QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday, August 30, 2000

Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
2. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc in the complex plane.
3. For points $x$ and $y$ in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
4. If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.
5. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,

$$||f||_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}.$$ 

Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

6. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then

$$||f||_{\infty} = \inf \{ t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0 \}.$$ 

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $||f||_{\infty} < +\infty$.

7. $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \subset \mathbb{R}$.

8. If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f * g$ is defined to be the function

$$f * g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt$$

whenever the integral converges.

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Problem I \hspace{1em} \text{Let } \{a_n\}, n \geq 0, \text{ be an infinite sequence of complex numbers. Recall that the infinite series } \sum_{n=0}^{\infty} a_n \text{ converges to a complex number } S \text{ if and only if the sequence of partial sums } \{S_N = \sum_{n=0}^{N} a_n\} \text{ converges to } S. \text{ The infinite series } \sum_{n=0}^{\infty} a_n \text{ is said to be } \textit{Abel summable} \text{ to a complex number } A \text{ if for all real numbers } x \text{ with } 0 \leq x < 1, \text{ the infinite series } \sum_{n=0}^{\infty} a_n x^n \text{ converges, and } \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = A.

(a) Suppose that \sum_{n=0}^{\infty} a_n \text{ converges to } S. \text{ Prove that } \sum_{n=0}^{\infty} a_n \text{ is Abel summable to } S.

(b) Give an example that shows that a series \sum_{n=0}^{\infty} a_n \text{ can be Abel summable, but can fail to converge.}

(c) Prove that if \( a_n \geq 0 \) and if \( \sum_{n=0}^{\infty} a_n \) is Abel summable to a complex number \( A \), then the infinite series \( \sum_{n=0}^{\infty} a_n \) converges to \( A \).

Problem II \hspace{1em} \text{Let } \{f_n\}, n \geq 1, \text{ be an infinite sequence of real valued continuous functions defined for all real numbers } x \in \mathbb{R}. \text{ Suppose there is a function } g \text{ so that } f_n \text{ converges uniformly to } g \text{ on } \mathbb{R}.

(a) Either prove that \( f_n^2 \) converges uniformly to \( g^2 \) on \( \mathbb{R} \), or give a counter-example to this assertion.

(b) Suppose that \( \varphi \) is a uniformly continuous function defined on \( \mathbb{R} \). Prove that \( \varphi(f_n) \) converges uniformly to \( \varphi(g) \) on \( \mathbb{R} \).

(c) Suppose that \( \varphi \) is a bounded continuous function defined on \( \mathbb{R} \). Either prove that \( \varphi(f_n) \) converges uniformly to \( \varphi(g) \), or give a counter-example to this assertion.

(d) Suppose that each \( f_n \) is continuously differentiable, and that \( f'_n \) converges uniformly to a function \( h \). Prove directly from the definitions that \( g \) is differentiable, and that \( g'(z) = h(z) \) for all \( z \in \mathbb{R} \).

Problem III \hspace{1em} \text{Let } \varphi \text{ be an infinitely differentiable function on } \mathbb{R}^3 \text{ with compact support. Recall that the Laplacian is the second order differential operator } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.

(a) For which \( p > 0 \) does \( \lim_{\epsilon \to 0} \iint_{|x^2+y^2+z^2| > \epsilon^2} \frac{\varphi(x, y, z)}{|x^2+y^2+z^2|^{p/2}} \, dx \, dy \, dz \) exist for all such functions \( \varphi \)?

(b) For which \( p > 0 \) is it true that \( \Delta \left[ (x^2+y^2+z^2)^{-p/2} \right] = 0 \) for \( (x, y, z) \neq (0, 0, 0) \)?

(c) Prove that \( \varphi(0) = \lim_{\epsilon \to 0} \frac{1}{4\pi} \iint_{|x^2+y^2+z^2| > \epsilon^2} \frac{\Delta \varphi(x, y, z)}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz \).
Problem IV  Suppose \( \{f_n\} \) is a sequence of complex valued measurable functions on the interval \([0, 1]\) and suppose that \( \lim_{n \to \infty} f_n(x) = g(x) \) for almost every \( x \in [0, 1] \).

(a) Prove that \( g \) is a measurable function.

(b) Prove the following version of Egoroff’s Theorem: Given any \( \epsilon > 0 \), there exists a measurable set \( E \subset [0, 1] \) with the Lebesgue measure \( |E| < \epsilon \) such that \( f_n \to g \) uniformly on \([0, 1] \setminus E\).

(You may use without proof the basic properties of Lebesgue measure, such as countable additivity. However, your proof should not depend on quoting results about convergence theorems for Lebesgue integrals.)

Problem V  Let \( \{f_n\} \) be a sequence of functions belonging to \( L^1(\mathbb{R}) \), the set of integrable functions on the real line \( \mathbb{R} \). Suppose that there is a measurable function \( f \) so that \( \lim_{n \to \infty} f_n(x) = f(x) \) for almost every \( x \in \mathbb{R} \). Consider the following statements:

1. \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = A \) exists.
2. \( \int_{\mathbb{R}} |f(x)| \, dx < +\infty \).
3. \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = \int_{\mathbb{R}} |f(x)| \, dx < +\infty \).
4. \( \lim_{n \to \infty} \int_{E} |(f_n - f)(x)| \, dx = 0 \) for every measurable set \( E \subset \mathbb{R} \).

Discuss (with proofs or examples) which of these four statements do or do not imply other statements in the list.

Problem VI  Let \( f \in L^1(\mathbb{R}) \). For \( y > 0 \), define \( f_y(x) = \frac{2}{\pi} \int_{-\infty}^{+\infty} f(x-t) \frac{y^3}{(t^2 + y^2)^2} \, dt \).

(a) Prove that for each \( y > 0 \) the function \( f_y \in L^1(\mathbb{R}) \).

(b) Prove that \( \lim_{y \to 0} \|f - f_y\|_1 = 0 \). (You may use the fact that \( \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(t^2 + 1)^2} = 1 \).)

(c) Prove that there is a sequence \( y_j \to 0 \) so that for almost all \( x \in \mathbb{R} \), \( \lim_{j \to \infty} f_{y_j}(x) = f(x) \).

(d) A stronger true statement than that in part (c) is that for almost all \( x \in \mathbb{R} \), \( \lim_{y \to 0} f_y(x) = f(x) \). Without giving proofs, discuss some of the steps that are involved in proving this result.
**Problem VII** Evaluate the convergent improper Riemann integral

$$\int_{-\infty}^{+\infty} \frac{\cos \left( \frac{\pi x}{2} \right)}{x^4 - 1} \, dx.$$ 

Be sure to justify all your calculations.

**Problem VII**

(a) For a complex number $z \in \mathbb{C}$, let

$$f_N(z) = \prod_{n=0}^{N} \left( 1 - e^{-\pi n} e^{2\pi i z} \right) \left( 1 - e^{-\pi n} e^{-2\pi i z} \right).$$

Show the sequence $\{f_N\}$ converges, as $N \to \infty$ to an entire holomorphic function $f(z)$ and prove that $f(z) = 0$ if and only if $z = a + ib$ for where $(a, b)$ is a pair of integers.

(b) Does there exist an entire holomorphic function $g$ such that $g(z) = g(z + 1) = g(z + i)$ for all complex numbers $z$ and such that $g(z) = 0$ if and only if $z = a + ib$ where $(a, b)$ is a pair of integers? Why or why not?

(c) Show that the sum

$$\varphi(z) = \frac{1}{z^2} + \sum_{m+n \neq 0} \left( \frac{1}{(z + m + ni)^2} - \frac{1}{(m + ni)^2} \right)$$

converges to a meromorphic function.

(d) Prove that $\varphi'(z) = \varphi'(z + 1) = \varphi'(z + i)$ for all complex numbers $z$, and then show that the same identities are true for the function $\varphi(z)$.

**Problem IX** Let $f$ be a holomorphic function in the unit disk $\mathbb{D}$.

(a) Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{D}$. Prove that $|f'(z)| \leq M \left( 1 - |z| \right)^{-1}$ for all $z \in \mathbb{D}$.

(b) Suppose that $f(0) = 0$ and that $\iint_{\mathbb{D}} |f'(x + iy)|^2 \, dx \, dy = A^2 < +\infty$. Show that for all $z \in \mathbb{D}$,

$$|f(z)| \leq A \log(1 - |z|).$$
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(a) Suppose that \( \sum_{n=0}^{\infty} a_n \) converges to \( S \). Prove that \( \sum_{n=0}^{\infty} a_n \) is Abel summable to \( S \).

(b) Give an example that shows that a series \( \sum_{n=0}^{\infty} a_n \) can be Abel summable, but can fail to converge.

(c) Prove that if \( a_n \geq 0 \) and if \( \sum_{n=0}^{\infty} a_n \) is Abel summable to a complex number \( A \), then the infinite series \( \sum_{n=0}^{\infty} a_n \) converges to \( A \).

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(a) Either prove that \( f_n^2 \) converges uniformly to \( g^2 \) on \( \mathbb{R} \), or give a counter-example to this assertion.

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(a) For which \( p > 0 \) does \( \lim_{\varepsilon \to 0} \int \int \int_{|x^2+y^2+z^2|>\varepsilon} \frac{\varphi(x,y,z)}{|x^2+y^2+z^2|^{p/2}} \ dx \ dy \ dz \) exist for all such functions \( \varphi \)?

(b) For which \( p > 0 \) is it true that \( \Delta \left[ (x^2+y^2+z^2)^{-p/2} \right] = 0 \) for \( (x,y,z) \neq (0,0,0) \)?

(c) Prove that \( \varphi(0) = \lim_{\varepsilon \to 0} -\frac{1}{4\pi} \int \int \int_{|x^2+y^2+z^2|>\varepsilon} \frac{\Delta \varphi(x,y,z)}{\sqrt{x^2+y^2+z^2}} \ dx \ dy \ dz \).
Problem IV  Suppose \( \{f_n\} \) is a sequence of complex valued measurable functions on the interval \([0, 1]\) and suppose that \( \lim_{n \to \infty} f_n(x) = g(x) \) for almost every \( x \in [0, 1] \).

(a)  Prove that \( g \) is a measurable function.

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(You may use without proof the basic properties of Lebesgue measure, such as countable additivity. However, your proof should not depend on quoting results about convergence theorems for Lebesgue integrals.)

Problem V  Let \( \{f_n\} \) be a sequence of functions belonging to \( L^1(\mathbb{R}) \), the set of integrable functions on the real line \( \mathbb{R} \). Suppose that there is a measurable function \( f \) so that \( \lim_{n \to \infty} f_n(x) = f(x) \) for almost every \( x \in \mathbb{R} \). Consider the following statements:

1. \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = A \) exists.
2. \( \int_{\mathbb{R}} |f(x)| \, dx < +\infty \).
3. \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = \int_{\mathbb{R}} |f(x)| \, dx < +\infty \).
4. \( \lim_{n \to \infty} \int_E |f_n - f|(x) \, dx = 0 \) for every measurable set \( E \subset \mathbb{R} \).

Discuss (with proofs or examples) which of these four statements do or do not imply other statements in the list.

Problem VI  Let \( f \in L^1(\mathbb{R}) \). For \( y > 0 \), define \( f_y(x) = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{f(x-t) \, y^3}{(t^2 + y^2)^2} \, dt \).

(a)  Prove that for each \( y > 0 \) the function \( f_y \in L^1(\mathbb{R}) \).

(b)  Prove that \( \lim_{y \to 0} ||f - f_y||_1 = 0 \). (You may use the fact that \( \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(t^2 + 1)^2} = 1 \).)

(c)  Prove that there is a sequence \( y_j \to 0 \) so that for almost all \( x \in \mathbb{R} \), \( \lim_{j \to \infty} f_{y_j}(x) = f(x) \).

(d)  A stronger true statement than that in part (c) is that for almost all \( x \in \mathbb{R} \), \( \lim_{y \to 0} f_y(x) = f(x) \). Without giving proofs, discuss some of the steps that are involved in proving this result.
Problem VII  Let $H$ be a real Hilbert space with inner product $\langle \ , \ \rangle$. A sequence $\{x_n\}_{n \geq 1}$ converges weakly to $x_0$ if and only if $\lim_{n \to \infty} \langle x_n, y \rangle = \langle x_0, y \rangle$ for all $y \in H$. A sequence $\{x_n\}_{n \geq 1}$ converges strongly to $x_0$ if and only if $\lim_{n \to \infty} ||x_n - x_0|| = 0$. A bounded linear operator $T : H \to H$ is compact if for every bounded sequence $\{x_n\}$ of vectors in $H$, there is a subsequence of integers $n_j \to \infty$ so that $T(x_{n_j})$ converges strongly in $H$.

(a) Prove that if a sequence $\{x_n\}$ converges weakly to $x_0$ and if a sequence $\{y_n\}$ converges strongly to $y_0$, then $\lim_{n \to \infty} \langle x_n, y_n \rangle = \langle x_0, y_0 \rangle$.

(b) Prove that if $T$ is a compact operator on the Hilbert space $H$, then there is a vector $x_0 \in H$ with $||x_0|| = 1$ and $\sup_{||x|| \leq 1} \langle T(x), x \rangle = \langle T(x_0), x_0 \rangle$.

(c) Let $T$ be a compact operator on a Hilbert space $H$. Let $\lambda \in \mathbb{C}$ be a non-zero complex number, and let $E_\lambda = \{x \in H \mid T(x) = \lambda x\}$. Prove that $E_\lambda$ is a finite dimensional subspace of $H$.

Problem VIII  A distribution $T$ on $\mathbb{R}^2$ is a continuous linear functional on the space $C_0^\infty(\mathbb{R}^2)$.

(a) Let $T$ be a distribution on $\mathbb{R}^2$. Give a precise definition of what it means that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on an open set $\Omega \subset \mathbb{R}^2$.

(b) Let $T$ be a distribution on $\mathbb{R}^2$, and suppose that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on all of $\mathbb{R}^2$. Prove that there is a constants $A$ so that for every function $\varphi \in C_0^\infty(\mathbb{R}^2)$, $T[\varphi] = A \int_{\mathbb{R}^2} \varphi(x) \, dx$.

(c) Suppose that $T$ is a distribution on $\mathbb{R}^2$ and suppose that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on the open set $\mathbb{R}^2 \setminus \{0\}$. Is the conclusion of part (b) still true? Either give a proof, or give a counter-example.

(d) Define a real valued function $g$ on $\mathbb{R}^2$ by setting $g(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1 \\ 1 & \text{if } x^2 + y^2 > 1 \end{cases}$. Show that $\Delta g$ (in the sense of distributions) is a finite measure on $\mathbb{R}^2$, and calculate what that measure is.

Problem IX  A real valued function $f$ defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\sqrt{|x - y|}} < +\infty.$$  

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there is a constant $C$ so that for every $\epsilon > 0$ there is a bounded function $\varphi \in C^\infty(\mathbb{R})$ such that

$$\sup_{x \in \mathbb{R}} |f(x) - \varphi(x)| \leq C \sqrt{\epsilon} \quad \text{and} \quad \sup_{x \in \mathbb{R}} \sqrt{\epsilon} |\varphi'(x)| \leq C.$$
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(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
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Problem I

(a) Show that \( n^n e^{-n} \leq n! \leq n^n \) for all positive integers \( n \).

(b) Let \( \{c_n\} \), \( n = 1, 2, \ldots \), be a sequence of positive real numbers. Suppose there is a real number \( \alpha \in \mathbb{R} \) and a constant \( C > 0 \) so that for all \( n \geq 1 \)

\[
\frac{c_{n+1}}{c_n} = 1 + \frac{\alpha}{n} + R(n) \quad \text{where} \quad |R(n)| \leq \frac{C}{n^2}.
\]

Show that, depending on \( \alpha \), the sequence \( \{c_n\} \) has a limit which is either zero, positive, or infinite.

(c) Using the results of part (b), show that the sequence \( \left\{ \frac{n!}{n^n e^{-n} \sqrt{n}} \right\} \), \( n = 1, 2, \ldots \), has a finite non-zero limit.

Problem II (NOTE: This is an advanced calculus problem. Do not quote theorems from the theory of Lebesgue integration for its solution). For each positive integer \( n \) let \( a^{(n)} \) be an infinite sequence of complex numbers so that

\[
a^{(n)} = (a_0^{(n)}, a_1^{(n)}, \ldots, a_j^{(n)}, \ldots).
\]

(a) Suppose there is a positive real number \( M \) such that for all \( n \)

\[
\sum_{j=0}^{\infty} |a_j^{(n)}|^2 \leq M.
\]

Prove that

\[
\sum_{j=0}^{\infty} \lim \inf_{n \to \infty} |a_j^{(n)}|^2 \leq M.
\]

Is it also true that

\[
\sum_{j=0}^{\infty} \lim \sup_{n \to \infty} |a_j^{(n)}|^2 \leq M?
\]

Either prove this or show that it is false by giving a counter-example.

(b) Assume that for each \( j \), \( \lim_{n \to \infty} a_j^{(n)} = \alpha_j \) exists. If

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)}|^2 = \sum_{j=0}^{\infty} |\alpha_j|^2,
\]

prove that

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)} - \alpha_j|^2 = 0.
\]

Problem III For \( x > 0 \), let \( F(x) = \int_0^\infty \frac{1 - e^{-xt}}{t^2} \, dt \).

(a) Show that this improper integral converge.

(b) Show that the function \( F \) is differentiable, and find an explicit formula for \( F'(x) \) in terms of elementary functions.

(c) Use the results of part (b) to find an explicit expression for \( F(x) \) in terms of elementary functions.
Problem IV Let $E \subseteq \mathbb{R}$ be a proper non-empty measurable set, so that $\emptyset \neq E \neq \mathbb{R}$. Assume that $E$ is invariant under translation by rational numbers. Explicitly, this means that if $x \in E$ and if $r$ is any rational number, then $x + r \in E$. Show that either $E$ has Lebesgue measure zero or that $\mathbb{R} - E$ has Lebesgue measure zero. Give examples to show that both conclusions are possible.

Problem V Let $\{E_j\}_{j \in \mathbb{N}}$ be a countable collection of measurable subsets of $\mathbb{R}^d$.

(a) Show that the set $A$ of points $x \in \mathbb{R}^d$ that belong to all but finitely many of the sets $E_j$ is measurable.
(b) Show that if $\lim_{j \to \infty} |E_j| = 0$, the set $A$ defined in part (a) has measure zero.
(c) Show that the set $B$ of points $x \in \mathbb{R}^d$ that belong to infinitely many of the set $E_j$ is measurable.
(d) If $\lim_{j \to \infty} |E_j| = 0$, must the set $B$ defined in part (c) have measure zero? Either prove that this is true, or show that it is false by giving a counter-example.

Problem VI For $t \in \mathbb{R}$ let $g(t) = (1 + |t|)^{-1}$. Fix $x \in \mathbb{R}$ and, for each non-zero $h \in \mathbb{R}$ set

$$G_h(t) = \frac{g(x + h - t) - g(x - t)}{h}.$$

(a) Prove that each $G_h \in L^2(\mathbb{R})$.
(b) Prove that $\lim_{h \to 0} G_h(t)$ exists for almost every $t \in \mathbb{R}$.
(c) Prove that if $G_0$ is the limit function found in part (b), then

$$\lim_{h \to 0} \int_{\mathbb{R}} |G_h(t) - G_0(t)|^2 \, dt = 0.$$

(d) Let $f \in L^2(\mathbb{R})$, and define

$$f \ast g(x) = \int_{\mathbb{R}} f(t) g(x - t) \, dt.$$

Prove that $f \ast g$ is a continuously differentiable function on $\mathbb{R}$. Be sure to justify all your steps including the existence of the integral defining $f \ast g$, the continuity of this function, and the continuous differentiability of this function.

Problem VII Let $C^\infty_c(\mathbb{R})$ denote the space of infinitely differentiable complex valued functions with compact support on $\mathbb{R}$.

(a) Show that for any $\varphi \in C^\infty_c(\mathbb{R})$,

$$\lim_{\epsilon \to 0} \int_{|t| > \epsilon} \frac{\varphi(t)}{t} \, dt$$

exists. Denote this limit by $T[\varphi]$.

(b) Show that the linear functional $T$ defined in part (a) is a distribution on $\mathbb{R}$. In particular, check carefully that all the hypotheses in the definition of a distribution are satisfied.

(c) If $S$ is a distribution on $\mathbb{R}$, define carefully what is meant by the support of $S$.

(d) Find two distributions $T_1$ and $T_2$ on $\mathbb{R}$ whose supports are, respectively, $(-\infty, 0]$ and $[0, +\infty)$, such that if $T$ is the distribution defined in part (a), then $T = T_1 + T_2$. 
Problem VIII  In this problem $L^2$ stands for $L^2(0,1)$ and its norm is denoted simply by $|| \cdot ||$; also $W^{1,2} = \{ f \in L^2 \mid f' \in L^2 \}$, with norm $||f||_{W^1} = (||f||^2 + ||f'||^2)^{1/2}$; finally $C^1 = C^1([0,1])$, with norm $||f||_1 = \text{Sup} |f| + \text{Sup} |f'|$.

(a) Let $B$ be the closed unit ball of $W^{1,2}$; show that it is a compact subset of $L^2$.

(b) Let $B'$ be the closed unit ball of $C^1$; show that it is a relatively compact subset of $L^2$. Is it a compact subset of $L^2$?

(c) Let $E$ be a closed subspace of $L^2$, such that $E \cap W^{1,2} = \{ 0 \}$ (for example the linear span of a function in $L^2$ and not in $W^{1,2}$). Let $\varphi$ be a continuous linear form on $E$, continuous with respect to the $L^2$ norm. Show that for every $\epsilon > 0$, there exists a continuous linear form $\tilde{\varphi}$ on $L^2$, whose restriction to $E$ is $\varphi$, and such that

$$\sup_{f \in B} |\tilde{\varphi}(f)| \leq \epsilon$$

with $B$ as in part (a). Can one take $\epsilon = 0$?

You can use the following geometric form of the Hahn Banach Theorem: If $K$ is a convex compact subset of a normed vector space $E$, and $L$ is a closed affine subspace of $E$ that does not intersect $K$, there exists a close affine hyperplane containing $L$ and still not intersecting $K$.

Problem IX  Let $H$ be a separable Hilbert space with norm $|| \cdot ||_H$ and inner product $(\cdot, \cdot)_H$. Let $\{ \varphi_n \}$, $n = 1, 2, \ldots$, be a complete orthonormal basis for $H$. Let $0 \leq \delta < 1$ and let $\{ f_n \}$ be a sequence of elements of $H$ such that for every finite set of complex numbers $\{ a_n \}$ we have

$$|| \sum a_n(\varphi_n - f_n) ||_H^2 \leq \delta^2 \sum |a_n|^2.$$ 

(a) Prove that the series $K[x] = \sum_{n=1}^{\infty} (x, \varphi_n)_H (\varphi_n - f_n)$ converges in norm for every $x \in H$.

(b) Prove that $K$ defines a bounded linear transformation from $H$ to $H$, and show that if $K^*$ is the adjoint operator, then $||K^*|| \leq \delta$.

(c) Prove that for each $n \geq 1$, $(I - K)[\varphi_n] = f_n$, and there exists a unique element $g_n \in H$ such that $(I - K^*)[g_n] = \varphi_n$. Hint: Prove that if an operator $T$ has operator norm less than 1, then $I - T$ is invertible.

(d) Prove that for $m, n \geq 1$, $(f_n, g_m)_H = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$

(e) Prove that for every $x \in H$, $x = \sum_{n=1}^{\infty} (x, g_n)_H f_n$ where the series converges in the norm in $H$. 
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday, August 29, 2001

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
(2) $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ denotes the unit disc in the complex plane.
(3) For points $x$ and $y$ in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
(4) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.
(5) If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,

$$||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.$$  

Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

(6) If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then

$$||f||_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.$$  

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $||f||_{\infty} < +\infty$.

(7) $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \mathbb{R}$.

(8) If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f \ast g$ is defined to be the function

$$f \ast g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt$$

whenever the integral converges.

(9) If $T$ is a distribution and $\varphi$ is a test function, then $\langle T, \varphi \rangle$ denotes the value of the distribution applied to the test function.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.
Problem I  Let \( f \) be a positive decreasing function defined on \((0, \infty)\). This means that if \(0 < a < b < \infty\), then \( f(a) \geq f(b) > 0 \). Let \( \epsilon > 0 \) be a fixed positive number.

(a) Suppose that for all \( 0 < x < \infty \), \( f(2x) \leq 2^{-1-\epsilon} f(x) \). Prove that there is a constant \( C \) depending only on \( \epsilon \) so that for \( a > 0 \),
\[
\int_a^\infty f(x) \, dx \leq C a f(a).
\]

(b) Suppose that for all \( 0 < x < \infty \), \( f(x) \leq 2^{1+\epsilon} f(2x) \). Prove that there is a constant \( C \) depending only on \( \epsilon \) so that for \( a > 0 \),
\[
\int_0^a f(x) \, dx \leq C a f(a).
\]

(c) Suppose that for all \( 0 < x < \infty \), \( f(2x) \geq 2^{-1} f(x) \). Prove that the improper integral \( \int_1^\infty f(x) \, dx \) diverges.

Problem II  For \( a, b > 0 \), let
\[
F(a, b) = \int_{-\infty}^{+\infty} \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}. 
\]
For which \( p > 0 \) is it true that
\[
\int_0^1 \int_0^1 F(a, b)^p \, da \, db < +\infty? 
\]

HINT: Do not try to evaluate the integral defining \( F(a, b) \) directly. Instead, first suppose \( a \leq b \) and show that there are positive constants \( C_1 \) and \( C_2 \) so that
\[
C_1 \leq b^p F(a, b) \leq C_2. 
\]

Problem III  Let \( \{z_1, z_2, \ldots, z_n, \ldots\} \) be a sequence of complex numbers, and suppose that
\[
\lim_{n \to \infty} z_n = L
\]
eexists. Prove that
\[
\lim_{n \to \infty} \frac{1}{n^2} \left( z_1 + 3z_2 + 5z_3 + \cdots + (2n-1)z_n \right) = L.
\]

Problem IV  Let \( \{E_n\}, n = 2, 3, 4, \ldots \) be a sequence of open subsets of \( \mathbb{R} \) defined as follows:
\[
E_2 = (0, 1) \\
E_3 = \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right) \\
E_4 = \left(0, \frac{3}{24}\right) \cup \left(\frac{5}{24}, \frac{19}{24}\right) \cup \left(\frac{21}{24}, 1\right) \quad \text{etc.}
\]
where \( E_n \) is the union of open intervals, all of the same length \( l_n \), and \( E_{n+1} \) is obtained from \( E_n \) by removing a closed interval of length \( \frac{l_n}{n+1} \) from the center of each interval of \( E_n \). Set \( E = \bigcap_{n=2}^{\infty} E_n \).

(a) Prove that \( E \) is an uncountable Borel set, and find the Lebesgue measure of \( E \).

(b) The set \( E_n \) is the disjoint union of \( k_n \) intervals, each of length \( l_n \). For \( \alpha \in [0, 1] \) find \( \lim_{n \to \infty} k_n l_n^\alpha \).

(c) What does the result in part (b) indicate about the Hausdorff dimension of the set \( E \)? (You are not asked for a complete proof, which may be somewhat technical. You are only asked to show understanding of the definitions of Hausdorff dimension and Hausdorff measure.)
Problem V  Let \( \{f_n\} \) be a sequence of real-valued continuous functions on \( \mathbb{R}^3 \). Suppose that the sequence \( \{f_n\} \) converges pointwise to a function \( f \). Assume that each function \( f_n \) and the limit function \( f \) are integrable. Assume that there are real numbers \( m \) and \( p \) so that for all \( x \in \mathbb{R}^3 \)
\[
-\frac{1}{1+|x|^m} \leq f_n(x) \leq \frac{1}{1+|x|^p}.
\]

(a)  Consider the statement— "If for each \( n \), \( \int\int f_n(x) \, dx = 0 \), then \( \int\int f(x) \, dx = 0 \)." For which \( m \) and \( p \) is this statement true.

(b)  Consider the statement— "If for each \( n \), \( \int\int f_n(x) \, dx \geq 0 \), then \( \int\int f(x) \, dx \geq 0 \)." For which \( m \) and \( p \) is this statement true.

Be sure to give proofs of the positive statements you make in parts (a) and (b). You only need to give counter-examples for one of the two parts.

Problem VI  Let \( f \) be a measurable, integrable function on \( \mathbb{R}^2 \).

(a)  Show that you can approximate \( f \) by a sequence of continuous functions \( \{f_n\} \) in such a way that for almost every \( y \),
\[
\int_{\mathbb{R}} |f_n(x, y) - f(x, y)| \, dx \to 0.
\]

Now set \( \hat{f}(x, y) = \int_{x-1}^{x+1} f(t, y) \, dt \).

(b)  Show that \( \hat{f} \) is defined almost everywhere.

(c)  Use the result of part (a) to show that the function \( \hat{f} \) is measurable.

In doing this problem you may use without proof the following version of Fubini's theorem: If \( f \) is a measurable, integrable function on \( \mathbb{R}^2 \), set \( f_x(y) = f(x, y) = f(y)(x) \). Then:

(1) For almost all \( x \in \mathbb{R} \), \( f_x \in L^1(\mathbb{R}) \), and for almost all \( y \in \mathbb{R} \), \( f_y \in L^1(\mathbb{R}) \).

(2) The functions \( g(x) = \int_{\mathbb{R}} f_x(y) \, dy \) and \( h(y) = \int_{\mathbb{R}} f_y(x) \, dx \) are defined almost everywhere, and belong to \( L^1(\mathbb{R}) \).

(3)  \[
\int\int_{\mathbb{R}^2} f(x, y) \, dA = \int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} h(y) \, dy.
\]

Problem VII  Let
\[
\Omega = \left\{ re^{i\theta} \in \mathbb{C} \mid 0 < r < 1, \quad \text{and} \quad -\pi < \theta < +\pi \right\} \subset \mathbb{D}.
\]
Assume that \( f : \Omega \to \mathbb{D} \) is holomorphic, one-to-one, and onto. Show that there is no continuous function \( g : \mathbb{D} \to \mathbb{C} \) such that \( f(z) = g(z) \) for all \( z \in \Omega \).

Problem VIII  Let \( \mathbb{D}^* = \mathbb{D} \setminus \{0\} \) denote the punctured unit disk. Suppose that \( f : \mathbb{D}^* \to \mathbb{D}^* \) is holomorphic, and has the property that the equation \( f(z) = w \) has exactly two solutions \( z \in \mathbb{D}^* \) for every \( w \in \mathbb{D}^* \), counting multiplicity. Prove that there is a complex number \( \lambda \) with \( |\lambda| = 1 \) so that \( f(z) = \lambda z^2 \).

