R$ and $S$ be rings with 1. Prove that

$$J(R \oplus S) = J(R) \oplus J(S).$$

(5 points)

(ii) Given $R$, let $T$ be the ring of all $2 \times 2$ matrices (over $R$) of the form

$$\begin{pmatrix}
a & c \\
0 & b
\end{pmatrix}$$

where $a, b, c \in R$. Show that

$$\begin{pmatrix}
a & c \\
0 & b
\end{pmatrix} \in J(T)$$

if and only if $a, b \in J(R)$.

(5 points)
4. Let $A$ be a finite-dimensional algebra over a field $F$ and let $T$ be an automorphism of the algebra $A$. For each element $\alpha \in F$ define the space of generalized eigenvectors to be

$$A_\alpha = \{ x \in A \mid (T - \alpha I)^n x = 0 \text{ for some integer } n \geq 0 \} .$$

Note that $\alpha$ is an eigenvalue of $T$ if $A_\alpha \neq 0$.

(i) Show that $A_\alpha A_\beta \subseteq A_{\alpha \beta}$ for all $\alpha, \beta \in F$. (6 points)

(ii) Show that when $A$ is a division algebra, those eigenvalues of $T$ which lie in $F$ are roots of unity. (4 points)

5. Let $G$ be a finite simple group and let $P$ be a Sylow $p$-subgroup with $|P| = p$. Let $G$ act on $\Omega = \text{Syl}_p(G)$ by conjugation and let $\pi : G \to \text{Sym}(\Omega)$ denote the corresponding homomorphism.

(i) If $x \in G$ has order $p$, show that $\pi(x)$ has precisely one fixed point. (2 points)

(ii) If $y \in G$ has order $pq$, where $q$ is a prime, show that $\pi(y)$ contains a $pq$-cycle. (3 points)

(iii) Show that $G$ cannot have order $2^3 \cdot 5 \cdot 7 \cdot 11$.

(Hint. Remember $G$ is simple. Try $p = 7$.) (5 points)
6. Let $A$ be the field of all algebraic numbers (i.e. all complex numbers which are algebraic over the rationals $\mathbb{Q}$) and let $G$ be the group of all field automorphisms of $A$.

(i) If $\alpha \in A$ with $\alpha \notin \mathbb{Q}$ show that there exists $g \in G$ with $g(\alpha) \neq \alpha$. (4 points)

(ii) Let $B$ be the subfield of $A$ generated by all roots of unity. If $E$ is any subfield of $B$ show that $g(E) = E$ for all $g \in G$. (6 points)

7. Let $R = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be a polynomial ring over the integers $\mathbb{Z}$ and let $S$ be a subring of $R$ containing $1$. Assume that $R$ is an integral extension of $S$.

(i) Show that there exists a finitely generated subring $T$ of $S$ (with $1 \in T$) such that $R$ is integral over $T$. (4 points)

(ii) Show that $T$ is a Noetherian ring and that $R$ is finitely generated as a $T$-module. (3 points)

(iii) Prove that $S$ is a Noetherian ring. (3 points)
8. Let $G$ be a finite (multiplicative) abelian group and let $\hat{G}$ denote the set of all homomorphisms $\lambda : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of the complex numbers. Note that $\hat{G}$ is a group under pointwise multiplication.

(i) Show that $|\hat{G}| = |G|$. (5 points)

(ii) If $H$ is a subgroup of $G$ prove that the restriction map $\rho : \hat{G} \to \hat{H}$ is onto. (5 points)
Qualifying Exam

ALGEBRA

August 25, 1986

Instructions: Do FOUR questions.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

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1. Let $G$ be a finite group with normal subgroup $N$ and assume that $G/N$ is nilpotent.
   
   i. Show that $G$ has a nilpotent subgroup $K$ such that $G = NK$. (5 pts)
   
   ii. If $N$ is abelian and $G$ has trivial center, prove that any nilpotent subgroup $K$ of $G$ with $G = NK$ satisfies $K \cap N = 1$ and $K = N_G(K)$. (5 pts)

2. i. Let $R$ be a commutative ring with 1 having exactly one maximal ideal $M$. Let $a,b \in R$ and suppose that the principal ideals $(a)$ and $(b)$ are equal. Show that $a = bu$ for some unit $u \in R$. (5 pts)

   ii. Let $Z$ be the ring of integers and let $p > 3$ be a prime. Let $R$ be the subring of $Z \oplus (Z/p^2Z)$ consisting of those elements $(u,v)$ such that $u \equiv v \mod p$. Consider the elements $a = (0,2p)$ and $b = (0,p)$ in $R$. Show that $(a) = (b)$ but that there is no unit $u$ of $R$ with $a = bu$. (5 pts)

3. Let $K$ be a field and let $f(x) \in K[x]$ be a separable polynomial. Let $E$ be a splitting field for $f(x)$ over $K$ and let $F$ be the subfield of $E$ generated over $K$ by all elements $\alpha - \beta$ where $\alpha$ and $\beta$ are roots of $f(x)$ in $E$.

   i. Prove that the degree $(E:F)$ equals 1 if char $K = 0$ and is a power of $p$ if char $K = p > 0$. (Hint: Determine the order of a field automorphism of $E$ over $F$.) (8 pts)

   ii. Give an example which shows that $E$ need not equal $F$. (2 pts)
4. Let $V$ be a complex vector space with basis $B = \{v_1, v_2, \ldots, v_n\}$ where $n \geq 5$. Let $G$ be the multiplicative group of those linear transformations of $V$ whose matrix with respect to the basis $B$ has precisely one nonzero entry in each row and column, and where each such nonzero entry is either $+1$ or $-1$.

i. Find the order of $G$ and list the factors in some composition series. (5 pts)

ii. Let $W$ be a nonzero subspace of $V$, which is mapped into itself by each element of $G$. Prove that $W = V$. (5 pts)

5. Let $G$ be a finite simple group.

i. If $G$ has a proper subgroup of index $\leq 9$, show that $G$ has no elements of order 21. (3 pts)

ii. If $|G| = 504$, find the number of Sylow 7-subgroups of $G$. Prove your answer. (4 pts)

iii. Again assume that $|G| = 504$. Show that $G$ has no elements of order 21. (3 pts)

6. Let $R$ be a commutative ring with 1 and let $S$ be a subring with the same 1.

i. Assume that $R$ is the direct sum $S \oplus T$ where $T$ is an $S$-submodule of $R$. If $N$ is a maximal ideal of $S$, show that $N = S \cap M$ for some maximal ideal $M$ of $R$. (5 pts)

ii. Show by example that (i) can fail if we do not assume the existence of $T$. (1 pt)

iii. Now suppose $R = K[x_1, x_2, \ldots, x_n]$ is a polynomial ring over the field $K$ in $n$ variables and that $S$ is a subring containing $K$. If $M$ is a maximal ideal of $R$, prove that $S \cap M$ is maximal in $S$. (4 pts)
7. We say that a field extension $T \supset F$ is purely transcendental if the only elements of $T$ algebraic over $F$ are in $F$. Let $E \supset T \supset F$ be fields with $T$ purely transcendental over $F$.

   i. Let $\alpha \in E$ be algebraic over $F$ with minimal polynomial $f(x) \in F[x]$. Show that $f(x)$ is irreducible in $T[x]$. (4 pts)

   ii. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$ be separable algebraic elements over $F$ which are linearly independent over $F$. Show that they are linearly independent over $T$. (6 pts)

8. Let $K \subseteq E$ be fields. Let $\delta: E \to E$ be a $K$-linear map satisfying

   $$\delta(ab) = a \delta(b) + \delta(a) b$$

   for all $a, b \in E$. (Such a map is called a $K$-derivation.)

   i. Show that $\delta(k) = 0$ for all $k \in K$. (2 pts)

   ii. Let $f(x) \in K[x]$ and let $f'(x)$ be its formal derivative. Show that $\delta(f(a)) = f'(a) \delta(a)$ for all $a \in E$. (3 pts)

   iii. If $E/K$ is algebraic and separable, prove that $\delta = 0$. (5 pts)
Qualifying Exam

ALGEBRA

January 12, 1987

Instructions: Do FOUR questions.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

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1. Let $G$ be a finite group and let $p$ be a prime. Assume that if \( q^a \) is a prime power dividing $|G|$ then $p$ does not divide $q^a - 1$.

(a) If $G$ is solvable, prove that it has a normal Sylow $p$-subgroup. (7 points)

(b) Show by example that a nonsolvable group $G$ need not have a normal Sylow $p$-subgroup. (3 points)

2. Let $R$ be a commutative ring with 1 and let $V$ be a right $R$-module.

(a) Let $A$ and $B$ be submodules of $V$ and let $P$ and $Q$ be prime ideals of $R$. Suppose that $A \cong R/P$ and $B \cong R/Q$ as $R$-modules. If $P = Q$, show that $A \cap B = 0$. (Hint. If $0 = a \in A$, compute $\text{ann}_R(a) = \{ r \in R \mid ar = 0 \}$. ) (5 points)

(b) Now assume that $R$ is Noetherian and $V = 0$. Show that there exists a submodule $A$ of $V$ and a prime ideal $P$ of $R$ with $A \cong R/P$. (Hint. Choose a maximal element in the set of all $\text{ann}_R(a)$ with $0 = a \in V$. ) (5 points)

3. Let $E$ and $F$ be subfields of the complex numbers each of which has finite degree over the rationals $Q$. Assume that $E \cap F = Q$ and let $\alpha \in E$.

(a) If $E$ is a normal extension of $Q$ prove that the degrees satisfy $|F[\alpha] : F| = |Q[\alpha] : Q|$. (Hint. Show that the coefficients of the minimal polynomial $m_F(\alpha)$ all lie in $E$. ) (5 points)

(b) Give an example to show that this equality of degrees can fail if $E$ is not normal over $Q$. (5 points)
4. Let $V$ be a vector space over the field $K$ with $\dim_K V$ not necessarily finite. Let $R$ be the ring of all linear operators on $V$ acting on the right.

(a) If $\alpha \in R$ show that there exists $\beta \in R$ with $\alpha \beta \alpha = \alpha$.
(Hint. If $U$ is a complement in $V$ for the kernel of $\alpha$, show that the map $U \rightarrow V\alpha$ induced by $\alpha$ is invertible.) (7 points)

(b) If $I = \alpha R$ is a principal right ideal of $R$ show that $I = eR$ for some idempotent $e \in R$. (3 points)

5. Let $p$ be a prime.

(a) Let $A$ be a finite abelian $p$-group having a cyclic subgroup $Z$ of index $p$. Let $X$ be a $p$-subgroup of $\text{Aut } A$ and assume that every element of $X$ maps $Z$ to itself. Prove that $X$ is abelian.
(Hint. If $A$ is not cyclic show that $A = ZW$ where $W$ is a characteristic subgroup of order $p^2$.) (5 points)

(b) Let $G$ be a finite $p$-group with commutator subgroup $G'$. Assume that the center $Z$ of $G'$ is cyclic. Prove that $G' = Z$. (Hint. If $G' = Z$, apply part (a) with $A$ a suitable subgroup of $G'$.) (5 points)
6. Let \( R \) be the set of all \( 6 \times 6 \) matrices over the field \( F \) of the form

\[
\begin{bmatrix}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & * \\
\end{bmatrix}
\]

Thus \( R \) is a subring of the matrix ring \( M_6(F) \).

(a) Find the Jacobson radical \( J(R) \) of \( R \) and determine the structure of \( R/J(R) \). Prove your answer. (7 points)

(b) How many nonisomorphic irreducible modules does \( R \) have and what are their dimensions over \( F \)? Explain. (3 points)

7. Let \( \alpha \) be the real number \( \alpha = \sqrt{2} + \sqrt{3} \) and let \( Q \) be the field of rational numbers. Find all the fields \( F \) with \( Q \subset F \subset Q(\alpha) \). How many are there? (10 points)

8. Let \( A = 0 \) be a finitely generated abelian group written additively and let \( G = \text{Aut} \, A \) be its full group of automorphisms. Show that \( A \) is noncyclic with no nonzero elements of finite order if and only if each nonzero element of \( A \) lies in an infinite orbit under the action of \( G \). (5 points for each direction)
1. Let $G$ be a finite group and let $P \in \text{Syl}_p(G)$ be a Sylow $p$-subgroup of $G$. Let $Q$ be a characteristic subgroup of $P$.
   i. Show that the number of conjugates of $Q$ in $G$ is congruent to 1 mod $p$. (5 points)
   ii. Show that the number of these conjugates which happen to be contained in $P$ is also congruent to 1 mod $p$. (5 points)

2. Let $R$ be a commutative Noetherian ring with 1.
   i. Let $A$ and $B$ be ideals of $R$ such that $R/A$ and $R/B$ are finite. Prove that $R/(AB)$ is finite. (5 points)
   ii. Assume that $R/P$ is finite for every prime ideal $P$ of $R$. Show that $R$ is finite. (5 points)

3. Let $F \subset E$ be a finite degree Galois extension with Galois group $G$.
   i. Suppose $E$ is the splitting field of the irreducible polynomial $f(x) \in F[x]$ and let $e \in E$ be a root of $f(x)$. If $H$ is the subgroup of $G$ given by $H = \{ \sigma \in G \mid \sigma(e) = e \}$, show that $H$ contains no nonidentity normal subgroup of $G$. (5 points)
   ii. Now, let $H$ be any subgroup of $G$ which contains no nonidentity normal subgroup of $G$. Show that $E$ is the splitting field of an irreducible polynomial $f(x) \in F[x]$ with $\deg f(x)$ equal to the index $|G:H|$. (5 points)
4. Let $K$ be a field and fix an integer $n \geq 2$. Let $M$ be the ring of $n \times n$ matrices over $K$ and let $S \subseteq M$ be the subset consisting of all matrices of trace 0. Assume that either $n \geq 3$ or $n = 2$ and the characteristic of $K$ is not 2. Prove that every element of $S$ is a $K$-linear combination of matrices of the form $\alpha \beta - \beta \alpha$ with $\alpha, \beta \in S$.

5. Let $G$ be a finite group and let $N$ be a normal subgroup of $G$.
   i. If $N$ is nilpotent and $M$ is a maximal subgroup of $G$, show that $M \cap N$ is normal in $G$. (5 points)
   ii. Conversely suppose that $M \cap N$ is normal in $G$ for every maximal subgroup $M$ of $G$. Prove that $N$ is nilpotent. (5 points)

6. Let $R$ be a ring with 1 and let $J(R)$ be its Jacobson radical. Suppose $S$ is a subring of $R$ such that $R = S + J(R)$.
   i. Let $1_S$ be the identity element of $S$. Show that $1_S = 1$. (3 points)
   ii. Prove that $J(S) \subseteq J(R) \cap S$. (4 points)
   iii. If $R$ is right Artinian deduce that $J(S) = J(R) \cap S$. (3 points)
7. Let $F$ be a field of prime characteristic $p$ and let $f(x) \in F[x]$ be a polynomial of degree $n$. (Recall that a polynomial in $F[x]$ is *separable* if no irreducible factor has repeated roots.)

i. If $p \nmid n$ show that there are at most finitely many elements $c \in F$ such that the polynomial $f(x) + c$ is not separable. (5 points)

ii. Give an example (with proof) where $p \mid n$ and $f(x) + c$ is not separable for infinitely many values $c \in F$. (5 points)

8. Let $G$ be a collection of pairwise commuting $n \times n$ matrices over the complex numbers $\mathbb{C}$. Prove that there exists a vector which is a common eigenvector for all matrices in $G$. 
ALGEBRA QUALIFYING EXAM
January, 1988

Instructions: Do precisely 4 of the following 8 problems.

1. Let $G$ be a group (not necessarily finite). A subgroup $N$ of $G$ is said to be minimal normal in $G$ if $1 ≠ N < G$ and there is no nonidentity normal subgroup of $G$ properly contained in $N$.

Suppose $N$ is a minimal normal subgroup of $G$ and $M$ is a minimal normal subgroup of $N$.

i. Let $g ∈ G$. If $M^g ≠ M$, prove that $M^g$ centralizes $M$. (4 points)

ii. Show that $M$ is a simple group. (6 points)

2. Let $R$ be a ring (with 1) and $V$ a right $R$-module. $V$ is said to be Noetherian (respectively, Artinian) if its set of submodules satisfies the ascending (respectively, descending) chain condition. A submodule $W$ of $V$ is complemented if there exists a submodule $U$ of $V$ with $V$ the internal direct sum $V = W + U$.

Suppose now that $V$ is a Noetherian $R$-module and that every maximal submodule of $V$ is complemented.

i. Prove that $V$ is a direct sum of a finite number of simple submodules. (7 points)

ii. Prove that $V$ is Artinian. (3 points)

3. Let $f(x) ∈ Q[x]$ be an irreducible polynomial over the field of rationals $Q$. Suppose there exist two distinct complex numbers $α$ and $β$ with $f(α) = 0 = f(β)$ and $α/β ∈ Q$.

i. Show that $β = -α$. (5 points)

ii. If $γ$ is any complex root of $f(x)$, prove that $-γ$ is also a root. (5 points)

4. Let $V$ be a vector space over a field $F$ and let $( , )$ be a bilinear form from $V × V$ to $F$. Assume that $(v, w) = 0$ implies $(w, v) = 0$ for $v, w ∈ V$.

i. Suppose $z ∈ V$ with $(z, z) ≠ 0$ and let $U = \{ u ∈ V \mid (z, u) = 0 \}$. Show that $(x, y) = (y, x)$ for all $x, y ∈ U$. (5 points)

ii. Prove that either $(v, w) = (w, v)$ for all $v, w ∈ V$ or $(v, v) = 0$ for all $v ∈ V$. (5 points)
5. Let $G$ be a finite nonsolvable group and suppose that all proper subgroups of $G$ are solvable. If $N$ is a proper normal subgroup of $G$, show that $N$ is nilpotent. (10 points)

6. Let $R$ be a Noetherian commutative ring (with 1) and let $M$ be a nonzero right $R$-module.
   i. If $m \in M$, show that $mR \cong R/I$ for some ideal $I \subseteq R$ and that $m$ can be chosen so that $I$ is prime. (7 points)
   ii. If $M$ is finitely generated, show that there exist finitely many submodules $M_i$ and prime ideals $P_i \subseteq R$ such that

   $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$

   with $M_i/M_{i-1} \cong R/P_i$ for $1 \leq i \leq k$. (3 points)

7. Let $F$ be a field extension of the rationals $\mathbb{Q}$ and let $S$ be the set of all prime numbers $p$ which are squares in $F$ (that is, $\sqrt{p} \in F$).
   i. If the degree $[F : \mathbb{Q}] = 2$, prove that $|S| \leq 1$. (6 points)
   ii. If $[F : \mathbb{Q}] < \infty$, prove that $|S| < \infty$. (4 points)

8. Let $R$ be a (commutative) unique factorization domain with field of fractions $F$. Let $f(x)$ be a polynomial in $F[x]$ with the property that $f(r) \in R$ for all $r \in R$.
   i. If all nonzero homomorphic images of $R$ are infinite rings, prove that $f(x) \in R[x]$. (8 points)
   ii. Give an example to show that, in general, $f(x)$ need not belong to $R[x]$. (2 points)
ALGEBRA QUALIFYING EXAM
August, 1988

Instructions: Do precisely 4 of the following 8 problems.