Hint: Consider the mapping which takes \( z_0 \in \mathbb{D} \) to the second root of the equation \( f(z) = f(z_0) \).

Problem IX  
(a)  Show that on the complex plane slit along the positive real axis from 1 to \( +\infty \) and along the negative real axis from \(-1\) to \(-\infty \), there is a holomorphic function \( h \) satisfying \( h(z)^2 = (z^2-1)^{-1} \) such that \( h(0) = -i \). Compute: \( \lim_{y \to 0} h(2 + iy) \), \( \lim_{y \to 0} h(2 - iy) \), \( \lim_{y \to 0} h(-2 + iy) \), and \( \lim_{y \to 0} h(-2 - iy) \).

(b)  Evaluate \( \int_{-1}^{\infty} \frac{x^2 - 1}{x} \, dx \) (You may use any method you choose.)
QUALIFYING EXAM
in
ANALYSIS
Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 29, 2001

Versions for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( \mathcal{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
||f||_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( ||f||_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
||f||_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( ||f||_{\infty} < +\infty \).

(7) \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x-t) \, g(t) \, dt
\]

whenever the integral converges.

(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Let \( f \) be a positive decreasing function defined on \((0, \infty)\). This means that if \( 0 < a < b < \infty \), then \( f(a) \geq f(b) > 0 \). Let \( \epsilon > 0 \) be a fixed positive number.

(a)  Suppose that for all \( 0 < x < \infty \), \( f(2x) \leq 2^{-1-\epsilon} f(x) \). Prove that there is a constant \( C \) depending only on \( \epsilon \) so that for \( a > 0 \),

\[
\int_a^\infty f(x) \, dx \leq C \cdot a f(a).
\]

(b)  Suppose that for all \( 0 < x < \infty \), \( f(x) \leq 2^{1-\epsilon} f(2x) \). Prove that there is a constant \( C \) depending only on \( \epsilon \) so that for \( a > 0 \),

\[
\int_0^a f(x) \, dx \leq C \cdot a f(a).
\]

(c)  Suppose that for all \( 0 < x < \infty \), \( f(2x) \geq 2^{-1} f(x) \). Prove that the improper integral \( \int_1^\infty f(x) \, dx \) diverges.

Problem II  For \( a, b > 0 \), let

\[
F(a, b) = \int_{-\infty}^{+\infty} \frac{dx}{x^4 + (x - a)^4 + (x - b)^4}.
\]

For which \( p > 0 \) is it true that

\[
\int_0^1 \int_0^1 F(a, b)^p \, da \, db < +\infty?
\]

**HINT:** Do not try to evaluate the integral defining \( F(a, b) \) directly. Instead, first suppose \( a \leq b \) and show that there are positive constants \( C_1 \) and \( C_2 \) so that

\[
C_1 \leq b^3 F(a, b) \leq C_2.
\]

Problem III  Let \( \{z_1, z_2, \ldots, z_n, \ldots\} \) be a sequence of complex numbers, and suppose that

\[
\lim_{n \to \infty} z_n = L
\]

exists. Prove that

\[
\lim_{n \to \infty} \frac{1}{n^2} \left( z_1 + 3z_2 + 5z_3 + \cdots + (2n - 1)z_n \right) = L.
\]

Problem IV  Let \( \{E_n\} \), \( n = 2, 3, 4, \ldots \) be a sequence of open subsets of \( \mathbb{R} \) defined as follows:

\[
E_2 = (0, 1)
\]
\[
E_3 = \left( 0, \frac{1}{3} \right) \cup \left( \frac{2}{3}, 1 \right)
\]
\[
E_4 = \left( 0, \frac{3}{24} \right) \cup \left( \frac{5}{24}, \frac{1}{3} \right) \cup \left( \frac{2}{3}, \frac{19}{24} \right) \cup \left( \frac{19}{24}, 1 \right)
\]

where \( E_n \) is the union of open intervals, all of the same length \( l_n \), and \( E_{n+1} \) is obtained from \( E_n \) by removing a closed interval of length \( \frac{l_n}{n+1} \) from the center of each interval of \( E_n \). Set \( E = \bigcap_{n=2}^{\infty} E_n \).

(a)  Prove that \( E \) is an uncountable Borel set, and find the Lebesgue measure of \( E \).

(b)  The set \( E_n \) is the disjoint union of \( k_n \) intervals, each of length \( l_n \). For \( \alpha \in [0, 1] \) find \( \lim_{n \to \infty} k_n l_n^\alpha \).

(c)  What does the result in part (b) indicate about the Hausdorff dimension of the set \( E \)? (You are not asked for a complete proof, which may be somewhat technical. You are only asked to show understanding of the definitions of Hausdorff dimension and Hausdorff measure.)
Problem V Let \( \{f_n\} \) be a sequence of real-valued continuous functions on \( \mathbb{R}^3 \). Suppose that the sequence \( \{f_n\} \) converges pointwise to a function \( f \). Assume that each function \( f_n \) and the limit function \( f \) are integrable. Assume that there are real numbers \( m \) and \( p \) so that for all \( x \in \mathbb{R}^3 \)

\[
-\frac{1}{1+|x|^m} \leq f_n(x) \leq \frac{1}{1+|x|^p}.
\]

(a) Consider the statement—"If for each \( n \), \( \iiint_{\mathbb{R}^3} f_n(x) \, dx = 0 \), then \( \iiint_{\mathbb{R}^3} f(x) \, dx = 0 \)." For which \( m \) and \( p \) is this statement true.

(b) Consider the statement—"If for each \( n \), \( \iiint_{\mathbb{R}^3} f_n(x) \, dx \geq 0 \), then \( \iiint_{\mathbb{R}^3} f(x) \, dx \geq 0 \)." For which \( m \) and \( p \) is this statement true.

Be sure to give proofs of the positive statements you make in parts (a) and (b). You only need to give counter-examples for one of the two parts.

Problem VI Let \( f \) be a measurable, integrable function on \( \mathbb{R}^2 \).

(a) Show that you can approximate \( f \) by a sequence of continuous functions \( \{f_n\} \) in such a way that for almost every \( y \), \( \int \|f_n(x,y) - f(x,y)\| \, dx \to 0 \).

Now set \( \widehat{f}(x,y) = \int_{x-1}^{x+1} f(t,y) \, dt \).

(b) Show that \( \widehat{f} \) is defined almost everywhere.

(c) Use the result of part (a) to show that the function \( \widehat{f} \) is measurable.

In doing this problem you may use without proof the following version of Fubini's theorem: If \( f \) is a measurable, integrable function on \( \mathbb{R}^2 \), set \( f_x(y) = f(x,y) = f^y(x) \). Then:

1. For almost all \( x \in \mathbb{R} \), \( f_x \in L^1(\mathbb{R}) \), and for almost all \( y \in \mathbb{R} \), \( f^y \in L^1(\mathbb{R}) \).

2. The functions \( g(x) = \int f_x(y) \, dy \) and \( h(y) = \int f^y(x) \, dx \) are defined almost everywhere, and belong to \( L^1(\mathbb{R}) \).

3. \[
\int \int f(x,y) \, dA = \int \int g(x) \, dx = \int h(y) \, dy.
\]

Problem VII If \( f \) is continuous with compact support in \( (0, \infty) \) set

\[
H[f](x) = \int_0^\infty \frac{f(y)}{x+y} \, dy.
\]

(a) Prove that there does not exist a constant \( C \) such that for every \( f \) continuous with compact support in \( (0, \infty) \) we have

\[
\|H[f]\|_{L^\infty(0, \infty)} \leq C \|f\|_{L^\infty(0, \infty)}.
\]

(b) Prove that there does not exist a constant \( C \) such that for every \( f \) continuous with compact support in \( (0, \infty) \) we have

\[
\|H[f]\|_{L^1(0, \infty)} \leq C \|f\|_{L^1(0, \infty)}.
\]

HINT: Consider the adjoint of the operator \( H \) and use duality.

(c) Prove that for every \( 1 < p < \infty \) there is a constant \( C_p \) such that

\[
\|H[f]\|_{L^p(0, \infty)} \leq C_p \|f\|_{L^p(0, \infty)}.
\]

HINT: In the integral defining \( H[f](x) \), make the change of variables \( y = x s \) and apply Minkowski's inequality for integrals.

(d) Prove that \( \left\{ x > 0 \mid |H[f](x)| > \lambda \right\} \leq \lambda^{-1} \|f\|_{L^1(0, \infty)}. \)
Problem VIII  Let $F$ be a real-valued continuous function with compact support defined on the real line $\mathbb{R}$. We say that $F$ satisfies a Hölder condition of order $\alpha$ for some $0 < \alpha < 1$ if there is a constant $C$ so that for all $a, b \in \mathbb{R}$ we have
\[ |F(b) - F(a)| \leq C |b - a|^{\alpha} \]

(a) Suppose that $F$ is a continuous function with compact support on $\mathbb{R}$, and suppose for every $\varepsilon > 0$ there is a continuously differentiable function $G$ such that for all $x \in \mathbb{R}$,
\[ |F(x) - G(x)| \leq \varepsilon^{\alpha} \]
\[ |G'(x)| \leq C \varepsilon^{\alpha-1}. \]
Prove that $F$ satisfies a Hölder condition of order $\alpha$.

(b) Suppose that $F$ is a continuous function with compact support on $\mathbb{R}$ and that $F$ satisfies a Hölder condition of order $\alpha$ with $0 < \alpha < 1$. Prove that there is an infinitely differentiable function $G$ with compact support on $\mathbb{R}$ and a constant $C$ so that for all $x \in \mathbb{R}$,
\[ |F(x) - G(x)| \leq C \varepsilon^{\alpha} \]
\[ |G'(x)| \leq C \varepsilon^{\alpha-1}. \]

Problem IX  As usual, let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denote the Laplace operator on $\mathbb{R}^2$. Also, set
\[ f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}. \]

(a) What is meant by the distribution on $\mathbb{R}^2$ defined by the function $f$, and what is the definition of the distribution $\Delta f$?

(b) Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ and assume that $\varphi(0,0) = \frac{\partial \varphi}{\partial x}(0,0) = \frac{\partial \varphi}{\partial y}(0,0) = 0$. Prove that $\frac{\varphi(x, y)}{(x^2 + y^2)^{\frac{3}{2}}}$ is an integrable function on $\mathbb{R}^2$, and then prove that
\[ \langle \Delta f, \varphi \rangle = \iint_{\mathbb{R}^2} \frac{\varphi(x, y)}{(x^2 + y^2)^{\frac{3}{2}}} \, dx \, dy. \]
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday, January 16, 2002

Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.
1. (i) Let \( \{f_n\} \) be a sequence of \( C^1 \) functions on a compact interval \( I \) such that 
\[ |f_n(x)| + |f'_n(x)| \leq M \]
for all \( x \in I \) and \( n = 1, 2, 3, \ldots \). Show that there is a subsequence \( \{f_{n_k}\} \) which converges uniformly on \( I \).

(ii) Is the preceding statement still true if we drop the assumption that \( I \) is compact? (Proof or counterexample)

(iii) Can one also show that under the assumptions in (i) the sequence \( f_n \) has a subsequence whose derivatives converge uniformly? (Proof or counterexample)

2. Let \( \{a_n\}_{n=1}^{\infty} \) be a numerical sequence and let 
\[ b_n = \frac{1}{n^6} \sum_{k=1}^{n} k^5 a_k \]

(i) Prove or disprove: If \( a_n \) converges then \( b_n \) converges.

(ii) Prove or disprove: If \( b_n \) converges then \( a_n \) converges.

Hint: Relate \( \sum_{k=1}^{n} k^5 \) to an integral.

3. (i) Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is open, \( f : \mathcal{O} \to \mathbb{R} \) is a \( C^\infty \) function and \( n > 1 \). Show that \( f \) is not a one to one function.

(ii) Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is open, \( f : \mathcal{O} \to \mathbb{R}^k \) is a \( C^\infty \) function and \( n > k \). Show that \( f \) is not a one to one function.

Hint: Use induction on \( k \).

4. Suppose that the sequence \( \{f_n\} \) of nonnegative Lebesgue measurable functions on \( \mathbb{R} \) converges to \( f \) pointwise, and suppose that \( \int_{\mathbb{R}} f_n(x)\,dx < \infty \) for all \( n \), 
\( \int_{\mathbb{R}} f(x)\,dx < \infty \) and \( \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} f(x)\,dx \).

Prove that for all measurable sets \( E \) we have 
\[ \lim_{n \to \infty} \int_{E} f_n(x)\,dx = \int_{E} f(x)\,dx \]

5. (i) Let \( f : \mathbb{R} \to \mathbb{C} \) be continuous on \([0,1]\) and assume that \( f \) is 1-periodic, i.e. \( f(x+1) = f(x) \) for all \( x \). Let \( \beta \) be an irrational number in \((0,1)\) and define
\[ G_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f(x + k\beta) \]

Show that for all \( x \in \mathbb{R} \)
\[ \lim_{n \to \infty} G_n f(x) = \int_{0}^{1} f(t)\,dt \]

and that the convergence is uniform on \( \mathbb{R} \).

Hint: Show first that the formula for the limit is correct for \( f_m(x) = e^{2\pi imx} \), \( m \in \mathbb{Z} \).

(ii) Formulate and prove a generalization for \( f \in L^p \), for \( 1 \leq p < \infty \).
6. (i) Show that for nonnegative scalars \( a, b \in \mathbb{R} \) and \( p \geq 2 \) we have

\[
a^p + b^p \leq (a^2 + b^2)^{p/2}
\]

and

\[
\left( \frac{a^2 + b^2}{2} \right)^{p/2} \leq \frac{a^p}{2} + \frac{b^p}{2}.
\]

Hint: For the second inequality use the convexity of \( t \mapsto t^{p/2} \) for \( t > 0 \).

(ii) Show that for \( f, g \in L^p(X, d\mu) \) and \( 2 \leq p < \infty \)

\[
\left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}.
\]

(iii) Show that each closed convex set in \( L^p \) (\( 2 \leq p < \infty \)) has an element \( f \) of minimal norm.

7. Suppose \( u \) is real-valued and harmonic in \( \mathbb{C} \) and suppose that for all \( r > 0 \)

\[
\max_{0 \leq \theta \leq 2\pi} u(re^{i\theta}) \leq M(r),
\]

with \( 0 \leq M(r) < \infty \).

a) Show that

\[
\frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|d\theta + u(0) \leq 2M(r).
\]

b) Suppose that \( f \) is an entire function and there are non-negative constants \( A, B \) and \( \lambda \) so that

\[
\text{Re} \ f(re^{i\theta}) \leq Ar^\lambda + B
\]

for all \( r \) and \( \theta \). Show that \( f \) is a polynomial of degree at most \( \lambda \).

8. Evaluate

\[
\int_0^\infty \frac{(\ln x)^2}{1 + x^2} \, dx.
\]

9. Fix \( 0 < r < R \) and let \( A = \{ z \in \mathbb{C} : r < |z| < R \} \). Suppose that \( f \) is holomorphic and non vanishing on \( A \) and continuous on the closure of \( A \) and that \( |f(re^{i\theta})| \equiv \alpha \) and \( |f(Re^{i\theta})| \equiv \beta \) for some constants \( \alpha \) and \( \beta \) and for all \( 0 \leq \theta \leq 2\pi \).

Show that \( f(z) = cz^n \) for some \( c \in \mathbb{C} \) and some \( n = 0, \pm 1, \pm 2, \cdots \).

Hint: Consider the function \( u(z) = \log|f(z)| - \gamma \log|z| \) for an appropriate value of \( \gamma \).
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\[ |f_n(x)| + |f'_n(x)| \leq M \text{ for all } x \in I \text{ and } n = 1, 2, 3, \ldots \] 
Show that there is a subsequence \( \{f_{n_k}\} \) which converges uniformly on \( I \).

(ii) Is the preceding statement still true if we drop the assumption that \( I \) is compact? (Proof or counterexample)

(iii) Can one also show that under the assumptions in (i) the sequence \( f_n \) has a subsequence whose derivatives converge uniformly? (Proof or counterexample)

2. Let \( \{a_n\}_{n=1}^{\infty} \) be a numerical sequence and let

\[ b_n = \frac{1}{n^5} \sum_{k=1}^{n} k^5 a_k. \]

(i) Prove or disprove: If \( a_n \) converges then \( b_n \) converges.

(ii) Prove or disprove: If \( b_n \) converges then \( a_n \) converges.

**Hint:** Relate \( \sum_{k=1}^{n} k^5 \) to an integral.

3. (i) Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is open, \( f : \mathcal{O} \to \mathbb{R} \) is a \( C^\infty \) function and \( n > 1 \). Show that \( f \) is not a one to one function.

(ii) Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is open, \( f : \mathcal{O} \to \mathbb{R}^k \) is a \( C^\infty \) function and \( n > k \). Show that \( f \) is not a one to one function.

**Hint:** Use induction on \( k \).

4. Suppose that the sequence \( \{f_n\} \) of nonnegative Lebesgue measurable functions on \( \mathbb{R} \) converges to \( f \) pointwise, and suppose that \( \int_{\mathbb{R}} f_n(x)dx < \infty \) for all \( n \), \( \int_{\mathbb{R}} f(x)dx < \infty \) and \( \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx \).

Prove that for all measurable sets \( E \) we have

\[ \lim_{n \to \infty} \int_{E} f_n(x)dx = \int_{E} f(x)dx \]

5. (i) Let \( f : \mathbb{R} \to \mathbb{C} \) be continuous on \([0, 1]\) and assume that \( f \) is 1-periodic, i.e. \( f(x + 1) = f(x) \) for all \( x \). Let \( \beta \) be an irrational number in \((0, 1)\) and define

\[ \mathcal{G}_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f(x + k\beta) \]

Show that for all \( x \in \mathbb{R} \)

\[ \lim_{n \to \infty} \mathcal{G}_n f(x) = \int_{0}^{1} f(t)dt \]

and that the convergence is uniform on \( \mathbb{R} \).

**Hint:** Show first that the formula for the limit is correct for \( f_m(x) = e^{2\pi imx} \), \( m \in \mathbb{Z} \).

(ii) Formulate and prove a generalization for \( f \in L^p \), for \( 1 \leq p < \infty \).
6. (i) Show that for nonnegative scalars $a, b \in \mathbb{R}$ and $p \geq 2$ we have

$$a^p + b^p \leq (a^2 + b^2)^{p/2}$$

and

$$\left(\frac{a^2 + b^2}{2}\right)^{p/2} \leq \frac{a^p}{2} + \frac{b^p}{2}.$$ 

*Hint:* For the second inequality use the convexity of $t \mapsto t^{p/2}$ for $t > 0$.

(ii) Show that for $f, g \in L^p(X, d\mu)$ and $2 \leq p < \infty$

$$\left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \left\| f \right\|_p^p + \left\| g \right\|_p^p.$$ 

(iii) Show that each closed convex set in $L^p$ $(2 \leq p < \infty)$ has an element $f$ of minimal norm.

7. Prove that there is a unique $C^\infty$ function $f$ defined on $[0, 1]$ which satisfies the integral equation

$$f(x) + \int_0^1 \frac{t \cos(tx)f(t)}{1 + f(t)^2} \, dt = 1$$

for all $x \in [0, 1]$.

8. State and prove Baire's theorem, and describe a concrete application.

9. Let $E$ be a Banach space and let $E'$ its dual. For a sequence $\{x_n\}$ in $E$ we say that $x_n$ converges weakly to $x$ if $\lambda(x_n) \to \lambda(x)$ for all $\lambda \in E'$.

(i) Show that if $x_n$ converges weakly to $x$ then $\|x_n\|$ is a bounded sequence and

$$\|x\| \leq \liminf_{n \to \infty} \|x_n\|.$$ 

(ii) Show that $f_n(x) = \chi_{[n,n+1]}(x)$ defines a sequence which converges weakly in the space $L^p(\mathbb{R})$ if $1 < p < \infty$.

(iii) Let $\{a_n\}$ be a numerical sequence with $\lim_{n \to \infty} a_n = \infty$, and let $g_n(x) = a_n \chi_{[n,n+1]}(x)$. Does the sequence $g_n$ converge weakly in $L^p$?
QUALIFYING EXAM
in
ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 28, 2002

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
2. \( D = \{ z \in \mathbb{C} ||z|| < 1 \} \) denotes the unit disc in the complex plane.
3. For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
4. If \( E \subseteq \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
5. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
\|f\|_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).

6. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_\infty < +\infty \).

7. \( L^\infty_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subseteq \mathbb{R} \).

8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_\mathbb{R} f(x - t) g(t) \, dt
\]

whenever the integral converges.

9. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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1. (i) Let \( \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \) be sequences of real numbers such that

\[
\lim_{n\to\infty} \frac{a_n}{b_n} = 1.
\]

(a) Prove: If \( a_n > 0 \) for all \( n \) then \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \sum_{n=1}^{\infty} b_n \) converges.

(b) Is the statement in (a) still correct if one drops the assumption of positivity of the \( a_n^2 \)? Give a proof or a counterexample.

(ii) Suppose that \( \sum a_n \) converges and \( \sum_{n=1}^{\infty} a_n = A \). Let \( \rho_n = \sum_{k=1}^{n} (1 - \frac{k}{n}) a_k \). Prove that \( \lim_{n\to\infty} \rho_n = A \).

2. Let \( a \in \mathbb{R} \) and define \( f_a : \mathbb{R}^n \to \mathbb{R} \) by

\[
f_a(x) = (x_1^4 + x_2^4 + \cdots + x_{n-1}^4 + x_n^2)^a
\]

(i) Let \( B = \{ x \in \mathbb{R}^n : |x| \leq 1 \} \). For every \( a \in \mathbb{R} \) determine whether

\[
\int_B f_a(x) \, dx < \infty.
\]

(ii) Let \( S_\varepsilon \) be the unit sphere centered at \( c = (0, \ldots, 0, 1) \) and let \( d\sigma \) be surface measure on \( S_\varepsilon \).

For every \( a \in \mathbb{R} \) determine whether

\[
\int_{S_\varepsilon} f_a \, d\sigma < \infty.
\]

3. Let \( p > 0 \). Suppose that \( |f|^p \) is integrable on every compact interval in \( \mathbb{R} \) and \( f(x) \) satisfies the equation

\[
f(x) = \int_0^x \cos(xt) \sin(f(t))|f(t)|^p \, dt
\]

for all \( x \in \mathbb{R} \). Show that \( f(x) = 0 \) for all \( x \in \mathbb{R} \).
4. For \( f \in L^\infty(\mathbb{R}) \) and \( t > 0 \) define

\[
K(t, f) = \inf_{f = g + h} \{ \| g \|_\infty + t \| h' \|_\infty \}
\]

so that the infimum is taken over all possible decompositions of \( f = g + h \) where \( g \in L^\infty \) and \( h \) is a \( C^1 \) function with bounded derivative. Moreover let

\[
\omega(t, f) = \sup_x \sup_{|s| \leq t} |f(x + s) - f(x)|.
\]

Prove that there exists two positive constants \( C_1, C_2 \) (independent of \( f \) and \( t \)) so that for all \( f \in L^\infty, t > 0 \)

\[
C_1 K(t, f) \leq \omega(t, f) \leq C_2 K(t, f)
\]

(i.e. \( K(t, \cdot) \) and \( \omega(t, \cdot) \) are uniformly equivalent seminorms).

**Hint:** For one inequality one has to efficiently decompose \( f \); try a convolution \( h = \phi_t \ast f \) for suitable \( \phi_t \).

5. (i) Prove that for \( 1 < p < \infty, \alpha > 1/p \)

\[
(*) \quad \int_0^\infty x^{-\alpha p} \left| \int_0^x f(t) dt \right|^p dx \leq C_p \int_0^\infty |f(x)x^{1-\alpha}|^p dx.
\]

(ii) Is there a \( p \in (1, \infty) \) so that \( (*) \) remains true for \( \alpha = 1/p \)?

6. Let \( f \) be a measurable function on \( I = [0, 1] \) and assume that \( f \notin L^\infty(I) \).

a) Prove that

\[
\lim_{p \to \infty} \| f \|_p = \infty.
\]

b) Can \( \| f \|_p \) tend to \( \infty \) arbitrarily slowly? The precise question is: Is it true that to every positive function \( \Phi \) on \( (0, \infty) \) with \( \Phi(p) \to \infty \) there is a measurable \( f \) with \( \| f \|_p \to \infty \) but \( \| f \|_p \leq \Phi(p) \) for sufficiently large \( p \)?
7C. (i) State the Arzela-Ascoli theorem (which deals with compact subsets of $C(K)$ for compact $K$).

(ii) Let $\{f_n\}$ be a sequence of holomorphic functions in $\{z : |z| < R\}$ and assume that for all $n = 0, 1, 2, \ldots$

$$|f_n(z)| \leq C(|z|)$$

where $C(r) < \infty$ for every $r < R$.

Prove using part (i) that there is a subsequence of $\{f_n\}$ which converges uniformly on every compact subset of $\{z : |z| < R\}$.

8C. Let $\Omega = \{z : 0 < |z| < 1\}$ and let $f$ be holomorphic in $\Omega$ so that

$$\iint_{\Omega} |f(x + iy)|^2 dx dy < \infty.$$ 

Show that $f$ has a removable singularity at 0, i.e. can be extended to a holomorphic function on $\{z : |z| < 1\}$.

9C. (i) Compute

$$\int_0^\infty e^{-x^2} dx$$

(by a method of your choice).

(ii) Show that the improper integrals

$$\int_0^\infty e^{-(a+i\lambda)x^2} dx$$

exist for $\lambda \neq 0$ and $a \geq 0$ and use contour integrals to deduce their values from (i).

(iii) Show that

$$\lim_{a \to 0^+} \int_0^\infty x^2 e^{-(a+i\lambda)x^2} dx$$

exists for $\lambda \neq 0$ and compute this limit.

Be careful to justify all steps.
QUALIFYING EXAM
in
ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 28, 2002

Versions for Math 725

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(1) \(\mathbb{R}\) and \(\mathbb{C}\) denote the fields of real and complex numbers respectively.
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(3) For points \(x\) and \(y\) in \(\mathbb{R}^n\), \(|x - y|\) denotes the Euclidean distance between the points.
(4) If \(E \subset \mathbb{R}^n\) is a Lebesgue measurable set, then \(|E|\) denotes its Lebesgue measure.
(5) If \(\mu\) is a positive measure on a set \(X\), and \(f\) is a complex valued measurable function on \(X\), then for \(1 \leq p < +\infty\),

\[
\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}.
\]

Two functions on \(X\) are said to be equivalent if they are equal except on a set of \(\mu\) measure zero. For \(1 \leq p < +\infty\), \(L^p(X) = L^p(X, d\mu)\) is the space of equivalence classes of complex valued measurable functions such that \(\|f\|_p < +\infty\).
(6) If \(\mu\) is a positive measure on a set \(X\), and \(f\) is a complex valued measurable function on \(X\), then

\[
\|f\|_{\infty} = \inf \{t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0\}.
\]

\(L^\infty(X)\) is the space of equivalence classes of measurable, complex valued functions on \(X\) such that \(\|f\|_{\infty} < +\infty\).
(7) \(L^p_{\text{loc}}(\mathbb{R})\) is the space of measurable, complex valued functions on \(\mathbb{R}\) which belong to \(L^p(K)\) for every compact set \(K \subset \subset \mathbb{R}\).
(8) If \(f\) and \(g\) are measurable functions on \(\mathbb{R}\), the convolution \(f \ast g\) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) \, g(t) \, dt
\]

whenever the integral converges.
(9) If \(T\) is a distribution and \(\varphi\) is a test function, then \(\langle T, \varphi \rangle\) denotes the value of the distribution applied to the test function.

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\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]

(a) Prove: If \( a_n > 0 \) for all \( n \) then \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \sum_{n=1}^{\infty} b_n \) converges.

(b) Is the statement in (a) still correct if one drops the assumption of positivity of the \( a_n \)? Give a proof or a counterexample.

(ii) Suppose that \( \sum a_n \) converges and \( \sum_{n=1}^{\infty} a_n = A \). Let \( \rho_n = \sum_{k=1}^{n} (1 - \frac{k}{n})a_k \). Prove that \( \lim_{n \to \infty} \rho_n = A \).

2. Let \( a \in \mathbb{R} \) and define \( f_a : \mathbb{R}^n \to \mathbb{R} \) by

\[
f_a(x) = (x_1^4 + x_2^4 + \cdots + x_{n-1}^4 + x_n^2)^a
\]

(i) Let \( B = \{x \in \mathbb{R}^n : |x| \leq 1 \} \). For every \( a \in \mathbb{R} \) determine whether

\[
\int_B f_a(x) \, dx < \infty.
\]

(ii) Let \( S_\varepsilon \) be the unit sphere centered at \( \varepsilon = (0, \ldots, 0, 1) \) and let \( d\sigma \) be surface measure on \( S_\varepsilon \).

For every \( a \in \mathbb{R} \) determine whether

\[
\int_{S_\varepsilon} f_a \, d\sigma < \infty.
\]

3. Let \( p > 0 \). Suppose that \( |f|^p \) is integrable on every compact interval in \( \mathbb{R} \) and \( f(x) \) satisfies the equation

\[
f(x) = \int_0^x \cos(xt) \sin(f(t))|f(t)|^p \, dt
\]

for all \( x \in \mathbb{R} \). Show that \( f(x) = 0 \) for all \( x \in \mathbb{R} \).
4. For \( f \in L^\infty(\mathbb{R}) \) and \( t > 0 \) define

\[
K(t, f) = \inf_{f=g+h} \{ \|g\|_\infty + t\|h'|_\infty \}
\]

so that the infimum is taken over all possible decompositions of \( f = g + h \) where \( g \in L^\infty \) and \( h \) is a \( C^1 \) function with bounded derivative. Moreover let

\[
\omega(t, f) = \sup_{x} \sup_{|s| \leq t} |f(x+s) - f(x)|.
\]

Prove that there exists two positive constants \( C_1, C_2 \) (independent of \( f \) and \( t \)) so that for all \( f \in L^\infty, t > 0 \)

\[
C_1 K(t, f) \leq \omega(t, f) \leq C_2 K(t, f)
\]

(i.e. \( K(t, \cdot) \) and \( \omega(t, \cdot) \) are uniformly equivalent seminorms).

**Hint:** For one inequality one has to efficiently decompose \( f \); try a convolution \( h = \phi_t \ast f \) for suitable \( \phi_t \).

5. (i) Prove that for \( 1 < p < \infty, \alpha > 1/p \)

\[
(*) \quad \int_0^\infty x^{-\alpha p} \left( \int_0^x f(t) dt \right)^p dx \leq C_p \int_0^\infty |f(x)x^{1-\alpha}|^p dx.
\]

(ii) Is there a \( p \in (1, \infty) \) so that (*) remains true for \( \alpha = 1/p \)?

6. Let \( f \) be a measurable function on \( I = [0,1] \) and assume that \( f \notin L^\infty(I) \).
   a) Prove that

\[
\lim_{p \to \infty} \|f\|_p = \infty.
\]

b) Can \( \|f\|_p \) tend to \( \infty \) arbitrarily slowly? The precise question is: Is it true that to every positive function \( \Phi \) on \((0, \infty)\) with \( \Phi(p) \to \infty \) there is a measurable \( f \) with \( \|f\|_p \to \infty \) but \( \|f\|_p \leq \Phi(p) \) for sufficiently large \( p \)?
7R.
(i) State the Arzela-Ascoli theorem (which deals with compact subsets of $C(K)$ for compact $K$).
(ii) Prove using (i):

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of 1-periodic\footnote{Here $f$ is said to be 1-periodic if $f(x+1) = f(x)$ for almost every $x \in \mathbb{R}$} functions and assume that

$$\int_0^1 |f_n(t)|^2 dt \leq 1.$$

Assume that the derivative $f'_n$ in the sense of distributions belongs to $L^2[0,1]$ and

$$\int_0^1 |f'_n(t)|^2 dt \leq 1$$

for all $n$. Show that there is a subsequence of $\{f_n\}$ which converges uniformly on $\mathbb{R}$.

8R. State and prove Baire’s theorem, and describe a concrete application.

9R. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$u(x, y, z) = \begin{cases} 
x & \text{if } x^2 + y^2 + z^2 \leq 1 \text{ and } z > 0 \\
0 & \text{if } x^2 + y^2 + z^2 > 1 \text{ or } z \leq 0
\end{cases}$$

(i) Compute the derivative $\partial u/\partial y$ in the sense of distributions and show that it can be identified with a bounded Borel measure $\mu$.
(ii) Compute $\mu(E)$ for $E = \{(x, y, z) : 0 < x < y < 1, -\infty < z < 1/2\}$. 


QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday January 15, 2003

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( D = \{ z \in \mathbb{C} | |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_{\infty} = \inf \{ t > 0 | \mu(\{ x \in X | |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_{\infty} < +\infty \).

(7) \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) \, g(t) \, dt
\]

whenever the integral converges.

(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.
Problem I  Does the series

$$\sum_{k=1}^{\infty} \frac{\sin \sqrt{k}}{k}$$

converge?

[Hint: compare $\sum_{k=M}^{N} \frac{\sin \sqrt{k}}{k}$ with a similar integral.]

Problem II  Let $E \subset \mathbb{Q}$ be the set of $x$ whose decimal expansion is of the form $x = 0.d_1d_2 \cdots d_N$ for some $N \in \mathbb{N}$, and where $d_1, \ldots, d_N \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ (so $d_i \neq 0$ and $d_i \neq 9$ for $i = 1, \ldots, N$). Show that any compact subset of $E$ is finite.
Can we drop the hypotheses that $d_i \neq 0$ and $d_i \neq 9$?

Problem III  Let $Q = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n$ be the unit cube, and consider the function

$$f(x_1, \ldots, x_n) = \frac{x_1 x_2 \cdots x_n}{x_1^{a_1} + \cdots + x_n^{a_n}},$$

where the $a_j$ are positive constants. For which $a_1 > 0, \ldots, a_n > 0$ is the integral $\int_Q f(x)dx$ finite?

Problem IV  Let $1 \leq p < \infty$, and let $f_n \in L^p(\mathbb{R})$ be a sequence of functions. Suppose

$$\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_{L^p} < \infty.$$

Show that the sequence $f_n$ converges pointwise almost everywhere.

Problem V  Define

$$\Lambda(x) = \int_{0}^{\infty} \frac{e^{-t}}{\log t} (t^{x-1} - 1)dt.$$

(1) For which $x \in \mathbb{R}$ is the integrand a Lebesgue integrable function?
(2) Show that $\Lambda(x)$ is a continuous function for $x \in \mathbb{R}_+$. 
(3) Show that $\Lambda(x)$ is differentiable for $x \in \mathbb{R}_+$, and that the derivative is given by $\Lambda'(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt$. 
Give complete proofs.
Problem VI
Let $K : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative measurable function for which
\[
\int_0^\infty \frac{K(t)}{\sqrt{t}} \, dt = A < \infty.
\]
In this problem $L^2(0, \infty) = \{ f : (0, \infty) \to \mathbb{R} : f \text{ is measurable and } \int_0^\infty f(x)^2 \, dx < \infty \}$, with the usual convention that functions which differ only on a set of measure zero are identified.

1. Show that for any two functions $f, g \in L^2(0, \infty)$ one has
\[
\int_0^\infty \int_0^\infty K(xy) f(x) g(y) \, dx \, dy \leq A \| f \|_{L^2} \| g \|_{L^2}.
\]
[Hint: try the substitution $x = z/y$; or you could try to solve (b) first...]

2. Prove that for any $f \in L^2(0, \infty)$ the integral
\[
Tf(x) \overset{\text{def}}{=} \int_0^\infty K(xy) f(y) \, dy
\]
converges for almost every $x \in \mathbb{R}_+$, and that $T$ defines a bounded operator on $L^2(0, \infty)$ (i.e. there is a finite constant $C$ such that $\| Tf \|_{L^2} \leq C \| f \|_{L^2}$ for all $f \in L^2(0, \infty)$.)

Problem VII
Let $D$ be the open unit disk in $\mathbb{C}$.

1. Let $(h_n)$ be a sequence of holomorphic maps from $D$ into $D$. Assume that $|h'_n(0)|$ tends to $0$ as $n$ tends to $\infty$. Show that $h_n(0)$ tends to $0$.

2. Let $\mathcal{F}$ be the set of holomorphic maps from $D$ into $D - \{ \frac{1}{2} \}$. Show that there is a constant $M < 1$ such that for every $f \in \mathcal{F}$, $|f'(0)| \leq M$.

The second question can be treated, assuming the result of the first question.

Problem VIII
Let $0 < \epsilon < \infty$, $0 < R < \infty$ and let $D_1$, $D_2$ be two closed disjoint disks in $\mathbb{C}$. Show that there is an entire function $f$ so that $f(D_1)$ is contained in $\{ z : |z| < \epsilon \}$ and $f(D_2)$ contains $\{ z : |z| < R \}$.

[Hint: Use Runge’s theorem.]

Problem IX
Use complex methods to find the value of
\[
\sum_{n=1}^\infty \frac{(-1)^n}{n^2}.
\]
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday January 15, 2003

Versions for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_\infty = \inf \{ t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_\infty < +\infty \).
(7) \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).
(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
\]

whenever the integral converges.
(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Does the series
\[ \sum_{k=1}^{\infty} \frac{\sin \sqrt{k}}{k} \]
converge?

[Hint: compare \( \sum_{k=M}^{\infty} \frac{\sin \sqrt{k}}{k} \) with a similar integral.]

Problem II  Let \( E \subset \mathbb{Q} \) be the set of \( x \) whose decimal expansion is of the form \( x = 0.d_1d_2 \cdots d_N \) for some \( N \in \mathbb{N} \), and where \( d_1, \ldots, d_N \in \{1, 2, 3, 4, 5, 6, 7, 8\} \) (so \( d_i \neq 0 \) and \( d_i \neq 9 \) for \( i = 1, \ldots, N \)). Show that any compact subset of \( E \) is finite.
Can we drop the hypotheses that \( d_i \neq 0 \) and \( d_i \neq 9 \)?

Problem III  Let \( Q = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n \) be the unit cube, and consider the function
\[ f(x_1, \ldots, x_n) = \frac{x_1x_2 \cdots x_n}{x_1^{a_1} + \cdots + x_n^{a_n}}, \]
where the \( a_j \) are positive constants. For which \( a_1 > 0, \ldots, a_n > 0 \) is the integral \( \int_Q f(x) \, dx \) finite?

Problem IV  Let \( 1 \leq p < \infty \), and let \( f_n \in L^p(\mathbb{R}) \) be a sequence of functions. Suppose
\[ \sum_{n=1}^{\infty} \| f_{n+1} - f_n \|_{L^p} < \infty. \]
Show that the sequence \( f_n \) converges pointwise almost everywhere.

Problem V  Define
\[ \Lambda(x) = \int_0^\infty \frac{e^{-t}}{\log t} (t^{x-1} - 1) \, dt. \]

(1) For which \( x \in \mathbb{R} \) is the integrand a Lebesgue integrable function?
(2) Show that \( \Lambda(x) \) is a continuous function for \( x \in \mathbb{R}_+ \).
(3) Show that \( \Lambda(x) \) is differentiable for \( x \in \mathbb{R}_+ \), and that the derivative is given by \( \Lambda'(x) = \int_0^\infty e^{-t} t^{x-1} \, dt. \)
Give complete proofs.
Problem VI Let $K : \mathbb{R}^+ \to \mathbb{R}$ be a nonnegative measurable function for which
\[ \int_0^\infty \frac{K(t)}{\sqrt{t}} dt = A < \infty. \]
In this problem $L^2(0, \infty) = \{ f : (0, \infty) \to \mathbb{R} : f \text{ is measurable and } \int_0^\infty f(x)^2 dx < \infty \}$, with the usual convention that functions which differ only on a set of measure zero are identified.

1. Show that for any two functions $f, g \in L^2(0, \infty)$ one has
\[
\int_0^\infty \int_0^\infty K(xy) f(x)g(y) \, dx \, dy \leq A \|f\|_{L^2} \|g\|_{L^2}.
\]
[Hint: try the substitution $x = z/y$; or you could try to solve (b) first...]

2. Prove that for any $f \in L^2(0, \infty)$ the integral
\[
Tf(x) \overset{\text{def}}{=} \int_0^\infty K(xy) f(y) dy
\]
converges for almost every $x \in \mathbb{R}^+$, and that $T$ defines a bounded operator on $L^2(0, \infty)$ (i.e. there is a finite constant $C$ such that $\|Tf\|_{L^2} \leq C \|f\|_{L^2}$ for all $f \in L^2(0, \infty)$.)

Problem VII

Let $X$ and $Y$ be Banach spaces, and let $T_{jk}, j, k \in \mathbb{N}$ be a family of bounded operators from $X$ to $Y$. Suppose that for every $k \in \mathbb{N}$ there exists an $x \in X$ such that $\sup_{j \in \mathbb{N}} \|T_{jk} x\|_Y = \infty$. Then prove that an $x \in X$ exists such that $\sup_{j \in \mathbb{N}} \|T_{jk} x\|_Y = \infty$ holds for all $k \in \mathbb{N}$.

Problem VIII

Let $1 \leq p \leq \infty$. Consider the operators $T_\epsilon : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ given by
\[
T_\epsilon f(x) = \frac{1}{2} \int_{-1}^1 f(x + \epsilon t) dt.
\]

1. Show that $|T_\epsilon f(x) - T_\epsilon f(y)| \leq C|x-y|^{1-1/p}$ for all $x, y \in \mathbb{R}$ and for some finite constant $C$, depending on $f$ and $\epsilon$ but independent of $x$ and $y$. Give $C$ explicitly in terms of $f$ and $\epsilon$.

2. It is known that $\lim_{\epsilon \searrow 0} \|T_\epsilon f - f\|_{L^p(\mathbb{R})} = 0$ for each $f \in L^p(\mathbb{R})$. Is it true that
\[
\lim_{\epsilon \searrow 0} \|T_\epsilon - I_{L^p}\|_{\mathcal{L}(L^p)} = 0
\]
where $\|\cdots\|_{\mathcal{L}(L^p)}$ is the operator norm on the space $\mathcal{L}(L^p)$ of bounded operators on $L^p(\mathbb{R})$, and $I_{L^p}$ is the identity operator on $L^p$?

Problem IX

Let $E = \{(x, y) \in \mathbb{R}^2 : y \geq |x|\}$, let $f : \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function of $E$, and let $\delta$ be the Dirac measure at 0. Show that in the sense of distributions one has
\[
\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2} = 2\delta.
\]
ANALYSIS QUALIFYING EXAM – 722 version
Wednesday, August 27, 2003

Advanced Calculus

Problem I.
(1) For which \( x \in \mathbb{R} \) does the Taylor series of \( f(x) = \log(1 - x) \) centered at 0 converge to \( f(x) \)?
(2) For which \( a > -1 \) does the Taylor series converge uniformly on the interval \([-1, a)\)?
(3) Evaluate the sum
\[
\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} \quad \text{where} \ |x| \leq 1.
\]
Include all the details of your derivation.

Problem II. Let \( a > 0 \) and \( b > 0 \). Prove that there is a unique differentiable function \( f \) defined on \((-\infty, \infty)\) satisfying \( f(0) = 0 \) and
\[
f'(x) = a - b|f(x)|^{3/2}
\]
for all \( x \). Also show that \( \lim_{x \to -\infty} f(x) \) exists and determine this limit.

Problem III. Consider a differentiable function \( f : \mathbb{R} \to \mathbb{R} \).
(1) Suppose the second derivative of \( f \) exists at \( x_0 \) (but not necessarily anywhere else). Show that
\[
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).
\]
(2) Suppose that
\[
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}
\]
eexists. Is it true that the second derivative of \( f \) exists at \( x_0 \)? Give a proof or a counterexample!
721 problems

**Problem IV.**

(1) Let $E, F \subset \mathbb{R}^N$ be Lebesgue measurable subsets both of which have finite and positive measure. For $x \in \mathbb{R}^N$ we define the translates $E_x = \{ x + y : y \in E \}$. Show that the set

$$G = \{ x \in \mathbb{R}^N : E_x \cap F \neq \emptyset \}$$

has positive Lebesgue measure. [Hint: Several solutions are possible: you could consider the convolution of suitable characteristic functions; or you could apply Lebesgue’s differentiation theorem to a characteristic function.]

(2) If $E \subset \mathbb{R}$ is an open and dense subset of the real line, must its Lebesgue measure be infinite? (Prove, or give a counterexample.)

**Problem V.** For $f \in L^1_{loc}(\mathbb{R})$, $j \geq 0$ define the so-called “conditional expectation operator” $E_j$ by

$$E_j f(x) = 2^j \int_{n2^{-j}}^{(n+1)2^{-j}} f(\xi) d\xi \quad \text{if} \quad x \in [n2^{-j}, (n+1)2^{-j}), \quad n \in \mathbb{Z}.$$ 

(1) Show that for $f \in L^1(\mathbb{R})$ we have

$$\lim_{j \to \infty} E_j f(x) = f(x) \text{ almost everywhere.}$$

(2) Show that for $f \in L^2(\mathbb{R})$ one has

$$\lim_{j \to \infty} \| E_j f - f \|_{L^2} = 0.$$ 

**Problem VI.** Let $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$. Show that for every $x \in \mathbb{R}$ the convolution integral

$$A(x) = \int f(x - y)g(y) dy$$

is well defined and that $A$ is in fact a continuous function satisfying

$$\lim_{|x| \to \infty} A(x) = 0.$$
Problem VII.

(1) Let $F$ and $G$ be functions defined on $\mathbb{C}$, assume that
\[ F(z) = z^{10} + z^9 + f(z), \quad G(z) = z^{10} + 9z^9 + g(z), \]
where $f$ and $g$ are continuous, but not necessarily holomorphic, functions such that for some constant $C > 0$,
\[ |f(z)| + |g(z)| \leq C(1 + |z|^8), \quad \text{for all } z \in \mathbb{C}. \]
Find the limit as $R \to \infty$ of
\[ \int_{|z|=R} \frac{F(z)}{G(z)} \, dz \]
(integral over the circle $|z| = R$, with counterclockwise orientation).
(2) For all values of $R$ for which it makes sense, evaluate:
\[ \int_{|z|=R} \frac{z^{10} + z^9 + 1}{z(z+1)^9} \, dz. \]

Problem VIII. Let $\Delta$ be the open unit disc in $\mathbb{C}$, and $\Delta_{\frac{1}{2}}$ be the open disc of radius $\frac{1}{2}$, with center at 0. Let $h$ be a holomorphic map from $\Delta$ into itself.

(1) Assume that $h(0) \in (-1, 0)$. Show that $\frac{1}{2} \notin h(\Delta_{\frac{1}{2}})$.
(2) Assume that $h(\Delta_{\frac{1}{2}}) \supset \Delta_{\frac{1}{2}}$. Show that there exists $\theta \in \mathbb{R}$, such that $h(z) = e^{i\theta}z$.

Problem IX.

(1) Let $u$ be a harmonic function defined on a neighborhood $V$ of $[-i, +i]$ in $\mathbb{C}$. Assume that $u(x) = 0$ for all $x \in V \cap \mathbb{R}$. Show that $u(-i) = -u(i)$.
(2) Let $\varphi$ be the map defined on $\mathbb{C}$ by $\varphi(z) = z + iz^2$. Describe $\varphi([-i, +i])$.
Let $w$ be a harmonic function defined on a neighborhood $V_1$ of $[-2i, \frac{i}{4}]$.
Show that if $w(x + ix^2) = 0$ if $x \in \mathbb{R}$ and $x + ix^2 \in V_1$ (i.e. $w$ vanishes on the parabola $y = x^2$), then $w(-2i) = 0$.
(3) Give examples of harmonic functions defined either on $\mathbb{C}$, or just on a neighborhood of 0, that vanish identically on the parabola $y = x^2$. 

Problem I.

(1) For which \( x \in \mathbb{R} \) does the Taylor series of \( f(x) = \log(1 - x) \) centered at 0 converge to \( f(x) \)?

(2) For which \( a > -1 \) does the Taylor series converge uniformly on the interval \([-1, a]\)?

(3) Evaluate the sum
\[
\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} \text{ where } |x| \leq 1.
\]

Include all the details of your derivation.

Problem II. Let \( a > 0 \) and \( b > 0 \). Prove that there is a unique differentiable function \( f \) defined on \((-\infty, \infty)\) satisfying \( f(0) = 0 \) and
\[
f'(x) = a - b|f(x)|^{3/2}
\]
for all \( x \). Also show that \( \lim_{x \to \infty} f(x) \) exists and determine this limit.

Problem III. Consider a differentiable function \( f : \mathbb{R} \to \mathbb{R} \).

(1) Suppose the second derivative of \( f \) exists at \( x_0 \) (but not necessarily anywhere else). Show that
\[
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).
\]

(2) Suppose that
\[
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}
\]
exists. Is it true that the second derivative of \( f \) exists at \( x_0 \)?
Give a proof or a counterexample!
721 problems

Problem IV.

(1) Let $E, F \subset \mathbb{R}^N$ be Lebesgue measurable subsets both of which have finite and positive measure. For $x \in \mathbb{R}^N$ we define the translates $E_x = \{x + y : y \in E\}$. Show that the set

$$G = \{x \in \mathbb{R}^N : E_x \cap F \neq \emptyset\}$$

has positive Lebesgue measure. [Hint: Several solutions are possible: you could consider the convolution of suitable characteristic functions; or you could apply Lebesgue's differentiation theorem to a characteristic function.]

(2) If $E \subset \mathbb{R}$ is an open and dense subset of the real line, must its Lebesgue measure be infinite? (Prove, or give a counterexample.)

Problem V. For $f \in L^1_{\text{loc}}(\mathbb{R})$, $j \geq 0$ define the so-called “conditional expectation operator” $E_j$ by

$$E_j f(x) = 2^j \int_{n2^{-j}}^{(n+1)2^{-j}} f(\xi) d\xi \quad \text{if } x \in [n2^{-j}, (n + 1)2^{-j}), \ n \in \mathbb{Z}.$$

(1) Show that for $f \in L^1(\mathbb{R})$ we have

$$\lim_{j \to \infty} E_j f(x) = f(x) \text{ almost everywhere.}$$

(2) Show that for $f \in L^2(\mathbb{R})$ one has

$$\lim_{j \to \infty} \|E_j f - f\|_{L^2} = 0.$$

Problem VI. Let $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$. Show that for every $x \in \mathbb{R}$ the convolution integral

$$A(x) = \int f(x - y)g(y) dy$$

is well defined and that $A$ is in fact a continuous function satisfying

$$\lim_{|x| \to \infty} A(x) = 0.$$
725 Problems

Problem VII.

1. Let $E \subset L^2(\mathbb{R})$ be a linear subspace which is translation invariant, i.e. if $f \in E$, then for any $x \in \mathbb{R}$ the function $f_x(t) \overset{\text{def}}{=} f(x + t)$ also belongs to $E$.

   Prove: if $\dim E < \infty$ then $E$ is trivial, i.e. $E = \{0\}$.

2. Let $L^2(\mathbb{R}/\mathbb{Z})$ be the space of all functions in $L^2_{\text{loc}}$ which are periodic with period 1, and suppose $E \subset L^2(\mathbb{R}/\mathbb{Z})$ is a finite dimensional, translation invariant subspace. Must $E = \{0\}$ hold?

Problem VIII. Denote the unit disc in $\mathbb{R}^2$ by $\Omega$. Let

$$f(x, y) = \begin{cases} 
1 - x^2 - y^2 & \text{for } (x, y) \in \Omega \\
0 & \text{on } \mathbb{R}^2 \setminus \Omega.
\end{cases}$$

Prove that in the sense of distributions one has

$$\Delta f + 4\chi_{\Omega} \geq 0,$$

i.e. show that

$$\langle \Delta f + 4\chi_{\Omega}, \psi \rangle \geq 0$$

for any test function $\psi \in C^\infty_c(\mathbb{R}^2)$ with $\psi(x, y) \geq 0$ everywhere.

(Here $\chi_{\Omega}$ is the characteristic function of the set $\Omega$.)

Problem IX. Let $N \geq 3$. Show that there is one and only one value of $p \in [1, \infty)$ such that for all $f \in C^\infty_c(\mathbb{R}^N)$ one has

$$\left( \int_{\mathbb{R}^N} |f(x)|^p \, dx \right)^{1/p} \leq C \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \, dx$$

where $C$ is a constant which does not depend on $f$. 

QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday January 14, 2004
Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( D = \{ z \in \mathbb{C} | |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_\infty < +\infty \).
(7) \( L^\infty_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).
(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
\]

whenever the integral converges.
(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Prove or disprove the following
(a) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} a_n^3$ converges.
(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges.

Problem II  Show that
\[ \int_{0}^{\infty} e^{-tx} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - \arctan t, \quad t > 0. \]
*(Justify all steps.)*

Problem III
(a) Let $f$ be a differentiable function defined on $[-1, 1]$. Assume that $f' = |f|^{1/2}$. Prove that if $f(0) > 0$ then $f(1) > 1/4$ and that if $f(0) < 0$ then $f(-1) < -1/4$.

(b) Let $\epsilon > 0$. Find a differentiable function $g$ defined on $[-1, 1]$ such that $g'(x) = x|g(x)|^{1/2}$, $g(0) \neq 0$ but $|g(x)| \leq \epsilon$ for $x \in [-1, 1]$.

Problem IV  Let $p \in [1, \infty)$. For $f \in L^p(\mathbb{R})$ define the functions
\[ g_n(x) = \frac{1}{n} \sum_{k=1}^{n} f \left( x + \frac{k}{n} \right). \]
Show that the sequence $g_n$ converges in $L^p(\mathbb{R})$, and determine the limit function.

Problem V  Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function which vanishes for $x^2 + y^2 > R^2$.
(a) Show that for every $\theta \in [0, 2\pi]$ one has
\[ |f(0, 0)| \leq \int_{0}^{\infty} |\nabla f(r \cos \theta, r \sin \theta)| \, dr. \]

(b) Let $p > 2$. Show that there exists $C_{p,R} > 0$ (depending only on $p$ and $R$) such that
\[ |f(0, 0)| \leq C_{p,R} \left( \int_{\mathbb{R}^2} |\nabla f|^p \, dx \, dy \right)^{1/p}. \]
*(Hint: Integrate the inequality from part (1) over all $\theta \in [0, 2\pi]$.)

(c) Show that there is no constant $C < \infty$ such that
\[ |f(0, 0)| \leq C \int_{\mathbb{R}^2} |\nabla f| \, dx \, dy \]
for all continuously differentiable $f : \mathbb{R}^2 \to \mathbb{R}$ which vanish for $x^2 + y^2 > 1$.

Problem VI  Assume that $(\Omega, \Sigma, \mu)$ is a measure space with $\mu(\Omega) < \infty$. A sequence $f_n$ of complex measurable functions is said to converge in measure to a complex measurable function $f$, if for every $\epsilon > 0$ there exists $N$ such that
\[ \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon, \quad \text{for } n \geq N. \]
Prove or disprove (with a counter-example) the following statements:
(a) If $f_n \to f$, a.e. then $f_n \to f$ in measure.
(b) If $f_n \to f$ in $L^p$, with $1 \leq p \leq \infty$, then $f_n \to f$ in measure.
(c) If $f_n$ is a sequence in $L^2$ such that for every $g \in L^2$, $\int_{\Omega} f_ng \to 0$ as $n \to \infty$, then $f_n \to 0$ in measure.
Problem VII  Let $0 < \alpha < 1$. Let $f(z)$ be the determination of $z^\alpha$ on $\mathbb{C} \setminus (-\infty, 0]$ with $f(1) > 0$. Let $g(z)$ be the determination of $(1 + z)^{1-\alpha}$ on $\mathbb{C} \setminus (-\infty, 0]$ with $g(1) > 0$.

(a) What are the limits as $t \to 0^+$, and as $t \to 0^-$, of $f(-\frac{1}{2} + it)$ and $g(-\frac{1}{2} + it)$?

(b) Show that $fg$ extends holomorphically to $\mathbb{C} \setminus [-1, 0]$.

(c) Evaluate

$$\int_{-1}^{0} \frac{dx}{x^\alpha(1 + x)^{1-\alpha}}.$$

Problem VIII  Let $f(z)$ be holomorphic on the unit disk in $\mathbb{C}$. Fix $r \in (0, 1)$. Assume that $f(r) = \max\{|f(z)| : |z| = r\}$.

(a) Show that $f'(r) > 0$, if $f$ is non constant.

(b) Show that if $f(0) = 0$, then $f'(r) \geq \frac{f(r)}{r}$ and equality holds if and only if $f(z) = cz$ for some non-negative constant $c$.

Problem IX

(a) Show that there is no holomorphic function $f(z)$ on $\{z \in \mathbb{C} : 1 < |z| < 3\}$ satisfying

$$\left| \frac{f(z)^2}{z} - 1 \right| < 1.$$

(b) Show that there exists $\epsilon > 0$ so that no holomorphic function on $\{z \in \mathbb{C} : 1 < |z| < 3\}$ satisfies

$$\left| \frac{|f(z)|^2}{|z|} - 1 \right| < \epsilon.$$
Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty 

\[ ||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}. \]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( ||f||_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[ ||f||_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}. \]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( ||f||_\infty < +\infty \).

(7) \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \mathbb{R} \).

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f * g \) is defined to be the function

\[ f * g(x) = \int_{\mathbb{R}} f(x-t) g(t) \, dt \]

whenever the integral converges.

(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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(b) If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} a_n^3 \) converges.

Problem II  Show that
\[
\int_{0}^{\infty} e^{-tx} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - \arctan t, \quad t > 0.
\]
(Justify all steps.)

Problem III  
(a) Let \( f \) be a differentiable function defined on \([-1, 1]\). Assume that \( f' = |f|^{1/2} \). Prove that if \( f(0) > 0 \) then \( f(1) > 1/4 \) and that if \( f(0) < 0 \) then \( f(-1) < -1/4 \).

(b) Let \( \epsilon > 0 \). Find a differentiable function \( g \) defined on \([-1, 1]\) such that \( g'(x) = x|g(x)|^{1/2}, g(0) \neq 0 \) but \( |g(x)| \leq \epsilon \) for \( x \in [-1, 1] \).

Problem IV  Let \( p \in [1, \infty) \). For \( f \in L^p(\mathbb{R}) \) define the functions
\[
g_n(x) = \frac{1}{n} \sum_{k=1}^{n} f \left( x + \frac{k}{n} \right).
\]
Show that the sequence \( g_n \) converges in \( L^p(\mathbb{R}) \), and determine the limit function.

Problem V  Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a continuously differentiable function which vanishes for \( x^2 + y^2 > R^2 \).

(a) Show that for every \( \theta \in [0, 2\pi] \) one has
\[
|f(0,0)| \leq \int_{0}^{\infty} |\nabla f(r \cos \theta, r \sin \theta)| \, dr.
\]

(b) Let \( p > 2 \). Show that there exists \( C_{p,R} > 0 \) (depending only on \( p \) and \( R \)) such that
\[
|f(0,0)| \leq C_{p,R} \left( \iint_{\mathbb{R}^2} |\nabla f|^p \, dx \, dy \right)^{\frac{1}{p}}.
\]
(Hint: Integrate the inequality from part (1) over all \( \theta \in [0, 2\pi] \).)

(c) Show that there is no constant \( C < \infty \) such that
\[
|f(0,0)| \leq C \iint_{\mathbb{R}^2} |\nabla f| \, dx \, dy
\]
for all continuously differentiable \( f : \mathbb{R}^2 \to \mathbb{R} \) which vanish for \( x^2 + y^2 > 1 \).

Problem VI  Assume that \( (\Omega, \Sigma, \mu) \) is a measure space with \( \mu(\Omega) < \infty \). A sequence \( f_n \) of complex measurable functions is said to converge in measure to a complex measurable function \( f \), if for every \( \epsilon > 0 \) there exists \( N \) such that
\[
\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon, \quad \text{for } n \geq N.
\]
Prove or disprove (with a counter-example) the following statements:

(a) If \( f_n \to f, \text{a.e.} \) then \( f_n \to f \) in measure.

(b) If \( f_n \to f \) in \( L^p \), with \( 1 \leq p \leq \infty \), then \( f_n \to f \) in measure.

(c) If \( f_n \) is a sequence in \( L^2 \) such that for every \( g \in L^2, \iint_{\Omega} f_n g \to 0 \) as \( n \to \infty \), then \( f_n \to 0 \) in measure.
Problem VII

(a) (Formula for integration by parts) Let \( u, v \) be smooth functions on \( \mathbb{R}^2 \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) whose boundary is piecewise smooth. Define \( R_1, R_2 \) by
\[
\int_\Omega u(x,y) \frac{\partial v}{\partial x}(x,y) \, dx\, dy = -\int_\Omega v(x,y) \frac{\partial u}{\partial x}(x,y) \, dx\, dy + R_1,
\]
\[
\int_\Omega u(x,y) \frac{\partial v}{\partial y}(x,y) \, dx\, dy = -\int_\Omega v(x,y) \frac{\partial u}{\partial y}(x,y) \, dx\, dy + R_2.
\]
Express \( R_1 \) and \( R_2 \) by integrals on the boundary \( \partial \Omega \).