1. Let $G$ be a finite group and suppose $G = AB$ where $A$ and $B$ are normal subgroups of $G$ with $A \cap B = 1$. Show that $|A|$ and $|B|$ are coprime (that is, relatively prime) if and only if for every subgroup $H \subseteq G$ we have $H = (H \cap A)(H \cap B)$.

2. a. Let $R$ be a ring with 1 and let $M$ be a right $R$-module. If the endomorphism ring $\text{End}_R(M)$ is infinite, prove that the module $M \oplus M$ has infinitely many submodules. (5 points)
   
   b. Let $R$ be a simple ring with 1 having only finitely many right ideals. Show that either $R$ is finite or it is a division ring. (5 points)

3. a. Let $p$ be a prime. Show that there exists a Galois extension $K$ of the rationals $\mathbb{Q}$ with degree $(K : \mathbb{Q}) = p$. (5 points)
   
   b. Let $p_n$ denote the $n$th prime. Prove that there exist fields $F_i$ satisfying

   $$ \mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots $$

   and such that $F_i$ is Galois over $\mathbb{Q}$ with $(F_i : F_{i-1}) = p_i$. (5 points)

4. Let $K$ be a field.
   
   a. Let $S$ be a $K$-vector subspace of the space of $n \times n$ matrices $M_n(K)$. Assume that every nonzero matrix in $S$ is nonsingular. Prove that $\dim_K S \leq n$. (5 points)
   
   b. Suppose there exists a field extension $E \supseteq K$ with degree $(E : K) = n$. Show that a subspace $S$ of $M_n(K)$ as above exists with $\dim_K S = n$. (5 points)

5. a. Let $G$ be a finite simple group. If $H$ is a subgroup of $G$ with index 12 and if $G$ contains an element of order 15, prove that $H$ has an element of order 15. (5 points)
   
   b. Show that any simple group of order 660 has exactly 66 Sylow-5 subgroups. (Hint. First show that the Sylow 11 normalizer has index 12.) (5 points)

6. Let $R$ be a commutative ring with 1.
   
   a. Suppose $P_1$ and $P_2$ are prime ideals of $R$ and that $P_1 \cap P_2$ is a primary ideal. Show that either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. (3 points)
   
   b. Let $J$ be the Jacobson radical of $R$ and let $0 \neq a \in R$. If the ideal $aJ$ is primary, prove that its radical $\sqrt{aJ}$ contains $J$. (4 points)
   
   c. If all ideals of $R$ are primary, prove that $R$ has at most one nonzero prime ideal. (3 points)

7. Let $K$ be a field and let $E$ be a splitting field over $K$ for some polynomial $f(x) \in K[x]$ of degree $n$. Suppose the roots of $f$ in $E$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$ where the $\alpha_i$ are distinct. If $E$ is not generated over $K$ by any $n - 2$ of these roots, determine the Galois group $\text{Gal}(E/K)$ (up to isomorphism) and prove that $f(x)$ is irreducible in $K[x]$.

8. Let $V$ be a vector space over the field $F$ with $\dim_F V = n < \infty$ and let $T : V \to V$ be a linear transformation. Let $k$ be an integer with $1 \leq k < n$ and assume that $T(W) \subseteq W$ for all subspaces $W \subseteq V$ with $\dim_F W = k$. Prove that $T$ is multiplication by some scalar $\alpha \in F$. 
ALGEBRA QUALIFYING EXAM
January, 1989

Instructions: Do precisely 4 of the following 8 problems.

1. Let $H$ be a finite group with trivial center. Show that the following are equivalent. (5 points for each implication)
   i. Every automorphism of $H$ is inner.
   ii. Whenever $H$ is a normal subgroup of a group $G$, then

   $$G = H \times C_G(H)$$

   where $C_G(H)$ is the centralizer of $H$ in $G$.

2. Let $R$ be a right Artinian ring with 1 and assume that every unit (that is, element of $R$ with a two-sided inverse) is central. If $J(R)$ is the Jacobson radical of $R$, prove that $R/J(R)$ is commutative.

3. Let $F$ be the finite field $GF(7) = \mathbb{Z}/7\mathbb{Z}$. Prove that the polynomial $x^9 - 2$ is an irreducible factor of $x^{27} - 1$ in $F[x]$.

4. Let $V$ be a finite dimensional vector space over some field $F$.
   a. If $T : V \to V$ is an $F$-linear transformation such that $T^2 = T$, show that $V$ is the direct sum $V = V_0 + V_1$ where

   $$T(v) = \begin{cases} 
   0 & \text{if } v \in V_0 \\
   v & \text{if } v \in V_1.
   \end{cases}$$

   (3 points)
   b. Now assume $F$ is a finite field $GF(q)$ and that $\dim_F V = 3$. Determine (in terms of $q$) the number of linear transformations $T$ with $T^2 = T$. (7 points)
5. Let $G$ be a nonabelian finite simple group of order divisible by some prime $p$. If $G$ has no more than $2p$ Sylow $p$-subgroups, determine (in terms of $p$) the number of elements of $G$ whose order is a power of $p$.

6. Let $M$ be a right $R$-module, where $R$ is a ring with 1. Suppose $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is the internal direct sum of the simple submodules $M_1, M_2, \ldots, M_n$. Show that the following are equivalent.
   i. The $M_i$ are pairwise nonisomorphic.
   ii. $M$ has precisely $n$ simple submodules.
   iii. $M$ has precisely $2^n$ submodules.

7. Let $K$ be a finite degree extension field of the rationals $\mathbb{Q}$. Show that the following two statements are equivalent. (5 points for each implication)
   i. $K$ is a splitting field for some polynomial $g(x) \in \mathbb{Q}[x]$.
   ii. For every irreducible polynomial $f(x) \in \mathbb{Q}[x]$, all irreducible factors of $f(x)$ in $K[x]$ have equal degrees.

8. Let $\mathbb{Q}^n$ denote the $n$-dimensional row vector space over the rationals $\mathbb{Q}$ and let $\mathbb{Z}^n$ denote the subset of those vectors with entries in the integers $\mathbb{Z}$. Notice that for each prime $p$ there is a natural map $\theta_p: \mathbb{Z}^n \to (\mathbb{Z}/p\mathbb{Z})^n$. Show that for each $\mathbb{Q}$-linearly independent subset $S$ of $\mathbb{Z}^n$ there are at most finitely many primes $p$ such that the image $\theta_p(S)$ is not $\mathbb{Z}/p\mathbb{Z}$-linearly independent.
ALGEBRA QUALIFYING EXAM
August, 1989

Instructions: Do precisely 4 of the following 8 problems.

1. a. Let $A$ be an abelian normal subgroup of the finite group $G$ and let $a \in A$ have prime order $p$. Suppose $g \in G$ has order $n$ and set

$$B = \langle a, a^g, a^{g^2}, \ldots, a^{g^{n-1}} \rangle.$$ 

Show that either $|B| < p^n$ or $g$ centralizes some nonidentity element of $B$. (5 points)

b. Let $G$ be a solvable group of order $2^r3^s$ with $r, s \geq 1$. Show that $G$ has a subgroup of order 6 or one of order 12. (5 points)

2. Let $K$ be a field and let $I$ be a maximal ideal of the polynomial ring $R = K[x_1, x_2, \ldots, x_n]$.

a. Show that $\dim_K R/I$ is finite. (4 points)

b. For every ideal $J$ of $R$, show that $\dim_K J/JI$ is also finite. (6 points)

3. Let $F \subseteq E$ be an extension of fields and assume that $E = F[\alpha]$ where $\alpha^n \in F$ for some integer $n \geq 1$. Assume that $F$ contains a primitive $n$-th root of unity. Show that $\alpha^m \in F$ where $m$ is the degree $[E : F]$.

4. Let $M_n(F)$ be the ring of $n \times n$ matrices over the field $F$ and let $E_{i,j}$ denote the matrix with 1 in the $(i,j)$-position and 0 elsewhere.

a. If $i \neq j$, show that $E_{i,j}$ and $E_{i,i} - E_{j,j}$ can each be written in the form $XY - YX$ for some $X,Y$ in $M_n(F)$. (5 points)

b. Let $A \in M_n(F)$ and suppose that $\text{tr}(XYA) = \text{tr}(YXA)$ for all $X,Y \in M_n(F)$. Here $\text{tr}$ denotes the usual matrix trace. Show that $A$ is a scalar matrix. (5 points)
5. Let $G$ be a finite group and let $p$ be a prime. Assume that for every maximal subgroup $M$ of $G$ we have $|G : M| \not\equiv 1 \pmod{p}$.
   a. Show that $G$ has a normal Sylow $p$-subgroup. (6 points)
   b. If $M$ is a maximal subgroup of $G$ with $|G : M| \equiv 0 \pmod{p}$, show that $|G : M|$ is a power of $p$. (4 points)

6. Let $R$ be a ring with 1 and let
   \[ 0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = R \]
   be a chain of right ideals. Assume that the factors $M_k = I_k/I_{k-1}$ are pairwise nonisomorphic simple $R$-modules.
   a. Prove that $R$ is a right Artinian ring with trivial Jacobson radical. (7 points)
   b. Show that $R$ is a finite direct sum of division rings. (3 points)

7. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 5 over the rationals $\mathbb{Q}$ and assume that $f(x)$ is not solvable by radicals over $\mathbb{Q}$. If $E$ is a splitting field for $f(x)$ over $\mathbb{Q}$, show that $E$ is also a splitting field for some irreducible polynomial $g(x) \in \mathbb{Q}[x]$ with $\deg g(x) = 60$ or 120.

8. Let $T: V \to V$ be a linear transformation on the finite dimensional $F$-vector space $V$ where $F$ is a finite field. Assume that no proper nonzero subspace of $V$ is mapped into itself by $T$ and let $\mathcal{X}$ be the set of all linear transformations $S: V \to V$ which commute with $T$. Show that $|\mathcal{X}| \leq |V|$ where $| |$ denotes cardinality.
ALGEBRA QUALIFYING EXAM
January, 1990

Instructions: Do precisely 4 of the following 8 problems.

1. Let $A$ be a subgroup of the symmetric group on $n$ points and assume that $A$ is abelian. Suppose $O_1, O_2, \ldots, O_k$ are the orbits of $A$.
   a. If $x \in A$ fixes an element of $O_i$, show that it fixes all elements of $O_i$. (3 points)
   b. Prove that $|A| \leq \prod_i |O_i|$. (5 points)
   c. If $|A| = 16$, what is the smallest that $n$ could be? Justify your answer. (2 points)

2. Let $\alpha$ and $\beta$ be complex numbers and suppose that $\alpha + \beta$ and $\alpha^2 + \beta^2$ are algebraic integers (that is, they are integral over $\mathbb{Z}$).
   a. Show that $\sqrt{2} \alpha$ and $\sqrt{2} \beta$ are also algebraic integers. (7 points)
   b. Show by example that $\alpha$ and $\beta$ need not be algebraic integers. (3 points)

3. Let $F$ be a field and let $g(x)$ be a polynomial in $F[x]$. If $1 + g(x)^2$ has an irreducible factor of odd degree, prove that $-1$ is a square in $F$.

4. Let $V$ be a countably infinite dimensional vector space over the field $K$ and let $R$ be the ring of all $K$-linear transformations from $V$ to $V$. (Multiplication in $R$ is defined so that $rs$ means first do $r$ and then $s$.)
   a. Show that $R$ has an element which has a left inverse and yet is not invertible. (3 points)
   b. Show that $R$ is not a simple ring. (Hint. Consider linear transformations of finite rank.) (3 points)
   c. Show that $R$ contains an infinite collection of nonzero right ideals whose sum is direct. (Hint. Note that the annihilators of subsets of $V$ are right ideals of $R$.) (4 points)
5. Let $G$ be a group of order 27. Prove that the automorphism group $A$ of $G$ has order divisible by 3. Is the order of $A$ necessarily divisible by 9? Justify your answer.

6. Let $R$ be a ring with 1 and let $V$ and $W$ be isomorphic right $R$-modules. Suppose $\varphi: V \to W$ is an $R$-module epimorphism.
   a. Assume $V$ has the property that for every simple $R$-module $S$ there are at most finitely many submodules $M$ of $V$ with $V/M \cong S$. Prove that the kernel of $\varphi$ is contained in every maximal submodule of $V$. (7 points)
   b. Show, by example, that the kernel of $\varphi$ need not be contained in all maximal submodules of $V$ if we drop the finiteness assumption of part (a). Find such an example for each ring $R$. (3 points)

7. Let $F$ be a field of characteristic $p > 0$ and let $f(x) \in F[x]$ be an irreducible polynomial. Define $g(x) = f(x^p)$ and let $E$ be a splitting field for $g(x)$ over $F$ with $G = \text{Gal}(E/F)$. (Note that $E \supseteq F$ need not be a Galois extension.)
   a. Show that $G$ transitively permutes the roots of $f(x)$ and also the roots of $g(x)$. (5 points)
   b. Show that $g(x) = h(x)^m$ where $h(x)$ is an irreducible polynomial in $F[x]$ and $m \geq 1$. (3 points)
   c. Show that $m$ above is either 1 or $p$. (2 points)

8. Let $F$ be a field of characteristic $\neq 2$ and let $V$ be a 2-dimensional $F$-vector space with basis \{ $x, y$ \}. Suppose $(,): V \times V \to F$ is a bilinear form with $(x, x) = (x, y) = (y, x) = 1$ and $(y, y) = 0$. Now let $G$ be the group of all invertible $F$-linear transformations $T: V \to V$ such that $(vT, wT) = (v, w)$ for all $v, w \in V$.
   a. Show that there are precisely two 1-dimensional subspaces of $V$ consisting of vectors $z$ with $(z, z) = 0$. (5 points)
   b. For each nonzero $z \in V$ with $(z, z) = 0$ show that there exists a unique $T \in G$ with $yT = z$. (5 points)
1. Let $H$ be a normal subgroup of the finite group $G$ and assume that $H$ is a nonabelian group of order $pq$ where $p > q$ are primes. If $G/H$ is a $p$-group, prove that $G$ is the direct product of $H$ with the centralizer of $H$ in $G$.

2. Let $F$ be a field, let $F[x]$ be the polynomial ring over $F$ and let $F(x)$ be the field of rational functions in the variable $x$. Suppose $f \in F[x]$ is a nonconstant polynomial and, as usual, write $F[f]$ for the subring of $F[x]$ consisting of all polynomial expressions in $f$ with coefficients in $F$. Furthermore, let $F(f) \subseteq F(x)$ be the field of fractions of $F[f]$.
   a. (5 points) Show that $F[x]$ is an integral extension of $F[f]$.
   b. (5 points) Prove that $F(f) \cap F[x] = F[f]$.

3. Let $F \subseteq \mathbb{C}$ be the splitting field for the polynomial $x^5 - 2$ over the rationals $\mathbb{Q}$.
   a. (2 points) If $i = \sqrt{-1} \in \mathbb{C}$, show that $F[i]$ contains a primitive 20-th root of 1.
   b. (5 points) Prove that $i \notin F$.
   c. (3 points) Prove that the field $\mathbb{Q}[i + \sqrt[5]{2}]$ is not isomorphic to $\mathbb{Q}[\sqrt[5]{2}]$.

4. Let $f \in \mathbb{Q}[x]$ be a polynomial which is not solvable by radicals. Suppose that $E$ is the splitting field of $f$ over $\mathbb{Q}$ and that the degree $|E : \mathbb{Q}| = 168$.
   a. (5 points) Show that the Galois group $G = \text{Gal}(E/\mathbb{Q})$ has a subgroup of order 21.
   b. (5 points) Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ of degree 8 whose splitting field is also $E$.

5. Let $S = \text{Sym}_n$ be the symmetric group on $n$ letters and let $C_1$ and $C_2$ be cyclic subgroups of $S$ of the same order $m$. Suppose $\theta : C_1 \rightarrow C_2$ is an isomorphism.
   a. (5 points) If $C_1 = C_2$, so that $\theta$ is an automorphism of $C_1$, prove that there exists $s \in S$ such that
   \begin{equation}
   \theta(c) = s^{-1}cs \quad \text{for all } c \in C_1.
   \end{equation}
   
   b. (5 points) Give an example to show that (*) need not hold if $C_1 \neq C_2$. Hint. Cycle structures are relevant for both parts.

6. Let $R$ be a right Artinian ring (with 1) and let $J$ denote its Jacobson radical. Show that $J$ is the set of all nilpotent elements of $R$ if and only if $R/J$ is a ring direct sum of division rings.
7. Suppose $E$ is a field and $g(x) \in E[x]$ is a monic irreducible polynomial.
   a. (4 points) Prove that each root of $g$ has multiplicity which is equal to 1 or a power of the characteristic of $E$.
   b. (3 points) Suppose $F$ is a subfield of $E$ and that some power of $g$ is contained in $F[x]$. If $n \geq 1$ is minimal with $g^n \in F[x]$, prove that $g^n$ is an irreducible polynomial of $F[x]$.
   c. (3 points) If $n$ is as in part (b), prove that $n$ is either equal to 1 or to a power of the characteristic of $E$.