(b) Let \( Q = \{ (x,y) : y \geq |x|, x \leq 0 \} \). Let \( \varphi \) be a smooth function on \( \mathbb{R}^2 \) with compact support. Show that
\[
\int_Q (y + x) \left( \frac{\partial^2 \varphi(x,y)}{\partial y^2} - \frac{\partial^2 \varphi(x,y)}{\partial x^2} \right) \, dx\, dy = \int_0^\infty \varphi(0,y) \, dy - \int_0^\infty y \frac{\partial \varphi}{\partial x}(0,y) \, dy.
\]

(c) Consider the function
\[
f(x,y) \overset{\text{def}}{=} \begin{cases} y - |x| & \text{for } y \geq |x| \\ 0 & \text{otherwise} \end{cases}
\]
Show that \( S = \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2} \) is a nonnegative distribution, i.e. show that \( \langle S, \varphi \rangle \geq 0 \) for any nonnegative test function \( \varphi \in \mathcal{D}(\mathbb{R}^2) \).

Problem VIII

Let \( E \) be a Banach space whose closed unit ball will be denoted by \( \overline{B} \). A linear operator \( T \) from \( E \) into itself is said to be a compact operator if \( T(\overline{B}) \) is a relatively compact subset of \( E \).

(a) Give a non-trivial example of compact operators and give an example of application.

(b) Assume that \( E \) is a Hilbert space. Show that if \( T \) is a compact operator then \( T(\overline{B}) \) is a compact subset of \( E \) (not only relatively compact).

(Hint: Use weak convergence.)

(c) Find a more general hypothesis on \( E \) in order that the conclusion in (b) still holds.

Problem IX

(a) Let \( C_0^\infty(\mathbb{R}) \) be the set of smooth functions on \( \mathbb{R} \) with compact support. Let \( \varphi \in C_0^\infty(\mathbb{R}) \). Assume that \( g \in C^3(\mathbb{R}) \) and that \( g(0) = g'(0) = g''(0) = g'''(0) = 0 \). Set \( g_k(x) = g(x)\varphi(kx) \).

Prove that \( g_k \) and its derivatives of order \( \leq 3 \) tend to 0 uniformly on \( \mathbb{R} \) as \( k \to \infty \).

(b) What are all the distributions \( T \) on \( \mathbb{R} \) supported by \{0\}, and such that for some constant \( K \) and for all \( \varphi \in C_0^\infty(\mathbb{R}) \)
\[
|T\varphi| \leq K \sup_{x \in [-1,1]} \{|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi'''(x)|\}?
\]

(c) Find a distribution \( S \) on \( \mathbb{R} \) that agrees with the function \( \frac{1}{x^2} \) on \( \mathbb{R} \setminus \{0\} \).

(d) What are all the distributions on \( S \), satisfying (c) and the additional property that, for some constant \( K \) and any \( \varphi \in C_0^\infty(\mathbb{R}) \),
\[
|S\varphi| \leq K \sup_{x \in \mathbb{R}} \{|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi'''(x)|\}?
\]
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison

August 30, 2004

Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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3. For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
4. If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
5. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).

6. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_{\infty} = \inf \left\{ t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0 \right\}.
\]

\( L^{\infty}(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_{\infty} < +\infty \).

7. \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).

8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f * g \) is defined to be the function

\[
f * g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
\]

whenever the integral converges.

9. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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ADVANCED CALCULUS

Problem I
Let $U \subset \mathbb{R}^3$ be a nonempty open subset.
Use differential calculus to show that a continuously differentiable map $f : U \to \mathbb{R}^2$ cannot be injective.

Problem II
Let $\epsilon_n > 0$ be a sequence with $\sum_{n=1}^{\infty} \epsilon_n < \infty$.
(a) Suppose that $u_n$ is a sequence of real numbers satisfying
$$u_{n+1} \leq u_n + \epsilon_n$$
for all $n \geq 1$. Show that $\lim_{n \to \infty} u_n$ exists (the possibility $\lim_{n \to \infty} u_n = -\infty$ is allowed.)
(b) Suppose that $v_n > 0$ are real numbers satisfying $u_n \leq 1 + \epsilon_n$. Show that $\lim_{n \to \infty} \prod_{k=1}^{n} v_k$ exists.

Problem III
According to a Theorem of Weierstrass, every continuous function on $[-1, +1]$ can be uniformly approximated by a sequence of polynomials. Here we study the question of approximation by polynomials of fixed degree.

Let $f$ be a $C^4$ function defined on $[-1, +1]$ (i.e. $f$ and its derivatives of order $\leq 4$ are continuous functions on $[0, 1]$.) Show that there is a constant $C > 0$ such that for every polynomial $P$ of degree $\leq 4$
$$\sup_{|x| \leq 1} |f(x) - P(x)| \geq C \left| \int_{-1}^{1} x(x^2 - 1)^4 f^{(m)}(x) \, dx \right|.$$
Either give an explicit value of $C$ or indicate very clearly an easy computation that would lead to such a value. Give full justifications.

Problem IV
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which vanishes outside the unit circle and define
$$I(f) = \int_{0}^{2\pi} \int_{0}^{1} f(r \cos \theta, r \sin \theta) \, dr \, d\theta.$$
For which $p \geq 1$ is there a constant $C_p$ such that
$$I(f) \leq C_p \|f\|_{L^p(\mathbb{R}^2)}$$
for all such functions $f$.

Problem V
Let $f \in L^\infty(\mathbb{R})$ with $f(x + 1) = f(x)$.
(a) Show that for every measurable subset $E \subset [0, 1]$ we have
$$\lim_{n \to \infty} \int_E f(nx) \, dx = |E| \int_0^1 f(x) \, dx.$$
Hint: one approach is to first show that it is true for the functions $f_k(x) = e^{2\pi i k x}$, $k = 0, \pm 1, \pm 2 \ldots$.
(b) Suppose that there is a measurable set $E \subset [0, 1]$ with $|E| > 0$ such that for some sequence of integers $n_k \to \infty$,
$$\lim_{k \to \infty} f(n_k x) = g(x)$$
exists for all $x \in E$. Show that there is a constant $C$ such that $f(x) = C$ almost everywhere on $[0, 1]$.
Hint: First use part (a) to show that there is a constant $C$ such that $g(x) = C$ almost everywhere on $E$.
Then use part (a) again (with a different $f$) to show that $f(x) = C$ almost everywhere on $[0, 1]$. 
Problem VI

(a) True or false? In other words, prove or disprove the following statements:
(i) If \( \mu \) is a finite Borel measure on \( \mathbb{R} \), then \( \lim_{x \to x_0} \mu((-\infty, x]) = \mu((-\infty, x_0]) \) holds for any \( x_0 \in \mathbb{R} \).
(ii) If \( \mu \) is a finite Borel measure on \( \mathbb{R} \), then \( \lim_{x \to x_0} \mu((-\infty, x]) = \mu((-\infty, x_0]) \) holds for any \( x_0 \in \mathbb{R} \).

Let \( E \subset \mathbb{R} \). Let \( f \) be a continuous function on \( \mathbb{R} \). Assume that \( f \) is differentiable at any point \( x \in \mathbb{R} \setminus E \), and that for any such point \( f'(x) = 0 \).

(b) Assume that the set \( E \) has Lebesgue measure 0, must \( f \) be constant?
(c) Assume that the set \( E \) is countable. Show that \( f \) is constant. Although the result is true in this generality, full credit will be given for a proof in the special and easier case when \( E \) is a closed countable set.

Problem VII

Let \( f_n(z) \) be a sequence of polynomials. Assume that for some function \( h : \mathbb{C} \to \mathbb{C} \) one knows that
\[
\lim_{n \to \infty} f_n^2(z) + f_n(z) = h(z)
\]
uniformly on each compact subset of \( \mathbb{C} \).

(a) Show that \( h(z) \) is not the polynomial \( h(z) = z \).
(b) If \( h(z) = az^2 + bz + c \), find all possible values of \( a, b, c \) (or a necessary and sufficient condition on \( a, b, c \)).

Problem VIII

Suppose that \( f \) is holomorphic in the unit disc in \( \mathbb{C} \) and that
\[
\int_0^{2\pi} |f(re^{it})|^p dt \leq \frac{C}{(1-r)^A}
\]
for some \( 1 < p < \infty \) and some constants \( C > 0 \) and \( A \geq 0 \). Show that \( |f(z)| \leq \frac{D}{(1 - |z|)^B} \) for some positive constants \( D \) and \( B \). Try to find the best value of \( B \).

Problem IX

Let \( g \) be a continuous function defined on the interval \([-1, +1]\) in \( \mathbb{R} \). It is a classical result that if one sets
\[
g_{\tau}(x) = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi \tau}} g(t) e^{-\frac{(x-t)^2}{\tau}} dt,
\]
then for any \( x \in (-1, +1) \), \( g_{\tau}(x) \) tends to \( g(x) \) as \( \tau \to 0^+ \), and the convergence is uniform on smaller intervals.

(It is just a matter of classical approximate identity kernels, it shows up naturally when solving the heat equation and it was used by Weierstrass in proving his approximation theorem.)

Now, let \( g \) be a holomorphic function defined on \( \mathbb{C} \). For \( x \in \mathbb{C} \) and \( \tau > 0 \) set:
\[
g_{\tau}(x) = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi \tau}} g(t) e^{-\frac{(x-t)^2}{\tau}} dt.
\]

(a) Prove that \( g_{\tau} \) is an entire function, i.e. a holomorphic function defined on all of \( \mathbb{C} \).
(b) Find a region \( U \subset \mathbb{C} \) containing a neighborhood of 0 such that for all \( x \in U \), \( g_{\tau}(x) \) tends to \( g(x) \) as \( \tau \to 0^+ \).

Hint: If \( z = x + iy \), switch from integration on \([-1,+1]\) to integration on the line segment \([-1+iy, 1+iy]\).
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

August 30, 2004

Version for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( D = \{ z \in \mathbb{C} | |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for

\[
\|f\|_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_\infty = \inf \{ t > 0 | \mu(\{x \in X | |f(x)| > t\}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_\infty < +\infty \).

(7) \( L^p_c(K) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) \, g(t) \, dt
\]

whenever the integral converges.

(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Advanced Calculus

Problem I

Let $U \subset \mathbb{R}^3$ be a nonempty open subset.

Use differential calculus to show that a continuously differentiable map $f : U \to \mathbb{R}^2$ cannot be injective.
Problem VI
(a) True or false? In other words, prove or disprove the following statements:
(i) If $\mu$ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \to x_0} \mu((\infty, x]) = \mu((\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.
(ii) If $\mu$ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \to x_0} \mu((\infty, x]) = \mu((\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.

Let $E \subseteq \mathbb{R}$. Let $f$ be a continuous function on $\mathbb{R}$. Assume that $f$ is differentiable at any point $x \in \mathbb{R} \setminus E$, and that for any such point $f'(x) = 0$.

(b) Assume that the set $E$ has Lebesgue measure 0, must $f$ be constant?
(c) Assume that the set $E$ is countable. Show that $f$ is constant. Although the result is true in this generality, full credit will be given for a proof in the special and easier case when $E$ is a closed countable set.

7.2.5
Problem VII Show that there is a distribution $U$ on $\mathbb{R}$ such that for all test functions $\varphi \in C_0^\infty(\mathbb{R})$ that vanish identically near 0:
\[ (U, \varphi) = \int_\mathbb{R} \frac{\varphi(x)}{x^4} \, dx \, . \]
Show that there is no distribution $V$ on $\mathbb{R}$ such that for all test functions $\varphi \in C_0^\infty(\mathbb{R})$ that vanish identically near 0:
\[ (V, \varphi) = \int_\mathbb{R} \varphi(x)e^x \, dx \, . \]

Problem VIII For $f \in L^2(\mathbb{R})$ let
\[ Tf(x) = \int_0^1 f(x + y) \, dy \, . \]
(a) Show that $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.
(b) Show that $\|Tf\|_2 \leq \|f\|_2$, and equality holds if and only if $f = 0$ almost everywhere.
(c) Prove or disprove that the map $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by $S : f \mapsto f - Tf$ is onto.

Problem IX Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions whose derivatives $f'_n$ in the sense of distributions belong to $L^2(0, 1)$. The functions also satisfy $f_n(0) = 0$.
(a) Assume that
\[ \lim_{n \to \infty} \int_0^1 f'_n(x)g(x) \, dx \]
exists for all $g \in L^2(0, 1)$. Show that the $f_n$ converge uniformly as $n \to \infty$.
(b) Assume that
\[ \lim_{n \to \infty} \int_0^1 f''_n(x)g(x) \, dx \]
exists for all $g \in C([0, 1])$. Is it still necessarily true that the $f_n$ converge uniformly?
QUALIFYING EXAM
in
ANALYSIS
Department of Mathematics
University of Wisconsin-Madison
Tuesday January 11, 2005
Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subseteq \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( ||f||_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
||f||_\infty = \inf \{ t > 0 : \mu(\{ x \in X : |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( ||f||_\infty < +\infty \).

(7) \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subseteq \mathbb{R} \).

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
\]

whenever the integral converges.

(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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**Problem I**
(a) Show that if \( \sum a_n \) converges then there exists a sequence \( b_n \rightarrow \infty \) so that \( \sum a_n b_n \) is still convergent.

(b) Let \( b_n \) be an unbounded sequence. Show that there exists a convergent \( \sum a_n \) so that \( \sum a_n b_n \) is divergent.

**Problem II**
(a) For which real values of \( p \) does the integral
\[
I_p = \int_0^\infty \int_0^\infty \frac{dx 

\text{dy}}{\sqrt[3]{x^2 + y^3 + x^2y^2}}
\]
converge?

(b) For which real values of \( p \) does the sum
\[
\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{1 + m^2 + np + m^2 n^2}
\]
converge?

**Problem III** Let \( a_n \) be a convergent sequence. Define
\[
F(\lambda) = \sum_{n=1}^\infty \lambda e^{-\lambda n} a_n, \quad \lambda > 0.
\]

(a) Show that \( \lim_{\lambda \downarrow 0} F(\lambda) \) exists.

(b) Show that \( F : (0, \infty) \rightarrow \mathbb{R} \) is a continuously differentiable function.

(c) Does the limit
\[
\lim_{\lambda \downarrow 0} \frac{F(\lambda) - F(0)}{\lambda}
\]
exist? (Here we interpret \( F(0) \) as \( \lim_{\lambda \downarrow 0} F(\lambda) \).)

**Problem IV** For \( f \in L^1_{\text{loc}}(\mathbb{R}^3) \), define
\[
(Kf)(x) = \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} |f(y)| dy, \quad x \in \mathbb{R}^3
\]
and
\[
\|f\|_K = \|Kf\|_{L^\infty(\mathbb{R}^3)}.
\]
Let \( X = \{ f \in L^1_{\text{loc}}(\mathbb{R}^3) : \|f\|_K < \infty \} \).

(a) Prove or disprove that the normed vector space \((X, \| \cdot \|_K)\) is complete.

(b) For which \( p \in [1, \infty) \) is \( L^p(\mathbb{R}^3) \) a subset of \( X \)?

**Problem V** Let \( K \) be a non-empty compact subset of \( \mathbb{R} \), of Lebesgue measure 0.

(a) Show that there is a continuous non negative function \( f \) such that \( f = 0 \) on \( K \) and \( f > 0 \) off \( K \), and such that \( \int_{\mathbb{R}} \frac{dx}{f(x)} < +\infty \).

(b) Show that there is no function \( f \) as above that is continuously differentiable.
Problem VI  Let \( f \in L^2(\mathbb{R}) \).

Are the following true statements? Prove or disprove by a counter example.

(a) If \( f \) is continuous then \( f(x) \) tends to 0 as \( |x| \) tends to \(+\infty\).

(b) \( \int_n^{n+1} |f(t)| \, dt \) tends to 0 as \( n \to \infty \).

(c) \( \sqrt{n} \int_n^{n+1} |f(t)| \, dt \) tends to 0 as \( n \to \infty \).

(d) \( \lim \inf_{n \to \infty} \sqrt{n} \int_n^{n+1} |f(t)| \, dt = 0 \).

Problem VII

(a) Show that there is no holomorphic function \( g(z) \) on \( \mathbb{C} \setminus [-2,2] \) satisfying

\[
g(z)^3 - 3g(z) = z, \quad z \in \mathbb{C} \setminus [-2,2].
\]

(b) Let \( D = \mathbb{C} \setminus ((-\infty, -2] \cup [2, \infty)) \). It is a fact that there is a holomorphic function \( f(z) \) on \( D \) satisfying

\[
f(z)^3 - 3f(z) = z, \quad z \in D, \quad f(0) = 0.
\]

(You can accept this fact.) Prove that \( -f(-z) = f(z) \) holds on \( D \).

(c) For \( \epsilon > 0 \) let \( \gamma_\epsilon \) be the oriented path consisting of: the horizontal half line connecting \(+\infty - i\epsilon \) to \( 2 - \epsilon - i\epsilon \), the vertical segment connecting \( 2 - \epsilon - i\epsilon \) to \( 2 - \epsilon + i\epsilon \), and the horizontal half line connecting \( 2 - \epsilon + i\epsilon \) to \(+\infty + i\epsilon \).

For the function \( f \) in (b), evaluate

\[
\lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} \frac{dz}{z^2 f(z)}.
\]

Problem VIII  Let \( u(z) \) be a real harmonic function on \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).

(a) Show that there is a constant \( c \) so that

\[
u(z) - c \log |z| = f(z) + \overline{f(z)}
\]

for some holomorphic function \( f \) defined on \( \mathbb{C}^* \).

(b) Show that \( u(z) \) has a finite limit as \( z \to 0 \), if

\[
\lim_{z \to 0} \frac{u(z)}{\log |z|} = 0.
\]

(c) Show that \( u(z) \) has a finite limit as \( |z| \to \infty \), if

\[
\lim_{|z| \to \infty} \frac{u(z)}{\log |z|} = 0.
\]

Problem IX  Let \( D \) be a domain in \( \mathbb{C} \), with non-empty boundary \( \partial D = (\overline{D} \cap \mathbb{C}) \setminus D \). Let \( f(z) \) be holomorphic on \( D \) and continuous on \( \overline{D} \cap \mathbb{C} \). Assume that there are two constants \( A \) and \( B \) so that

\[
\sup_{z \in \partial D} |f(z)| \leq A, \quad \sup_{z \in D} |f(z)| \leq B.
\]

Show that \( |f(z)| \leq A \) on \( D \).

Hint: Take \( a \in \partial D \) and consider

\[
\frac{f(z)^n}{z - a}, \quad z \in D \setminus \{z \in |z - a| \leq \epsilon\}, \quad n = 1, 2, 3, \ldots.
\]
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Tuesday January 11, 2005

Versions for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
2. \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
3. For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
4. If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
5. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
   \[
   ||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
   \]
   Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( ||f||_p < +\infty \).
6. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then
   \[
   ||f||_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
   \]
   \( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( ||f||_\infty < +\infty \).
7. \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).
8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f * g \) is defined to be the function
   \[
   f * g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
   \]
   whenever the integral converges.
9. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I

(a) Show that if $\sum a_n$ converges then there exists a sequence $b_n \to \infty$ so that $\sum a_n b_n$ is still convergent.

(b) Let $b_n$ be an unbounded sequence. Show that there exists a convergent $\sum a_n$ so that $\sum a_n b_n$
Problem VI  Let $f \in L^2(\mathbb{R})$.
Are the following true statements? Prove or disprove by a counter example.

(a) If $f$ is continuous then $f(x)$ tends to 0 as $|x|$ tends to $+\infty$.

(b) $\int_1^{n+1} |f(t)| \ dt$ tends to 0 as $n \to \infty$.

(c) $\sqrt{n} \int_1^{n+1} |f(t)| \ dt$ tends to 0 as $n \to \infty$.

(d) $\liminf_{n \to \infty} \sqrt{n} \int_1^{n+1} |f(t)| \ dt = 0$.

Problem VII  Let $S(\mathbb{R}^d)$ be the set of smooth functions $f$ satisfying
\[ \rho_N(f) = \sup_{|\alpha| \leq N} (1 + |x|^2)^N |D^\alpha f(x)| < \infty, \quad N = 0, 1, 2, \ldots, \]
where $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}.$ $S(\mathbb{R}^d)$ is a vector space, in which the topology is defined by semi-norms $\rho_N, N = 0, 1, 2, \ldots.$ Let $S'(\mathbb{R}^d)$ be the set of continuous linear functionals on $S(\mathbb{R}^d)$. Recall that each element in $S'(\mathbb{R}^d)$ is a distribution on $\mathbb{R}^d$.

Let $T \in S'(\mathbb{R}^d)$ satisfy
\[ D^\alpha T = 0, \]
for any multi-index $\alpha$ with $|\alpha| = 2$. Show that $T$ is a polynomial in $x$ of degree at most 1. (The statement is true, under the assumption that $T$ is a distribution. But you don’t have to prove it.)

Problem VIII  Construct a function $f$ in $L^2(\mathbb{R}^2)$, with distribution derivative $f'$ in $L^2(\mathbb{R}^2)$, that diverges to infinity $(\lim_{x \to r} |f(x)| = \infty)$ at every rational point $r$ in the unit square.

Problem IV  A subset $S$ of a (complex) Banach space $X$ is called weakly bounded if $\sup_{\lambda \in S} |\lambda(x)| < \infty$ for any $\lambda \in X^*$. The set $S$ is called strongly bounded if $\sup_{\lambda \in S} ||\lambda|| < \infty$. Prove that $S$ is strongly bounded if and only if $S$ is weakly bounded.
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 24, 2005

Versions for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
(2) $D = \{ z \in \mathbb{C} | |z| < 1 \}$ denotes the unit disc in the complex plane.
(3) For points $x$ and $y$ in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
(4) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.
(5) If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,

$$
\|f\|_p = \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p}.
$$

Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $\|f\|_p < +\infty$.

(6) If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then

$$
\|f\|_{L^\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
$$

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $\|f\|_{L^\infty} < +\infty$.

(7) $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \mathbb{R}$.

(8) If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f \ast g$ is defined to be the function

$$
f \ast g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
$$

whenever the integral converges.

(9) If $T$ is a distribution and $\varphi$ is a test function, then $\langle T, \varphi \rangle$ denotes the value of the distribution applied to the test function.

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Problem I  Assume that \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are sequences of nonnegative real numbers such that

(i) \(a_n \leq a_{n+1}\) for any \(n = 1, 2, \ldots\).
(ii) \(b_n \geq b_{n+1}\) for any \(n = 1, 2, \ldots\), and \(\lim_{n \to \infty} b_n = 0\).
(iii) \(\sum_{n=1}^{\infty} a_n (b_n - b_{n+1})\) is convergent.

(a) Prove that \(\lim_{n \to \infty} a_n b_n = 0\).
(b) Show that conclusion (a) may fail if assumption (i) is omitted.

Problem II  Let

\[ F(t) = \int_{0}^{\infty} e^{-x^2} \cos(tx) \, dx, \quad t \in \mathbb{R}. \]

(a) Prove that \(F'(t) = -\frac{1}{2} F(t)\). (Justify all steps.)
(b) Find \(F(1)\).

Problem III  Prove that

\[ xy + x^4 - y^4 = 0 \]

admits a continuous solution \(y = f(x)\) for \(|x| < \frac{1}{100}\).

Problem IV  Let

\[ F(x) = \int_{[0, \infty)} \frac{e^{-xt}}{\sin t} \, dt, \quad x > 0. \]

(a) Find the values of the parameter \(a\) for which the function \(F\) is well-defined on \((0, \infty)\) (i.e. the integrand is in \(L^1\)).
(b) Show that \(F \in C^\infty(0, \infty)\) for these values of \(a\). (Hint: Use the Lebesgue Dominated Convergence Theorem.)

Problem V  Assume that \((\Omega, \Sigma, \mu)\) is a measure space and \(f \in L^p(\Omega)\) for some \(0 < p < \infty\).

(a) Show that

\[ (*) \quad \lim_{q \to \infty} \|f\|_{L^q} = \|f\|_{L^\infty}. \]
(b) Does the conclusion (\(*)\) still hold if we omit the assumption \(f \in L^p(\Omega)\) (proof or counterexample)?

Problem VI  Construct a sequence of continuous functions \(f_n\) on \([0, 1]\) such that \(0 \leq f_n \leq 1\),

\[ \lim_{n \to \infty} \int_{0}^{1} f_n(x) \, dx = 0, \]

but the sequence \(f_n(x)\) does not converge for any \(x \in [0, 1]\).
Problem VII Recall that if $T$ is a distribution on $\mathbb{R}^2$, $(\frac{\partial}{\partial x} T)(\varphi) = -T(\frac{\partial}{\partial x} \varphi)$, for any smooth function $\varphi$ with compact support.

Let $D \subset \mathbb{R}^2$ be the domain $x > |y|$. Let $\chi_D$ be the characteristic function of $D$. (So $\chi_D = 1$ on $D$ and $\chi_D = 0$ on $\mathbb{R}^2 \setminus D$.) Let $\varphi$ be a smooth function on $\mathbb{R}^2$ with compact support. Assume $\varphi \geq 0$ and $\varphi(0) > 0$. Let $\varphi_n(x, y) = \varphi(nx, ny)$. Determine values of $\alpha$ for which $\lim_{n \to \infty} (\frac{\partial}{\partial x} \chi_D)(n^\alpha \varphi_n)$ exists (and is finite).

Problem VIII Let $f$ be a continuous function on $\mathbb{R}^2$. Assume that the distributional partial derivative $\frac{\partial f}{\partial x}$ is in $L^\infty_{loc}(\mathbb{R}^2)$, that is that for some $a \in L^\infty_{loc}(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} f \frac{\partial \varphi}{\partial x} \, dx \, dy = - \int_{\mathbb{R}^2} a \varphi \, dx \, dy$$

for all smooth functions $\varphi$ on $\mathbb{R}^2$ with compact support. Prove that the distributional derivative $\frac{\partial}{\partial x}(f(x, 0))$ is in $L^\infty_{loc}(\mathbb{R})$.

Problem IX Let $0 < \alpha \leq 1$. Let $Lip_\alpha$ be the set of functions $f$ on $[0, 1]$ satisfying

$$|f|_\alpha = \sup_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x < y \leq 1} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty.$$ 

Let $S \subset Lip_\alpha$ be a closed linear subspace of $L^2([0, 1])$.

(a) Prove that there is a positive constant $c$ such that

$$|f|_\alpha \leq c\|f\|_{L^2}, \quad f \in S.$$ 

(b) Prove that $S$ is finite-dimensional.
Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
2. $D = \{z \in \mathbb{C} \mid |z| < 1\}$ denotes the unit disc in the complex plane.
3. For points $x$ and $y$ in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
4. If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.
5. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,
   \[ \|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}. \]
   Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $\|f\|_p < +\infty$.
6. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then
   \[ \|f\|_\infty = \inf \{ t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0 \}. \]
   $L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $\|f\|_\infty < +\infty$.
7. $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \subset \mathbb{R}$.
8. If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f \ast g$ is defined to be the function
   \[ f \ast g(x) = \int_{\mathbb{R}} f(x - t) \, g(t) \, dt \]
   whenever the integral converges.
9. If $T$ is a distribution and $\varphi$ is a test function, then $\langle T, \varphi \rangle$ denotes the value of the distribution applied to the test function.

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Problem I  Give an example of a Riemann integrable function $f: [0,1] \to [0,1]$ which has a dense set of discontinuities. Verify all conclusions.

Problem II  Let

$$f(x,y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + y_n^2}, \quad y > 0.$$  

(a) Show that for each $y > 0$, $g(y) = \lim_{x \to +\infty} f(x,y)$ exists. Evaluate the limit function $g(y)$.

(b) Determine if $f(x,y)$ converges to $g(y)$ uniformly for $y \in (0, \infty)$ as $x \to +\infty$.

(Justify all steps.)

Problem III  Let

$$s_n(x) = \sum_{k=1}^{n} \sin(kx).$$

Show that there exists a constant $C$, independent of $N,x$, such that

$$\sum_{n=1}^{N} \frac{|s_n(x)|}{n^2} < C, \quad 0 < x < \pi, \quad N = 1, 2, 3, \ldots.$$  

(Hint: Estimate $s_n(x)$ for $n \leq \frac{1}{x}$ and for $n > \frac{1}{x}$ separately.)

Problem IV  Give an example of a sequence $f_k$ such that $f_k$ converges weakly to zero in $L^2[0,1]$ and strongly to zero in $L^{3/2}[0,1]$, but does not converge strongly in $L^2[0,1]$. Verify all conclusions.

Problem V  Fix a function $g \in L^1(\mathbb{R})$ such that $\int g(x) \, dx = 0$. Denote $g_\epsilon(x) = \epsilon^{-1}g(\epsilon^{-1}x)$. Consider an operator

$$T_\epsilon f(x) = \int_{\mathbb{R}} g_\epsilon(y)f(x - y) \, dy.$$  

(a) Prove that there exists a constant $C$ such that $\|T_\epsilon f\|_p \leq C\|f\|_p$, for all $\epsilon \neq 0$ and $1 \leq p < \infty$.

(b) Prove that $\lim_{\epsilon \to 0} \|T_\epsilon f\|_p = 0$ for any $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$.

Problem VI  Fix $\alpha > 0$.

(a) Suppose that $f_n \in L^\infty(\mathbb{R})$ satisfy $\|f_n\|_{L^\infty(\mathbb{R})} \geq n^{1+\alpha}$, $n = 1, 2, \ldots$. Show that there is a function $g \in L^1(\mathbb{R})$ such that $\lim_{n \to \infty} \|f_n g\|_{L^1(\mathbb{R})} = \infty$.

(b) Prove or disprove that (a) holds when $\alpha = 0$. 
Problem VII. Let \( f(z) = 10z + z^2 + iz^4 \).

(a) Show that for each \( w \) with \(|w| < 8\), \( f(z) = w \) has a unique solution \( z \) satisfying \(|z| < 1\).

(b) Show that there exist distinct \( z_1, z_2 \) in \( \{z : |z| < 2\} \) such that \( f(z_1) = f(z_2) \).

Problem VIII

(a) Let \( f(z) \) be the branch of \( \sqrt{z(1-z)} \) on \( \mathbb{C} \setminus [0,1] \) with \( f(2) = \sqrt{2}i \). Determine the values of 
\[
\lim_{y \to 0, y > 0} f\left(\frac{1}{2} + iy\right), \quad \lim_{y \to 0, y < 0} f\left(\frac{1}{2} + iy\right). 
\]

(b) Evaluate 
\[
\int_0^1 \frac{x^2}{\sqrt{x(1-x)}} \, dx.
\]

(Justify all steps.)

Problem IX. Let \( \Delta = \{z : |z| < 1\} \).