8. Let $V$ be a vector space of finite dimension $n \geq 1$ over a field $F$ of characteristic $\neq 2$ and let $\langle x, y \rangle$ be a nondegenerate symmetric bilinear form on $V$.
   a. (4 points) Show that there exists $v \in V$ with $\langle v, v \rangle \neq 0$.
   b. (4 points) Prove that $V$ has a basis $\{ v_1, v_2, \ldots, v_n \}$ such that $\langle v_i, v_j \rangle = 0$ for $i \neq j$ and that $\langle v_i, v_i \rangle \neq 0$ for all $i$.
   c. (2 points) If $F$ is algebraically closed and $n \geq 2$, prove that there exists $0 \neq v \in V$ with $\langle v, v \rangle = 0$. 
Qualifying Exam

ALGEBRA

January 14, 1991

Instructions: Do all five questions.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
Algebra Qualifying Exam
January 1991

Do all 5 problems.

1. Let $G$ be a finite group having exactly $n$ Sylow $p$-subgroups for some prime $p$. Show that there exists a subgroup $H$ of the symmetric group $\text{Sym}_n$ such that $H$ also has exactly $n$ Sylow $p$-subgroups.

2. Let $R$ be a commutative integral domain and let $R[x_1, x_2, \ldots, x_n]$ be the polynomial ring over $R$ in the $n$ variables $x_1, x_2, \ldots, x_n$. If $a = (a_1, a_2, \ldots, a_n)$ is an $n$-tuple of elements of $R$, then there is an evaluation homomorphism $\varphi_a : R[x_1, x_2, \ldots, x_n] \to R$ given by $\varphi_a(f) = f(a_1, a_2, \ldots, a_n)$.
   a. (5 points) If $R$ is the field of complex numbers and if $I$ is a proper ideal of $R[x_1, x_2, \ldots, x_n]$, show that there exists an $n$-tuple $a = (a_1, a_2, \ldots, a_n)$ with $\varphi_a(I) \neq R$.
   b. (5 points) Now let $R$ be the ring of integers and let $I$ be the ideal of the polynomial ring $R[x]$ in one variable generated by 3 and $x^2 + 1$. Show that $I$ is a proper ideal of $R[x]$ but that $\varphi_a(I) = R$ for all 1-tuples $a = (a)$ with $a \in R$.

3. Let $F \subseteq E$ be an extension of fields of characteristic $\neq 2$ and assume that the degree $[E : F] = 4$.
   a. (3 points) Show that $E = F[\alpha]$ for some $\alpha$.
   b. (2 points) If $E = F[\alpha]$, where $\alpha$ is a root of a polynomial of the form $x^4 + ax^2 + b \in F[x]$, prove that there exists an intermediate field properly between $E$ and $F$.
   c. (5 points) Now let $E = F[\beta]$ with no assumption on $\beta$ and let $L \supseteq E$ be a splitting field for the minimal polynomial of $\beta$ over $F$. If $\text{Gal}(L/F)$ is isomorphic to the symmetric group $\text{Sym}_4$, show that there is no intermediate field properly between $E$ and $F$.

4. Let $F$ be an algebraically closed field of prime characteristic $p$ and let $V$ be an $F$-vector space of dimension precisely $p$. Suppose $A$ and $B$ are linear operators on $V$ such that $AB - BA = B$. If $B$ is nonsingular, prove that $V$ has a basis $\{v_1, v_2, \ldots, v_p\}$ of eigenvectors of $A$ such that $Bv_i = v_{i+1}$ for $1 \leq i \leq p - 1$ and $Bv_p = \lambda v_1$ for some $0 \neq \lambda \in F$.

5. Let $G \neq (1)$ be a possibly infinite group whose subgroups are linearly ordered by inclusion. In other words, if $H$ and $K$ are subgroups of $G$, then either $H \subseteq K$ or $K \subseteq H$.
   a. (5 points) Prove that $G$ is an abelian group and that the orders of the elements of $G$ are all powers of the same prime $p$.
   b. (5 points) If $G_n = \{ g \in G \mid g^{p^n} = 1 \}$, prove that $|G_n| \leq p^n$. 
Algebra Qualifying Exam
September 1991

Do all 5 problems.

1. Let $p$ and $q$ be distinct primes and suppose $G$ is a finite group having precisely $p+1$ Sylow $p$-subgroups and $q+1$ Sylow $q$-subgroups. Prove that there exist $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ such that the subgroup of $G$ generated by $P$ and $Q$ is $PQ = P \times Q$.

2. Let $R$ be a commutative ring with 1. If $a \in R$, we write $\text{ann}(a) = \{ r \in R \mid ar = 0 \}$ for the annihilator of $a$ in $R$. Thus $\text{ann}(a)$ is an ideal of $R$ and we let $S \subseteq R$ be the set of all elements $a \in R$ such that $\text{ann}(a)$ is a prime ideal of $R$.
   i. (4 points) If $R$ is Noetherian, show that $S$ is nonempty.
   ii. (4 points) If $a \in S$ and $r \in R$, show that either $ar = 0$ or $ar \in S$.
   iii. (2 points) If $a, b \in S$ and $\text{ann}(a) \neq \text{ann}(b)$, prove that $ab = 0$.

3. Let $F \subseteq E$ be an algebraic extension of fields. We say that an element $\alpha$ of $E$ is abelian if $F[\alpha]$ is a Galois extension of $F$ with abelian Galois group $\text{Gal}(F[\alpha]/F)$. Prove that the set of abelian elements of $E$ is a subfield of $E$ containing $F$.

4. Let $V$ be a finite-dimensional vector space over a field $K$ and let $(\ , \ )$ be a bilinear form on $V$. Suppose $T: V \rightarrow V$ is a linear transformation satisfying $(v, Tw) = (Tv, w)$ for all $v, w \in V$. Write $N = \ker(T) = \{ v \in V \mid Tv = 0 \}$.
   i. (5 points) Assume that the form restricted to $N$ is nondegenerate – that is, if $v \in N$ and $(v, N) = 0$, then $v = 0$. If $T$ is nilpotent, prove that $T = 0$.
   ii. (5 points) Find a 2-dimensional example with $T$ a nonzero nilpotent transformation and with the form $(\ , \ )$ nondegenerate on the whole vector space $V$.

5. Let $R$ be a ring with 1 and let $M$ be a right $R$-module. Suppose

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors $M_i/M_{i-1}$ are simple and pairwise nonisomorphic. If $X$ and $Y$ are isomorphic submodules of $M$, prove that $X = Y$. 
Algebra Qualifying Exam
January 1992

Do all 5 problems.

1. Let $G$ be a finite group and fix a prime number $p$. Define the function $f$ on the set of subgroups $H \subseteq G$ by

$$f(H) = |\{ P \in \text{Syl}_p(G) \mid P \supseteq H \}|.$$ 

In other words, $f(H)$ is the number of Sylow $p$-subgroups of $G$ which contain $H$. Prove that if $f(H) > 0$, then $f(H) \equiv 1 \text{ mod } p$.

2. Let $F$ be a field and let $R$ be the ring of all $3 \times 3$ matrices over $F$ with $(3,1)$ and $(3,2)$ entry equal to 0. Thus,

$$R = \begin{pmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{pmatrix}.$$ 

Prove that the Jacobson radical of $R$ is a minimal left ideal, but not a minimal right ideal.

3. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose $E$ is a splitting field for $f(x)$ over $F$ and assume that there exists an element $\alpha \in E$ such that both $\alpha$ and $\alpha + 1$ are roots of $f(x)$.

i. Show that the characteristic of $F$ is not zero. (5 points)

ii. Prove that there exists a field $L$ between $F$ and $E$ such that the degree $|E : L|$ is equal to the characteristic of $F$. (5 points)

4. Let $V$ be a finite dimensional complex vector space and suppose $(\cdot, \cdot) : V \times V \to \mathbb{C}$ is an inner product on $V$, that is, $(\cdot, \cdot)$ is a positive definite Hermitian form on $V$.

i. Suppose $T : V \to V$ is a linear transformation such that $(Tv, v) = 0$ for all $v \in V$. Prove that $V = 0$. (7 points)

ii. Does the result of part (i) hold if $V$ is assumed to be a real inner product space? Justify your answer. (3 points)

5. Let $\mathbb{Z}$ denote the ring of integers and let $\mathbb{Q}$ and $\mathbb{C}$ be the rational and complex fields, respectively. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, then we let $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n]$ denote the ring generated by these elements over $\mathbb{Z}$. In particular, note that $\mathbb{Z}[1/2]$ is the set of all rational numbers with denominator a power of 2. Now suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the integer polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{Z}[x]$ with $a_0 = 2$.

i. Prove that $2\alpha_i$ is an algebraic integer for all $i = 1, 2, \ldots, n$. (3 points)

ii. Show that $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n] \cap \mathbb{Q} \subseteq \mathbb{Z}[1/2]$. (4 points)

iii. If some $a_j$ with $j \geq 1$ is odd, prove that $1/2 \in \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n] \cap \mathbb{Q}$ and deduce that the latter intersection is equal to $\mathbb{Z}[1/2]$. What happens if all $a_j$ are even? (3 points)
Algebra Qualifying Exam
September 1992

Do all 5 problems.

1. Let $x$ and $y$ be elements of a finite $p$-group $P$ and let $z = [x, y]$ be the commutator $x^{-1}y^{-1}xy$ of $x$ and $y$. Suppose that $x$ lies in every normal subgroup of $P$ which contains $z$. Prove that $x = 1$.

2. Let $K[x]$ be a polynomial ring over the field $K$ and let $R$ be the subring of $K[x]$ consisting of all polynomials whose $x$-coefficient is equal to 0. Thus a typical element of $R$ has the form

$$a_0 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

with $a_i \in K$. Show that the principal ideal $(x^2) = x^2R$ is a primary ideal of $R$ which is not equal to a power of its radical.

3. Let $E$ be a finite degree field extension of the rationals $\mathbb{Q}$ and suppose that $f(x)$ is a monic irreducible polynomial in $E[x]$.

i. Show that there exists a unique monic irreducible polynomial $g(x) \in \mathbb{Q}[x]$ such that $f(x)$ divides $g(x)$ in $E[x]$. (4 points)

ii. Let $g(x)$ be as above. If $E$ is a splitting field over $\mathbb{Q}$ for some polynomial in $\mathbb{Q}[x]$, show that the degree of $f(x)$ divides the degree of $g(x)$. (4 points)

iii. Give an example to show that the degree of $f(x)$ need not divide the degree of $g(x)$ in general. (2 points)

4. Let $V$ be a finite dimensional vector space over a field $K$ and let $B: V \times V \to K$ be a bilinear form. Suppose that for all $x, y \in V$ we have $B(x, y) = 0$ if and only if $B(y, x) = 0$.

i. If $v, w \in V$ with $B(v, v) \neq 0$, prove that $B(v, w) = B(w, v)$. (5 points)

ii. Deduce that either $B$ is symmetric or $B(v, v) = 0$ for all $v \in V$. (5 points)

5. Let $G$ be the multiplicative group of all $2 \times 2$ matrices over the integers $\mathbb{Z}$ whose determinant is equal to 1. Notice that $G$ acts by right multiplication on the set $\Omega$ of all 1-dimensional subspaces of the 2-dimensional row space $\mathbb{Q}^2$ over the rational numbers $\mathbb{Q}$.

i. Find all elements of $G$ which act trivially, that is which fix every element of $\Omega$. (4 points)

ii. Prove that $G$ acts transitively. In other words, show that $\Omega$ is an orbit under the action of $G$. (6 points)
Algebra Qualifying Exam  
January 1993

Do all 5 problems.

1. Let $G = A \times B$ be the internal direct product of finite subgroups $A$ and $B$. Suppose $H$ is a subgroup of $G$ with $A \cap H = 1$.
   i. Show that $B$ contains a subgroup isomorphic to $H$, but that $B$ need not contain $H$ in general. (5 points)
   ii. If $A$ and $B$ have relatively prime orders, prove that $H \subseteq B$. (5 points)

2. Let $K \subseteq E$ be an extension of fields and let $R$ be the subring of the polynomial ring $E[x]$ consisting of all polynomials with constant term in $K$. In other words,
   $$R = K + Ex + Ex^2 + Ex^3 + \cdots$$

Now let $I$ be a nonzero ideal of $R$ and let $m$ be the minimal degree of the nonzero elements of $I$. Define
   $$I_m = \{ f(x) \in I \mid \deg f(x) = m \} \cup \{0\}$$
so that $I_m$ is clearly a nonzero $K$-subspace of $R$.
   i. If $\dim_K I_m = 1$, prove that $I$ is a principal ideal. (4 points)
   ii. If $\dim_K I_m > 1$, show that $I_m$ contains a nonzero polynomial with constant term equal to 0. (2 points)
   iii. If $I$ is a prime ideal which is not principal, prove that $m = 1$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $K = \mathbb{Q}[\sqrt{2}]$. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic irreducible polynomial of odd degree $n \geq 1$ and notice that $f(x + \sqrt{2})$ is a monic polynomial of degree $n$ in $K[x]$.
   i. Show that the coefficient of $x^{n-1}$ in $f(x + \sqrt{2})$ is not rational. (2 points)
   ii. Show that the polynomial $f(x + \sqrt{2})$ is irreducible in $K[x]$. (4 points)
   iii. Prove that the polynomial $g(x) = f(x + \sqrt{2})f(x - \sqrt{2})$ is irreducible in the ring $\mathbb{Q}[x]$. (4 points)

4. Let $V \neq 0$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $X, Y, Z$ be linear operators on $V$ which satisfy
   $$XY - YX = Z \quad XZ = ZX \quad YZ = ZY.$$ 
If $V$ has no proper subspace invariant under all three operators, prove that $\dim_{\mathbb{C}} V = 1$.

5. Let $V$ be a vector space over the rational numbers $\mathbb{Q}$ and let $v_1, v_2, \ldots, v_n \in V$. Show that there exist elements $w_1, w_2, \ldots, w_m \in V$ which are linearly independent over $\mathbb{Q}$ and which satisfy
   $$\sum_{i=1}^{n} v_i Z = \sum_{j=1}^{m} w_j Z,$$
where $Z$ is the ring of integers.
1. Let $G$ be a group of order $|G| = 504 = 7 \cdot 8 \cdot 9$.
   i. If there exists an element $x \in G$ of order 21, show that there exists a subgroup $H \subset G$ having index $|G : H| = 8$. Hint. Find the possibilities for $n_7$, the number of Sylow 7-subgroups of $G$. (6 points)
   ii. If $G$ is simple, show that it contains no element of order 21. (4 points)

2. Let $R$ be a ring and let $M$ be a right $R$-module which has a composition series. Assume that $M$ has a unique simple submodule $N$ and that $N$ is not isomorphic to any composition factor of $M/N$. Prove that the ring $\text{End}_R(M)$ is a division ring. In other words, prove that every nonzero $R$-endomorphism of $M$ is one-to-one and onto.

3. Let $K \subseteq E$ be fields and let $f(x) \in K[x]$ be a polynomial of degree $n \geq 2$ having $n$ distinct roots $a_1, a_2, \ldots, a_n$ in $E$. Suppose that the field extension $K[a_1, a_2]/K$ has degree $|K[a_1, a_2] : K| = n(n - 1)$.
   i. Find the degrees of the irreducible factors of $f(x)$ over the field $K$ and over the field $K[a_1]$. (3 points)
   ii. If $g(x)$ is the minimal polynomial of $a_1 + a_2$ over $K$, prove that $a_i + a_j$ is a root of $g(x)$ for all $i \neq j$. Hint. First consider the case $i = 1$. (7 points)

4. Let $A$ be an $m \times n$ matrix over the integers $\mathbb{Z}$ and consider the system of homogeneous linear equations $AX = 0$ where $X$ is the column vector of unknowns $x_1, x_2, \ldots, x_n$. Suppose that every integer solution of this system has all $x_i$'s equal. Prove that the same is true for every real solution of this system of equations.

5. Let $K = \mathbb{Q}[i]$ be the field generated over the rationals $\mathbb{Q}$ by $i = \sqrt{-1}$ and let $R$ be the subring of $K$ defined by
   \[ R = \{ a + bi \mid a, b \in \mathbb{Z} \} \]
   where $\mathbb{Z}$ is the ring of integers. Suppose $\alpha$ is a complex number which is the root of a monic polynomial in $R[x]$. Prove that the minimal monic polynomial of $\alpha$ over $K$ has all coefficients in $R$. 

Algebra Qualifying Exam
January 1994

Do all 5 problems.

1. A finite group is said to be perfect if it has no nontrivial abelian homomorphic image.
   i. Show that a perfect group has no nontrivial solvable homomorphic image. (3 points)
   ii. Let $H \triangleleft G$ with $G/H$ perfect. If $\theta : G \to S$ is a homomorphism from $G$ to a solvable
       group $S$ and if $N = \ker \theta$, prove that $G = NH$ and deduce that $\theta(H) = \theta(G)$. (7 points)

2. Let $R$ be a ring and let $V$ be a right $R$-module. Assume that every simple submodule of $V$ is a direct summand of $V$.
   i. If $W$ is any submodule of $V$, show that any simple submodule of $W$ is a direct summand of $W$. (5 points)
   ii. If $V$ is an Artinian module, that is if its submodules satisfy the minimal condition, prove that $V$ is a direct sum of finitely many simple submodules. (5 points)

3. Let $\alpha$ be the real positive 16th root of 3 and consider the field $F = \mathbb{Q}[\alpha]$ generated by $\alpha$
   over the rationals $\mathbb{Q}$. Notice that we have the chain of intermediate fields

   \[ \mathbb{Q} \subseteq \mathbb{Q}[\alpha^8] \subseteq \mathbb{Q}[\alpha^4] \subseteq \mathbb{Q}[\alpha^2] \subseteq \mathbb{Q}[\alpha] = F. \]

   i. Compute the degrees of these five intermediate fields over $\mathbb{Q}$ and conclude that these fields are all distinct. (4 points)
   ii. Show that every intermediate field between $\mathbb{Q}$ and $F$ is one of the above. Hint. If $\mathbb{Q} \subseteq K \subseteq F$, consider the constant term of the minimal polynomial of $\alpha$ over $K$. (6 points)

4. Let $X$ be a subspace of $M_n(C)$, the $C$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in $X$ is invertible. Prove that $\dim_C X \leq 1$.

5. Let $E$ be an algebraic extension of the rational numbers $\mathbb{Q}$ and let $\alpha \in E$.
   i. Prove that there exists a nonzero integer $n \in \mathbb{Z}$ such that $n\alpha$ is an algebraic integer. (4 points)
   ii. Show that $\mathbb{Z}[\alpha]$ does not contain $\mathbb{Q}$ and hence conclude that $\mathbb{Z}[\alpha]$ is not a field. (6 points)
1. Let $G$ be a finite group and let $P \in \text{Syl}_p(G)$ for some prime $p$. Suppose $N$ is a normal subgroup of $G$ with $|G : N| = |P| > 1$.
   i. Prove that $N$ is the subset of $G$ consisting of all elements of order not divisible by $p$. (4 points)
   ii. If the elements of $G$ outside of $N$ all have $p$-power order, prove that $P$ is its own normalizer. (6 points)

2. Let $R$ be a commutative ring and let $P$ be a prime ideal of $R$. If $V$ is a (right) $R$-module, define
   $$W = \{ v \in V \mid va = 0 \text{ for some } a \in R \setminus P \}$$
   where $R \setminus P$ is the set of elements of $R$ not in $P$.
   i. Show that $W$ is an $R$-submodule of $V$. (2 points)
   ii. If $R$ is Noetherian and $V$ is a finitely generated $R$-module, prove that $Wb = 0$ for some $b \in R \setminus P$. (3 points)
   iii. If $V$ is a simple $R$-module and $W = 0$, prove that $P$ is a maximal ideal. (5 points)

3. Let $E$ be a splitting field of the polynomial $x^3 - 2$ over the rationals $\mathbb{Q}$, and assume that $E$ is contained in the complex numbers $\mathbb{C}$. Let $F = E \cap \mathbb{R}$ be the real subfield of $E$, and note that $F = \mathbb{Q}[\sqrt[3]{2}]$.
   i. Show that $G = \text{Gal}(E/\mathbb{Q})$ contains an element $\sigma$ with the property that the only elements of $F$ fixed by $\sigma$ are rational. (4 points)
   ii. Let $a \in F$ and suppose that $a^3 \in \mathbb{Q}$. Show that one of $a$, $a\sqrt[3]{2}$, or $a\sqrt[3]{4}$ is contained in $\mathbb{Q}$. (4 points)
   iii. Prove that $\sqrt[3]{3} \notin E$. (2 points)

4. Let $A$ be an $n \times n$ matrix over a field $K$ and assume that the characteristic polynomial of $A$ has distinct roots in the algebraic closure of $K$. Prove that any two $n \times n$ $K$-matrices which commute with $A$ must commute with each other.

5. Let $S = M_n(F)$ be the ring of $n \times n$ matrices over the field $F$.
   i. If $s \in S$ is nilpotent, prove that the trace of $s$ is zero. (4 points)
   ii. Suppose $R$ is a ring and that $\theta: R \to S$ is a surjective ring homomorphism. Let $I$ be an ideal of $R$ with the property that every element of $I$ is a sum of nilpotent elements of $R$. Show that $\theta(I) = 0$. (6 points)
Algebra Qualifying Exam
January 1995

Do all 5 problems.

1. Let $P$ be a Sylow $p$-subgroup of a finite group $G$ and let $N = \text{N}_G(P)$ be its normalizer.
   i. Show that $N$ is not contained in any proper normal subgroup of $G$. (5 points)
   ii. If the commutator subgroup $G'$ is abelian, prove that $G' \cap N$ is normal in $G$. (5 points)

2. Let $R$ be a commutative Noetherian domain and suppose that $P$ is the unique nonzero prime ideal of $R$.
   i. Show that every element of $R$ not in $P$ is a unit of $R$. (3 points)
   ii. If $Q$ is a nonzero ideal of $R$, prove that $Q$ is primary and that $Q \supseteq P^n$ for some integer $n \geq 1$. (3 points)
   iii. If $P = (\pi)$ is principal, prove that every nonzero element of $R$ is a product of a unit and a power of $\pi$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathbb{Q}[x]$. Suppose $E$ is a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = \text{Gal}(E/\mathbb{Q})$.
   i. Find $|E : \mathbb{Q}|$ and determine the Galois group $G$ up to isomorphism. (5 points)
   ii. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of $G$ on $\Omega$. (5 points)

4. Let $A$ be an $n \times n$ matrix over an algebraically closed field $K$ and let $K[A]$ denote the $K$-linear span of the matrices $I = A^0, A, A^2, A^3, \ldots$. Show that $A$ is diagonalizable if and only if $K[A]$ contains no nonzero nilpotent element.

5. Let $G$ be a (not necessarily finite) group and denote the operation in $G$ by multiplication. Let $\mathbb{Z}[G]$ denote the group ring of $G$ over the integers $\mathbb{Z}$. Thus, every element of $\mathbb{Z}[G]$ is a finite $\mathbb{Z}$-linear combination of elements of $G$, and the multiplication in $\mathbb{Z}[G]$ is built naturally from the multiplication in $G$. Let $I$ be a right ideal of $\mathbb{Z}[G]$ and define
   $$\text{gp}(I) = \{ g \in G \mid 1 - g \in I \}.$$  
   i. Prove that $\text{gp}(I)$ is a subgroup of $G$. (4 points)
   ii. If $I$ is a 2-sided ideal of $\mathbb{Z}[G]$, show that $\text{gp}(I)$ is normal in $G$. (3 points)
   iii. If $\text{gp}(I) = G$, prove that $I$ is a 2-sided ideal of $\mathbb{Z}[G]$. (3 points)
1. Let $G$ be a finite group. We say that a subgroup $M$ of $G$ has property $(\ast)$ if $M$ is abelian, maximal, and not normal in $G$.
   
i. If $M$ and $N$ are distinct subgroups of $G$ with property $(\ast)$, prove that $M \cap N = Z$, where $Z = Z(G)$ is the center of $G$. (2 points)
   
   ii. Let $M$ have property $(\ast)$ and let $S(M)$ denote the set of all noncentral elements of $G$ which are conjugate to elements of $M$. Note that
   
   $$S(M) = \bigcup_{x \in G} (M \setminus Z)x.$$
   
   Compute the cardinality $|S(M)|$ of $S(M)$ in terms of $|M| = m$, $|Z| = z$, and $|G| = g$. Deduce that $g - z > |S(M)| > (g - z)/2$. (5 points)
   
   iii. Show that any two subgroups of $G$ having property $(\ast)$ must be conjugate in $G$. (3 points)

2. Let $R$ be a ring. If $V$ and $W$ are right $R$-modules, we write $V \sim W$ when $V$ is isomorphic to a submodule of $W$ and $W$ is isomorphic to a submodule of $V$.
   
i. If $V \sim W$ and if $V$ satisfies the minimum condition, prove that $V$ and $W$ are isomorphic. (4 points)
   
   ii. Suppose $R = \mathbb{Z}$ is the ring of integers. If $V \sim W$ and if $V$ is finitely generated, prove that $V$ and $W$ are isomorphic. (3 points)
   
   iii. Suppose $R$ is a commutative integral domain and let $I$ be a nonzero ideal of $R$. Show that $R \sim I$ when we view $R$ and $I$ as right $R$-modules. Conclude that if $R$ is not a PID, then there exist nonisomorphic $R$-modules $V$ and $W$ with $V \sim W$. (3 points)

3. Let $E$ be the subfield of the real numbers generated over $\mathbb{Q}$ by $\sqrt{2}$ and $\sqrt[3]{2}$.
   
i. Show that $|E : \mathbb{Q}| = 6$. (2 points)
   
   ii. If $K$ is a field with $\mathbb{Q} \subseteq K \subseteq E$, show that $K$ is one of the fields $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt[3]{2}]$, or $E$. (5 points)
   
   iii. Prove that $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{2}]$. (3 points)
4. Let $V$ and $W$ be finite-dimensional vector spaces over an algebraically closed field $F$ and let $A: V \rightarrow V$ and $B: W \rightarrow W$ be linear operators. Suppose $T: V \rightarrow W$ is a nonzero linear transformation such that $T(A(v)) = B(T(v))$ for all $v \in V$, and let $N = \ker T$.

i. Show that $A(N) \subseteq N$. (2 points)

ii. Show that there exists $\lambda \in F$ and a vector $v \in V$ with $v \notin N$ such that $A(v) - \lambda v \in N$. (4 points)

iii. If $\lambda$ is as in part (ii), show that $\lambda$ is an eigenvalue for both $A$ and $B$. (4 points)

5. Let $S$ be the set of all $2 \times 2$ complex matrices of the form

$$
\begin{bmatrix}
    a & \bar{b} \\
    b & \bar{a}
\end{bmatrix}
$$

with $a, b \in \mathbb{C}$ and where, as usual, $\bar{\quad}$ denotes complex conjugation.

i. Show that $S$ is a subring of the ring $M_2(\mathbb{C})$ of all $2 \times 2$ matrices over $\mathbb{C}$. (2 points)

ii. Determine the center $Z$ of $S$ and show that $Z$ is isomorphic to the real numbers $\mathbb{R}$. (3 points)

iii. Prove that

$$
I = \left\{ \begin{bmatrix} x & \bar{x} \\ x & \bar{x} \end{bmatrix} \mid x \in \mathbb{C} \right\}
$$

is a minimal right ideal of $S$ and that it is faithful as a right $S$-module. (3 points)

iv. Show that $\dim_Z I = 2$ and conclude that $S \cong M_2(\mathbb{R})$. (2 points)
Algebra Qualifying Exam
January 1996

Do all 5 problems.

1. A finite group $G$ is said to have property $*$ if there exists a conjugacy class $\mathcal{K}$ of $G$ such $G$ is generated by the elements of $\mathcal{K}$.
   i. If $G$ has property $*$, prove that $G/G'$ is cyclic, where $G'$ is the commutator subgroup of $G$. (3 points)
   ii. Show that $G$ has property $*$ if and only if $G$ is not the set-theoretic union of its proper normal subgroups. (3 points)
   iii. Suppose $G = N \times M$ where $N$ is nonabelian simple and $M$ has property $*$. Prove that $G$ has property $*$. (Hint. First show that every normal subgroup of $G$ that does not contain $N$ must be contained in $M$.) (4 points)

2. Let $R$ be a commutative ring with 1 and suppose that $M$ is an ideal of $R$.
   i. If $M$ is both maximal and principal, show that there is no ideal $I$ of $R$ satisfying $M > I > M^2$, where $>$ denotes strict inclusion. (6 points)
   ii. Give examples to show that neither of the two conditions on $M$ in part (i) can be removed. (4 points)

3. Let $Q \subseteq L \subseteq E$ be fields with $Q$ the rational numbers and with $|E : L| < \infty$. Let $K$ be the subfield of $E$ consisting of all those elements of $E$ which are algebraic over $Q$, and assume that $K \cap L = Q$.
   i. If $\alpha \in K$, show that its minimal polynomial over $Q$ is irreducible over $L$ and deduce that $|Q[\alpha] : Q| \leq |E : L|$. (5 points)
   ii. Show that $|K : Q| \leq |E : L|$. (Hint. Start with a subfield $M$ of $K$ maximal with the property that $|M : Q| \leq |E : L|$.) (5 points)

4. Let $V$ be a vector space over a field $K$ and let $S$ and $T$ be $K$-linear operators on $V$. Suppose that $S$ is one-to-one, that $T(v) = 0$ for some $0 \neq v \in V$, and that $TS - ST = S$.
   i. For every integer $n \geq 0$, prove that $S^n(v)$ is an eigenvector for $T$ and determine the corresponding eigenvalue. (4 points)
   ii. If $K$ has characteristic 0, prove that $\dim_K V = \infty$. (4 points)
   iii. If $K$ has characteristic $p > 0$, then $\dim_K V$ can be finite. Give a concrete example of such a finite-dimensional situation when $p = 3$. (2 points)

5. Let $q$ be a prime power and let $F$ be the finite field of size $q$. Let
   \[ f(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \in F[x]. \]
   i. If $f(x)$ has a root in $F$, show that $f(x)$ splits completely over $F$, and prove that this happens precisely when $q \equiv 0$ or 1 mod 5. (6 points)
   ii. If $f(x)$ has an irreducible monic factor $g(x)$ of degree 2, show that $g(x)$ has constant term equal to 1. (2 points)
   iii. Factor $f(x)$ explicitly into irreducible quadratic factors when $q = 29$. (2 points)
5. The goal of this problem is to prove:

**Theorem.** Let $M_3(F)$ denote the space of $3 \times 3$ matrices over the field $F$. The following are equivalent.

i. $F$ has an extension field of degree 3.

ii. $M_3(F)$ contains a 3-dimensional subspace whose nonzero members are all invertible matrices.

iii. $M_3(F)$ contains a 2-dimensional subspace whose nonzero members are all invertible matrices.

a. If $E$ is an extension field of $F$ of degree 3, show that the ring $M_3(F)$ contains an isomorphic copy of $E$. Deduce that (i) implies (ii). (5 points)

b. Let $A$ and $B$ be linearly independent invertible matrices in $M_3(F)$. If the characteristic polynomial of $AB^{-1}$ is not irreducible over $F$, show that some nonzero $F$-linear combination of $A$ and $B$ is not invertible. Deduce that (iii) implies (i). (5 points)
Algebra Qualifying Exam
January 1997

Do all 5 problems.

1. Let $G$ be a finite group having the property that for every choice of two subgroups $X \subseteq G$ and $Y \subseteq G$, either $X \cap Y = 1$ or $X \subseteq Y$ or $Y \subseteq X$.
   i. If $H \subseteq G$, show that either $|H|$ is a prime power or else that $|H|$ and $|G : H|$ are relatively prime. (4 points)
   ii. If $1 < N \triangleleft G$, prove that $G/N$ is nilpotent. (2 points)
   iii. If $N \triangleleft G$ and $N \neq G$, show that $N$ is nilpotent. (4 points)

2. Let $R$ be a ring, let $V$ be a right $R$-module, and suppose that $V = V_1 + V_2 + V_3 + \cdots$ is the (internal) direct sum of its submodules $V_1, V_2, V_3, \ldots$. Show that $V$ is an Artinian module if and only if each $V_i$ is Artinian and only finitely many of the $V_i$'s are nonzero.

3. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 5 over the rational numbers $\mathbb{Q}$ that is not solvable by radicals, and let $S$ be the splitting field of $f(x)$ over $\mathbb{Q}$ which is contained in the complex numbers.
   i. Show that there exists at most one subfield $E$ of $S$ such that $|E : \mathbb{Q}| = 2$. (7 points)
   ii. If $\alpha, \beta \in S$ are irrational elements which satisfy $\alpha^2 \in \mathbb{Q}$ and $\beta^2 \in \mathbb{Q}$, prove that $\alpha \beta \in \mathbb{Q}$. (3 points)

4. If $K$ is a field, then the general linear group $G = \text{GL}_n(K)$ is the multiplicative group of $n \times n$ invertible matrices over $K$.
   i. If the characteristic of $K$ is not equal to 2, show that $G$ has precisely $n$ conjugacy classes of elements of order 2. (5 points)
   ii. If $\text{char } K = 2$, show that $G$ has precisely $[n/2]$ (the greatest integer in $n/2$) conjugacy classes of elements of order 2. (5 points)

5. Let $S$ be a commutative integral domain and let $R$ be a subring of $S$ with the same identity 1. Suppose that there exist finitely many elements $s_1, s_2, \ldots, s_n \in S$ such that $S = s_1R + s_2R + \cdots + s_nR$. Show that $R$ is a field if and only if $S$ is a field.
1. Suppose that $G$ is a finite group that has exactly 50 Sylow 7-subgroups. Let $P \in \text{Syl}_7(G)$ and write $N = N_G(P)$.
   a. Show that $N$ is a maximal subgroup of $G$. (5 points)
   b. If $N$ has a normal Sylow 5-subgroup $Q$, prove that $Q \triangleleft G$. (5 points)

2. Let $R$ be a commutative domain with 1.
   a. Let $a, b \in R$ and assume that the principal ideal $(ab)$ is primary. If $a \neq 0$ and $b$ is not a unit, prove that $b^n \in (a)$ for some integer $n \geq 1$. (4 points)
   b. Now assume that every principal ideal of $R$ is primary and let $P$ be any nonzero prime ideal of $R$. Show that $P$ contains every nonunit of $R$. Deduce that $P$ is the unique nonzero prime ideal of $R$. (6 points)

3. Let $\alpha$ be a nonzero real number and suppose that $\alpha^n \in \mathbb{Q}$, the rational numbers, for some positive integer $n$. Let $g(x)$ be the minimal (monic) polynomial of $\alpha$ over $\mathbb{Q}$, and suppose that $\deg g = m$.
   a. Show that $g(0) = \pm \alpha^m$. (5 points)
   b. Deduce that $g(x) = x^m - b$ for some rational number $b$. (2 points)
   c. Prove that $m$ divides $n$. (3 points)

4. Let $V$ be a finite dimensional vector space over an algebraically closed field $K$ and let $T : V \rightarrow V$ be a linear operator. Also let $I : V \rightarrow V$ denote the identity operator. Show that $V$ has a basis consisting of eigenvectors of $T$ if and only if the kernel of $(\lambda I - T)^2$ is equal to the kernel of $\lambda I - T$ for all choices of $\lambda \in K$. (5 points for each direction)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Suppose $X$ and $Y$ are $R$-submodules of $V$ such that $V = X \oplus Y$, the (internal) direct sum. If $\theta$ is any $R$-homomorphism from $X$ to $Y$, define $W_\theta \subseteq V$ to be the set of elements $x - \theta(x) \in V$ for all $x \in X$.
   a. Show that $W_\theta$ is an $R$-submodule of $V$ and that $V = W_\theta \oplus Y$. (4 points)
   b. Conversely, suppose $U$ is an $R$-submodule of $V$ such that $V = U \oplus Y$. Prove that $U = W_\theta$ for some $R$-homomorphism $\theta : X \rightarrow Y$. (6 points)
1. Fix a prime $p$ and let $G$ be a finite group with the property that every nonidentity $p$-subgroup of $G$ is contained in a unique Sylow $p$-subgroup of $G$. Suppose $N < G$ and $|N|$ is divisible by $p$.
   i. If $P$ and $Q$ are Sylow $p$-subgroups of $G$, show that $Q = P^n$ for some element $n \in N$. (6 points)
   ii. Prove that $G/N$ has a unique Sylow $p$-subgroup. (4 points)

2. Let $R$ be a commutative domain and write $(a)$ for the principal ideal generated by $a \in R$. Recall that an element of $R$ is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.
   i. Show that $(a) \subseteq (b)$ if and only if $b|a$, and that $(a) = (b)$ if and only if $b = au$ for some unit $u \in R$. (2 points)
   ii. If $R$ is a UFD (unique factorization domain), prove that the set of principal ideals of $R$ satisfies the maximal condition. (4 points)
   iii. If the set of principal ideals of $R$ satisfies the maximal condition, show that every nonzero, nonunit element of $R$ can be written as a finite product of irreducible elements. (4 points)

3. Let $p$ be a prime, let $F \subseteq K$ be fields of characteristic 0, and assume that $F$ contains a primitive $p$th root of unity. Fix $a \in K$.
   i. Prove that there exists a field $E \supseteq K$ such that $E$ contains a $p$th root of $a$ and $|E : K| = 1$ or $p$. (4 points)
   ii. Now assume that $K$ is a finite degree Galois extension of $F$. Show that there exists a field $E \supseteq K$ such that $E$ contains a $p$th root of $a$, $E$ is Galois over $F$, and $|E : K|$ is a power of $p$. (6 points)

4. Let $V$ be a finite dimensional vector space over a field of characteristic 0. Suppose $T:V \to V$ is a linear operator such that the trace $\text{tr} T^n = 0$ for all integers $k \geq 1$.
   i. Show that the constant term of the characteristic polynomial of $T$ is zero, and deduce that $T(V) \neq V$. (5 points)
   ii. Let $S$ denote the restriction of $T$ to the subspace $T(V)$, so that $S$ is a linear operator on $T(V)$. Prove that $\text{tr} S^n = 0$ for all integers $k \geq 1$. (4 points)
   iii. Show that $T$ is nilpotent. (1 point)

5. Let $G$ be a (not necessarily finite) group and let $\theta:G \to G$ be a homomorphism such that $\theta^n(G) = \{1\}$ for some integer $n \geq 1$.
   i. If the kernel of $\theta$ is finite, prove that the kernel of $\theta^2$ is finite, and deduce that $G$ is finite. (5 points)
   ii. If $\theta(G)$ has finite index in $G$, prove that $\theta^2(G)$ has finite index in $G$, and deduce that $G$ is finite. (5 points)
Algebra Qualifying Exam
August 1998

Do all 5 problems.