(a) Let \( 0 < a_n < 1 \) such that \( \sum_{n=1}^{\infty} (1 - a_n) \) is convergent. Show that the limit function
\[
f(z) = \lim_{n \to +\infty} \prod_{k=1}^{n} \frac{a_k - z}{1 - a_k z}, \quad z \in \Delta
\]
is holomorphic, and that \( f \) has zeros at \( a_n \) only.

(b) Give an example of a bounded holomorphic function \( g \) on \( \Delta \) and a sequence \( b_n \) in \( \Delta \) such that \( g \) has simple zeros at \( b_n, n = 1, 2, \ldots \) and 
\[
\lim_{n \to \infty} g'(b_n)(1 - |b_n|) = 0.
\]

Verify all conclusions.
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 23, 2006
Versions for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
2. \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
3. For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
4. If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
5. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
6. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
\|f\|_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_{\infty} < +\infty \).
7. \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \mathbb{R} \).
8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
(f \ast g)(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt
\]

whenever the integral converges.
9. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Give an example of a Riemann integrable function $f : [0, 1] \to [0, 1]$ which has a dense set of discontinuities. Verify all conclusions.

Problem II  Let

$$f(x, y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + yn^2}, \quad y > 0.$$ 

(a) Show that for each $y > 0$, $g(y) = \lim_{x \to +\infty} f(x, y)$ exists. Evaluate the limit function $g(y)$.

(b) Determine if $f(x, y)$ converges to $g(y)$ uniformly for $y \in (0, \infty)$ as $x \to +\infty$.

(Justify all steps.)

Problem III  Let

$$s_n(x) = \sum_{k=1}^{n} \sin(kx).$$

Show that there exists a constant $C$, independent of $N, x$, such that

$$\sum_{n=1}^{N} \frac{|s_n(x)|}{n^2} < C, \quad 0 < x < \pi, \quad N = 1, 2, 3, \ldots$$

(Hint: Estimate $s_n(x)$ for $n \leq \frac{1}{x}$ and for $n > \frac{1}{x}$ separately.)

Problem IV  Give an example of a sequence $f_k$ such that $f_k$ converges weakly to zero in $L^2[0, 1]$ and strongly to zero in $L^{3/2}[0, 1]$, but does not converge strongly in $L^2[0, 1]$. Verify all conclusions.

Problem V  Fix a function $g \in L^1(\mathbb{R})$ such that $\int g(x) \, dx = 0$. Denote $g_\epsilon(x) = \epsilon^{-1} g(\epsilon^{-1} x)$.

Consider an operator

$$T_\epsilon f(x) = \int_{\mathbb{R}} g_\epsilon(y) f(x - y) \, dy.$$ 

(a) Prove that there exists a constant $C$ such that $\|T_\epsilon f\|_p \leq C \|f\|_p$, for all $\epsilon \neq 0$ and $1 \leq p < \infty$.

(b) Prove that $\lim_{\epsilon \to 0} \|T_\epsilon f\|_p = 0$ for any $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$.

Problem VI

(a) Fix $\alpha > 0$. Suppose that $f_n \in L^\infty(\mathbb{R})$ satisfy $\|f_n\|_{L^\infty(\mathbb{R})} \geq n^{1+\alpha}$, $n = 1, 2, \ldots$. Show that there is a function $g \in L^1(\mathbb{R})$ such that $\lim_{n \to \infty} \|f_n g\|_{L^1(\mathbb{R})} = \infty$.

(b) Prove or disprove that (a) holds when $\alpha = 0$. 
Problem VII

(a) Find all distributions $T \in \mathcal{S}'(\mathbb{R}^2)$ with the property that
$$x_1 T = x_2 T = 0.$$ 

(b) Give an example of a distribution $T \in \mathcal{S}'(\mathbb{R}^2)$ which does not have compact support and has the property that
$$x_1 x_2 T = 0.$$

Problem VIII  
Show that if $L \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$ then for $\psi_x(y) = \phi(x - y)$ the function
$$f(x) = L(\psi_x)$$
is continuous on $\mathbb{R}$, and
$$|f(x)| \leq C(1 + |x|)^N$$
for any $x \in \mathbb{R}$, for some constants $C$ and $N$.

Problem IX  
Assume that $H$ is a Hilbert space, $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set, and $x \in H$. Find
$$\inf_{c_1, \ldots, c_n \in \mathbb{C}} ||x - \sum_{i=1}^{n} c_i v_i||,$$
in terms of $\|x\|$ and $(x, v_i)$, $i = 1, \ldots, n$. 

QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison

Wednesday, August 24, 2005

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \(\mathbb{R}\) and \(\mathbb{C}\) denote the fields of real and complex numbers respectively.
(2) \(D = \{z \in \mathbb{C} \mid |z| < 1\}\) denotes the unit disc in the complex plane.
(3) For points \(x\) and \(y\) in \(\mathbb{R}^n\), \(|x - y|\) denotes the Euclidean distance between the points.
(4) If \(E \subset \mathbb{R}^n\) is a Lebesgue measurable set, then \(|E|\) denotes its Lebesgue measure.
(5) If \(\mu\) is a positive measure on a set \(X\), and \(f\) is a complex valued measurable function on \(X\), then for \(1 \leq p < +\infty\),
   \[
   \|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
   \]
   Two functions on \(X\) are said to be equivalent if they are equal except on a set of \(\mu\) measure zero. For \(1 \leq p < +\infty\), \(L^p(X) = L^p(X, d\mu)\) is the space of equivalence classes of complex valued measurable functions such that \(\|f\|_p < +\infty\).
(6) If \(\mu\) is a positive measure on a set \(X\), and \(f\) is a complex valued measurable function on \(X\), then
   \[
   \|f\|_{\infty} = \inf \{t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0\}.
   \]
   \(L^\infty(X)\) is the space of equivalence classes of measurable, complex valued functions on \(X\) such that \(\|f\|_{\infty} < +\infty\).
(7) \(L^p_{\text{loc}}(\mathbb{R})\) is the space of measurable, complex valued functions on \(\mathbb{R}\) which belong to \(L^p(K)\) for every compact set \(K \subset \mathbb{R}\).
(8) If \(f\) and \(g\) are measurable functions on \(\mathbb{R}\), the convolution \(f \ast g\) is defined to be the function
   \[
   f \ast g(x) = \int_{\mathbb{R}} f(x-t) g(t) \, dt
   \]
   whenever the integral converges.
(9) If \(T\) is a distribution and \(\varphi\) is a test function, then \(\langle T, \varphi \rangle\) denotes the value of the distribution applied to the test function.

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Problem I  Assume that \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are sequences of nonnegative real numbers such that

(i) \(a_n \leq a_{n+1}\) for any \(n = 1, 2, \ldots\)
(ii) \(b_n \geq b_{n+1}\) for any \(n = 1, 2, \ldots\), and \(\lim_{n \to \infty} b_n = 0\).
(iii) \(\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})\) is convergent.

(a) Prove that \(\lim_{n \to \infty} a_nb_n = 0\).
(b) Show that conclusion (a) may fail if assumption (i) is omitted.

Problem II  Let

\[ F(t) = \int_{0}^{\infty} e^{-s^2} \cos(tx) \, ds, \quad t \in \mathbb{R}. \]

(a) Prove that \(F'(t) = -\frac{t}{2} F(t)\). (Justify all steps.)
(b) Find \(F(1)\).

Problem III  Prove that

\[ xy + x^4 - y^4 = 0 \]

admits a continuous solution \(y = f(x)\) for \(|x| < \frac{1}{100}\).

Problem IV  Let

\[ F(x) = \int_{[0,\infty)} \frac{e^{-xt}}{|\sin t|^a} \, dt, \quad x > 0. \]

(a) Find the values of the parameter \(a\) for which the function \(F\) is well-defined on \((0, \infty)\) (i.e. the integrand is in \(L^1\)).
(b) Show that \(F \in C^\infty(0, \infty)\) for these values of \(a\). (Hint: Use the Lebesgue Dominated Convergence Theorem.)

Problem V  Assume that \((\Omega, \Sigma, \mu)\) is a measure space and \(f \in L^p(\Omega)\) for some \(0 < p < \infty\).

(a) Show that

\[ \lim_{q \to \infty} \|f\|_{L^q} = \|f\|_{L^\infty}. \]

(b) Does the conclusion (\(\ast\)) still hold if we omit the assumption \(f \in L^p(\Omega)\) (proof or counterexample)?

Problem VI  Construct a sequence of continuous functions \(f_n\) on \([0, 1]\) such that \(0 \leq f_n \leq 1,\)

\[ \lim_{n \to \infty} \int_{0}^{1} f_n(x) \, dx = 0, \]

but the sequence \(f_n(x)\) does not converge for any \(x \in [0, 1]\).
Problem VII  Let \( \Delta \) be the unit disc in \( \mathbb{C} \).

(a) Find a sequence of holomorphic functions \( f_n : \Delta \to \mathbb{C} \setminus \{0\} \) such that \( f_n(0) = \frac{1}{2} \) and \( \lim_{n \to \infty} f_n(\frac{1}{2}) = 0 \).

(b) Prove that if \( f : \Delta \to \mathbb{C} \setminus \{0\} \) is holomorphic and \( f(0) = \frac{1}{2} \), then \( |f(\frac{1}{2})| > c > 0 \) for some constant \( c \) independent of \( f \).

Problem VIII  Evaluate

\[
\int_0^\infty \frac{\log x}{x^3 + 1} \, dx.
\]

(Justify all steps.)

Problem IX  Let \( f(z) \) be holomorphic on \( \text{Re} \, z > 0 \) and continuous on \( \text{Re} \, z \geq 0 \). Assume that \( |f(z)| < e^{-|z|} \). Show that \( f(z) \equiv 0 \). (Hint: Consider \( F(z) = f(z) e^{\frac{1}{z} - \alpha z} \) as \( \alpha \) tends to \( 1^- \).)
Qualifying Exam in Analysis

Do any five of the seven listed.

Give proofs in detail.

9-78
Problem 1.

Let \( f \) be a real valued function on the real line.

Suppose there is a constant \( \alpha \) between zero and one and a positive constant \( C \) such that for all \( x, y \),
\[
|f(x) - f(y)| \leq C |x - y|^\alpha.
\]

Prove that for all \( h > 0 \) there is a continuously differentiable function \( g_h \) such that for all \( x \),
1. \( |f(x) - g_h(x)| \leq Kh^\alpha \),
2. \( |g'_h(x)| \leq Kh^{\alpha - 1} \),

\( K \) is a constant not depending on \( h \).  **Hint:** Try convolution.

Problem 2.

Let \( f \) be a continuously differentiable function on \( [0, \infty) \) such that
\[
\int_0^\infty |f'(t)|^2 \, dt < \infty.
\]

Suppose that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt = L
\]

Prove that
\[
\lim_{T \to \infty} f(t) = L.
\]

Problem 3.

Prove: If \( f \in L^1(\mathbb{R}) \) then
\[
\int e^{i\lambda t^2} f(t) \, dt \to 0 \quad \text{as} \quad \lambda \to \infty.
\]
Problem 4.

Evaluate: \[ \sum_{1}^{\infty} \frac{1}{1 + n^2}. \]

Problem 5.

For \( \Re(s) > 1 \), define \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \).

Show that \((s - 1) \zeta(s)\) has an analytic continuation to \( \Re(s) > 0 \).

Problem 6.

Show: \[ \left| \int_{-N}^{N} e^{is^2h} \, ds \right| \leq \frac{C}{\sqrt{h}}. \]

Problem 7.

Give an example of a function \( f(z) \) which satisfies:

1) \( f(z) \) is holomorphic for \( |z| < 1 \) and continuous for \( |z| \leq 1 \).

2) \( f(e^{i\theta}) \) is \( C^{\infty} \) in \( \theta \) and

3) \( f(z) \) is not analytic in any disk centered at the origin with radius bigger than one.
Do any five problems.

Problem 1. Let \( f \) be twice continuously differentiable on \( (-\infty, \infty) \).

Suppose that \( |f(x)| \leq \frac{1}{1 + x^2} \) and

that \( |f''(x)| \leq 1 + |x| \) for all \( x \).

What can be said about the behavior of \( |f'(x)| \) for large \( |x| \)?

Note: If you are able to conjecture certain properties of \( f'(x) \) but
not able to prove them, state the conjectures carefully and completely.

If you find examples that add information, give them.

Problem 2. Suppose \( \theta \) is positive and irrational and that \( f \) is a continuous
periodic function with period one (i.e. \( f(x+1) = f(x) \) for all \( x \)).

Consider the averages:

\[
\sigma_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(n\theta) \quad (N = 1, 2, \ldots).
\]

Show that \( \lim_{N \to \infty} \sigma_N(f) \) exists. What is it?

Hint: Try it first for \( e^{2\pi ipx} \), \( p \) integral.
Problem 3. Suppose $a(t)$ and $b(t)$ are continuous on $[0, \infty)$ and that
\[ \int_0^\infty |b(s)| \, ds < \infty. \]

Suppose all functions $y(t)$ satisfying $\frac{dy}{dt}(t) = a(t) y(t)$ for $t \geq 0$ are bounded.

Does it follow that all functions satisfying $\frac{dy}{dt}(t) = a(t) y(t) + b(t)$ for $t \geq 0$ are bounded?

Give proof or counterexample (or conjecture and argument).

Problem 4. Suppose $f$ is the conformal map of the unit disk onto the wedge $|\theta| < \pi/4$ which carries $-1$ to $0$, $0$ to $1$ and $1$ to $\infty$.

Find $f^{(4/5)}$.

Problem 5. There are at least two ways in which the improper integral
\[ \int_0^{2\pi} \frac{d\theta}{1 - e^{i\theta}} \]
can be interpreted:

(1) \[ \lim_{\epsilon \searrow 0} \int_0^{2\pi - \epsilon} \frac{d\theta}{1 - e^{i\theta}} \]

(2) \[ \lim_{r \to 1} \int_0^{2\pi} \frac{d\theta}{1 - re^{i\theta}}. \]

Surprise! They're different.

Compute these two limits. Can you "explain" your answer?
Problem 6. Let $\{\phi_n\}_1^\infty$ be orthonormal in $L^2[0,1]$.

Let $\{a_n\}$ be a real sequence with $\sum a_n^2 \log n < \infty$.

Let $S_n = \sum_{1}^{n} a_k \phi_k$.

Prove that $\{S_{2^k}\}$ converges a.e. on $[0,1]$.

Problem 7. Evaluate

$$\int_0^\infty \frac{\log x}{1 + x^2} \, dx .$$

Problem 8. Let $f$ be a twice continuously differentiable function on some open set in the plane.

Suppose that if the disk of radius $r$ about $x_0$ is in the domain of $f$, then the average of $f$ on the boundary of the disk is equal to $f$ at the center of the disk.

Use the divergence theorem to show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 .$$
Analysis Qualifying Examination

Do any 5 of the following 10 problems:

1) Prove
\[
\frac{1}{e^z - 1} = \frac{1}{2} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2 \pi^2}.
\]

2) Show
\[
\int_{0}^{\infty} \frac{\log^2 t}{1 + t^2} \, dt = \frac{\pi^2}{8}.
\]

3) Let \( u(x, t) \) be a function of two variables such that

i) \( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 u = 0 \) \( \quad (m \neq 0) \)

ii) \( u_0(x) = u(x, 0) \) is in \( L^2 \)

iii) \( u_1(x) = \left( \frac{\partial u}{\partial t} \right)(x, 0) = 0 \).

Prove \( u(x, t) \to 0 \) as \( t \to \infty \).

4) Suppose \( a(t) \) is a function such that any solution of the differential equation

\[
u'(t) + a(t)u(t) = 0\]

is bounded as \( t \to +\infty \). Is every solution of the equation

\[
u'(t) + a(t)u(t) = b(t)\]

bounded as \( t \to +\infty \) for every \( b(t) \) in \( L^1 \)?
5) Suppose \( f \) is analytic in the whole complex plane and for some \( \rho > 0 \)

\[
|f(z)| \leq e^{\frac{1}{\rho} |z|^\rho}.
\]

Suppose

\[
\lim_{r \to \infty} \frac{\log \log \max_{|z|=r} |f(z)|}{\log r}
\]

is not an integer. Show \( f(z) = 0 \) for some value of \( z \).

6) \( F(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha} \), \( 0 < \alpha < 1 \)

is a function holomorphic for \( |z| < 1 \). Show that \( F(z) \) can be analytically continued across some arc of the unit circle.

Hint: \( \frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-nt} t^{\alpha-1} \frac{dt}{t} \).

7) Let \( f(z) \) be holomorphic in \( |z| < 1 \) and continuous in \( |z| \leq 1 \).

Let \( \phi(z) \) be holomorphic in \( |z| > 1 \) and continuous for \( |z| > 1 \).
9) Let $V$ be a subspace of $L^2[0,1]$ such that

$$\sup_{x \in [0,1]} |f(x)| \leq C(\int_0^1 |f(x)|^2 \, dx)^{1/2}$$

for some constant $C$. Show $V$ is finite dimensional.

10) Let $f(x) \in L^1[\mathbb{R}]$. Show that the function

$$t \mapsto \int_{-\infty}^{\infty} e^{i \pi t} f(x) \, dx$$

is continuous.
Qualifying Exam - Analysis
January 1980

Do 5 of the following 7 problems.

1) Let \( r_k \) be an enumeration of the rational numbers. Prove

\[
\sum_{k=1}^{\infty} 2^{-k} |x - r_k|^{-1/2}
\]

converges for almost every real number \( x \).

2) Let \( f(z) \) be holomorphic in \(|z| < 1\). Assume

\[
|f(z)| \leq \frac{1}{(1 - |z|)^{1/2}}.
\]

Prove

\[
|f'(z)| \leq \frac{c}{(1 - |z|)^{3/2}}.
\]

3) Let

\[
f(x) = \begin{cases} 
  x \sin 1/x & 0 < x \leq 1 \\
  0 & x = 0
\end{cases}
\]

a) Is \( f(x) \) absolutely continuous on \( 0 \leq x \leq 1 \)?

b) Is \( f(x) \) of bounded variation on \( 0 \leq x \leq 1 \)?
4) Let \( f(z) \) be analytic in \( 0 < |z| < 1 \), and assume \( |f(z)| \leq \frac{c}{\sqrt{|z|}} \).

Prove that there is a function \( g(z) \) such that \( g(z) = f(z) \) for \( 0 < |z| < 1 \) and \( g(z) \) is regular for \( |z| < 1 \).

5) If \( f(x) \) and \( g(x) \) are absolutely continuous functions of a real variable \( x \), is

\[
h(x) = f(g(x))
\]

absolutely continuous.

6) Let \( f(t) \) be in \( L^1(R) \). Show

\[
\lim_{x \to \infty} \int e^{ixt} f(t) \, dt = 0.
\]

7) Prove

\[
\int_{|z| = 1} \frac{dz}{\sqrt{4z^2 + 4z + 3}} \, dz = 2\pi
\]

if the integral is taken in the counterclockwise sense and \( \sqrt{4x^2 + 4x + 3} \) is to be positive for \( x = 1 \).
Analysis Qualifying Examination

Do any 5 of the following 10 problems:

1) Prove
\[
\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2 \pi^2}.
\]

2) Show
\[
\int_{0}^{\infty} \frac{\log t}{1 + t^2} \, dt = \frac{\pi^2}{8}.
\]

3) Let \( u(x, t) \) be a function of two variables such that
   i) \( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 u = 0 \quad (m \neq 0) \)
   ii) \( u_0(x) = u(x, 0) \) is in \( L^2 \)
   iii) \( u_1(x) = \left( \frac{\partial u}{\partial t} \right)(x, 0) = 0 \).

Prove \( u(x, t) \to 0 \) as \( t \to \infty \).

4) Suppose \( a(t) \) is a function such that any solution of the differential equation
\[
u'(t) + a(t) u(t) = 0
\]

is bounded as \( t \to +\infty \). Is every solution of the equation
\[
u'(t) + a(t) u(t) = b(t)
\]

bounded as \( t \to +\infty \) for every \( b(t) \) in \( L^1 \)?
5) Suppose $f$ is analytic in the whole complex plane and for some $\rho > 0$

$$|f(z)| \leq e^{\rho |z|}.$$

Suppose

$$\lim_{r \to \infty} \log \log \max_{|z|=r} |f(z)| \log r$$

is not an integer. Show $f(z) = 0$ for some value of $z$.

6) $F(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$, $0 < \alpha < 1$

is a function holomorphic for $|z| < 1$. Show that $F(z)$ can be analytically continued across some arc of the unit circle.

Hint: $\frac{1}{n^\alpha} = \frac{1}{F(\alpha)} \int_{0}^{\infty} e^{-nt} t^\alpha \frac{dt}{t}$.

7) Let $f(z)$ be holomorphic in $|z| < 1$ and continuous in $|z| \leq 1$.
Let $g(z)$ be holomorphic in $|z| > 1$ and continuous for $|z| \geq 1$.
Suppose $f(z) = g(z)$ if $|z| = 1$. Prove $g(z)$ is an analytic continuation of $f(z)$.

8) Let $f_n$ and $f$ be $L^1$ functions on the unit interval, $[0,1]$.
Suppose

$$f_n \to f \text{ a.e.,}$$

and

$$\int_{0}^{1} |f_n| dx \to \int_{0}^{1} |f(x)| dx.$$

Prove

$$\int_{0}^{1} |f_n(x) - f(x)| dx \to 0.$$
9) Let $V$ be a subspace of $L^2[0,1]$ such that

$$\sup_{x \in [0,1]} |f(x)| \leq C \left( \int_0^1 |f(x)|^2 \, dx \right)^{1/2}$$

for some constant $C$. Show $V$ is finite dimensional.

10) Let $f(x) \in L^1[\mathbb{R}]$. Show that the function

$$t \mapsto \int_{-\infty}^{\infty} e^{ixt} f(x) \, dx$$

is continuous.
Qualifying Exam in Analysis
August, 1981

Do any 4 of the following problems.

1. Show that there is one and only one function $f$ with the following properties: $f$ is holomorphic in a region containing $\{z : |z| \leq 1\}$, $f$ has a simple zero at $z = \frac{1}{2}$, $f$ has a double zero at $z = \frac{(1+1)}{2}$, and $f(0) = \frac{1}{4}$.

2. Let $C_0$ be the set of continuous function on $\mathbb{R}^2$ that vanish off some compact set. Let $L(f) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{s} t^2 f(s, t) \, dt \right] \, ds$, for $f \in C_0$.

Then there is a Borel measure $\mu$ on $\mathbb{R}^2$ such that $L(f) = \int f \, d\mu$.

a) Find $\mu(E)$ where $E = (0, 1) \times (0, 1)$.

b) Find the Lebesgue decomposition of two dimensional Lebesgue measure $m$ with respect to $\mu$.

3. Show that the polynomials $P_n(x) = \frac{d^n}{dx^n} \left[ (x^2 - 1)^n \right]$, $n = 0, 1, 2, \cdots$, suitably normalized, form an orthonormal basis for $L^2(-1, 1)$.
4. a) Suppose $f$ is an entire function and there are positive constants $A, B, C$ such that $|f(z)| \leq A|z|^B + C$, for all $z \in \mathbb{C}$. Show that $f$ is a polynomial.

b) Suppose $f$ is an entire function and there are polynomials $p_0(z), \ldots, p_n(z)$ with $p_n(z) \neq 0$ such that

$$\sum_{k=0}^{n} p_k(z) [f(z)]^k = 0.$$ 

Show that $f$ is a polynomial.

5. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a smooth curve such that $\gamma(0) = 0$, $\gamma(1) = 1$ and $\gamma(t) \neq \pm 1$ for $0 \leq t \leq 1$. Show that there is an integer $k$ such that

$$\int_{\gamma} \frac{dz}{1 + z^2} = \frac{\pi}{4} + k\pi.$$ 

6. Suppose that $f$ is a non-negative measurable function on the real line such that $\int_{-\infty}^{\infty} f(x) \, dx < \infty$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subseteq \mathbb{R}$ is measurable and $\int_{E} 1 \, dx < \delta$ then $\int_{E} f(x) \, dx < \varepsilon$.

7. Compute the maximum of $\left| \int_{-1}^{1} f(x) e^x \, dx \right|$ where $f$ ranges over all measurable functions on $(-1, 1)$ such that $\int_{-1}^{1} |f(x)|^2 \, dx \leq 1$, $\int_{-1}^{1} f(x) \, dx = 0$, and $\int_{-1}^{1} f(x) x \, dx = 0$. 
QUALIFYING EXAM in ANALYSIS
January, 1982

Do any 5 of the following problems.

1. Let $(X, M, \mu)$ be a measure space, $g$ a nonnegative Borel measurable function on $X$. Define a measure $\nu$ on $M$ by

$$\nu(E) = \int_E g \, d\mu \quad (E \in M).$$

Show that if $f$ is a Borel measurable function on $X$, then

$$\int_X f \, d\nu = \int_X fg \, d\mu,$$

in the sense that if one of the integrals exists, then so does the other, and the two integrals are equal. (Intuitively: $\frac{d\nu}{d\mu} = g$, so $d\nu = g \, d\mu$.)

2. Find all entire functions $f$ such that $|f(z)| = 1$ whenever $|z| = 1$.

3. Let $m$ denote Lebesgue measure on $[0,1]$. Suppose that the sequence

$$\{f_n\} \quad (n = 1, 2, \ldots) \quad \text{and} \quad f \quad \text{satisfy:}$$

$$f_n \in L^1(m) \quad \text{for each} \quad n, \quad f \in L^1(m),$$

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{a.e. on} \quad [0,1],$$

and

$$\lim_{n \to \infty} \int_0^1 |f_n| \, dm = \int_0^1 |f| \, dm.$$

Does it follow that $f_n \to f$ in $L^1(m)$?
4. Let \( U \) be the open unit disc. How many fixed points can a holomorphic \( f : U \to U \) have without being the identity map?

5. Let \( f : (0,1) \to \mathbb{R} \) \((n \in \mathbb{N})\) be differentiable functions with \( f(0) = 0 \).
7. Denote $S = \{0,1,\ldots\}$. Let $A = [\alpha(x,y); x,y \in S]$ be a symmetric strictly positive matrix, and define

$$L^2(A) = \{f : S \times S \to \mathbb{R} : \|f\|_{L^2(A)} < \infty\},$$

where

$$\|f\|_{L^2(A)} = \left[ \sum_{x,y \in S} \alpha(x,y) f^2(x,y) \right]^{1/2}.$$

Given $f : S \to \mathbb{R}$, define $\nabla f : S \times S \to \mathbb{R}$ by

$$\nabla f(x,y) = f(x) - f(y).$$

Introduce the norm

$$\Phi(f) = \|\nabla f\|_{L^2(A)},$$

and the space

$$H = \{f : \Phi(f) < \infty\}.$$

Show that $H$ is a Hilbert space with norm $\Phi$.

8. Suppose that $f$ is entire, and real only on the real axis. Argue that $f$ is linear.
Qualifying Exam

ANALYSIS

August 24, 1982

Do ALL Problems.

1. Let \( \mu \) be a positive \( \sigma \) finite measure. Suppose \( 1 < p < q < \infty \) and that \( M \) is a subspace of \( L^p(\text{d}\mu) \). Suppose further that there is a constant \( C \) such that

\[
\left( \int |f|^q \text{d}\mu \right)^{1/q} \leq C \left( \int |f|^p \text{d}\mu \right)^{1/p}
\]

for all \( f \in M \). Show that for each \( F \in L^{q'}(\text{d}\mu) \) there is a \( G \in L^{p'}(\text{d}\mu) \) such that

\[
\left( \int |G|^{p'} \text{d}\mu \right)^{1/p'} \leq C \left( \int |F|^{q'} \text{d}\mu \right)^{1/q'}
\]

and \( \int F \text{d}\mu = \int G \text{d}\mu \) for all \( f \in M \). Here \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \).

2. Let \( \{a_n\} \) be a sequence of complex numbers such that \( \lim_{n \to \infty} na_n = 0 \).

Suppose that \( \lim_{r \to 1, \; 0 < r < 1} \sum_{k=1}^{\infty} a_k r^k = L \). Show that

\[
\lim_{n \to \infty} \sum_{k=1}^{n} a_k = L.
\]

(Hint: Compare \( \sum_{k=1}^{n} a_k \) with \( \sum_{k=1}^{\infty} a_k (1 - \frac{r}{n})^k \)).
3. Show that if $\alpha > 0$; the function $F(z) = \sum_{n=1}^{\infty} z^n \frac{n}{n^\alpha}$ defined originally for $|z| < 1$ has an analytic continuation into the complex plane cut from 1 to $\infty$ along the real axis. Hint: consider the integral

$$\int_{0}^{\infty} e^{-nx} x^{\alpha-1} \, dx.$$ 

4. Evaluate:

$$\lim_{T \to \infty} \int_{\gamma_T} e^{-z^2} \, dz,$$

where $\gamma_T(t) = te^{i \pi/4}$, $0 \leq t \leq T$.

5. Suppose $f$ is holomorphic in $H = \{z : \text{Im } z > 0\}$, $f(H) \subseteq H$ and $f(i) = i + 1$. Find a sharp upper bound for $|f(2i)|$. 
Qualifying Exam

ANALYSIS

January 18, 1983

Do as many problems as possible.

1. $f(t)$ decreases monotonically to 0 as $t \to \infty$ and is in the class $C^\infty$. Show

$$\lim_{N \to \infty} \int_0^N e^{itx} f(t) \, dt$$ exists for $x \neq 0$.

2. Evaluate $\int_0^\infty \frac{x^{-a}}{1+x} \, dx$ for $0 < a < 1$. 
3. Let \((X,M,\mu)\) be a \(\sigma\)-finite measure space. For every measurable \(f : X \to \mathbb{C}\) and \(1 < p < \infty\) let \(\|f\|_p\) be the least number in \([0,\infty)\) for which
\[
\int \frac{|f|}{\mu(K)^{1/q}} \, d\mu \leq \left\| f \right\|_p \mu(K)^{1/q} \quad \text{for all } K \in M,
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\). Let \(M_p\) be the set of measurable \(f\) such that \(\|f\|_p < \infty\).

Prove:

a) \(L_p(\mu) \subseteq M_p\).

b) \(\mu\{x : |f(x)| > \lambda\} < \frac{\|f\|_p}{\lambda^p}\) for \(\lambda > 0\).

c) If \(\{f_n\}\) is a sequence of measurable functions then
\[
\liminf_{n \to \infty} \left\| f_n \right\|_p \leq \liminf_{n \to \infty} \left\| f_n \right\|_p.
\]

d) If \(X = \mathbb{R}\) and \(\mu\) is Lebesgue measure then for \(f \in L_1\) and \(g \in M_p\) it follows that \(f \ast g \in M_p\) and \(\|f \ast g\|_p \leq \|f\|_1 \|g\|_p\).
4. For each of the following determine if there is a function \( f(z) \), holomorphic in the open unit disc, which satisfies the given property. Either give an example or a proof.

a) \( f\left(\frac{1}{n^2}\right) = \frac{1}{n} \) for \( n = 2, 3, 4, \ldots \)

b) \( f \) bounded, \( f(0) = 1 \) and \( f(1 - \frac{1}{n^2}) = 0 \) for \( n = 2, 3, 4, \ldots \)

c) \( f^{(n)}(0) = \frac{n!}{2^n} \) for \( n = 1, 2, 3, \ldots \)

5. Suppose \( f \) and \( g \) are real valued measurable functions on \( \mathbb{R} \) which are periodic, with period 1. If \( g \) is bounded and
\[
\int_0^1 |f| < \infty \quad \text{then show}
\]
\[
\lim_{n \to \infty} \int_0^1 f(x) g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx
\]

Hint: First assume \( f \) is continuous.

6. Let \( f \) be holomorphic in the unit disc and \( |f| \leq M \). If \( |f(0)| = a > 0 \) and \( n \) is the number of zeroes of \( f \) in the disc of radius 1/3 then show
\[
n \leq \frac{1}{\log 2} \log \left( \frac{M}{a} \right)
\]

Hint: Consider \( g(z) = f(z) \prod_{k=1}^n \left( \frac{z_k}{z_k - z} \right) \) where \( z_1, z_2, \ldots, z_n \) are the zeroes of \( f \) in the disc of radius 1/3.
Give complete solutions to any six of the following problems.

You may appeal to standard theorems as needed.

1. Let $A$ be a bounded linear operator from the Banach space $X$ onto the Banach space $Y$. Show that there is a positive number $k$ such that for each $y \in Y$ there is an $x \in X$ with $y = Ax$ and $\|x\| \leq k \|y\|$.

2. Give an example of a sequence of functions $f_n$ on $[0,1]$ such that each $f_n$ is Riemann integrable, $|f_n| < 1$ for all $n$, $f_n \to f$ everywhere, but $f$ is not Riemann integrable.

3. Suppose

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}$$

and let $D_R$ denote the disc with center $0$ and radius $R$. Compute

$$\int_{D_R} |f'|^2 \, dm, \quad 0 < R < 1,$$

where $m$ is 2-dimensional Lebesgue measure.
4. Let $\mu$ be a (positive) measure on $\mathbb{R}^1$, with $\mu(\mathbb{R}^1) < \infty$. Define

$$F(x) = \mu((\infty, x]), \quad x \text{ real}.$$ 

Evaluate:

$$\int_{-\infty}^{\infty} (F(x+c)-F(x)) \, dx, \quad c \text{ real}.$$ 

5. Show that $\{\varphi_n; n \geq 1\}$ is a complete orthonormal set for $L^2[0,1]$ (Lebesgue measure) if and only if

$$\sum_{n=1}^{\infty} \left[ \int_0^x \varphi_n(t) \, dt \right]^2 = x \quad \text{for all } x \in [0,1].$$

6. Let $\mu$ be a probability measure on $\mathbb{R}^1$, i.e. $\mu(\mathbb{R}^1) = 1$. Write

$$\psi(u) = \int_{-\infty}^{\infty} e^{iux} \, d\mu(x), \quad u \text{ complex}.$$ 

Suppose $|\psi(u)| = 1$ for some $u \neq 0$. What can one conclude about $\mu$?
7. Let \( Q \) be the square with vertices at \( 1, i, -1, -i \).

Show that there is a conformal map \( f \) of the unit disc onto \( Q \) that fixes each of the points \( 1, i, -1, -i \). By comparing \( f(iz) \) and \( f(z) \), prove a precise statement to the effect that \( 3/4 \) of the coefficients in the expansion \( f(z) = \sum a_n z^n \) are 0.

8. An integer sequence \( A_k \) is determined inductively by the convolution equation

\[
A_n = \sum_{k=1}^{n-1} \binom{n}{k} A_k A_{n-k}, \quad n \geq 2,
\]

\( A_1 = 1. \)

Show that \( A_n \leq 4^n n! \)
Qualifying Exam

ANALYSIS

January 17, 1984

**Instructions:** Do five problems.

**Policy on Misprints**

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Evaluate by contour integration.

\[ \int_0^\infty \frac{\sin x}{x(x^2+4)} \, dx \ . \]

2. Let \( f \) be a non-constant meromorphic function defined on \( \mathbb{C} \), such that \( f(z+1) = f(z) \) and \( f(z+i) = f(z) \) for all \( z \in \mathbb{C} \). Prove that there is no \( \alpha \in \mathbb{C} \) such that

a) \( f \) has a simple pole at \( \alpha \) (and hence at \( \alpha + m + ni, \ m, n \in \mathbb{Z} \)) and

b) \( f \) has no poles other than at \( \alpha + m + ni, \ m, n \in \mathbb{Z} \).

3. Let \( \lambda > 1 \). Show that \( \lambda - z - e^{-z} = 0 \) has exactly one solution in the half plane \( \text{Re} \, z > 0 \), and that it is real. What happens to this solution as \( \lambda \to 1 \)?
4. For \( x, y \in \mathbb{R} \) and \( u > 0 \) define
\[
f_u(x,y) = (2\pi u)^{-1/2} \exp\{-|y-x|^2/(2u)\}.
\]

Let \( \varphi \) be a continuous function with compact support on \( \mathbb{R} \).
Show that for any \( x \neq 0 \),
\[
\lim_{t \to \infty} \int_{0}^{1} \int_{\mathbb{R}} f_u(x,y) \ t \varphi(ty) \ dy \ du
= \int_{0}^{1} f_u(x,0) \ du \ \int_{\mathbb{R}} \varphi(y) \ dy.
\]

5. Define \( h_s(t), \ 0 \leq s \leq \pi \), on \([-\pi, \pi]\) by
\[
h_s(t) = \begin{cases} 
  t & |t| \leq s \\
  -s & t < -s \\
  s & t \geq s
\end{cases}
\]

Use these functions to check that
\[
\min(s,t) = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k^2},
\]
\[
0 \leq s, t \leq \pi.
\]
6. Suppose that \( f \in L^1_{\text{loc}}(R) \) and that for every interval \( I \) we have

\[
\lim_{h \to 0} \int_I \frac{|f(x+h) - f(x)|}{h} \, dx = 0.
\]

Show that there is a constant \( c \) such that \( f = c \) almost everywhere.

7. Let \( H \) be a complex Hilbert space and let \( T : H \to H \) be a bounded linear operator. Suppose that \( \langle x,y \rangle = 0 \) implies that \( \langle Tx,Ty \rangle = 0 \). Show that there is a constant \( c > 0 \) such that \( \| Tx \| = c \| x \| \) for all \( x \in H \).

(Hint: use an orthonormal basis.)
Qualifying Exam

ANALYSIS

August 28, 1984

Instructions: Do five problems.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $\mathbb{U}$ be an open set in the complex plane, and $f(z)$ a function holomorphic in $\mathbb{U}$, but not a polynomial. Prove that there is a point $z_0$ in $\mathbb{U}$ such that the Taylor series for $f$ at $z_0$ is a power series in $(z-z_0)$ which does not have any zero coefficients.

2. Let $f$, $g$, and $h$ be real valued functions defined for $-\infty < t < \infty$, one of which has a graph that is a closed set. Suppose that $f(x+y) = g(x) + h(y)$ for all $x$ and $y$. Find all possible choices for these functions.

3. Justify the formula: $\arctan(z) = \frac{1}{2i} \log \left( \frac{1 + iz}{1 - iz} \right)$

and discuss its range of validity and domain of definition.

4. Let $\mu$ be Lebesgue measure on $[0, \infty)$ and $f \in L^1_{\mu}$.

(a) Show that it is not necessarily true that $\lim_{x \to \infty} f(x) = 0$ off a set of zero-measure.

(b) Show that for any $\epsilon > 0$, there is a set $E \subset [0, \infty)$ such that $\mu(E) < \epsilon$, and $\lim_{x \to \infty} f(x) = 0 \quad x \notin E$.

(c) Show that there exist points $x$ such that $\lim_{n \to \infty} f(x+n) = 0$. 

5. DO ANY FIVE PROBLEMS
5. Let $c_n > 0$, and \( \lim_{n \to \infty} \frac{1}{N} \sum_{k=1}^{N} c_k = 0 \). Show that
\[
\lim_{n \to \infty} \frac{1}{N} \sum_{k=1}^{N} (c_k)^{1/p} = 0
\]
for every \( p, 1 \leq p < \infty \), but that this assertion cannot necessarily be made for \( p = 1/2 \) or \( 3/4 \), or in fact for any \( p, 0 < p < 1 \).

6. Consider the class of all functions \( f(z) \), holomorphic in the upper half plane, and such that \( f(i) = 0 \) and such that \( \{f(z)\}^2 \) is never a real number \( \geq 4 \). How large can \( |f'(i)| \) be?

7. Let \( \mathcal{D} \) be the class of all real non-negative continuous functions \( f \) defined on \([-1,1]\) such that
\[
\int_{-1}^{1} f(x)dx = 1 + \int_{-1}^{0} f(x)dx
\]
Let \( \|f\| = \max_{-1 \leq x \leq 1} |f(x)| \).

Find \( \inf \|f\| \), and show that this value cannot be attained by any \( f \in \mathcal{D} \).

8. Let \( \mu \) be Lebesgue measure in \( \mathbb{R}^3 \), and \( A_1, A_2, A_3 \) be bounded open sets with \( \bigcap_i A_i = \emptyset \). Prove that
\[
\mu \left( \bigcup_i A_i \right) \geq \frac{3}{2} \min_i \mu(A_i)
\]
Can the number \( 3/2 \) be increased?
Qualifying Exam

ANALYSIS

January 15, 1985

Instructions: Do five problems.
Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $f \in L'(\mathbb{R})$, $f \geq 0$.

Suppose $E_n \subset \mathbb{R}$ are measurable sets, and

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \emptyset.$$  

Prove

$$\lim_{n \to \infty} \int_{E_n} f(x) dx = 0.$$  

(Here $|A|$ denotes the Lebesgue measure of $A$.)

2. In each of the following, either give an example of a function holomorphic in $U = \{ |z| < 1 \}$ which satisfies the required conditions, or prove that no such function exists.

(a) $f(0) \neq 0$, $f(1 - \frac{1}{n^2}) = 0$ for $n = 2, 3, 4, \cdots$

(b) $|f(z)| \leq 1$ for all $z \in U$, $f''(0) = 3$

(c) $f^{(n)}(0) = n!$ for $n = 0, 1, 2, \cdots$

(d) $\lim_{r \to 1} \Re f(re^{i\theta}) = 0$ for $0 < \theta < \pi$, $f(0) = 1/2$.

(e) $f(0) = 1$, $f(1/2) = 4$, and $\Re f(z) > 0$ for $z \in U$. 


3. Let \( f \) be continuous on \( \mathbb{R} \) and \( \lim_{x \to \infty} f(x) = L \).

Show \( \lim_{y \to 0^+} \int_0^\infty e^{-xy} f(x) \, dx = L \).

4. If \( f \) is an entire function such that for all \( x \) and \( y \),

\[ |f(x+iy)| \leq e^{xy} \quad \text{and} \quad f(0) = 1, \quad \text{find} \ f. \]

5. If \( f \in L'(\mathbb{R}) \) show that

\[ \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} e^{i\lambda t^2} f(t) \, dt = 0. \]
6. For \( m = 1, 2, 3, \ldots \), define

\[
f_m(z) = \frac{z}{1 + z^m}.
\]

(a) Show that \( f_m \) is one-to-one in the open unit disc \( U \) if and only if \( m \) is 1 or 2.

(b) Find the region \( f_2(U) \).

7. If \( f : [0,1] \rightarrow (0,\infty) \) is measurable, must

\[
\lim_{n \to \infty} \int_0^1 \frac{f^n(x)}{1 + f^{2n}(x)} \, dx
\]

exist?

When it exists, what is it?
8. Let $f$ and $g$ be measurable functions on $\mathbb{R}$ of period one with
\[
\int_0^1 |f|^2 < \infty, \quad \int_0^1 |g|^2 < \infty.
\]
Find
\[
\lim_{k \to \infty} \int_0^1 f(kx)g(x)\,dx.
\]

9. Let $M = \{\text{All } F : \mathbb{R} \to [0,1], \text{ such that } F \text{ is non-decreasing and right continuous}\}$. Let $F_n \in M$, $n = 1,2,3,\ldots$. Show that there is a subsequence $\{F_{n_k}\}$ and an element $F \in M$ such that $\lim_{k \to \infty} F_{n_k}(x) = F(x)$ at all points $x$ where $F$ is continuous.

10. Find the residue at $z = 0$ of the function
\[
f(z) = \frac{e^{1/z}}{z^2(1+z)^2}.
\]
Qualifying Exam

ANALYSIS

August 27, 1985

Instructions: Do five problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Suppose that $u$ is harmonic in the complex plane and that

$$\lim_{|z| \to \infty} \frac{|u(z)|}{|z|} = 0.$$ Show that $u$ is constant.

2. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in the closed unit disc except at $z_0, |z_0| = 1$, where $f$ has a simple pole.

Show that $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0$.

3. Let $\psi$ be a continuous function on $\mathbb{R}$ with $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$ and $\int_{-\infty}^{\infty} \psi(t) dt = 0$. Show that $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \psi\left(\frac{t}{\varepsilon}\right) g(t) dt = 0$ for every $g$ which is continuous on $\mathbb{R}$ with compact support.

4. Let $\mu$ be a positive measure. $M \subseteq L^1(\mu)$ is a closed subspace such that $M \subseteq \bigcup_{p>1} L^p(\mu)$. Show that there is a $p > 1$ such that $M \subseteq L^p(\mu)$.
5. a) Suppose \( f \in L^\infty(-\pi, \pi) \) with Fourier series
\[
f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(k) e^{ikx}.
\] Suppose that \( \hat{f}(k) > 0 \) for all \( k \).
Show that \( \sum_{k=-\infty}^{\infty} \hat{f}(k) \) converges. (Hint: Consider the Poisson integral of \( f \).)

b) What if \( f \in L^2(-\pi, \pi) \)?

6. Let \( \{f_n\}_{n=1}^{\infty} \) be an orthonormal set in \( L^2(d\mu) \); \( \mu \) is a positive measure. Show that
\[
\lim_{k \to \infty} \frac{f_1 + f_2 + \cdots + f_{n_k}}{\sqrt{n_k(n_k + 1)}} = 0 \quad \text{a.e.}\ d\mu.
\]
Here \( n_k = 2^{k^2} \).
(Hint: To get started, prove the inequality
\[
\mu(\{x: |F(x)| > \varepsilon\}) \leq \frac{1}{\varepsilon^2} \int |F|^2 \, d\mu,
\]
valid for any \( F \in L^2(d\mu) \).
7. Let $L_1, L_2$ be a pair of straight lines in the plane that intersect at angle $\alpha$. Let $u$ be a harmonic function defined on $\mathbb{C}$ such that $u \equiv 0$ on $L_1 \cup L_2$.

a) Assuming that $\alpha$ is not a rational multiple of $\pi$, show that $u \equiv 0$.

b) Show that if $\alpha$ is a rational multiple of $\pi$ then there is such a $u$ with $u \not\equiv 0$ in $\mathbb{C}$.

8. Let $Y$ be the set of real valued non-negative measurable functions $f$ on $(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} f^2(x)dx = 2, \quad \int_{-\infty}^{\infty} f^5(x)dx = 5.$$ 

a) Calculate $m = \sup_{-\infty}^{\infty} f^3(x)dx$.

b) Is there $f \in Y$ such that $\int_{-\infty}^{\infty} f^3(x)dx = m$?
9. Find a sequence \( \{c_n\}_{n=1}^{\infty} \) so that the integrals

\[
c_n \int_{\mathbb{R}^2} f(x-y)g(ny) \, dm(y)
\]

converge to a limit as \( n \to \infty \), provided that \( f \) and \( g \) satisfy some reasonable conditions.

(Here \( m \) denotes Lebesgue measure in \( \mathbb{R}^2 \)). State such conditions, identify the limit and prove the resulting theorem.
Qualifying Exam

ANALYSIS

January 14, 1986

Instructions: Do six problems.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $\mathcal{S}$ be a countable collection of nonnegative continuous functions $f(x)$ defined for $0 \leq x < \infty$. Show that there exists a continuous function $\phi(x)$ with the property that for each $f \in \mathcal{S}$, there is a choice of $c$ such that $f(x) < \phi(x)$ for all $x > c$. Then discuss the situation when $\mathcal{S}$ is noncountable.

2. The improper integral $\int_0^{\infty} \frac{d\theta}{1 - e^{i\theta}}$ was interpreted in three different ways by Tom, Dick, and Harry:

\[ T: \lim_{\delta \to 0} \int_{\pi - \delta}^{\pi} \frac{d\theta}{1 - e^{i\theta}} \]

\[ D: \lim_{r \uparrow 1} \int_0^{2\pi} \frac{d\theta}{1 - r e^{i\theta}} \]

\[ H: \lim_{r \downarrow 1} \int_0^{2\pi} \frac{d\theta}{1 - r e^{i\theta}} \]

Calculate these three limits; is there a good explanation for the discrepancies?

3. Suppose $f(z)$ is holomorphic for $|z| < 1$, and $f(0) = 0$. Where is the function defined by $g(z) = f(z) + f(z^2) + f(z^3) + \ldots$ holomorphic?

4. Suppose that $f$ is a real valued measurable function defined on $[0, \infty)$ with compact support. Let $g(\beta) = \text{meas}\left\{ \text{all } x \text{ with } |f(x)| > \beta \right\}$ and suppose that there is $C > 0$ with $g(\beta) \leq C/\beta$. Show that if $0 < p < 1$ then $\int_0^\infty |f(x)|^p dx < \infty$.

5. Let $f$ and $g$ be continuous complex valued functions defined on $-\infty < x < \infty$, and let $f$ be periodic with period $\alpha$ and $g$ periodic with period $\beta$. Determine the precise conditions under which $f + g$ is a periodic function.
6. Find a conformal mapping that takes the region \( D \) onto the upper half plane, with 0 going to 1 and 1 to \( \infty \).

7. Let \( \{\phi_n\} \) be an orthonormal basis for \( L^2[-1,1] \), and let \( s_n = \sum_{k=1}^{\infty} a_k \phi_k \) where \( \sum_{k} |a_k|^2 < \infty \).

   (a) Show that there is a function \( f \in L^2 \) such that \( \sum_{n=1}^{\infty} \int_{-1}^{1} |f - s_n|^2 \) converges.

   (b) Must \( s_n(x) \) converge almost everywhere to \( f(x) \)?

8. Let \( E \) be a set of real numbers whose Lebesgue measure is finite and positive. Let \( A \) be the set \( \{ \text{all} \ x - y \ , \text{where} \ x \in E \ \text{and} \ y \in E \} \).

   Prove that \( A \) contains a non-empty neighborhood of 0.

9. Define a function \( H(r) \) by \( \sum_{n=1}^{\infty} \frac{1}{4n^2 - r^2} \).

   Find the exact value of \( H(5) \).
Qualifying Exam

ANALYSIS

August 26, 1986

Instructions: Do five questions.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Evaluate:

\[ \int_{-\infty}^{\infty} \frac{\cos(x)}{\cosh(x)} \, dx \]

here \( \cosh(x) = \frac{e^x + e^{-x}}{2} \),

2. Define \( T : C[0,1] \to C[0,1] \) by \( Tf(x) = \int_0^x f(t) \, dt \). (\( C[0,1] \) is the space of continuous complex valued functions defined on \([0,1]\)).

a) What is the adjoint \( T^* \) of \( T \)?

b) If \( \mu \) is the unit point mass located at \( 1/2 \), what is \( T^*(\mu) \)?

3. Suppose that the series \( \sum_{n=0}^{\infty} a_n z^n \) converges when \( z = 1 \).

Prove the strongest theorem you can about regions in which this series converges uniformly.
4. A function $f$, holomorphic for $|z| < 1$, is said to belong to $B$ if $|f'(z)| \leq \frac{C}{1-|z|}$ for some constant $C$, all $|z| < 1$.

a) Suppose that $\sum_{k=0}^{N} k|a_k| \leq A \cdot N$ for all $N$ and some constant $A$. Show that $f(z) = \sum_{k=0}^{\infty} a_k z^k \in B$.

b) Show that if $|b_k| \leq A$ for all $k = 0, 1, 2, \cdots$ and some constant $A$, then $f(z) = \sum_{k=0}^{\infty} b_k z^{2k} \in B$.

5. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, $f(0) = 0$, $f(1) = 1$, and $f(2) = 3$. Show that $f$ cannot be one-to-one on $\mathbb{C}$.
6. Show that there is a sequence \( \{f_n\} \) of entire functions such that

(i) no \( f_n \) is 0 at any point of \( \mathbb{C} \),

(ii) \( f(z) = \lim_{n \to \infty} f_n(z) \) exists for every \( z \in \mathbb{C} \),

(iii) \( f(0) = 0 \), \( f(\frac{1}{2}) = 1 \).

Show also that no such sequence converges uniformly on the unit disc.

7. State the Radon-Nikodym theorem and use it to prove that to every bounded linear functional \( \Lambda \) on \( L^1([0,1]) \) corresponds a \( g \in L^\infty([0,1]) \) so that the representation

\[
\Lambda f = \int_0^1 f(x) g(x) \, dx
\]

holds for every \( f \in L^1([0,1]) \).
8. Suppose that \( f \) is continuous on \( \mathbb{R} \), \( f(x) > 0 \) if \( 0 < x < 1 \) and \( f(x) = 0 \) otherwise. Let \( h_c(x) = \sup \{n^c f(nx) : n=1,2,3,\ldots\} \). Show that

a) \( h_c \in L^1(\mathbb{R}) \) if \( 0 < c < 1 \).

b) \( h_1 \not\in L^1(\mathbb{R}) \) but there is a constant \( C \) so that \( m(\{x:h_1(x) > t\}) \leq \frac{C}{t} \), for all \( t > 0 \). (Here \( m \) denotes Lebesgue measure.)
Qualifying Exam

ANALYSIS

January 13, 1987

Instructions: Do five questions.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Compute
\[ \int_{-\infty}^{\infty} \frac{dx}{x^3 + 8i}. \]

2. Why is it not possible to have
\[ \int_{X} f d\mu = 1, \quad \int_{X} f^2 d\mu = 2, \quad \int_{X} f^3 d\mu = 3, \]
with \(\mu\) a positive measure and \(f\) a positive function on some set \(X\)?

For which positive \(f\) is it true that
\[ \int_{X} f d\mu = 1, \quad \int_{X} f^2 d\mu = 2, \quad \int_{X} f^3 d\mu = 4? \]
3. Suppose $1 < t_1 < t_2 < \cdots$, $t_j \to \infty$ as $j \to \infty$, and

$$f_n(z) = \prod_{j=1}^{n} \frac{t_j - z}{t_j + z} \quad (n = 1, 2, 3, \cdots).$$

Prove that

$$f(z) = \lim_{n \to \infty} f_n(z)$$

exists for all $z$ in the right half-plane $P = \{z : \text{Re } z > 0\}$, and that either

(a) $f(z) = 0$ for every $z$ in $P$, or

(b) $f(z) \neq 0$ for every $z$ in $P$ which is not one of the $t_j$'s.

For which sequences $\{t_j\}$ will (a) happen, and for which will (b) happen?
4. Let $X$ be the set of all holomorphic functions $f$ in the open unit disc $U$ that have $f(0) = 0$ and

$$\frac{1}{\pi} \int_{U} |f'|^2 \, dm < \infty$$

where $m$ is 2-dimensional Lebesgue measure.

If $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$, and both $f$ and $g$ are in $X$, calculate

$$(f, g) = \frac{1}{\pi} \int_{U} f' \overline{g} \, dm$$

in terms of the coefficients $a_n$, $b_n$.

Show that $(f, g)$ is an inner product that makes $X$ a Hilbert space.
5. Suppose \( \int_{0}^{1} |f_n(x)|^2 \, dx \leq 10 \) for \( n = 1, 2, 3, \cdots \), and
\[
\lim_{n \to \infty} f_n(x) = 0
\]
for every \( x \) in \([0,1]\).

(a) Does it follow that \( \lim_{n \to \infty} \int_{0}^{1} |f_n(x)|^2 \, dx = 0 \)?

(b) Does it follow that \( \lim_{n \to \infty} \int_{0}^{1} |f_n(x)| \, dx = 0 \)?
6. Let $K = K_0 \cup K_1$, where $K_0$ and $K_1$ are disjoint compact sets, as in the picture,

and let the path $\Gamma$ surround $K_1$ as indicated.

Assume $m(K) > 0$, where $m$ is 2-dimensional Lebesgue measure. Define

$$f(z) = \int_K \frac{dm(w)}{w - z} \quad (z \in \mathbb{C}).$$

Prove that $f$ is continuous and bounded on $\mathbb{C}$, holomorphic outside $K$, not constant, and calculate

$$\frac{1}{2\pi i} \int_\Gamma f(z) \, dz.$$
7. Compute
\[ \lim_{n \to \infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin(\frac{x}{n}) \, dx. \]
Prove your answer is correct.

8. Let \( f_n \) be integrable with respect to Lebesgue measure on \([-1,1]\) for \( n = 1, 2, \cdots \). Assume that
\[ \lim_{n \to \infty} \int_0^1 f_n(x) \, g(x) \, dx \]
eexists for every continuous \( g \) on \([-1,1]\).

(i) Prove then that there is a Borel measure \( \mu \) on \([0,1]\) such that
\[ (*) \quad \lim_{n \to \infty} \int_0^1 f_n(x) \, g(x) \, dx = \int_{[0,1]} g \, d\mu \]
for all continuous functions \( g \).

(ii) Is it always the case that \( (*) \) holds for all bounded Borel measurable functions \( g \)?
DO ANY SIX PROBLEMS

1. J.E. Littlewood, in the context of Lebesgue measure, said:
   "There are three principles, roughly expressible in the
   following terms: Every measurable set is nearly a finite
   union of intervals, every measurable function is nearly
   continuous, and every pointwise convergent sequence of
   measurable functions is nearly uniformly convergent."
   State three theorems which support these statements, and
   prove one of the last two.

2. A real valued function $\alpha(x)$ defined on $(-\infty, \infty)$ is
   said to be convex if
   \[ f(\beta x + (1-\beta)y) \leq \beta f(x) + (1-\beta)f(y) \]
   for every $x$ and $y$, and every $\beta$, $0 \leq \beta \leq 1$.
   Prove:
   (a) Any convex function is absolutely continuous on any
       compact set.
   (b) A convex function is differentiable on $(-\infty, \infty)$ at
       all points except possibly a countable set.

3. Let $f(z)$ be holomorphic for $|z| < 1$, and obey $f''(0) = 0$, and
   $|f''(z)| < 1$ there. Let $C = f(1/2)$. Obtain as sharp an
   estimate for $|C|$ as you can. $f(0) = f'(0) = 0$

4. Let $f(z)$ be holomorphic for $|z| < 1$. Let $\{r_n\}$ and $\{C_n\}$ be
   positive increasing sequences with $\lim r_n = 1$, $\lim C_n = \infty$.
   Suppose that $|f(z)| > C_n$ for all $z$ with $|z| = r_n$.
   Must $f$ have a zero in $|z| < 1$? [ Either give a counter-
   example or a proof. ]

5. Give a careful statement of the Hahn-Banach theorem, and explain
   either how it can be used in a proof of the Runge Theorem, or
   in a proof of the Weierstrass approximation theorem.
6. Let \( g(z) \) be an entire function, and \( A \) and \( B \) two complex numbers such that
\[
g(z + A) = g(z) = g(z + B)
\]
for all values of \( z \).

(a) If \( A = 1 \) and \( B = i \), prove that \( g \) is a constant.

(b) What general hypothesis on \( A \) and \( B \) leads to the same conclusion?

7. Let \( \{ g_n \} \) be a finite valued real measurable function on \((-\infty, \infty)\) and for each real number \( \lambda \), let
\[
S(n, \lambda) = \left\{ x : g_n(x) < \lambda \right\}
\]
Define a function \( f \) by
\[
f(x) = \inf_{n=1,2,\ldots} g_n(x)
\]
Choose a positive number \( \beta \) and define a set by
\[
A = \left\{ x : f(x) > \beta \right\}
\]

(a) Express the set \( A \) in terms of the sets \( S(n, \lambda) \)

(b) What is the topological nature of the set \( A \) if all the functions \( g_n \) are in fact continuous?

8. Let \( \mathcal{U} \) be a connected open set in the plane, and \( \{ f_n \} \) a sequence of functions, each of which is holomorphic and one-to-one in \( \mathcal{U} \). Suppose that \( \{ f_n \} \) converges to \( f \), uniformly on each compact subset of \( \mathcal{U} \). Prove that \( f \) is either constant on \( \mathcal{U} \), or is one-to-one on \( \mathcal{U} \), and that both cases can occur.
QUALIFYING EXAMINATION IN ANALYSIS
January 1988

GENERAL INSTRUCTIONS

We expect you to do six of the following eleven problems. Do not hand in more than six problems. Please use a separate packet of paper for each question since each question will be graded by a different person.

If you think that there is a misprint in the exam or that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
Problem .1 Find a continuously differentiable function
\[ f : [0, \infty) \longrightarrow [0, \infty) \]
such that
\[ 0 < f(x)^2 \leq f'(x) \]
for all \( x \geq 0 \) or prove that no such function exists.

Problem .2 The function
\[ f(z) = \exp\left(\frac{1}{1-z}\right) \]
has a Laurent expansion
\[ f(z) = \sum_{n=0}^{\infty} A_n z^{-n} \]
valid for \( |z| > 1 \). Find the following:
(1) \( A_0 \)
(2) \( \sum_{n=0}^{\infty} |A_n|^2 \)
(3) \( \sum_{n=0}^{\infty} |A_n| \)
(Justify all limit operations.)

Problem .3 Maximize
\[ \int_0^1 xf(x) \, dx \]
subject to the constraints
(1) \( f \) is measurable.
(2) \( \int_0^1 |f(x)|^2 \, dx = 1. \)
(3) \( \int_0^1 f(x) \, dx = 0. \)
Problem 4 Let $\mathcal{F}$ be the class of all functions
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]
satisfying
\[ \sum_{n=0}^{\infty} |a_n|^2 \leq 1. \]
For each complex number $w$ with $|w| < 1$ define
\[ \mu(w) = \sup\{|f(w)| : f \in \mathcal{F}\} \]
(1) Find a more explicit expression for $\mu(w)$.
(2) Is it the case that for every $w$ with $|w| < 1$ there is an $f \in \mathcal{F}$ for which $f(w) = \mu(w)$?