1. If $G$ is a finite group, we define $\text{soc}(G)$ to be the subgroup generated by all the minimal normal subgroups of $G$.
   a. If $(1) \neq N \triangleleft G$, show that $N \cap \text{soc}(G) \neq \langle 1 \rangle$. (2 points)
   b. Prove that $\text{soc}(\text{soc}(G)) = \text{soc}(G)$. (4 points)
   c. If $\text{soc}(G) = G$, show that every minimal normal subgroup of $G$ is simple. (4 points)

2. Let $R$ be a commutative domain with 1 having $F$ as its field of fractions. For any element $q \in F$, we define $I_q = \{ r \in R \mid rq \in R \}$.
   a. Show that each $I_q$ is a nonzero ideal of $R$. (2 points)
   b. If $M$ is a maximal ideal of $R$, let $R_M$ denote the subring of $F$ containing $R$ and given by $R_M = \{ a/b \in F \mid a \in R, b \in R \setminus M \}$. Prove that $R = \bigcap_M R_M$, where the latter intersection is over all maximal ideals $M$ of $R$. (5 points)
   c. Now suppose that $R = \mathbb{Z}[\sqrt{-3}]$ and let $q = (1 - \sqrt{-3})/2 = 2/(1 + \sqrt{-3}) \in F$. Show that $I_q$ is not a principal ideal. (Hint. Use the fact that the norm map $N(\alpha) = |\alpha|^2$ is multiplicative.) (3 points)

3. Let $F$ be a field and suppose $f(x) \in F[x]$ is an irreducible polynomial. Fix an integer $n$ and let $g(x) = f(x^n)$.
   a. If $h(x)$ is any irreducible factor of $g(x)$ in $F[x]$, show that $\text{deg } h(x)$ is a multiple of $\text{deg } f(x)$. (5 points)
   b. Now suppose that $F$ has characteristic 0 and that it contains a primitive $n$th root of unity. Show that all irreducible factors of $g(x)$ in $F[x]$ have equal degrees. (5 points)

4. Let $V$ be a finite dimensional vector space over the field $K$ and let $( , ) : V \times V \to K$ be a symmetric bilinear form. For any subspace $U$ of $V$, we let $U^\perp = \{ v \in V \mid (U, v) = 0 \}$. Thus $U^\perp$ is also a subspace of $V$, and the form is nonsingular precisely when $V^\perp = 0$.
   a. Show that $\dim U + \dim U^\perp \geq \dim V$ for any subspace $U$ of $V$. (4 points)
   b. If the form is nonsingular and if $V = X + Y$ is the sum of subspaces $X$ and $Y$, prove that $\dim X^\perp + \dim Y^\perp \leq \dim V$. (2 points)
   c. If the form is nonsingular, show that $\dim U + \dim U^\perp = \dim V$ for any subspace $U$ of $V$. (4 points)

5. Let $A$ be a (not necessarily finite) abelian group and let $B$ be a subgroup of $A$.
   a. If $B$ is a direct factor of $A$, show that $B$ is a direct factor of every subgroup $C$ satisfying $B \subseteq C \subseteq A$. (4 points)
   b. Conversely, assume that $B$ is a direct factor of every subgroup $C$ such that $B \subseteq C \subseteq A$ and $C/B$ is cyclic. If $|A : B| < \infty$, show that $B$ is a direct factor of $A$. (6 points)
Algebra Qualifying Exam
January 1999

Do all 5 problems.

1. Let $G$ be a finite group and let $A \subseteq G$ be a maximal (proper) subgroup. Assume that $A$ is abelian, that $|G : A| = p^n$ for some prime $p$, and that $A$ contains no nonidentity normal subgroup of $G$.
   i. Show that $p$ does not divide $|A|$. (4 points)
   ii. Prove that the set $S$ of elements of $G$ not conjugate to any nonidentity element of $A$ has cardinality precisely $p^n$. (5 points)
   iii. Show that $G$ is not simple. (1 point)

2. Let $R$ be a ring with 1, and recall that $R$ is naturally a right $R$-module with respect to right multiplication. We denote this right regular $R$-module by $R_R$.
   i. Prove that $R$ is a division ring if and only if $R_R$ is a simple $R$-module. (5 points)
   ii. Prove that $R$ is a division ring if and only if every nonzero right $R$-module contains a submodule isomorphic to $R_R$. (5 points)

3. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial over the rational numbers $\mathbb{Q}$, and let $\alpha$ and $\beta$ be roots of $f(x)$ in the complex numbers $\mathbb{C}$. Suppose $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$ where $E$ is a (finite) Galois extension of $\mathbb{Q}$.
   i. Show that $\mathbb{Q}[\alpha] \cap E$ is isomorphic to $\mathbb{Q}[\beta] \cap E$. (5 points)
   ii. Now assume that $E = \mathbb{Q}[\varepsilon]$ where $\varepsilon$ is a root of unity. Prove that $\mathbb{Q}[\alpha] \cap E = \mathbb{Q}[\beta] \cap E$. (5 points)

4. Let $V$ be a finite-dimensional vector space over an algebraically closed field $F$, and let $S$ and $T$ be commuting linear operators on $V$. Assume that the characteristic polynomial of $S$ has distinct roots.
   i. Show that every eigenvector of $S$ is an eigenvector for $T$. (5 points)
   ii. If $T$ is nilpotent, prove that $T = 0$. (5 points)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Assume that $M_1, M_2, \ldots, M_n$ are finitely many $R$-submodules of $V$ with $M_1 \cap M_2 \cap \cdots \cap M_n = 0$, and let $W$ be the (external) direct sum
   $$ W = V/M_1 \oplus V/M_2 \oplus \cdots \oplus V/M_n. $$
   i. Show that $V$ is isomorphic to an $R$-submodule of $W$. (4 points)
   ii. Now suppose in addition that the modules $V/M_i$ are simple and pairwise nonisomorphic. Prove that $V$ is isomorphic to $W$. (Hint. First observe that $W$ has a composition series of length $n$.) (6 points)
Algebra Qualifying Exam
August 1999

Do all 5 problems.

1. Let $G$ be a group and let $K \subseteq H$ be subgroups of $G$ with $K \triangleleft H$.
   a. Prove that $H$ normalizes $C_G(K)$. (3 points)
   b. If $H \triangleleft G$ and $C_H(K) = \langle 1 \rangle$, prove that $H$ centralizes $C_G(K)$. (7 points)

2. In this problem, the word ideal always means two-sided ideal. Now let $R$ be a (not necessarily commutative) ring with 1. An ideal $P$ of $R$ is said to be prime if, for all ideals $A$ and $B$ of $R$, the inclusion $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.
   a. If an ideal $Q$ is not prime, show that there exist ideals $A > Q$ and $B > Q$ with
      $AB \subseteq Q$. (2 points)
   b. Let $I$ and $Q$ be ideals of $R$ and assume that $Q$ is maximal with the property
      that it contains no power of $I$. Show that $Q$ is prime. (4 points)
   c. Suppose $I$ is a nonnilpotent ideal of $R$. If $R$ satisfies the ascending chain
      condition on ideals, prove that there exists a prime ideal of $R$ which does not
      contain $I$. (4 points)

3. Let $K \subseteq L$ be a finite degree extension of fields. Suppose that $E$ and $F$ are
   intermediate fields, each Galois over $K$, and that $L = EF$ is the field generated by
   $E$ and $F$. (This means that no proper subfield of $L$ contains both $E$ and $F$.)
   a. Prove that $L$ is Galois over $K$. (4 points)
   b. If $Gal(E/K) = G$ and $Gal(F/K) = H$, show that $Gal(L/K)$ is isomorphic to
      a subgroup of $G \times H$. (6 points)

4. In this problem, all matrices are viewed over the complex numbers.
   a. For which complex numbers $x$, if any, is the matrix $\begin{bmatrix} 1 & -2 \\ 8 & x \end{bmatrix}$ not similar to a
      diagonal matrix? Explain. (5 points)
   b. Let $J$ be the $n \times n$ matrix all of whose entries are equal to 1. Find a diagonal
      matrix similar to $J$ or prove that none exists. (5 points)

5. Let $F[x, y]$ be the polynomial ring over the field $F$ in the two indeterminates $x$
   and $y$. Suppose $f(x) \in F[x] \subseteq F[x, y]$ and $g(y) \in F[y] \subseteq F[x, y]$ are polynomials of
   positive degree in the indeterminates $x$ and $y$, respectively. Let $I = (f(x), g(y))$ be
   the ideal of $F[x, y]$ generated by $f(x)$ and $g(y)$.
   a. Prove that $I \neq F[x, y]$. (5 points)
   b. If $f(x) = x - \alpha$ and $g(y) = y - \beta$ for some $\alpha, \beta \in F$, show that $I$ is a maximal
      ideal of $F[x, y]$. (5 points)
Algebra Qualifying Exam
January 2000

Do all 5 problems.

1. Let $G$ be a group of order $2^4 \cdot 3^3 \cdot 11$ and let $H$ be a group of order $5^3 \cdot 11$.
   a. Show that $H$ has a normal Sylow 11-subgroup. (2 points)
   b. If the number of Sylow 5-subgroups of $G$ is (strictly) less than 16, prove that $G$ has a proper normal subgroup of order divisible by 5. (4 points)
   c. If $G$ has exactly sixteen Sylow 5-subgroups, show that $G$ has a normal Sylow 11-subgroup. (4 points)

2. Let $R$ be a (not necessarily commutative) ring with 1 and suppose that $R$ can be written as the sum $R = \sum_{i=1}^{m} I_i$, where the $I_i$ are finitely many (two-sided) ideals of $R$ satisfying $I_i \cap I_j = 0$ whenever $i \neq j$.
   a. Prove that, for every simple right $R$-module $M$, there exists a unique subscript $k$ such that $MI_k \neq 0$. (5 points)
   b. Show that if $i \neq j$, then every right $R$-module homomorphism $\theta: I_i \to I_j$ is the zero map. (5 points)

3. Let $L/K$ be a finite degree Galois extension of fields with Galois group given by $\text{Gal}(L/K) = G$, and let $E$ be an intermediate field. Then $E$ is said to be a 2-tower over $K$ if there exists a chain of fields $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ such that $|E_i : E_{i-1}| = 2$ for all $i = 1, 2, \ldots, n$.
   a. If $G$ is abelian, prove that $E$ is a 2-tower over $K$ if and only if the degree $|E : K|$ is a power of 2. (7 points)
   b. Show by example that the characterization of 2-towers given in part (a) is false if $G$ is allowed to be a nonabelian group. (3 points)

4. Let $A$ be an $n \times n$ matrix over the complex numbers and assume that the rank of $A$ is equal to 1.
   a. What are the possible Jordan canonical forms for $A$? Justify your answer. (5 points)
   b. For each of the forms obtained in part (a), compute the characteristic polynomial of $A$ and the minimal polynomial of $A$. (5 points)

5. Let $R = F[x, y]$ be the polynomial ring over the field $F$ in the two indeterminates $x$ and $y$, and let $I = xR$ be the principal ideal of $R$ generated by $x$. Define $S = F + I$, so that $S$ is a subring of $R$, and observe that $I$ is an ideal of $S$.
   a. Show that $I$ is not finitely generated as an ideal of $S$. (5 points)
   b. Prove that there are infinitely many ideals of $S$ that are not ideals of $R$. (5 points)
Algebra Qualifying Exam
August 2000

Do all 5 problems.

1. Suppose that a group $G$ is the (internal) direct product of subgroups $S$ and $T$. Let $H$ be a subgroup of $G$ such that $SH = G = TH$.
   a. Prove that $S \cap H$ and $T \cap H$ are normal subgroups of $G$. (4 points)
   b. If $S \cap H = 1 = T \cap H$, prove that $S$ and $T$ are isomorphic. (3 points)
   c. If $S \cap H = 1 = T \cap H$ and $H$ is normal in $G$, show that $G$ is abelian. (3 points)

2. Let $A_1, A_2, \ldots, A_n$ be ideals of the commutative ring $R$, and let $D = \bigcap_{i=1}^{n} A_i$.
   a. Prove that $\sqrt{D} = \bigcap_{i=1}^{n} \sqrt{A_i}$. (3 points)
   b. Now suppose that $D$ is a primary ideal and that it is not the intersection of any proper subset of $\{A_1, A_2, \ldots, A_n\}$. Show that $\sqrt{A_i} = \sqrt{D}$ for all $i$. (7 points)

3. Let $K \subseteq E$ be a finite degree extension of fields of characteristic 0, and let $F_1$ and $F_2$ be intermediate fields. These intermediate fields are said to be linearly disjoint over $K$ if $|\langle F_1, F_2 : K \rangle| = |F_1 : K||F_2 : K|$, where $\langle F_1, F_2 \rangle$ is the subfield of $E$ generated by $F_1$ and $F_2$.
   a. Prove that $|\langle F_1, F_2 \rangle : F_1| \leq |F_2 : K|$ for any $F_1$ and $F_2$. (3 points)
   b. If $|F_1 : K|$ and $|F_2 : K|$ are relatively prime, prove that $F_1$ and $F_2$ are linearly disjoint over $K$. (2 points)
   c. Give an example with $|F_1 : K| = 2 = |F_2 : K|$ to show that fields can be linearly disjoint without having relatively prime degrees. (2 points)
   d. If $F_1$ and $F_2$ are linearly disjoint and Galois over $K$, prove that the Galois groups satisfy $\text{Gal}(\langle F_1, F_2 \rangle / K) \cong \text{Gal}(F_1 / K) \times \text{Gal}(F_2 / K)$. (3 points)

4. Let $V$ be a complex vector space, not necessarily of finite dimension. Suppose that $A, B : V \rightarrow V$ are nonzero C-linear transformations with $AB = \lambda BA$ for some fixed nonzero complex number $\lambda$. Assume that no proper subspace of $V$ is invariant under both $A$ and $V$. That is, if $W$ is a subspace of $V$ with $AW \subseteq W$ and $BW \subseteq W$, then $W = 0$ or $V$.
   a. Show that $A$ and $B$ are both one-to-one and onto. (5 points)
   b. If $V$ is finite dimensional, prove that $\lambda$ is a root of unity. (3 points)
   c. Show that a finite-dimensional example exists with $\lambda = -1$. (2 points)

5. Let $R$ be a ring and let $Z$ denote its center. A derivation $D : R \rightarrow R$ is a map satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.
   a. If $r \in R$, show that the map $A_r : R \rightarrow R$ given by $A_r(a) = ar - ra$, for all $a \in R$, is a derivation of $R$. (3 points)
   b. If $D$ is a derivation of $R$, prove that $D(Z) \subseteq Z$. (3 points)
   c. If $D$ is a derivation of $R$ and $e \in Z$ is an idempotent, prove that $D(e) = 0$. (Hint. You may need to evaluate $(1 - 2e)^2$.) (4 points)
Algebra Qualifying Exam
January 2001

Do all 5 problems.