Problem 5 Suppose $f_n : [0,1] \rightarrow [0,\infty)$ is a sequence of non-negative measurable functions satisfying
\[ \int_0^1 f_n(x)^2 \, dx \leq 5 \]
for all $n$ and
\[ \lim_{n \to \infty} f_n(x) = 0 \]
for all $x \in [0,1]$. Find all positive numbers $p$ such that it follows that
\[ \lim_{n \to \infty} \int_0^1 f_n(x)^p \, dx = 0 \]
(and prove your answer).
Problem 6 Exhibit a measurable function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ such that for each $t \in [0, 1]$ both functions

$$[0, 1] \rightarrow \mathbb{R} : x \mapsto f(x, t)$$

$$[0, 1] \rightarrow \mathbb{R} : y \mapsto f(t, y)$$

are integrable, with both functions

$$[0, 1] \rightarrow \mathbb{R} : x \mapsto \int_0^1 f(x, y) \, dy$$

$$[0, 1] \rightarrow \mathbb{R} : y \mapsto \int_0^1 f(x, y) \, dx$$

integrable and

$$\int_0^1 \int_0^1 f(x, y) \, dx \, dy \neq \int_0^1 \int_0^1 f(x, y) \, dy \, dx.$$

Problem 7 Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a continuous $2\pi$-periodic complex-valued function with Fourier expansion

$$f(\theta) = \sum_{n = -\infty}^{\infty} a_n e^{in\theta}.$$  

For $t > 0$ define

$$u(t, \theta) = \sum_{n = -\infty}^{\infty} a_n e^{-n^2 t} e^{in\theta}.$$  

(1) Prove that

$$\lim_{t \to 0} \int_0^{2\pi} |u(t, \theta) - f(\theta)|^2 \, d\theta = 0.$$  

(2) Prove that if $f$ has two continuous derivatives then

$$\lim_{t \to 0} \sup_{0 \leq \delta \leq 2\pi} |u(t, \theta) - f(\theta)| = 0.$$  

4
Problem 8 Evaluate
\[ \int_{-\infty}^{\infty} \frac{\ln(9 + x^2)}{(1 + x^2)} \, dx \]

Problem 9 Is there a function \( f = f(z) \) holomorphic in the unit disk \( |z| < 1 \) with the property that
\[ \lim_{n \to \infty} \inf_{|z|=r_n} |f(z)| = \infty \]
for some sequence of positive numbers \( r_n \) with
\[ \lim_{n \to \infty} r_n = 1? \]

Problem 10 Let \( X \) and \( Y \) be Banach spaces.

1. Show by example that a vector subspace \( V \subset Y \) can have codimension one and fail to be a closed subset of \( Y \).
2. Show by example that the image \( T(X) \subset Y \) of a continuous linear transformation \( T : X \to Y \) can fail to be a closed subset of \( Y \).
3. Prove that if the image \( T(X) \subset Y \) of a continuous linear transformation \( T : X \to Y \) has codimension one, then it is a closed subset of \( Y \).
Problem 11 Let \( f(z) = u(x, y) + iv(x, y) \) (where \( z = x + iy \)) be a holomorphic function defined in a region plane \( \Omega \) with real part \( u \) and imaginary part \( v \). Suppose that the gradient of \( u \) does not vanish in \( \Omega \). Let \( \kappa(x, y) \) be the curvature of the curve \( u^{-1}(u(x, y)) \) at \( (x, y) \) and let \( a = a(x, y) \) be the length of the gradient of \( u \):

\[
a = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}
\]

Show that the function \( h \) given by

\[
h(x, y) = \frac{\kappa(x, y)}{a(x, y)}
\]

is harmonic.

Hint: Let \( s \mapsto (x(s), y(s)) \) be a parameterization of a level curve \( u(x, y) = c \) (\( c \) a constant) with respect to arclength. Differentiate \( u(x(s), y(s)) \) twice to obtain an expression for \( \kappa \). Differentiate \( f'(z)^{-1} \).
Qualifying Exam

ANALYSIS

August 30, 1988

Instructions: The exam is in two parts, real analysis and complex analysis. Each part contains six problems. Please do three from each part. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $\mu_1, \mu_2, \text{ etc.}$ be a sequence of positive measures in a measurable space.

   (i) Suppose $\mu_n(E) \leq \mu_{n+1}(E)$ for every measurable set, and let $\mu(E) = \lim_{n \to \infty} \mu_n(E)$. Prove that $\mu$ is a measure.

   (ii) Does the same result follow if $\mu_n(E) \geq \mu_{n+1}(E)$ for every measurable $E$?

2. Let $\mu$ be a positive measure in a set $X$, and $f$, $0 \leq f \leq \infty$, a measurable function there. If $0 < p < \infty$, and

   $$0 < \int_X f d\mu < \infty,$$

   find

   $$\lim_{\lambda \to \infty} \int_X \lambda \ln(1 + \left(\frac{f}{\lambda}\right)^p) d\mu.$$
3. Is there a positive function \( f(x) \) such that

\[
\int_{-\infty}^{\infty} f(x) \, dx < \infty,
\]

while

\[
\int_{-\infty}^{\infty} f(x)^p \, dx = \infty \quad \text{if} \quad p \neq 1, \quad 0 < p < \infty.
\]

4. Let \( \mu \) be a positive measure in a set \( X \) with \( \mu(X) = 1 \).

If \( f \) is a positive measurable function, which is larger

\[
\int_X f \ln f \, d\mu \quad \text{or} \quad \int_X f \, d\mu \quad \int_X \ln f \, d\mu.
\]
5. Find a nonempty closed set in $L^2[0,1]$ that contains no element of least norm.

6. Let $f(x)$ and $g(x)$ be Lebesgue measurable functions.

   Suppose

   (i) $\int_{-\infty}^{\infty} |f(x)| \, dx = 1$

   (ii) $|g(x)| \leq 1$ for almost all $x$.

A. If

   $$f_n(x) = n \int_{x}^{x+\frac{1}{n}} f(y) \, dy,$$

   prove that

   $$\int_{-\infty}^{\infty} |f_n(x)| \, dx \leq 1.$$

B. Use the Lebesgue differentiation theorem to find

   $$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) g(x) \, dx.$$
COMPLEX ANALYSIS

1. Let \( A_1, A_2, \text{etc.} \) be complex numbers. Suppose

(i) \[ \sum_{k=1}^{\infty} |A_{k+1} - A_k| < \infty \]

(ii) \[ \lim_{k \to \infty} A_k = 0 \]

(iii) the radius of convergence of \( \sum_{k=1}^{\infty} A_k z^k \) is 1.

A. Where does the series converge?

B. Find the collection of sets on which it converges uniformly.

2. Suppose that \( f(z) \) is entire, and that in every power series

\[ f(z) = \sum_{n=0}^{\infty} A_n (z-\xi)^n \]

one of the coefficients vanishes. Prove that \( f(z) \) is a polynomial.
3. Find

\[ \int_{0}^{2\pi} \frac{\sin \theta}{1 + i \sin \theta} \, d\theta . \]

4. Let \( f(z) = \sum_{\lambda=0}^{\infty} A_{\lambda} z^\lambda \) be holomorphic in the open disc \(|z| < 1\). Also, let the point \( z = 1 \) be an isolated, but not an essential singularity of \( f(z) \).

(i) What, beyond not being essential, is true of the singularity if \( |A_{\lambda}| \leq 1 \) ?

(ii) What if \( \lim_{\lambda \to \infty} A_{\lambda} = 0 \) ?
5. Let $f(z)$ be continuous in the closed disc $|z| \leq 1$, and holomorphic in the open disc $|z| < 1$. Also, let

$$|f(z)| \geq 1 \quad \text{if} \quad |z| = 1,$$

and let $|f(0)| < 1$.

A. Prove that $f$ vanishes somewhere in the open disc $|z| < 1$.

B. If $|\xi| < 1$, prove that somewhere in the open disc, $f$ takes the value $\xi$.

6. Is the analytic function $\ln(z^2 - 1)$ single-valued in the $z$-plane less the closed interval $-1 \leq x \leq 1$? In other words, is there a function $g(z)$ holomorphic there and such that $z^2 - 1 = e^{g(z)}$?
Qualifying Exam

ANALYSIS

January 18, 1989

Instructions: The exam is in two parts, real analysis and complex analysis. Each part contains six problems. Please do three from each part. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

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Qualifying Exam
ANALYSIS
Real Analysis

1. Show \( \lim_{{n \to \infty}} n \int_{{1/n}}^{1} \frac{\cos(x + \frac{1}{n}) - \cos x}{x^{3/2}} \, dx \) exists.

2. Let \( A_1, A_2, \ldots \) be positive numbers such that
\[
\sum_{\ell=1}^{\infty} A_{\ell} < \infty.
\]
Let \( r_1, r_2, \ldots \) be the rationals and let \( E \) be the set of \( x \), \(-\infty < x < \infty\), for which
\[
\sum_{\ell=1}^{\infty} \frac{A_{\ell}^2}{|x - r_\ell|} < \infty.
\]
Show that \( E' \), the complement of \( E \), has Lebesgue measure 0.

3. Let \( A_1, A_2, \ldots \) and \( B_1, B_2, \ldots \) be sequences of positive numbers such that

(i) \( A_{\ell} \leq A_{\ell+1} \) and \( \lim_{{\ell \to \infty}} A_{\ell} = \infty \)

(ii) \( B_{\ell} > B_{\ell+1} \) and \( \lim_{{\ell \to \infty}} B_{\ell} = 0 \).

a) If \( \sum_{\ell=1}^{\infty} A_{\ell}(B_{\ell} - B_{\ell+1}) < \infty \) then prove that \( \lim_{{\ell \to \infty}} A_{\ell}B_{\ell} = 0 \).

Hint: This may be proved by dominated convergence.

b) If hypothesis (i) is omitted does the conclusion of (a) still hold?
4. \( f \) is a real valued measurable function defined on a measure space \( (S, \Sigma, \mu) \). For each \( y \in \mathbb{R} \), \( E_y \) is a set which differs from \( f^{-1}((y,\infty)) \) by a set of measure 0. Use the \( E_y \) to construct a function \( g(x) \) with \( f(x) = g(x) \) a.e.

5. A Lebesgue integrable function \( f \) defined on \([0,4]\) has the property that

\[
\int_{E} f(x)dx = 0 \quad \text{for all measurable } E \text{ with } m(E) = \pi. \quad \text{Must } f = 0 \text{ a.e.}?
\]

6. Let \( f_1, f_2, \ldots \) be a sequence of functions in \( L_2[0,\pi] \) such that

\[
\int_{0}^{1} |f_n(x)|^2 dx \leq M < \infty \quad \text{for all } n.
\]

Show there is a subsequence \( f_{n_k} \) such that

\[
\lim_{k \to \infty} \int_{0}^{1} f_{n_k}(x)g(x)dx \quad \text{exists for all } g \in L_2[0,1].
\]
Complex Analysis

1. Let \( f_1, f_2, \ldots, f_n \) be holomorphic in the open unit disc \( D \).

   (i) Find the \( f_\ell \) if

   \[
   |f_1|^2 + |f_2|^2 + \cdots + |f_n|^2 = 1 \quad \text{in} \quad D.
   \]

   (ii) Find the \( f_\ell \) if

   \[
   |f_1|^2 + |f_2|^2 + \cdots + |f_n|^2 = |z|^2 \quad \text{in} \quad D.
   \]

2. Let \( f \) be holomorphic in \( |z| < 1 + \delta \). Suppose \( |f(1)| \geq |f(z)| \) for \( |z| \leq 1 \).
   Show \( f'(1) \neq 0 \) unless \( f \) is constant.

3. Let \( f \) be holomorphic in an annulus \( r < |z| < R \). Suppose \( |f(z)| = |z|^\lambda \) for all \( z \) in the annulus. Show \( \lambda \) is an integer.

4. Let \( u \) be a \( C^3 \) real valued function defined for \( x^2 + y^2 < 1 \). Suppose that \( \Delta u(0,0) > 0 \). Show there is a harmonic function \( v \) such that \( v(0,0) = u(0,0) \) and \( v(x,y) \leq u(x,y) \) in some neighborhood of \( (0,0) \).
   Hint: Look at the Taylor expansion of \( u \).
5. Find a conformal map of the complement (in the sphere) of the line segment $[-1,1]$ onto the strip $\operatorname{Im} z > 0, \quad -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}$.

6. Find $\int_{|z|=1} \frac{dz}{\sqrt{2z^2 + 2z + 1}}$. 
Qualifying Exam

ANALYSIS

August 29, 1989

Instructions: The exam is in two parts, real analysis and complex analysis. Each part contains six problems. Please do three from each part. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Show that for every neighborhood $N$ of an isolated essential singularity for $f$, $f(N)$ is dense in $\mathbb{C}$.

2. Let $f(z)$ be non-constant and holomorphic in the closed annulus $A < |z| < B$, with $|f(z)| = 1$ if $|z| = A$ or $B$. Show that $f$ vanishes at least twice (counting multiplicities) in $A < |z| < B$.

3. Evaluate $\int_{0}^{\infty} \frac{dx}{x^2 + 3x + 2}$ by considering a contour integral of

$$\frac{\log z}{z^2 + 3z + 2}.$$
5. For each of the following determine if there is a function $f(z)$,
Real Analysis

1. Let the Borel set $E$ satisfy $0 < m(E \cap I) < m(I)$ for every nonempty, bounded open interval $I \subset \mathbb{R}$. Must $E$ be of infinite measure?

2. Let $f \in L_1(\mathbb{R})$. Show that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\int_E |f(x)\, dx| < \varepsilon$ whenever $m(E) < \delta$.

3. Fatou's Lemma states that under certain conditions, the $\lim \inf$ of the integrals of a sequence of functions is at least as large as the integral of the $\lim \inf$. State the condition and prove Fatou's Lemma in this form. Give a counter-example where the condition is absent.
4. Let \( f_1, f_2, \ldots \) be measurable functions in the open interval \( 0 < x < 1 \). Suppose

(i) \( 0 \leq f_n < \infty \)

(ii) \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0 \).

Must there be a point \( \xi \), \( 0 < \xi < 1 \), such that

\[ \limsup_{n \to \infty} f_n(\xi) < \infty ? \]

5. Let \( \mu \) be a positive measure, and \( f_1, f_2, \ldots \) functions in \( L_1(\mu) \). Which is larger,

\[ (\sum_{\ell=1}^{\infty} \left| f_\ell \right| \, d\mu)^{1/2} \quad \text{or} \quad \left( \int \sum_{\ell=1}^{\infty} |f_\ell|^2 \, d\mu \right)^{1/2} ? \]

6. Prove that \( L_\infty(0,1) \) is complete.
Qualifying Exam

ANALYSIS

January 17, 1990

Instructions: The exam is in two parts, real analysis and complex analysis. Each part contains six problems. Please do three from each part. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
Real Analysis

I. Let $f$ be a locally bounded function defined on $\mathbb{R}$ such that:

$$f(x+y) = f(x) + f(y) + xy$$

for all $x$ and $y \in \mathbb{R}$, and such that $f(1) = 1$. Determine what $f$ must be.

(A function $g$ is said to be locally bounded if for every $x \in \mathbb{R}$ there exists a neighborhood of $x$ on which $f$ is bounded.)

II. Let $S$ be a bounded set in $\mathbb{R}$ and let $G$ be a collection of intervals of $\mathbb{R}$ such that for every $x \in S$ and every $\delta > 0$, there exists an interval $I \in G$ with length less than $\delta$ and $x \in I$. Show that there is a subcollection $G_0$ of $G$ which is pairwise disjoint and such that almost every point of $S$ is included in one of the intervals in $G_0$.

Is it possible to cover every (and not only almost every) point of $S$?

III. Compute the maximum of

$$\left| \int_0^\pi f(x) \sin x \, dx \right|$$

where $f$ ranges over all measurable functions on $(0, \pi)$ with:

$$\int_0^\pi |f|^2 \leq 1, \quad \int_0^\pi f(x) \, dx = 0 \quad \text{and} \quad \int_0^\pi xf(x) \, dx = 0.$$
IV. For every function \( f \) defined (possibly only almost everywhere) on \((0, 1]\), we define \( \mathcal{F} \) to be the function defined (a.e.) on \( \mathbb{R} \), with period 1, which coincides with \( f \) on \((0, 1]\).

1) Let \( f \in L^1(0, 1) \). For \( k \in \mathbb{Z} \) set \( e_k(x) = e^{2\pi ikx} \). Let \( f \ast e_k \) be defined on \((0, 1]\) by:

\[
f \ast e_k(x) = \int_0^1 f(x-y)e_k(y)\,dy.
\]

(This is the convolution of periodic functions.) Show that \( f \ast e_k = ce_k \) for some constant \( c \).

2) Let \( M \) be a closed subspace of \( L^1(0, 1) \), \( M \neq \{0\} \). For \( f \in M \) and \( y \in \mathbb{R} \), let \( f_y \) be the function defined on \((0, 1]\) by \( f_y(x) = f(x+y) \). Suppose that for all \( f \in M \) and \( y \in \mathbb{R} \), \( f_y \in M \). Show that there is an integer \( k \) such that \( e_k \in M \).

V. Let \( f(x,y) \) be a \( C^\infty \) function defined on \( \mathbb{R}^2 \), 1 periodic in \( x \) and \( y \)
(i.e. \( f(x+k,y+\ell) = f(x,y) \) for every \((k,\ell) \in \mathbb{Z}^2) \). Find a necessary and sufficient condition in order that there exist \( C^\infty \) functions, \( g \) and \( h \), 1 periodic in \( x \) and \( y \) so that

\[
f = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}.
\]

VI. Let \( f \) be defined on \( Q = [0, 1] \times [0, 1] \) to be \( f \left( \frac{p_1}{q_1}, \frac{p_2}{q_2} \right) = \frac{1}{q_1 + q_2} \) at rational lattice points \((p_1/q_1, p_2/q_2 \text{ in lowest terms})\), \( f(x,y) = 0 \) elsewhere. Show that \( f \) is Riemann integrable on \( Q \).


Complex Analysis

I. Give an example of a holomorphic function \( f \), defined in a neighborhood of \( 0 \) in \( \mathbb{C} \), which is real along the curve \( \mathcal{P} \) defined by \( y = x^2 \). [Hint: It may be easier to get \( f \) as the inverse map to some mapping.]

Is it possible that a nonconstant holomorphic function \( g \), defined in a neighborhood of \( 0 \), be real on \( \mathcal{P} \) and on the real axis?

II. Let \( \Delta \) be the open disk of radius \( 1 \), with center at the point \( 1 \), in \( \mathbb{C} \).

1. Let \( u \) be a positive harmonic function on \( \Delta \). Why is \( \iint_{\Delta} u \, dx \, dy < \infty \)?

   Give an example to show that, however, \( u \) need not be bounded.

2. Let \( \Omega = \{ z \in \mathbb{C}, z^2 \in \Delta, \Re z > 0 \} \). Sketch \( \Omega \), showing very precisely the shape of \( \Omega \) in a neighborhood of \( 0 \).

   Let \( u \) be a positive harmonic function on \( \Omega \). What can you say (similar to what has been shown in 1)?

III. (1) Let \( P \) be a polynomial of degree \( n \). Show that for all \( z \in \mathbb{C} \) such that \( |z| \geq 1 \) one has:

\[
|P(z)| \leq \left[ \sup_{|\zeta| = 1} |P(\zeta)| \right] |z|^n .
\]

(2) Under which condition is it true that, for \( |z| \geq 1 \), we have also:

\[
\left[ \inf_{|\zeta| = 1} |P(\zeta)| \right] |z|^n \leq |P(z)| .
\]
IV. (1) Let $u$ be a harmonic function defined on $\mathbb{R}^2$, and $p \in [1, \infty)$ such that
\[ \iint_{\mathbb{R}^2} |u(x,y)|^p \, dx \, dy < +\infty. \] Show that $u \equiv 0$.

(2) Let $\Omega$ be an open set in $\mathbb{C}$ and $u$ be a harmonic function defined on $\Omega$, let $p \in [1, \infty)$. If $\iint_{\Omega} |u(x,y)|^p \, dx \, dy < +\infty$, estimate $u(x,y)$ in terms of
\[ \iint_{\Omega} |u|^p \] and $d$ the distance of $(x,y)$ to the boundary of $\Omega$.

(3) In 2 take for $\Omega$ the upper half plane defined by $y > 0$. Show that if $\iint_{\Omega} |u|^p < +\infty$ then $u(x,y)$ tends to 0 as $y$ tends to $\infty$.

V. Let $\Omega$ be a connected open set in $\mathbb{C}$. Let $f$ be a holomorphic function defined on $\Omega$. Show that the following are equivalent:

(i) There exists $g$ holomorphic on $\Omega$ so that $f = e^g$.

(ii) For every integer $n$, $n \neq 0$, there exists $g_n$ holomorphic on $\Omega$ such that $f = g_n^m$.

VI Let $U$ be the open unit disk in $\mathbb{C}$. Let $f$ be a holomorphic function defined on $U$. Assume that $f$ is integrable on $U$. Show that for all $z \in U$:
\[ f(z) = \frac{1}{\pi} \iint_{U} \frac{f(w)}{(1-\overline{z}w)^2} \, dx \, dy(w). \]

[If you want, consider first the case when $f$ is holomorphic on a neighborhood of $\overline{U}$.]
REAL ANALYSIS

1. Find a necessary and sufficient condition on a measure space \((S, \Sigma, \mu)\) in order that 
   
   \[ L^1(S, \Sigma, \mu) \subseteq L^2(S, \Sigma, \mu) \]

   Prove your assertion.

2. Prove Minkowski's inequality for \(1 < p < \infty\):
   \[ \|f + g\|_p \leq \|f\|_p + \|g\|_p \]

3. Show: if every set of positive measure in \((S, \Sigma, \mu)\) can be divided into two sets of
   positive measure, then for every \(A \in \Sigma\), there is a \(B \in \Sigma\), so that
   \[ \frac{1}{3} \mu(A) \leq \mu(A \cap B) \leq \frac{2}{3} \mu(A) \]

4. In \(L^2(0,1)\) one says that a sequence \(\{f_n\}\) converges weakly to \(f\), if for every
   \(\varphi \in L^2(0,1)\) we have
   \[ \lim_{n \to \infty} \int_0^1 f_n(x) \varphi(x) \, dx = \int_0^1 f(x) \varphi(x) \, dx \]

   (a) Determine the weak limits of the following sequences
   
   (i) \(f_n(x) = \sin(nx)\)
   
   (ii) \(f_n(x) = \sin^2(nx)\)

   (b) Assume that \(\{f_n\}\) is a sequence of real valued functions in \(L^2(0,1)\) such that
   \(f_n^2 \in L^2(0,1)\) for all \(n\). Assume that \(f_n\) converges weakly to \(f\), and \(f_n^2\) converges
   weakly to \(f^2\). Show that \(\int_0^1 (f_n - f)^2 \, dx \to 0\) as \(n \to \infty\).

   Is the similar result true in \(L^2(0,\infty)\)?

5. Suppose \(f_n \in L^1(0,1)\), \(f_n \geq 0\), \(\int_0^1 f_n(x) \, dx \leq 1\) and \(\lim_{n \to \infty} f_n(x) = 0\) a.e.

   (a) Is it possible that \(\lim_{n \to \infty} \int_0^1 f_n(x) g(x) \, dx = \int_0^1 g(x) \, dx\) for all \(g \in L^\infty(0,1)\)?

   (b) Is it possible that \(\lim_{n \to \infty} \int_0^1 f_n(x) g(x) \, dx = \int_0^1 g(x) \, dx\) for both \(g = \chi_{[0,\frac{1}{2}]}\) and
   \(g = \chi_{[\frac{1}{2},1]}\)?

   (c) Is it possible that \(\lim_{n \to \infty} \int_0^1 f_n(x) g(x) \, dx = \int_0^1 g(x) \, dx\) for all \(g \in C[0,1]\)?

6. Describe the set of \((p,q,r)\in\mathbb{R}^3\) such that \(p,q,r > 1\) and whenever \(f \in L^p[0,\infty]\),
   \(g \in L^q[0,\infty]\) and \(h \in L^r[0,\infty]\) then \(fgh \in L^1[0,\infty]\).
August 28, 1990

Do three from real and three from complex.

COMPLEX ANALYSIS

1. Show that there do not exist three real valued functions $u, v, w$, harmonic on the whole plane, such that
   (a) $u(0) = v(0) = u(1) = 0, v(1) = -1$, and
   (b) $w(z) \leq u(z), w(z) \leq v(z)$ for all $z$ in the plane.

2. Let $H(z) = \int_0^\infty \sin(xz)e^{-x^2}dx$.
   (a) Show that $H$ is an entire function.
   (b) Calculate $H'(0)$.

3. Let $f$ be holomorphic in $0 < |z| < 1$ except for poles $a_n$, with $a_n \to 0$ as $n \to \infty$.
   Show that for any $\omega \in \mathbb{C}$ there is a sequence $z_n \to 0$ such that $f(z_n) \to \omega$ as $n \to \infty$.

4. Prove that $\int_{-\infty}^{\infty} e^{-(t+iz)^2}dt = \sqrt{\pi}$, for all real $x$.

5. (a) Give an example of a non-constant entire function $f$ such that $f(z)$ is real where $z$ is real and real when $Im\ z = 1$.
   (b) What is the inverse image of the parabola $y = x^2$ under the mapping $\varphi(z) = z + iz^2$?
   (c) Find a non-constant entire function that is real on the parabola $y = x^2$.

6. (a) Suppose that $K \subseteq \mathbb{C}$ is compact and has a connected complement and that $z_0 \notin K$. Show that there is a (holomorphic) polynomial $p(z)$ so that $p(z_0) = 1$ and $|p(z)| \leq \frac{1}{2}$ for all $z \in K$.
   (b) Does there exist a function $F$, continuous on $|z| \leq 1$, and holomorphic for $|z| < 1$ such that the image of the unit circle, $|z| = 1$, under $f$ is precisely this “figure eight”, i.e. the union of the boundaries of two discs that touch at one point.
Qualifying Exam

ANALYSIS

January 15, 1991

Instructions: The exam is in two parts, real analysis and complex analysis. Each part contains six problems. Please do three from each part. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
REAL ANALYSIS
Throughout, $\mathbb{R} = (-\infty, \infty)$ and $m$ is Lebesgue measure.

1. Let $E$ be Lebesgue measurable and $B \subseteq \mathbb{R}$.
   (a) Suppose $A \cap B$ is measurable for all closed $A \subseteq E$. Is $E \cap B$ measurable?
   (b) Suppose $A \cup B$ is measurable for all closed $A \supseteq E$. Is $E \cup B$ measurable?

2. Let $r_n ; n = 1, 2, 3, \cdots$ be an enumeration of the rationals in $(0,1)$ and define
   \[ f(t) = \sum_{n: i \geq r_n} \frac{1}{2^n (1 - r_n)}. \]
   (a) Determine the $t$ in $(0, 1)$ where $f$ is continuous.
   (b) Let $F(x) = \int_0^x f(t) \, dt$. Where is $F$ differentiable?
   (c) Evaluate $F(1)$.

3. Suppose that $f$ is a bounded measurable function on $\mathbb{R}$ and $f(x + 1) = f(x)$ for all $x$.
   (a) Show that for all intervals $[a, b]$ :
   \[ \lim_{n \to \infty} \int_a^b f(nx) \, dx = (b - a) \int_0^1 f(x) \, dx. \]
   (b) Suppose there is an interval $[\alpha, \beta]$ with $\alpha < \beta$ and an increasing sequence of integers $n_k$ such that $\lim_{k \to \infty} f(n_k x) = g(x)$ for almost all $x \in [\alpha, \beta]$. Show that $g$ is constant a.e. on $[\alpha, \beta]$.
   (c) Under the hypotheses of (b) show that $f$ is constant a.e. on $\mathbb{R}$.
4. Let $\phi$ be a continuous function with compact support on $\mathbb{R}$ such that 
\[\int_{-\infty}^{\infty} \phi(t) dt = 1.\] Let $f$ be locally integrable on $\mathbb{R}$, and for $y > 0$ put
\[f_y(x) = \int_{-\infty}^{\infty} f(x - y t) \phi(t) dt.\]
(a) Prove that if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then
\[\lim_{y \to 0} ||f_y - f||_p = 0.
(b) Is the same statement true if $p = \infty$?

5. Let $f$ be a measurable function on the interval $[0, 1]$. For $t > 0$ put
\[\lambda_f(t) = m\{x \in [0, 1] : |f(x)| > t\}.
(a) Show that for $0 < p < \infty$,
\[\int_0^1 |f(x)|^p dx = p \int_0^\infty t^{p-1} \lambda_f(t) dt.
(b) For a given function $f$, show that if there exists a constant $C$ so that for all $t$,
\[\lambda_f(t) \leq \frac{C}{t},\] then
\[f \in \bigcap_{p < 1} L^p([0, 1]).

6. Let $f$ and $g$ be real valued Lebesgue measurable functions on $\mathbb{R}$ and define
\[D(a, b) = m\{x : f(x) < a, g(x) < b\} \text{ for } (a, b) \in \mathbb{R}^2.\] Show how $D$ determines
\[m\{x : (f(x), g(x)) \in B\} \text{ for every Borel set } B \subseteq \mathbb{R}^2.\]
COMPLEX ANALYSIS

1. Evaluate \[ \int_0^\infty \frac{[\log x]^2}{1 + x^2} \, dx. \]

2. In each of the following give an example of a function \( f(z) \), holomorphic in the open unit disc \( D = \{z \in \mathbb{C} : |z| < 1\} \), which satisfies the given conditions or else prove that no such function exists:

   (a) \( f(r e^{\pi i/n}) \) is real for \( 0 \leq r < 1 \) and \( n = 1, 2, 3, \ldots \) and \( f \) is not constant.

   (b) \( f \) is bounded on \( D \), \( f \) is not constant and
   \[ f \left( \frac{n^2 - 1}{n^2} \exp\left(\frac{n^2 - 1}{n^2} \pi i\right) \right) = 0 \text{ for } n = 1, 2, 3, \ldots \]

   (c) \( f(0) = 1 \), \( f'(0) = 3 \) and \( \text{Re} f(z) \geq 0 \) for \( z \in D \).