1. Let $X$ and $Y$ be distinct subgroups of a finite group $G$. We say that $X$ and $Y$ are a weird pair if $|X| = |Y|$ and if no subgroup of $G$ other than $X$ and $Y$ has this same order.
   a. If $G$ is a group having a weird pair of subgroups, show that some subgroup of $G$ has a weird pair of normal subgroups. (3 points)
   b. If $G = A \times B$ is a direct product of solvable groups, show that the subgroups $A \times 1$ and $1 \times B$ cannot be a weird pair. (4 points)
   c. Show that a solvable group cannot contain a weird pair of subgroups. (3 points)

2. Let $R$ be a ring with 1. Recall that an ideal $P$ of $R$ is said to be (right) primitive if there exists a simple right $R$-module $W$ with $P = \{ r \in R \mid Wr = 0 \}$. Furthermore, we recall that the Jacobson radical, $\text{Jrad}(R)$, of $R$ is defined to be the intersection of all primitive ideals of $R$.
   a. Let $V$ be a right $R$-module having a composition series of length $n$ and suppose that $R$ acts faithfully on $V$. Show that $J = \text{Jrad}(R)$ is an intersection of $n$ primitive ideals of $R$ and that $J^n = 0$. (7 points)
   b. Give an example of the situation in part (a) with $n = 2$ and with $\text{Jrad}(R) \neq 0$. Justify your answer. (3 points)

3. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients and suppose that $f(\alpha) = 0 = f(2\alpha)$ for some complex number $\alpha$.
   a. Show that $f(0)$, the constant term of $f$, is not equal to 1. (5 points)
   b. If $f$ is irreducible, prove that $\alpha = 0$. (5 points)

4. Let $A$ be an $n \times n$ matrix over the complex numbers and let $A^*$ denote the conjugate transpose of $A$.
   a. Prove that all eigenvalues of the product matrix $A^*A$ are real and nonnegative. (6 points)
   b. If $I$ is the $n \times n$ complex identity matrix, show that $\det(I + A^*A)$ is real and positive. (4 points)

5. Let $V$ be a finite-dimensional vector space over some field $F$ and let $T : V \to V$ be a linear operator. Write $F[T]$ to denote the ring of all linear operators on $V$ that can be expressed as polynomials in $T$. Assume that no nonzero proper subspace of $V$ is mapped into itself by $T$.
   a. If $0 \neq S \in F[T]$, show that $\{ v \in V \mid vS = 0 \}$ is the zero subspace. (3 points)
   b. Prove that $F[T]$ is a field. (4 points)
   c. Show that $|F[T] : F| = \dim_F V$. (3 points)
Algebra Qualifying Exam
August 2001

Do all 5 problems.

1. Let $G$ be a finite group of order $504 = 2^3 \cdot 3^2 \cdot 7$.
   a. Show that $G$ cannot be isomorphic to a subgroup of the alternating group $\text{Alt}_7$. (5 points)
   b. If $G$ is simple, determine the number of Sylow 3-subgroups of $G$. (5 points)

2. Let $R$ be a commutative ring with 1 and let $M$ be a maximal ideal of $R$.
   a. Show that the ring $R/M^2$ has no idempotents other than 0 and 1. (4 points)
   b. We know that $M/M^2$ is naturally an $R/M$-module. If $R$ is Noetherian, prove that this module is finitely generated. (2 points)
   c. Finally, assume that $R = K[x_1, x_2, \ldots, x_t]$ is a polynomial ring in finitely many variables over the field $K$. Prove that $\dim_K(R/M^2) < \infty$. (4 points)

3. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F[\alpha]$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   a. If $\alpha^m \in F$ with $m > 0$, show that $m$ is a multiple of $n$. (2 points)
   b. If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$. (4 points)
   c. If every root of unity in $E$ lies in $F$, show that $|E : F| = n$. (4 points)

4. Let $A$ be a real $n \times n$ matrix. We say that $A$ is a difference of two squares if there exist real $n \times n$ matrices $B$ and $C$ with $BC = CB = 0$ and $A = B^2 - C^2$.
   a. If $A$ is a diagonal matrix, show that it is a difference of two squares. (3 points)
   b. If $A$ is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
   c. Suppose $A$ is a difference of two squares, with corresponding matrices $B$ and $C$ as above. If $B$ has a nonzero real eigenvalue, prove that $A$ has a positive real eigenvalue. (4 points)

5. Let $K$ be a field of characteristic 0 and view the polynomial ring $V = K[x]$ as a $K$-vector space. Let $M: V \to V$ be the linear operator given by multiplication by $x$, so that $M(x^n) = x^{n+1}$ for all integers $n \geq 0$. In addition, let $D: V \to V$ be the linear operator given by differentiation with respect to $x$, so that $D(x^n) = nx^{n-1}$ for all $n \geq 0$. Let $L$ denote the set of all linear operators of the form $M^i D^j$ with $i, j \geq 0$, where $M^0 = D^0 = I$ is the identity operator on $V$.
   a. Prove that $DM - MD = I$. (3 points)
   b. Show that $L$ is a $K$-linearly independent set. (4 points)
   c. For all nonnegative integers $t$, prove that $DM^t$ is in the $K$-linear span of the set $L$. (3 points)
1. Let $N$ be a normal subgroup of the finite group $G$. A subgroup $H$ of $G$ is said to be a complement for $N$ in $G$ if $NH = G$ and $N \cap H = 1$.
   a. Show that all complements for $N$ in $G$ are isomorphic. (2 points)
   b. If $N$ has a complement in $G$ that is a $p$-group for some prime $p$, prove that every Sylow $p$-subgroup of $G$ contains a complement for $N$. (3 points)
   c. Assume that the center of $N$ is trivial, that is equal to the identity subgroup, and that every automorphism of $N$ is inner. Prove that $N$ has a unique complement $H$ that is normal in $G$. (5 points)

2. Let $R$ be a commutative integral domain with 1, and assume that $R$ is integrally closed in its field of fractions. Let $R[x]$ denote the ring of polynomials over $R$ in the variable $x$.
   a. Let $S \supseteq R$, where $S$ is a commutative integral domain with the same 1, and let $s \in S$ be an element integral over $R$. If $I$ is the ideal of $R[x]$ consisting of all polynomials $f(x)$ with $f(s) = 0$, prove that $I$ is principal. (5 points)
   b. Let $I$ be a prime ideal of $R[x]$ containing a monic polynomial. If $I \cap R = 0$, prove that $I$ is principal. (5 points)

3. Let $K$ be a field of prime characteristic $p$ and let $F = K(t)$ be the rational function field over $K$ in the indeterminate $t$. Write $f(x) = x^{2p} - tx^p + t \in F[x]$.
   a. Show that $f(x)$ is an irreducible polynomial in $F[x]$. (3 points)
   b. Let $E = F[s]$, where $s$ is a root of the polynomial $x^p - t \in F[x]$. If $L$ is the splitting field of $f(x)$ over $E$, prove that $|L : E| \leq 2$. (4 points)
   c. Show that $L = F[\alpha]$, where $\alpha$ is a root of $f(x)$. (3 points)

4. Let $V$ be an $n$-dimensional vector space over the field $K$ and let $T : V \to V$ be a linear operator. Write $K[T]$ to denote the ring of all linear operators on $V$ that can be expressed as polynomials in $T$, and let $C$ denote the $K$-vector space of all linear operators on $V$ that commute with $T$. Assume that there exists a vector $v_0 \in V$ that is contained in no proper $T$-invariant subspace of $V$.
   a. Show that $v_0K[T] = V$ and deduce that $\dim_K K[T] \geq n$. (3 points)
   a. If $S \in C$ with $v_0S = 0$, show that $S = 0$. Deduce that $\dim_K C \leq n$. (3 points)
   c. Show that $K[T] = C$, and deduce that the minimal polynomial of $T$ has degree equal to $n$. (4 points)

(over)
5. Let $R$ be a ring with 1, let $V$ be a right $R$-module with a composition series, and let $E = \text{End}_R(V)$ be the ring of $R$-endomorphisms of $V$.

a. If $\theta: V \to V$ is an element of $E$, prove that $\theta$ is one-to-one if and only if it is onto and hence if and only if it is invertible in $E$. (4 points)

b. If $V$ has a unique minimal $R$-submodule $U$, prove that $E$ has a unique maximal ideal $I$ and that every element of $E \setminus I$ is invertible. Be sure to verify that $I$ is indeed an ideal. (3 points)

c. Again, let $V$ have unique minimal submodule $U$, and suppose in addition that $U$ has multiplicity 1 as an $R$-composition factor of $V$. Prove that $E$ is a division ring. (3 points)
1. For any finite group $G$ and prime $p$, we let $n_p(G)$ denote the number of Sylow $p$-subgroups of $G$. Now suppose $K \triangleleft G$, and let $P$ be a Sylow $p$-subgroup of $G$.
   a. Show that $n_p(G/K)$ divides $n_p(G)$. (5 points)
   b. Prove that $n_p(G/K) = n_p(G)$ if and only if $P \triangleleft PK$. (5 points)

2. Let $R$ be a commutative ring with 1, and recall that a proper ideal $I \triangleleft R$ is said to be primary if, for all $r, s \in R$, the inclusion $rs \in I$ implies that either $r \in I$ or $s^n \in I$ for some integer $n \geq 1$. Assume now that every proper ideal of $R$ is primary.
   a. If $P$ is a prime ideal of $R$ and if $I \triangleleft R$, prove that either $I \subseteq P$ or $P = IP \subseteq I$. (4 points)
   b. If $M$ is a maximal ideal of $R$, prove that $M$ is precisely the set of nonunits of $R$. (3 points)
   c. Show that a proper ideal $J$ of $R$ is prime if and only if, for all $r \in R$, the inclusion $r^2 \in J$ implies that $r \in J$. (3 points)

3. Let $F$ be a field with algebraic closure $\overline{F}$, let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$, and let $E \supseteq F$ be the splitting field of $f(x)$ over $F$ with $E \subseteq \overline{F}$. Assume that $f(x)$ has $n$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $E$.
   a. Show that there exists an element $\beta \in E$ and $n$ polynomials $p_i(x) \in F[x]$ with $p_i(\beta) = \alpha_i$ for all $i = 1, 2, \ldots, n$. (3 points)
   b. Continuing with the notation of (a), let $g(x) \in F[x]$ be the minimal polynomial of $\beta$ over $F$. If $\gamma \in \overline{F}$ is any root of $g(x)$, show that $p_1(\gamma), p_2(\gamma), \ldots, p_n(\gamma)$ are equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$ in some order. (4 points)
   c. Continuing with the notation of (a) and (b), if $\gamma$ and $\gamma'$ are both roots of $g(x)$ and if $p_i(\gamma) = p_i(\gamma')$ for all $i = 1, 2, \ldots, n$, show that $\gamma = \gamma'$. (3 points)

4. Let $A$ be an $n \times n$ matrix over an algebraically closed field $K$.
   a. Show that $A = B + C$ where $B$ is a diagonalizable matrix, $C$ is nilpotent with $C^n = 0$, and $BC = CB$. (4 points)
   b. If char $K = p > 0$, prove that $A^{p^t}$ is diagonalizable for some integer $t \geq 0$. (2 points)
   c. If $K$ is the complex number field, prove that the exponential matrix $\exp(A) = \sum_{k=0}^{\infty} A^k/k!$ exists. (4 points)

(over)
5. Let $R$ be a ring with $1$, let $V$ be a right $R$-module, and let $W$ be a submodule of $V$. Suppose that $V = V_1 + V_2 + \cdots + V_n = \sum_i V_i$ is an internal direct sum of the simple (that is, irreducible) submodules $V_1, V_2, \ldots, V_n$. Furthermore, for each subscript $j$, let $V_j' = \sum_{i \neq j} V_i$ be the internal direct sum of those $V_i$ with $i \neq j$, so that $V = V_j + V_j'$.

a. If $W \neq 0$, prove that $W$ contains a minimal proper submodule and a maximal proper submodule. (3 points)

b. If $W$ is a maximal proper submodule of $V$, prove that there exists a subscript $k$ with $V = W + V_k$ and hence that $V/W \cong V_k$. (3 points)

c. If $W$ is simple, show that there exists a subscript $j$ with $W \cong V_j$ and $W + V_j' = V$. (4 points)
Algebra Qualifying Exam
January 2003

Do all 5 problems. In the following, \( \mathbb{Z} \) denotes the ring of integers, \( \mathbb{Q} \) is the field of rational numbers, and \( \mathbb{C} \) is the field of complex numbers.

1. Let \( N \) be a normal subgroup of the finite group \( G \) and suppose that \( G/N \) is a \( p \)-group for some prime \( p \).
   a. If \( N \subseteq \mathcal{Z}(G) \), the center of \( G \), show that the commutator subgroup \( G' \) of \( G \) is a \( p \)-group. (5 points)
   b. Now assume that \( N \) is cyclic (but not necessarily central in \( G \)). Prove that \( N \cap G' \subseteq \mathcal{Z}(G') \) and deduce that \( G'' \) is a \( p \)-group. (5 points)

2. Let \( R \) be a commutative integral domain with 1. A nonzero, nonunit element \( s \in R \) is said to be “special” if, for every element \( a \in R \), there exist \( q, r \in R \) with \( a = qs + r \) and such that \( r \) is either 0 or a unit of \( R \).
   a. If \( s \in R \) is special, prove that the principal ideal \((s)\) generated by \( s \) is maximal in \( R \). (3 points)
   b. Show that every polynomial in \( \mathbb{Q}[X] \) of degree 1 is special in \( \mathbb{Q}[X] \). (2 points)
   c. Prove that there are no special elements in the polynomial ring \( \mathbb{Z}[X] \). (Hint. Apply the definition of special with \( a = 2 \) and with \( a = X \).) (5 points)

3. Let \( F \) be a field with \( \mathbb{Q} \subseteq F \subseteq \mathbb{C} \), where \( F/\mathbb{Q} \) is a finite Galois extension. Let \( \alpha \in F \) and let \( f(X) \in \mathbb{Q}[X] \) be its minimal monic polynomial. Assume that \( 1 = |\alpha| \), the absolute value of \( \alpha \), and that Gal\((F/\mathbb{Q})\) is abelian.
   a. Show that \( F \) is closed under complex conjugation. (2 points)
   b. Prove that \( |\beta| = 1 \) for every complex root \( \beta \) of \( f(X) \). (3 points)
   c. Writing \( f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \), show that \( |a_i| \leq 2^n \) for all \( i \) with \( 0 \leq i < n \). (2 points)
   d. Prove that \( F \) contains only finitely many algebraic integers having absolute value 1 and deduce that each of these is a root of unity. (3 points)

(over)
4. Let $V$ be vector space over the field $K$ and let $(\ , \ ) : V \times V \rightarrow K$ be a bilinear form on $V$.
   
   a. If $V$ is finite dimensional and if $W$ is a proper subspace of $V$, show that there exists a nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$. (5 points)
   
   b. Now let $V$ have an infinite basis $\mathcal{B}$ and let $(\ , \ )$ be the unique bilinear form such that, for all $a, b \in \mathcal{B}$, we have $(a, b) = 0$ if $a \neq b$ and $(a, b) = 1$ if $a = b$. If $W$ is the subspace of $V$ spanned by all vectors of the form $a - b$ with $a, b \in \mathcal{B}$, show that $W$ is a proper subspace of $V$ and that there is no nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$. (5 points)

5. Let $R$ be a ring with 1. We say that a right $R$-module $W$ is "infinitely generated" if it is not finitely generated as an $R$-module.
   
   a. Let $V$ be a right $R$-module and let $W$ be a submodule of $V$. If $W$ is infinitely generated, prove that there exists a submodule $M$ with $W \subseteq M \subseteq V$ such that $M$ is infinitely generated, but such that all submodules of $V$ properly containing $M$ are finitely generated. (5 points)
   
   b. If $R$ is right Noetherian, show that $M = V$ in the above situation. (2 points)
   
   c. If $R$ is not right Noetherian, show that it is possible to choose $V$ and $W$ as in part (a) so that $M \neq V$. (3 points)
1. Let $G$ be a finite group of order $504 = 2^3 \cdot 3^2 \cdot 7$.
   a. Show that $G$ cannot be isomorphic to a subgroup of the alternating group $\text{Alt}_7$. (5 points)
   b. If $G$ is simple, determine the number of Sylow 3-subgroups of $G$. (5 points)

2. Let $R$ be a commutative integral domain with 1.
   a. Let $K$ be the field of fractions of $R$, let $t \in R$ be a nonzero element, and suppose that $K = R[1/t]$. In other words, every element of $K$ can be written as a polynomial in $1/t$ with coefficients in $R$. Show that $t$ is contained in every nonzero prime ideal of $R$. (5 points)
   b. Now suppose that $R$ is the polynomial ring $R = F[X_1, X_2, \ldots, X_n]$ where $F$ is an infinite field. If $f(X_1, X_2, \ldots, X_n)$ is contained in every nonzero prime ideal of $R$, show first that $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in F$. Then prove that the latter zero-value property implies that $f$ is the zero polynomial. (5 points)

3. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F[\alpha]$. Assume that some power of $\alpha$ lies in $F$ and let $n$ be the smallest positive integer such that $\alpha^n \in F$.
   a. If $\alpha^m \in E$ with $m > 0$, show that $m$ is a multiple of $n$. (2 points)
   b. If $E$ is a separable extension of $F$, prove that the characteristic of $F$ does not divide $n$. (4 points)
   c. If every root of unity in $E$ lies in $F$, show that $|E : F| = n$. (4 points)

4. Let $A$ be a real $n \times n$ matrix. We say that $A$ is a difference of two squares if there exist real $n \times n$ matrices $B$ and $C$ with $BC = CB = 0$ and $A = B^2 - C^2$.
   a. If $A$ is a diagonal matrix, show that it is a difference of two squares. (3 points)
   b. If $A$ is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
   c. Suppose $A$ is a difference of two squares, with corresponding matrices $B$ and $C$ as above. If $B$ has a nonzero real eigenvalue, prove that $A$ has a positive real eigenvalue. (4 points)

5. Let $K$ be a field of characteristic 0 and view the polynomial ring $V = K[x]$ as a $K$-vector space. Let $M : V \to V$ be the linear operator given by multiplication by $x$, so that $M(x^n) = x^{n+1}$ for all integers $n \geq 0$. In addition, let $D : V \to V$ be the linear operator given by differentiation with respect to $x$, so that $D(x^n) = nx^{n-1}$ for all $n \geq 0$. Let $L$ denote the set of all linear operators of the form $M^i D^j$ with $i, j \geq 0$, where $M^0 = D^0 = I$ is the identity operator on $V$.
   a. Prove that $DM - MD = I$. (3 points)
   b. Show that $L$ is a $K$-linearly independent set. (4 points)
   c. For all nonnegative integers $t$, prove that $DM^t$ is in the $K$-linear span of the set $L$. (3 points)
Do all 5 problems.

1. Let $G$ be a finite group and let $H \subseteq G$ be a subgroup of index $|G : H| = n$.
   a. Show that $|H : (H \cap H^g)| \leq n$ for all $g \in G$. (2 points)
   b. If $H$ is a maximal subgroup of $G$ and $H$ is abelian, show that $(H \cap H^g) \triangleleft G$ for all $g \notin H$. (3 points)
   c. Now suppose that $G$ is simple. If $H$ is abelian and $n$ is a prime, prove that $H = 1$. (5 points)

2. Let $K$ be a field and let $R$ be the subring of the polynomial ring $K[X]$ given by all polynomials with $X$-coefficient equal to 0.
   a. Prove that the elements $X^2$ and $X^3$ are irreducible but not prime in the ring $R$. (5 points)
   b. Show that $R$ is a Noetherian ring, and that the ideal $I$ of $R$ consisting of all polynomials in $R$ with constant term 0 is not principal. (5 points)

3. Recall that a field $K$ is algebraically closed if every polynomial $f \in K[X]$ splits over $K$ (is a product of linear factors in $K[X]$). Now let $F \subseteq E$ be an algebraic field extension.
   a. If every polynomial $f(X) \in F[X]$ splits over $E$, prove that $E$ is algebraically closed. (4 points)
   b. If every polynomial $f(X) \in F[X]$ has a root in $E$ and if $F$ has characteristic 0, prove that $E$ is algebraically closed. (6 points)

4. Let $V$ be a finite dimensional vector space over the field $F$. Suppose $T : V \to V$ is a linear operator and let $f(X) \in F[X]$ be its minimal polynomial.
   a. If $f(X)$ has a nonconstant polynomial factor of degree $m$, show that $V$ has a nonzero subspace $W$ of dimension $\leq m$ with $T(W) \subseteq W$. (5 points)
   b. Conversely, if $V$ has a nonzero subspace $W$ of dimension $n$ with $T(W) \subseteq W$, show that $f(X)$ has a nonconstant polynomial factor of degree $\leq n$. (5 points)

5. Let $R$ be a ring with 1 and let $V$ be a right $R$-module. Suppose that $V = X \oplus Y$ is the internal direct sum of the two nonzero submodules $X$ and $Y$.
   a. Show that 0, $X$, $Y$ and $V$ are the only $R$-submodules of $V$ if and only if $X$ and $Y$ are nonisomorphic simple $R$-modules. (6 points)
   b. If $X$ and $Y$ are nonisomorphic simple $R$-modules, prove that $\text{End}_R(V)$, the ring of $R$-endomorphisms of $V$, is isomorphic to the direct sum of two division rings. (4 points)
Algebra Qualifying Exam
August 2004

Do all 5 problems.

1. Let $G$ be a finite group of order $pm$, where $p$ is a prime that does not divide $m$, and let $n$ denote the number of Sylow $p$-subgroups of $G$.

   a. Show that the exists a homomorphism $\theta$ from $G$ to the symmetric group $\text{Sym}(n)$ such that, for all $x \in G$ of order $p$, the image $\theta(x)$ has exactly one fixed point. (4 points)

   b. Now suppose that $G$ is simple and contains an element $y$ of order $pq$, for some prime $q \neq p$. If $\theta$ is as in part (a), show that $\theta(y)$ must contain a cycle of length $pq$ in its cycle decomposition. (3 points)

   c. Now let $p = 5$ and suppose that $G$ is a simple group of order 660. Show that $G$ has no element of order 15. (3 points)

2. Let $R$ be a ring with 1, let $M$ be a finitely generated right $R$-module, and let $N < M$ be a proper submodule of $M$.

   a. Prove that there exists a maximal submodule $K$ of $M$ with $N \subseteq K < M$. (5 points)

   b. Show that $N + MJ < M$, where $J = J(R)$ denotes the Jacobson radical of $R$. (5 points)

3. In the field $\mathbb{C}$ of complex numbers, let $Q$ be the subfield of rational numbers, let $i = \sqrt{-1}$, and let $\sqrt[4]{2}$ be the positive real fourth root of 2.

   a. Prove that the polynomial $X^4 - 2$ is irreducible over the field $\mathbb{Q}[i]$. (4 points)

   b. If $\sqrt[4]{2} + i$ is a root of a polynomial $f(X) \in \mathbb{Q}[X]$, show that $i\sqrt[4]{2} + i$ is also a root of that polynomial. (3 points)

   c. Compute the degree of the minimal polynomial of $\sqrt[4]{2} + i$ over $\mathbb{Q}$. (3 points)

4. Let $V$ be a vector space of dimension $n$ over a field $K$. Suppose $V$ is spanned by the $n + 1$ vectors $v_0, v_1, \ldots, v_n$ where $v_0 + v_1 + \cdots + v_n = 0$. Now let $W$ be a second $K$-vector space and let $w_0, w_1, \ldots, w_n \in W$. Find necessary and sufficient conditions on the elements $w_0, w_1, \ldots, w_n$ so that there exists a linear transformation $T: V \to W$ with $T(v_i) = w_i$ for $i = 0, 1, \ldots, n$. (10 points)

5. Let $k$ be a field, let $K = k(x, y)$ be the rational function field over $k$ in the indeterminates $x$ and $y$, and let $\overline{K}$ denote the algebraic closure of $K$. Suppose $s$ and $t$ are elements of $\overline{K}$ with $s^2 = x + y$ and $t^3 = xy$, and let $R = k[s, t]$ be the subring of $\overline{K}$ generated by $k$, $s$ and $t$. Show that every element $r \in R$ is the root of some irreducible monic polynomial $f(Z) \in K[Z]$ of degree at most 6 with all coefficients in the polynomial ring $k[x, y]$. (10 points)
Algebra Qualifying Exam - January 2005

Do all 5 problems. Show all work.

1. Let $G$ be a finite group with $|G| = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$ and suppose that $E \subseteq G$ is a subgroup of order 11. Assume that $C_G(E) = E$.
   (a) Prove that $|N_G(E)| = 55$. (3 points)
   (b) If $M \triangleleft G$, show that either $E \subseteq M$ or $|M| \equiv 1 \mod 11$. (3 points)
   (c) Show that every minimal normal subgroup of $G$ contains $E$. (4 points)

2. All rings in this problem are commutative with 1. A ring $S$ is said to be finitely generated if there exist finitely many elements $s_1, s_2, \ldots, s_n \in S$ such that every element of $S$ can be written as a sum of products of these generators. Now let $R$ be a ring, let $G$ be a finite group of automorphisms of $R$, and let $R^G = \{ r \in R \mid r^g = r \text{ for all } g \in G \}$ be the fixed subring.
   (a) If $r \in R$, prove that $R^G$ contains a finitely generated subring $T$ such that $r$ is integral over $T$. (4 points)
   (b) If $R$ is finitely generated, show that $R^G$ contains a finitely generated subring $S$ such that $R$ is integral over $S$. (2 points)
   (c) Let $R$ and $S$ be as in (b). Deduce that $R$ is a finitely generated $S$-module and hence that $R^G$ is a finitely generated $S$-module. Conclude that $R^G$ is a finitely generated ring. (Hint. You can use the fact that any finitely generated ring is a homomorphic image of a polynomial ring in finitely many variables over the integers and hence is a Noetherian ring.) (4 points)

3. Let $F$ be a field and let $f(X) \in F[X]$ be an irreducible polynomial. Suppose $E \supseteq F$ is an extension field of $F$ containing a root $\alpha$ of $f(X)$ satisfying $f(\alpha^2) = 0$. Show that $f(X)$ splits over $E$. (10 points)

4. Let $F$ be an algebraically closed field and let $M_n(F)$ be the ring of $n \times n$ matrices over $F$. Describe those matrices $X \in M_n(F)$ with the property that all matrices that commute with $X$ are diagonalizable. (10 points)

5. An additive abelian group $U$ is said to be uniform if, for every two nonzero subgroups $X$ and $Y$, we have $X \cap Y \neq 0$. Let us also say that $U$ is max-uniform if $U$ is uniform and if $U$ is not contained in any properly larger uniform group.
   (a) If $U$ is uniform and has a nonzero element of finite order, show that there exists a prime $p$ such that every element of $U$ has order a power of $p$. (3 points)
   (b) Let $A$ be an abelian group and let $U$ be a uniform subgroup. Suppose $M$ is a subgroup of $A$ maximal with the property that $M \cap U = 0$. Show that $A/M$ is a uniform group. (3 points)
   (c) Let $A$ be an abelian group and let $U$ be a max-uniform subgroup. Prove that there exists a subgroup $M$ of $A$ with $A = U + M$, the internal direct sum of $U$ and $M$. Include details of the Zorn’s Lemma argument. (4 points)
Algebra Qualifying Exam - January 2006

Do all 5 problems. Show all work.

1. Let $A$, $B$ and $K$ be minimal normal subgroups of the group $G$ with $K \neq A$, $K \neq B$ and $K \subseteq AB$.
   (a) Show that $KA = AB = KB$. (4 points)
   (b) Prove that $A \cong K \cong B$. (3 points)
   (c) Show that $AB$ is abelian. (3 points)

2. Let $\mathbb{Z}[x]$ be the polynomial ring over the integers $\mathbb{Z}$ in the indeterminant $x$. Let $R$ be the subring of $\mathbb{Z}[x]$ consisting of all polynomials having their coefficients of $x$ and $x^2$ equal to 0.
   (a) Prove that $\mathbb{Q}(x)$ is the field of fractions of $R$, where $\mathbb{Q}$ is the field of rational numbers. (2 points)
   (b) Find the integral closure of $R$ in $\mathbb{Q}(x)$. (4 points)
   (c) Does there exist a polynomial $g(x) \in R$ such that $R$ is generated as a ring by 1 and $g(x)$? (4 points)

3. Let $n$ be a positive integer and let $F$ be a field of characteristic not dividing $n$. Let $f(x) \in F[x]$ be the polynomial $x^n - a$ for some $0 \neq a \in F$ and let $E$ be a splitting field for $f(x)$ over $F$.
   (a) Show that $E$ contains a primitive $n$th root of unity $\varepsilon$. (3 points)
   (b) If $\varepsilon \in F$, show that all irreducible factors of $f(x)$ in $F[x]$ have the same degree and that $|E : F|$ divides $n$. (3 points)
   (c) Now assume that $n$ is a power of 2, but do not assume that $\varepsilon \in F$. Prove that $|E : F|$ is a power of 2. (4 points)

4. Let $V$ be a finite-dimensional vector space over the real numbers $\mathbb{R}$.
   (a) If $\dim_{\mathbb{R}} V$ is odd, prove that every linear operator $A : V \to V$ has at least one real eigenvalue. (3 points)
   (b) Suppose $A_1, A_2, \ldots, A_n$ are finitely many pairwise commuting linear operators on $V$. Assume that none of the operators $A_i$ has a negative real eigenvalue. If the sum $A_1 + A_2 + \cdots + A_n$ is equal to the negative of the identity operator on $V$, show that $\dim_{\mathbb{R}} V$ is even. (Hint. Use induction on the dimension of $V$.) (7 points)

5. Let $R$ be a ring with 1 and let $M$ be a right $R$-module. We say that the module $M$ has property $(\ast)$ if every nonzero homomorphic image of $M$ has a simple submodule.
   (a) If $M$ is generated by its artinian submodules, show that $M$ has property $(\ast)$. (5 points)
   (b) If $M$ has property $(\ast)$ and is noetherian, show that it is artinian. (5 points)
Algebra Qualifying Exam  
August 2006  

Do all 5 problems.

1. Let $M$ be a minimal normal subgroup of the finite group $G$ and let $N/M$ be a nontrivial normal subgroup of $G/M$. Assume that $M$ is a $p$-group and that $N/M$ is a $q$-group for some primes $p$ and $q$, not necessarily distinct.
   a. Show that $G = MH$ where $H$ is a subgroup of $G$ having a nontrivial normal $q$-subgroup. (4 points)
   b. If $M$ is self-centralizing in $G$, prove that $p \neq q$. (3 points)
   c. If $M$ is self-centralizing and if $H$ is as in part a, prove that $M \cap H = 1$. (3 points)

2. Let $R$ be a ring with 1, not necessarily commutative. Recall that an element $e$ of $R$ is an idempotent if $e^2 = e$, and an element $0 \neq r \in R$ is a zero divisor if there exists $0 \neq s \in R$ with $rs = 0$ or $sr = 0$. Now assume that $R$ has a nil ideal $N$ such that $R/N$ has no zero divisors.
   a. Show that the only idempotents of $R$ are the elements 0 and 1. (5 points)
   b. If $R/N$ is a division ring, prove that every zero divisor in $R$ is nilpotent. (5 points)

3. Let $C \supseteq E \supseteq K \supseteq Q$ be a chain of fields, where $C$ is the field of complex numbers, $Q$ is the field of rational numbers, $E = Q[\alpha]$ with $\alpha^n \in Q$, and $K$ is generated by all roots of unity in $E$. Assume that $E$ is a Galois extension of $Q$.
   a. Show that the Galois group $\text{Gal}(E/K)$ is cyclic. (5 points)
   b. If the restriction $\tau$ of complex conjugation to $E$ is in the center of $\text{Gal}(E/Q)$, prove that $|\alpha|^2 \in Q$, where $| \cdot |$ denotes complex absolute value. (5 points)

4. Let $V \neq 0$ be a finite dimensional vector space over a field $F$ and let $T: V \to V$ be a linear transformation. We say that $T$ is regular if its characteristic polynomial and minimal polynomial are equal.
   a. If there exists a vector $v \in V$ such that $V$ is spanned by $v, T(v), T^2(v), \ldots$, prove that $T$ is regular. (5 points)
   b. Assume that $T$ is regular and let $W$ be a subspace of $V$ with $T(W) \subseteq W$. Show that $T_W$, the restriction of $T$ to $W$, and $T_{V/W}$, the induced action of $T$ on $V/W$, are both regular. (5 points)

5. Let $F = \text{GF}(q)$ be the finite field with $q$ elements and let $M_2(F)$ be the ring of $2 \times 2$ matrices over $F$.
   a. If $A \in M_2(F)$ has equal eigenvalues in the algebraic closure of $F$, show that the eigenvalues of $A$ actually belong to $F$. (4 points)
   b. Determine the number of nonzero nilpotent matrices in $M_2(F)$ as a function of $q$. (Hint. Use Jordan canonical form and note that the group $G$ of invertible $2 \times 2$ matrices over $F$ has order $(q^2 - 1)(q^2 - q)$.) (6 points)
Algebra Qualifying Exam
January 2007

Do all 5 problems.

1. Let $G$ be a finite group and let $\text{Syl}_p(G)$ denote its set of Sylow $p$-subgroups.
   a. Suppose that $S$ and $T$ are distinct members of $\text{Syl}_p(G)$ chosen so that $S \cap T$ is maximal among all such intersections. Prove that the normalizer $N_G(S \cap T)$ has more than one Sylow $p$-subgroup. (5 points)
   b. Show that $S \cap T = 1$ for all $S, T \in \text{Syl}_p(G)$, with $T \neq S$, if and only if $N_G(P)$ has exactly one Sylow $p$-subgroup for every nonidentity $p$-subgroup $P$ of $G$. (5 points)

2. Let $R$ be a commutative, Noetherian integral domain.
   a. If $P$ is a prime ideal of $R$, show that the radical of $P^n$ is $P$. (2 points)
   b. If $R$ has a unique nonzero prime ideal $P$, prove that all ideals of $R$ are primary. (3 points)
   c. Conversely, let us now assume that all ideals of $R$ are primary, and let $P$ and $Q$ be distinct prime ideals of $R$ with $Q \not\supseteq P$. Since $P^n \cap Q$ is primary, deduce first that $P^n \supseteq Q$ and then that $Q = 0$. (Hint. Consider whether the intersection $P^n \cap Q$ can be irredundant.) (5 points)

3. Let $F$ be a field of characteristic 0 and let $f \in F[X]$ be an irreducible polynomial of degree $> 1$ with splitting field $E \supseteq F$. Define $\Omega = \{\alpha \in E \mid f(\alpha) = 0\}$.
   a. Let $\alpha \in \Omega$ and let $m$ be a positive integer. If $g \in F[X]$ is the minimal polynomial of $\alpha^m$ over $F$, show that $\{\beta^m \mid \beta \in \Omega\}$ is the set of roots of $g$. (3 points)
   b. Now fix $\alpha \in \Omega$ and suppose that $\alpha r \in \Omega$ for some $r \in F$. Show that, for all $\beta \in \Omega$ and integers $i \geq 0$, we have $\beta r^i \in \Omega$. Conclude that $r$ is a root of unity. (3 points)
   c. If $\alpha$ and $r$ are as in (b) and if $m$ is the multiplicative order of the root of unity $r$, show that $f(X) = g(X^m)$, where $g$ is the minimal polynomial of $\alpha^m$ over $F$. (4 points)

4. Let $V$ be a finite dimensional vector space over a field $K$ and assume that $V$ is endowed with a not necessarily symmetric bilinear form $\langle \ , \ \rangle: V \times V \to K$. We Let $R$ and $L$ denote the right and left radicals of $\langle \ , \ \rangle$ given by $R = \{x \in V \mid \langle V, x \rangle = 0\}$ and $L = \{x \in V \mid \langle x, V \rangle = 0\}$, so that these are both subspaces of $V$.
   a. Use the bilinear form to construct a linear transformation $T$ from $V$ to the dual space $(V/R)^*$ of $V/R$ such that $\ker(T) = L$. (6 points)
   b. Show that $\dim_K L = \dim_K R$, and deduce that the map $T$ is surjective. (4 points)

5. Let $A$ be an additive abelian group and let $B$ be a subgroup. We say that $B$ is essential in $A$, and write $\text{B ess } A$, if and only if $B \cap X \neq 0$ for all nonzero subgroups $X$ of $A$.
   a. If $B_1 \text{ ess } A_1$ and $B_2 \text{ ess } A_2$, prove that $(B_1 \oplus B_2) \text{ ess } (A_1 \oplus A_2)$. (5 points)
   b. If $B \text{ ess } A$, and $B$ has no nonzero elements of finite order, prove that $A$ has no nonzero elements of finite order. (2 points)
   c. Let $Q$ denote the additive group of rational numbers and suppose that $Q \text{ ess } A$, for some abelian group $A$. Prove that $Q = A$. (3 points)
1. Let $G$ be a finite group of order $|G| = 504 = 2^3 \cdot 3^2 \cdot 7$.
   a. If $G$ has a normal subgroup $N$ of order 8, show that $G$ has at most 8 Sylow 7-subgroups, that is $|\text{Syl}_7(G)| \leq 8$. (5 points)
   b. If $|\text{Syl}_7(G)| \leq 8$, prove that $G$ has an element of order 21. (4 points)
   c. If $G$ is isomorphic to a subgroup of $\text{Sym}_9$, the symmetric group of degree 9, show that $G$ cannot have a normal subgroup of order 8. (1 point)

2. Let $R$ be a commutative integral domain with field of fractions $F$, and assume that $R$ is integrally closed.
   a. Suppose $K$ is a field containing $F$ and let $\alpha \in K$ be integral over $R$. Show that the minimal monic polynomial of $\alpha$ over $F$ is contained in $R[x]$. (5 points)
   b. Let $f(x) \in R[x]$ be a monic polynomial. Show that $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. (5 points)

3. Let $F$ be a field of characteristic 0 and let $E$ be a finite Galois extension of $F$.
   a. If $0 \neq \alpha \in E$ with $E = F[\alpha]$, show that $F[\alpha^2] \neq E$ if and only if there exists an automorphism $\sigma \in \text{Gal}(E/F)$ with $\alpha^\sigma = -\alpha$. (6 points)
   b. Prove that there exists an element $\alpha \in E$ with $E = F[\alpha^2]$. (4 points)

4. Let $V$ be a finite-dimensional vector space over the field $F$ with $\dim_F V = n$, and let $(\ , \ ): V \times V \to F$ be a symmetric bilinear form. If $X$ is a subset of $V$, write $X^\perp = \{v \in V \mid (X, v) = 0\}$ for the subspace of $V$ perpendicular to $X$.
   a. If $W$ is a subspace of $V$, show that $\dim_F W + \dim_F W^\perp \geq \dim_F V$. (Hint. If $w \in W$, note that $\{w\}^\perp$ has codimension $\leq 1$ in $V$.) (2 points)
   b. Now suppose $(\ , \ )$ is nonsingular, so that $V^\perp = 0$. If $A = \{a_1, a_2, \ldots, a_n\}$ is a basis for $V$, prove that there exists a unique dual basis $A' = \{a'_1, a'_2, \ldots, a'_n\}$. That is, $A'$ is a basis with $(a_i, a'_j) = 0$ if $i \neq j$ and $(a_i, a'_i) = 1$. (4 points)
   c. Again suppose $(\ , \ )$ is nonsingular, and let $B = \{b_1, b_2, \ldots, b_n\}$ be a second basis for $V$ with dual basis $B' = \{b'_1, b'_2, \ldots, b'_n\}$. Compare the change of basis matrix from $A$ to $B$ with the change of basis matrix from $B'$ to $A'$. (4 points)

5. Let $R$ be a not necessarily commutative ring with 1.
   a. If $V_1, V_2, \ldots, V_n$ are $n$ nonisomorphic irreducible right $R$-modules, show that there exists an $R$-module epimorphism from $R$, viewed as a right $R$-module, to the external direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$. (5 points)
   b. Suppose $R$, viewed as a right $R$-module, has a finite composition series with nonisomorphic composition factors. Prove that the Jacobson radical of $R$ is equal to 0. (5 points)
1. Let $G$ be a finite nonabelian group with center $Z$.
   a. If $G/Z$ is a $p$-group, for some prime $p$, show that $G$ has a normal Sylow $p$-subgroup and that $p$ divides $|Z|$. (5 points)
   b. If $G/Z$ is solvable, show that $G$ has a nonidentity normal $p$-subgroup for some prime dividing $|G : Z|$. (5 points)

2. Let $R \subseteq S$ be commutative rings with the same 1, and suppose that $S$ is finitely generated as an $R$-module.
   a. If an element $r \in R$ is not invertible in $R$, prove that it is not invertible in $S$.
      HINT. If $r$ is invertible in $S$, consider a polynomial in $R[X]$ having $1/r$ as a root. (5 points)
   b. If the ideals of $R$ satisfy the ascending chain condition, show that the ideals of $S$ satisfy the ascending chain condition. (5 points)

3. Working in the field of complex numbers, let $\varepsilon$ be a primitive 16th root of unity, and let $\alpha = \varepsilon\sqrt{2}$. Set $E = \mathbb{Q}[\varepsilon]$, where $\mathbb{Q}$ is the field of rational numbers, let $f(X) = X^8 + 16 \in \mathbb{Q}[X]$, and note that $\alpha$ is a root of $f(X)$.
   a. Show that $\sqrt{2} \in \mathbb{Q}[\varepsilon^2]$. (3 points)
   b. Conclude that $f(X)$ splits in $E[X]$. (2 points)
   c. If $G = \text{Gal}(E/\mathbb{Q})$, prove that no nonidentity element of $G$ fixes $\alpha$. Conclude that $f(X)$ is irreducible in $\mathbb{Q}[X]$. (5 points)

4. Let $V$ be a finite-dimensional vector space over the field $F$ of characteristic $p > 0$, let $T : V \to V$ be a linear operator on $V$, and set $W = \{v \in V \mid vT = v\}$. Suppose that $T^p = I$, the identity, and that $\dim_F W = 1$.
   a. Show that $(T - I)^p = 0$ and conclude that $\dim_F V \leq p$. (4 points)
   b. If $\dim_F V < p$, prove that $(T - I)^{p-1} = 0$. (3 points)
   c. If, for some vector $v \in V$, we have $v + vT + vT^2 + \cdots + vT^{p-1} \neq 0$, prove that $\dim_F V = p$. (3 points)

5. Let $R$ be a ring with 1. A (right) $R$-module $V$ is said to be strongly $n$-generated, for some integer $n$, if every submodule of $V$ is generated as an $R$-module by some set of $\leq n$ elements.
   a. If $V$ is strongly $n$-generated and if $W$ is a submodule of $V$, prove that both $W$ and $V/W$ are strongly $n$-generated. (3 points)
   b. Let $W$ be a submodule of $V$. If $W$ is strongly $n$-generated and if $V/W$ is strongly $m$-generated, prove that $V$ is strongly $(n + m)$-generated. (5 points)
   c. If $V$ has composition length $n$, prove that $V$ is strongly $n$-generated. (2 points)
1. In this problem we prove that a Sylow 2-subgroup of a simple group of order 168 is its own normalizer.
   a. If $G$ is a group of order 24 and $G$ has a normal Sylow 2-subgroup, show that $G$ contains an element of order 6. (4 points)
   b. If $G$ is a simple group and $H$ is a subgroup of $G$ with $|G : H| = 7$, show that $H$ contains no element of order 6. (3 points)
   c. Let $G$ be a simple group with $|G| = 168$ and let $P$ be a Sylow 2-subgroup of $G$. Prove that $N_G(P) = P$. (3 points)

2. Let $\mathbb{Z}$ be the ring of integers and let $S = \mathbb{Z} \oplus \mathbb{Z}$ be the ring external direct sum of two copies of $\mathbb{Z}$. Now let $R$ be the subring of $S$ given by

   $$ R = \{ (a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \mod 6 \}. $$

   a. Show that $R$ is a finitely generated $\mathbb{Z}$-module and conclude that $R$ is a Noetherian ring. (3 points)
   b. Prove that the ideal $P$ of $R$ given by

   $$ P = \{ (a, 0) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv 0 \mod 6 \} $$

   is prime. (2 points)
   c. If $Q$ is a primary ideal of $R$ with $P = \sqrt{Q}$, the radical of $Q$, show that $Q = P$. (5 points)

3. Let $\mathbb{C}$ denote the complex number field and let $E \subseteq \mathbb{C}$ be the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $x^3 - 2$.
   a. Show that $|E : \mathbb{Q}| = 6$. (2 points)
   b. If $\alpha \in E$ and $\alpha^5 \in \mathbb{Q}$, prove that $\alpha \in \mathbb{Q}$. (5 points)
   c. Show that there exists $\beta \in E$ with $\beta^2 \in \mathbb{Q}$, but $\beta \notin \mathbb{Q}$. (3 points)

(over)
4. Let $S$, $T$ and $M$ be $n \times n$ matrices over the complex numbers $\mathbb{C}$ and suppose that $SM = MT$.
   
   a. If $f(x) \in \mathbb{C}[x]$ is the minimal polynomial of $T$, show that $f(S)M = 0$. (4 points)
   
   b. If $M \neq 0$, deduce that $S$ and $T$ have a common eigenvalue. (3 points)
   
   c. Now suppose $n = 2$,

   $$S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}. $$

   Find a nonzero matrix $M$ with $SM = MT$ and show that it is impossible to find an invertible matrix $M$ with this property. (3 points)

5. Let $R$ be a subring of the ring $M_n(\mathbb{C})$ of all complex $n \times n$ matrices, and suppose that $R$ is finitely generated as module over the integers $\mathbb{Z}$. Let $M \in R$.
   
   a. Show that $M$ is contained in a commutative subring $S$ of $M_n(\mathbb{C})$ that is finitely generated as a $\mathbb{Z}$-module. (3 points)
   
   b. Deduce that there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(M) = 0$. (2 points)
   
   c. Prove that $\text{tr}(M)$, the matrix trace of $M$, is an algebraic integer. (5 points)
1. Let $G$ be a finite group of order $p(p + 1)$, where $p$ is an odd prime, and assume that $G$ does not have a normal Sylow $p$-subgroup.

(a) Find (with proof) the number of elements of $G$ with order different from $p$. (3 points)

(b) Show that each nonidentity conjugacy class of elements with order different from $p$ has size at least $p$, and conclude that there is precisely one such conjugacy class. (5 points)

(c) Prove that $p + 1$ is a power of 2. (2 points)

2. Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{C} \supseteq \mathbb{R}$ be the complex field. Define $S$ to be the subring of the polynomial ring $\mathbb{C}[X]$ consisting of all polynomials with real constant term so that

$$S = \mathbb{R} + \mathbb{C}X + \mathbb{C}X^2 + \mathbb{C}X^3 + \cdots.$$

(a) Show that the ideal of $S$ consisting of all polynomials with 0 constant term is not principal. (4 points)

(b) Let $I$ be a nonzero ideal of $S$ and choose $0 \neq f \in I$ to have minimal possible degree $n$. If $g \in I$, show that there exists $s \in S$ with $g - sf$ either equal to 0 or to a polynomial of degree $n$. Conclude that $I$ is generated by $f$ and perhaps one additional polynomial of degree $n$. (6 points)

3. Let $F$ be the field $\text{GF}(p)$ of prime order $p > 2$ and suppose that the polynomial $f(X) = X^m + 1 \in F[X]$ is irreducible.

(a) Show that every root of $f$ in a splitting field of the polynomial has multiplicative order $2m$. (4 points)

(b) Prove that $2m$ divides $p^m - 1$, but that $2m$ does not divide $p^n - 1$ for any integer $n$ with $0 < n < m$. (3 points)

(c) Show that $m \neq 4$. (3 points)
4. Let $V$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T: V \to V$ be a linear operator on $V$.

(a) If $T$ is diagonalizable on $V$ and if $W$ is a subspace of $V$ with $T(W) \subseteq W$, prove that $T$ is diagonalizable on $W$. (6 points)

(b) If $T$ has the matrix

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 0
\end{pmatrix}
$$

with respect to some basis of $V$, decide (with proof) whether $T$ is diagonalizable on $V$. (4 points)

5. In the following, all groups are additive abelian groups, and recall that an abelian group is said to be noetherian if its set of subgroups satisfies the ascending chain condition or equivalently the maximal condition. Furthermore, a nonzero group is said to be uniform if it contains no direct sum of nonzero subgroups.

(a) Show that every nonzero noetherian group contains a nonzero uniform subgroup. (4 points)

(b) Suppose $G = U \dot{+} V$ is the internal direct sum of the two subgroups $U$ and $V$ with $U$ uniform. If $G$ contains the direct sum $A \dot{+} B$ with $A$ and $B$ both nonzero, prove that $(A \dot{+} B) \cap V \neq 0$. (3 points)

(c) Let $G = U \dot{+} V$ be as above with $U$ uniform. If $G$ contains the direct sum $A \dot{+} B \dot{+} C$ with $A$, $B$ and $C$ all nonzero, prove that $V$ is not uniform. (3 points)
1. Let $H$ be a maximal subgroup of the finite group $G$ and let $\mathcal{X}$ be the set of normal subgroups $X$ of $G$ such that $X \neq 1$ and $X \cap H = 1$.
   a. Show that all members of $\mathcal{X}$ are minimal normal subgroups of $G$ of the same order. (3 points)
   b. If some member of $\mathcal{X}$ is abelian, show that all members of $\mathcal{X}$ are abelian $p$-groups for some prime $p$. (3 points)
   c. Let $U, V \in \mathcal{X}$ be distinct and assume that $\mathcal{X}$ contains at least one additional member different from $U$ and $V$. Show that $(UV \cap H) \trianglelefteq G$ and conclude that $(UV \cap H) \subseteq \mathbb{Z}(UV)$. (4 points)

2. Let $R \subseteq S$ be commutative rings with the same 1, and assume that every element of $S$ is integral over $R$.
   a. If $r \in R$ has an inverse in $S$, prove that this inverse is contained in $R$. (3 points)
   b. Suppose $R$ is a field and let $s \in S$ be a regular element (that is, if $sx = 0$ for some $x \in S$, then $x = 0$). Show that $s$ is invertible in $S$. (3 points)
   c. If $P$ is a prime ideal of $S$, prove that $P$ is a maximal ideal of $S$ if and only if $R \cap P$ is a maximal ideal of $R$. (4 points)

3. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial with splitting field $E$ over $F$. Choose $\alpha \in E$ with $f(\alpha) = 0$. Furthermore, for some fixed integer $n \geq 1$, let $g(x)$ be an irreducible polynomial in $F[x]$ with $g(\alpha^n) = 0$.
   a. Show that $\deg(g)$ divides $\deg(f)$ and that $\deg(f) / \deg(g) \leq n$. (5 points)
   b. If $\deg(f) / \deg(g) = n$ and if the characteristic of $F$ does not divide $n$, prove that $E$ contains a primitive $n$th root of unity. (5 points)

4. Let $V$ be a vector space over a field $F$ and let $(\cdot, \cdot): V \times V \to F$ be a bilinear form. For each $x \in V$ define $A(x) = \{ y \in V \mid (x, y) = -(y, x) \}$. Now suppose $v$ is a fixed element of $V$ with $(v, v) \neq 0$.
   a. For all $x \in V$, show that $A(x)$ is a subspace of $V$ of codimension at most 1. (4 points)
   b. If the characteristic of $F$ is different from 2, prove that $A(v)$ is a subspace of $V$ of codimension precisely 1. (1 point)
   c. If $F$ is algebraically closed and has characteristic different from 2, show that either $(a, a) = 0$ for every element $a \in A(v)$, or there exists $y \in V \setminus A(v)$ with $(y, y) = 0$. (5 points)

5. A multiplicative abelian group $A$ is said to be “divisible” if, for all $a \in A$ and positive integers $n$, there exists $b \in A$ with $b^n = a$.
   a. If $A$ is divisible and $\overline{A}$ is a homomorphic image of $A$, prove that $\overline{A}$ is divisible. (2 points)
   b. If $A$ is a finite divisible group, prove that $A = 1$. (3 points)
   c. Suppose $A$ is divisible and that $A$ is a subgroup of the abelian group $B$. If $A \cap X > 1$ for all nonidentity subgroups $X$ of $B$, prove that $A = B$. (5 points)
1. Let $S_7$ denote the symmetric group on seven points, and let $A_7$ be the corresponding alternating group.
   (a) Find the number of elements of order 7 in $S_7$, and find the order of the centralizer in $S_7$ of one of these elements. (3 points)
   (b) Find the order of the normalizer of a Sylow 7-subgroup in $A_7$. (3 points)
   (c) Prove that $S_7$ does not contain a simple subgroup $G$ of order $504 = 2^3 3^2 7$. (4 points)

2. Let $E \supseteq K$ be fields with $|E : K| < \infty$ and let $R$ be a subring (with 1) of $K$ having $K$ as its field of fractions.
   (a) Prove that there exists a ring $S$ with $R \subseteq S \subseteq E$ such that $S$ is a finitely generated $R$-module and such that $E$ is the field of fractions of $S$. (5 points)
   (b) Let $\alpha \in E$ be integral over $R$. If $R$ is integrally closed in $K$, prove that the minimal monic polynomial $f(X) \in K[X]$ of $\alpha$ over $K$ has all its coefficients in $R$. (5 points)

3. Let $F \subseteq E$ be finite fields, where $|F| = q < \infty$ and $|E : F| = n$.
   (a) Prove that every monic irreducible polynomial in $F[X]$ of degree dividing $n$ is the minimal polynomial over $F$ of some element of $E$. (4 points)
   (b) Compute the product of all the monic irreducible polynomials in $F[X]$ of degree dividing $n$. (2 points)
   (c) Suppose $|F| = 2$. Determine the number of monic irreducible polynomials of degree 10 in $F[X]$. (4 points)

4. Let $V$ be a finite dimensional vector space over the field $F$ and let $T : V \rightarrow V$ be a linear operator on $V$ with characteristic polynomial $f(X) \in F[X]$.
   (a) Show that $f(X)$ is irreducible in $F[X]$ if and only if there are no proper nonzero subspaces $W$ of $V$ with $T(W) \subseteq W$. (6 points)
   (b) If $f(X)$ is irreducible in $F[X]$ and if the characteristic of $F$ is 0, show that $T$ is diagonalizable when we extend the field $F$ to its algebraic closure. (4 points)

5. Let $R$ be a ring (with 1) and let $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$ be a chain of right ideals of $R$ such that each of the $n$ quotients $V_i = I_i/I_{i-1}$ is a simple right $R$-module.
   (a) If $M$ is a maximal right ideal of $R$, prove that $R/M$ is isomorphic as a right $R$-module to some $V_i$. (3 points)
   (b) Now assume that the $V_i$'s are pairwise nonisomorphic $R$-modules. Prove that the intersection of all the maximal right ideals of $R$ is equal to 0. (5 points)
   (c) Continue to assume that the $V_i$'s are pairwise nonisomorphic $R$-modules and deduce that $R$ is a finite ring direct sum of division rings. (2 points)