3. Let \( U = \{z = x + iy \in \mathbb{C} : y > 0\} \) be the upper half plane. Let \( f \) be holomorphic on \( U \) and suppose

   \[ \iint_U |f(x+iy)|^2 \, dx \, dy < \infty. \]

   (a) Prove that for every \( x \in \mathbb{R} \), \( \lim_{y \to 0^+} y \, |f(x+iy)| = 0. \)

   (b) Prove that for every \( y > 0 \), \( \int_{-\infty}^\infty |f(x+iy)|^2 \, dx < \infty. \)

   (c) Prove that \( \lim_{y \to 0^+} y \int_{-\infty}^\infty |f(x+iy)|^2 \, dx = 0. \)
4. Let $f$ and $g$ be entire functions. Suppose that for all $z \in \mathbb{C}$,
   \[ f(z)g(z + 1) = f(z + 1)g(z) \quad \text{and} \quad f(z)g(z + i) = f(z + i)g(z). \]
   Let $Q = \{ z = x + iy \in \mathbb{C} : 0 \leq x \leq 1, \ 0 \leq y \leq 1 \}$.

   (a) Show that if $g$ has no zeros on $Q$ then, for all $z, w \in \mathbb{C}$,
   \[ f(z)g(z + w) = f(z + w)g(z). \]

   (b) If $f$ and $g$ have no zeros on the boundary of $Q$, show that $f$ and $g$ have the same
   number of zeros on the interior of $Q$, counted with multiplicity.

5. Let $W \subseteq \mathbb{C}$ be a non empty, bounded open set. For $z \in \mathbb{C}$ let
   \[ F(z) = \frac{1}{\pi} \iint_{W} \frac{dA(\zeta)}{z - \zeta}. \]

   (a) Show that $F$ is continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} - \overline{W}$.

   (b) Show that $F$ is not identically zero on the unbounded component of $\mathbb{C} - \overline{W}$.

   (c) Show that $G(z) = F(z) - \overline{z}$ is holomorphic on $W$.

6. Let $U = \{ z = x + iy \in \mathbb{C} : x > 0 \}$ be the right half plane. Let $f$ be holomorphic on $U$
   and suppose
   \[ \lim_{z \to 0, \ z \in U} f(z) = L. \]

   Show that \[ \lim_{z \to 0, \ z \in U} z f'(z) = 0 \] uniformly on every angle $|\theta| \leq \alpha < \frac{\pi}{2}$. 
QUALIFYING EXAM – ANALYSIS

August 27, 1991

Instructions: Do 8 of the 10 problems. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1) Show that if \( f \in L^1(R) \)
\[
\int_{-\infty}^{\infty} |f(x + h) - f(x)| \, dx \to 0 \quad \text{as} \quad h \to 0.
\]
(You must do more than just quote a theorem.)

2) Evaluate
\[
\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx.
\]

3) Evaluate
\[
\oint_{|z|=1} e^{1/z^2} \, dz.
\]

4) Let \( E \) denote the space of continuous functions \( f(x) \) on \( R^2 \) such that
\[
\lim_{|x| \to \infty} f(x) = 0.
\]
On \( E \) define a continuous linear functional \( L \) by the formula
\[
Lf = \int_{0}^{2\pi} f(\cos \theta, \sin \theta) \, d\theta.
\]
According to the Riesz representation theorem \( L \) corresponds to a measure \( \mu \).
What is
\[
\mu\{(x_1, x_2) \mid x_2 > 0\}?
\]

5) How many zeros does \( z^5 + 2z^2 + 1 \) have in the disc \(|z| \leq 2|?\)
6) Say whether each statement is true or false. If the statement is true, give a proof. (This proof must contain more than just the statement of a theorem.) If the statement is false, give a counterexample.

(i) If \( \{f_n\} \) is a sequence of non-negative functions in \( L^1(R) \) and \( f_n(x) \) converges to \( f(x) \) uniformly on the real line,

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx.
\]

(ii) If \( \{f_n\} \) is a sequence of non-negative functions in \( L^1(R) \), and

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)dx = 0,
\]

\( f_n(x) \to 0 \) for almost every \( x \).

(iii) Let \( \{f_n\} \) be a sequence of \( L^1 \) functions. Suppose \( f_n(x) \to f(x) \) almost everywhere as \( n \to \infty \), and

\[
\int_0^1 |f_n(x)|dx \to \int_0^1 |f(x)|dx
\]

as \( n \to \infty \). Then

\[
\int_0^1 |f(x) - f_n(x)|dx \to 0
\]

as \( n \to \infty \).

7) Suppose \( \phi \) is a \( C^\infty \) function with compact support on \( R^1 \) and that \( f \) is continuous on \( R^1 \). Assume also that

\[
\int_{-\infty}^{\infty} \phi(x)dx = 1.
\]

Show

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(x - y) \phi \left( \frac{y}{\epsilon} \right) dy = f(x)
\]

for every \( x \).
8) Let \( C \) be the complex plane. Find a conformal mapping of the open first quadrant of \( C \) onto the open unit disc.

9) Let \( f(x) \) be Lebesgue measurable on \( R^1 \). Define

\[
g(x, y) = f(x - y)
\]

on \( R^2 \). Show that \( g \) is Lebesgue measurable on \( R^2 \).

10) (i) Show that if \( u \) is a bounded harmonic function in \( R^2 \) then \( u \) is constant. Suppose \( u \) is a \( C^2 \) function on \( R^2 \). A consequence of Green's Theorem is the formula

\[
(*) \quad u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})d\theta - \frac{1}{2\pi} \int \int_{|z| \leq 1} \Delta u(z) \log \frac{1}{|z|} dx dy.
\]

(Here \( z = x + iy \), with the standard identification of \( R^2 \) and \( C \).) You may use (*) in the remaining parts of this problem.

(ii) Let \( v \) be a \( C^2 \) function defined on \( R^2 \). Assume \( v \) is subharmonic (i.e. \( \Delta v \geq 0 \)) and that \( |v(z)| \leq 1 \) for \( |z| \leq 1 \). For \( \epsilon \) in \((0, 1)\) show that

\[
\int \int_{|z| \leq \epsilon} \Delta v(z) dx dy \leq \frac{(2\pi + 1)}{\log \frac{1}{\epsilon}}.
\]

(iii) For \( \lambda > 0 \), let \( u(z) \) be a \( C^2 \) function on \( R^2 \). For \( \lambda > 0 \) set \( v_\lambda(z) = u \left( \frac{z}{\lambda} \right) \). Find the relationship between

\[
\int \int_{|z| \leq \lambda R} \Delta v_\lambda(z) dx dy
\]

and

\[
\int \int_{|z| \leq R} \Delta u(z) dx dy.
\]

(iv) Let \( u \) be a \( C^2 \) subharmonic function on \( R^2 \). Show that if \( u \) is bounded on \( R^2 \), then \( u \) is constant.
Qualifying Exam

ANALYSIS

January 14, 1992

Instructions: Do 8 of the 10 questions. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Suppose that $f \in C^1(0, \infty)$ and
   
   a) $\int_0^\infty t|f'(t)|^2 dt < \infty$
   
   b) $\lim_{x \to -\infty} \frac{1}{x} \int_0^x f(t) dt = L$
   
   Show that $\lim_{x \to -\infty} f(x) = L$.
   
   Hint: First show that for $0 < s < t < \infty$
   
   $$\int_s^t f(x) dx = tf(t) - sf(s) - \int_s^t rf'(r) dr.$$

2. Suppose that $f \in L^p(0, \infty)$, $1 < p < \infty$.
   
   a) Show that $|\int_0^x f(t) dt| \leq \|f\|_p x^{1 - \frac{1}{p}}$.

   b) Show that $\lim_{x \to -\infty} \frac{1}{x^{1 - \frac{1}{p}}} \int_0^x f(t) dt = 0$.

3. Suppose $f \in L^p(0, \infty)$, $1 < p < \infty$, define $(Tf)(x) = \int_0^x f(tx) dt$.
   
   Show that there is a constant $C_p$ so that $\|Tf\|_p \leq C_p \|f\|_p$.

4. Suppose that $\varphi$ is a continuous increasing function of $R$, show that if $a < b$ then $\int_a^b \varphi'(x) dx \leq \varphi(b) - \varphi(a)$.

5. Suppose $1 < p < \infty$, $f_n \in L^p(-\infty, \infty)$ and $\|f_n\|_p \leq C < \infty$ and $f_n \to f$ a.e.
   
   Show that $\int_{-\infty}^{\infty} f_n g dx \to \int_{-\infty}^{\infty} fg dx$ for all $g \in L^q(-\infty, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$.

6. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^{1+1}}$.

7. For $0 < \alpha < 1$, evaluate
   
   $$\int_0^\infty \frac{x^\alpha}{1 + x^2} dx.$$

8. Suppose $F$ is holomorphic in $|z| < 1$ and $|F(z)| \leq 1$ for all $|z| < 1$. Show that $|F'(z)| \leq \frac{1}{1-|z|}$.

9. Show that there exist a sequence of holomorphic polynomials $p_n(z)$ in the complex variable $z$ such that $p_n(z) \to 1$ uniformly for all $z$ in the unit interval $[0,1]$ on the real axis and $p_n(z) \to 0$ for all other $z \in \mathbb{C}$.

10. Let $\gamma > 0$ and for $|z| < 1$ set $F(z) = \sum_{n=1}^{\infty} \frac{z^{2n}}{n^n}$. Show that $F$ has an analytic continuation into the complex plane minus the real axis cut from 1 to $+\infty$.

   (Hint: Show that $\frac{1}{n^n} = C_\gamma \int_0^\infty e^{-nt^\gamma} t^{-1} dt$, for some constant $C_\gamma \neq 0$.)

   1
Qualifying Exam

ANALYSIS

August 25, 1992

Instructions: Do 8 of the 10 questions. To facilitate grading, please use a separate packet of paper for each question.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $\mu$ be a positive Borel measure with compact support in $\mathbb{R}^n$ and $0 < \alpha < n$. Show that $\int \frac{d\mu(y)}{|x-y|^\alpha} < \infty$ for almost all $x$ in $\mathbb{R}^n$, with respect to Lebesgue measure.

2. Let $f \in L^1(\mathbb{R})$ and assume that $\lim_{h \to 0} \frac{1}{h} \int_{-\infty}^{\infty} |f(x+h) - f(x)|dx = 0$. Show that $f \equiv 0$ a.e.

3. Suppose $f_n$ is a sequence of measurable functions defined on $\mathbb{R}$ such that $f_n(x) \to 0$ as $n \to \infty$ for almost all $x$. Suppose further that $\exists g \in L^1(\mathbb{R})$ such that $|f_n(x)| \leq g(x)$ for all $n$ and $x$. Show that for any $\epsilon > 0 \exists$ a measurable set $E_\epsilon$ such that $|E_\epsilon| < \epsilon$ and $f_n \to 0$ uniformly on $\mathbb{R} \setminus E_\epsilon$.

4. For $f \in L^1(0, \infty)$ define
   \[ Tf(x) = \frac{1}{x} \int_0^x f(t)dt. \]
   Show that $\exists C > 0 \exists$
   \[ \int_0^\infty |Tf(x)|^2dx \leq C \int_0^\infty |f(x)|^2dx \]
   for all $f \in L^2(0, \infty)$.

5. Suppose $E \subseteq \mathbb{R}$ is measurable, $|E| > 0$
   a) Show that $\lim_{h \to 0} \frac{|E \cap (x, x+h)|}{h} = \chi_E(x)$ for almost all $x$.
   b) Show that $\exists E_0 \subseteq E, |E_0| > 0$ and $N_0 > 0$ such that
   \[ n|E \cap (x, x+\frac{1}{n})| \geq 1/2 \]
   for all $x \in E_0$ and all $n \geq N_0$.

6. Show that all of the zeros of
   \[ p(z) = 3z^3 - 2z^2 + 2iz - 8 \]
   lie in the annulus
   \[ 1 < |z| < 2. \]
7. Suppose that $f(z)$ is entire and not constant. Show that the range of $f$ is dense in the plane.

8. Suppose $f$ is holomorphic in the half plane $\text{Im} z > 0$ and that $\lim_{z \to 0} f(z) = L$ exists. Show that for each $\epsilon > 0 \quad \lim_{z \to 0} z f'(z) = 0$. 
\[ \epsilon < \arg z < \pi - \epsilon \]

9. Suppose $f(z) = u(z) + iv(z)$ is an entire function, $v(0) = 0$.
   a) Show that if $|z| < R$ then
   \[ f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta}) d\theta \]
   b) Suppose $\exists A, B, \alpha > 0 \ \exists$
   \[ |u(z)| \leq A + B|z|^{\alpha} \]
   Show $\exists C, D > 0 \ \exists$
   \[ |f(z)| \leq C + D|z|^{\alpha} \]

10. a) Show that $\sqrt{z^4 - 1}$ can be defined to be a single valued holomorphic function for $|z| > 2$.
    b) Calculate
    \[ \oint_{|z|=3} \frac{dz}{\sqrt{z^4 - 1}} \]
Qualifying Exam

ANALYSIS

January 12, 1993

Instructions: Do 6 of the 11 questions. To facilitate grading, please use a separate packet of paper for each question.

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case whether or not you feel there is a misstatement do not interpret any problem in such a way that it becomes trivial. If you have any doubts about the interpretation of a problem ask the proctor.
1. Let \( f_n \) be a sequence of functions in \( L^1(\mathbb{R}) \) that converges almost everywhere on \( \mathbb{R} \) to a function \( f \in L^1(\mathbb{R}) \) as \( n \to \infty \). Assume also that \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)|dx = \int_{\mathbb{R}} |f(x)|dx \). Prove that
\[
\lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x) - f(x)|dx = 0.
\]

2. Let \( f_n \geq 0 \) and \( f \geq 0 \) be functions in \( L^1(\mathbb{R}) \) such that \( f_n \to f \) almost everywhere, and \( \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx \). Prove that for every measurable set \( E \subset \mathbb{R} \) we have
\[
\lim_{n \to \infty} \int_{E} f_n(x)dx = \int_{E} f(x)dx.
\]

3. Set \( p(x) = x^2/(1 + x^2) \).
   (a) Show that for all \( \delta > 0 \) sufficiently small, the function \( x \to f(x + \delta p(x)) \) belongs to \( L^1(\mathbb{R}) \) for every \( f \in L^1(\mathbb{R}) \).
   (b) Show that for all \( f \in L^1(\mathbb{R}) \)
\[
\lim_{\delta \to 0} \int_{\mathbb{R}} |f(x + \delta p(x)) - f(x)|dx = 0.
\]

4. Let \( K \) be a continuous function on \( \mathbb{R} \times \mathbb{R} \) and let \( C < \infty \) be a constant such that \( \int_{\mathbb{R}} |K(x,y)|dx \leq C \) for all \( y \in \mathbb{R} \), and \( \int_{\mathbb{R}} |K(x,y)|dy \leq C \) for all \( x \in \mathbb{R} \). For \( f \in L^\infty(\mathbb{R}) \) we define
\[
(Tf)(x) = \int_{\mathbb{R}} K(x,y)f(y)dy.
\]
Prove that there is a finite constant \( M \), depending only on \( C \), such that for all \( f \in L^\infty(\mathbb{R}) \) we have
\[
||Tf||_2 \leq M||f||_2.
\]

5. Let
\[
f_n(x) = \frac{1 + x^{2n}}{(1 + x^2)^n}, \quad x \in \mathbb{R}, \quad n = 1, 2, 3, \ldots
\]
   (a) Find \( \lim_{n \to \infty} f_n(x) = f(x) \) for \( x \in \mathbb{R} \).
   (b) On what intervals is the convergence \( f_n \to f \) uniform?
6. Let \( D \subset \mathbb{R}^3 \) be a connected bounded domain with smooth boundary, and let \( f, g \in C^2(\overline{D}) \).

(a) Apply the divergence theorem to the vector field \( \mathbf{F} = f \nabla g \) to prove the identity

\[
\iiint_D \nabla f \cdot \nabla g \, dV + \iiint_D f \Delta g \, dV = \iint_{\partial D} f \frac{\partial g}{\partial n} \, dA.
\]

(b) Prove that, if \( g \in C^2(\overline{D}) \) is harmonic on \( D \) and the normal derivative of \( g \) on \( \partial D \) vanishes identically, then \( g \) is a constant.

7. Show that the integral

\[
F(a, b) = \int_0^\infty \frac{e^{-az} - e^{-bx}}{x} \, dx
\]

converges for every \( 0 < a < b < \infty \), and find its value.

8. Let \( D \subset \mathbb{C} \) be a domain such that \( \overline{D} \neq \mathbb{C} \). Suppose that \( f \) is a bounded holomorphic function on \( D \), and there is a constant \( C > 0 \) such that

\[
\limsup_{z \to \zeta} |f(z)| \leq C \quad \text{for all } \zeta \in bD.
\]

Prove that \( |f(z)| \leq C \) for all \( z \in D \).

(Hint: consider the function \( f(z)^n / (z - a) \) for some \( a \notin \overline{D} \).)

9. Evaluate the following integral:

\[
\int_0^\infty \frac{dx}{x^3 + 1}.
\]

(Hint: Apply the method of residues to the function \( F(z) = \log z / (z^3 + 1) \).

10. Construct in an explicit way (by an infinite series) a meromorphic function \( f(z) \) on \( \mathbb{C} \) that has a simple pole at every point \( z_n = in \ (n \in \mathbb{Z}) \), with \( \text{Res}_{in} f = n \), and has no other singularities. Justify the convergence!

11. Let \( F \) be an entire analytic function on \( \mathbb{C} \) that is periodic, with a complex period \( \omega \neq 0 \) (i.e., \( F(z + \omega) = F(z) \) for all \( z \)). If \( g \) is an analytic function with isolated singularities on a domain \( D \subset \mathbb{C} \), prove that all singularities of \( F \circ g \) on \( D \) are essential.
Qualifying Exam

ANALYSIS

August 25, 1993

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case whether or not you feel there is a misstatement do not interpret any problem in such a way that it becomes trivial. If you have any doubts about the interpretation of a problem ask the proctor.
1. For $n = 1, 2, \ldots$, let $f_n(x)$ be functions on $R$ defined by

$$f_n(x) = \frac{x^n}{1 + x^{2n}}.$$ 

i) For what values of $x$ does $\sum_{n=1}^{\infty} f_n(x)$ converge?

ii) In what intervals of $x$ does $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly?

2. Let $f(x)$ be a Lebesgue measurable function on $R$. For $(x, y) \in R^2$, define

$$g(x, y) = f(x - y).$$

Show $g(x, y)$ is Lebesgue measurable on $R^2$.

3. Let $f$ be a positive continuous function on the interval $[a, b] \subset R$. Show that the sequence

$$x_n = \left( \int_a^b f(x)^n \, dx \right)^{1/n}$$

converges to $\sup\{f(x) : a \leq x \leq b\}$.

4. Let $f$ be a holomorphic function on $\{z \in C : 0 < |z| < 1\}$ such that $\Re f(z) \geq 0$. What kind of singularity can $f$ have at the origin? Justify your answer!

5. Using residues calculate the integral

$$\int_0^\infty \left( \frac{\sin x}{x} \right)^2 \, dx.$$ 

6. Let $f : R \to R$ be a function of class $C^2$ such that for every harmonic function $g : C \to R$, the function $f \circ g$ is also harmonic. What can you say about $f$?
7. i) Show that if \( a_n \geq 0 \),
\[
\prod_{n=1}^{N} (1 + a_n)
\]
converges to a non-zero limit if and only if \( \sum_{n=1}^{\infty} a_n \) converges.

ii) Is the statement in part i) still valid if the hypothesis \( a_n \geq 0 \) is dropped? Give either a proof or counterexample.

8. Let \( f(x) \in L^p(\mathbb{R}) \) and assume
\[
\|f(x + y) - f(x)\|_{L^p(dx)} \leq C|y|.
\]
Show there are constants \( C_1 \) and \( C_2 \) and functions \( f_\epsilon(x) \) such that \( f_\epsilon(x) \in L^p, f_\epsilon(x) \in C' \) and \( f'_\epsilon(x) \in L^p \) such that
\[
\|f_\epsilon(x) - f(x)\|_{L^p} \leq C_1 \epsilon
\]
and
\[
\|f'_\epsilon(x)\|_{L^p} \leq \frac{C_2}{\epsilon}.
\]

Hint: Let
\[
f_\epsilon(x) = \frac{1}{\epsilon} \int f(x - y) \phi(y) \, dy
\]
for an appropriate \( \phi \).

9. Let \( D = \mathbb{R}^3 - L \) where \( L \) is the line \((0,0,z), z \in \mathbb{R}\). Let \( \vec{F} \) be the vector field
\[
\vec{F} = -\frac{y}{x^2 + y^2} \vec{i} + \frac{x}{x^2 + y^2} \vec{j} + xz \vec{k}.
\]
Determine all possible values of the path integral
\[
\int_{(0,2,0)}^{(2,0,0)} \vec{F} \cdot d\vec{R}
\]
along paths that lie in \( D \).
Qualifying Exam

ANALYSIS

January 19, 1994

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case whether or not you feel there is a misstatement do not interpret any problem in such a way that it becomes trivial. If you have any doubts about the interpretation of a problem ask the proctor.
1. Let \( f_n(x) = \sin(nx) \) for \( n \in \mathbb{Z}_+, x \in \mathbb{R} \). Show that there is no sequence \( \{n_k\} \subset \mathbb{Z}_+ \) such that the sequence \( f_{n_k}(x) \) converges for all \( x \) in a set of positive Lebesgue measure as \( k \to \infty \).

2. For \( n \in \mathbb{Z}_+ \) let

\[
f_n(x) = \int_1^n te^{it^3x} dt.
\]

Show that for every \( \epsilon > 0 \) the sequence of functions \( f_n(x) \) converges uniformly in \( x \geq \epsilon \) as \( n \to \infty \).

3. Let \( f \) be a holomorphic function in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) such that \( f(0) \neq 0 \). Prove that there is an \( \epsilon > 0 \) such that for every \( c \in \mathbb{C} \) satisfying \( 0 < |c| < \epsilon \) the equation \( z^m = cf(z) \) has precisely \( m \) distinct solutions in \( U \).

4. Let \( F = (f, g): U \to \mathbb{R}^2 \) be a smooth mapping, defined in a neighborhood \( U \subset \mathbb{R}^2 \) of the origin \( (0, 0) \), such that the image \( F(U) \) has Lebesgue measure zero and \( \partial f / \partial x(0,0) \neq 0 \). Prove that there is a smaller neighborhood \( V \subset U \) of \( (0,0) \) such that \( F(V) \) is contained in a smooth curve \( \gamma \subset \mathbb{R}^2 \).

5. Let

\[
F(x) = \int_0^\infty \frac{1-e^{-xt^2}}{t^2} dt, \quad x > 0.
\]

(a) Calculate \( F'(x) \).

(b) Use (a) to evaluate \( F(x) \). (Justify all calculations in both parts.)

6. Evaluate the integral

\[
\int_{-\infty}^{+\infty} \frac{e^{px}}{1+e^x} \, dx, \quad 0 < p < 1.
\]

(Hint: Integrate around the rectangle with vertices \( -R, R, R+2\pi i, -R+2\pi i \).)
7. Suppose $f \in L^1(0, 1)$ and $f(x) > 0$ for all $0 < x < 1$. Prove that for every $\alpha$, $0 < \alpha < 1$,

$$\inf_{|E|=\alpha} \int_E f(x) dx > 0.$$ 

8. For each invertible $n \times n$ matrix $A$ we set $\mathcal{F}(A) = A^{-1}$. Find the formula for the directional derivative $D\mathcal{F}(A) \cdot B$ of $\mathcal{F}$ at the matrix $A$ in the direction of the matrix $B$. (Hint: Consider first the case when $A$ is the identity matrix.)

9. For each $s \in \mathbb{C}$ with $\Re s > 0$ we define

$$H(s) = s \int_0^\infty t^s e^{-t} \frac{dt}{t}.$$ 

(i) Show that $H(s)$ is holomorphic in $\Re s > 0$.

(ii) Show that $H(s)$ continues to a holomorphic function in $\Re s > -1$.

(iii) Show that $H(s)$ does not continue to a holomorphic function in $\Re s > -2$.

Here, $\Re s$ denotes the real value of $s$. 
1. Let \( f_n(x) = \sin(nx) \) for \( n \in \mathbb{Z}_+ \), \( x \in \mathbb{R} \). Show that there is no sequence \( \{n_k\} \subset \mathbb{Z}_+ \) such that the sequence \( f_{n_k}(x) \) converges for all \( x \) in a set of positive Lebesgue measure as \( k \to \infty \).

2. For \( n \in \mathbb{Z}_+ \) let

\[
f_n(x) = \int_1^n e^{it^2x} \, dt.
\]

Show that for every \( \epsilon > 0 \) the sequence of functions \( f_n(x) \) converges uniformly in \( x \geq \epsilon \) as \( n \to \infty \).

3. Let \( f \) be a holomorphic function in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) such that \( f(0) \neq 0 \). Prove that there is an \( \epsilon > 0 \) such that for every \( c \in \mathbb{C} \) satisfying \( 0 < |c| < \epsilon \) the equation \( z^m = cf(z) \) has precisely \( m \) distinct solutions in \( U \).

4. Let \( F = (f, g) : U \to \mathbb{R}^2 \) be a smooth mapping, defined in a neighborhood \( U \subset \mathbb{R}^2 \) of the origin \( (0,0) \), such that the image \( F(U) \) has Lebesgue measure zero and \( \partial f/\partial x(0,0) \neq 0 \). Prove that there is a smaller neighborhood \( V \subset U \) of \( (0,0) \) such that \( F(V) \) is contained in a smooth curve \( \gamma \subset \mathbb{R}^2 \).

5. Let

\[
F(x) = \int_0^\infty \frac{1 - e^{-xt^2}}{t^2} \, dt, \quad x > 0.
\]

(a) Calculate \( F'(x) \).
(b) Use (a) to evaluate \( F(x) \). (Justify all calculations in both parts.)

6. Evaluate the integral

\[
\int_{-\infty}^{+\infty} \frac{e^{px}}{1 + e^x} \, dx, \quad 0 < p < 1.
\]

(Hint: Integrate around the rectangle with vertices \(-R, R, R + 2\pi i, -R + 2\pi i\).)
7. Suppose $f \in L^1(0,1)$ and $f(x) > 0$ for all $0 < x < 1$. Prove that for every $\alpha$, $0 < \alpha < 1$,

$$\inf_{|E|=\alpha} \int_E f(x)dx > 0.$$ 

8. For each invertible $n \times n$ matrix $A$ we set $\mathcal{F}(A) = A^{-1}$. Find the formula for the directional derivative $D\mathcal{F}(A) \cdot B$ of $\mathcal{F}$ at the matrix $A$ in the direction of the matrix $B$. (Hint: Consider first the case when $A$ is the identity matrix.)

9. For each $s \in \mathbb{C}$ with $\Re s > 0$ we define

$$H(s) = s \int_0^{\infty} t^s e^{-t} \frac{dt}{t}.$$ 

(i) Show that $H(s)$ is holomorphic in $\Re s > 0$.
(ii) Show that $H(s)$ continues to a holomorphic function in $\Re s > -1$.
(iii) Show that $H(s)$ does not continue to a holomorphic function in $\Re s > -2$.

Here, $\Re s$ denotes the real value of $s$. 

$$\Gamma(s) = \int_0^{\infty} t^s e^{-t} \frac{dt}{t}$$
INSTRUCTIONS: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

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1. Suppose that $a > 1$ is a real number, show that $f(z) = z - e^z + a$ has exactly one zero in the left half plane, $\text{Re } z < 0$.

2. Suppose that $f$ is a twice differentiable real valued function defined on $(0, \infty)$ and that
   \[ M_j = \sup_{0 < x < \infty} |f^{(j)}(x)|, \quad j = 0, 1, 2, \quad (f^{(0)} = f). \]
   Show that $M_1^2 \leq 4M_0M_2$.
   \textbf{Hint:} First show that if $x, h > 0$ that there is a $t$ so that
   \[ f'(x) = \frac{1}{2h} [f(x + 2h) - f(x)] - hf''(t). \]

3. Suppose that $f \in L^p(0, \infty), 1 < p < \infty$. Show that
   \[ \lim_{x \to \infty} x^{1/p} \int_x^\infty \frac{f(t)}{t} dt = 0. \]

4. Suppose that $\Delta = \{ z : |z| < 1 \}$, $f_n$ is holomorphic in $\Delta$ and that $f_n \to f$ uniformly on compact subsets of $\Delta$. Suppose that $f$ is one-to-one on $\Delta$. Show that for each $r < 1 \exists N \ni f_n$ is one-to-one on $\{ z : |z| \leq r \}$, for all $n \geq N$.

5. For $s, t \geq 0$ suppose $K(x, t) \geq 0$, and moreover that
   (a) $K(\lambda s, \lambda t) = \frac{1}{\lambda} K(x, t), \lambda > 0$,
   (b) $\int_0^\infty t^{-1/p} K(1, t) dt = \gamma < \infty$ for some $1 < p < \infty$.
   Define $Tf(s) = \int_0^\infty f(t) K(s, t) dt$. Show that
   \[ \|Tf\|_{L^p} \leq \gamma \|f\|_{L^p}. \]
6. Let \( E \subseteq \mathbb{R} \) be a measurable set of finite positive measure.

(a) Show that
\[
\lim_{n \to \infty} \frac{n}{2} \left| E \cap \left( x - \frac{1}{n}, x + \frac{1}{n} \right) \right| = 1 \quad \text{a.e. on } E
\]

(b) Show that \( \exists \) a subset \( E_0 \subseteq E \) and an \( N > 0 \) \( \ni |E_0| > \frac{|E|}{2} \) and
\[
\frac{n}{2} \left| E \cap \left( x - \frac{1}{n}, x + \frac{1}{n} \right) \right| \geq \frac{1}{2}
\]
for all \( n \geq N \) and all \( x \in E_0 \).

7. Evaluate:
\[
\sum_{n=1}^{\infty} \frac{1}{1 + n^2}
\]

8. Let \( \Delta = \{ z : |z| < 1 \} \). Suppose \( f \) is holomorphic in \( \Delta \) and that
\[
\lim_{\frac{z}{z \in \Delta}} f(z) = L.
\]
Show that
\[
\lim_{r \to 1} (1 - x)f'(x) = 0.
\]

9. Let \( r = \sqrt{x^2 + y^2 + z^2}, (x, y, z) \in \mathbb{R}^3 \). Define \( F(x, y, z) = \frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) \).
Find all possible values for
\[
\int_S \int \bar{F} \cdot \bar{N} dA
\]
where \( S \subseteq \mathbb{R}^3 \) is a smooth closed surface, and \( \bar{N} \) is the outward unit normal, and \((0,0,0) \not\in S\).
Qualifying Exam

ANALYSIS

January 18, 1995

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

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1. For any complex $n \times n$ matrix $A$ define

$$f_L(A) = \sum_{k=0}^{L} (-1)^k \frac{A^k}{(2k)!}.$$  

(i) Let $\| \cdot \|$ be a norm on the space $M_n$ of complex $n \times n$ matrices $A$. Show that $f_L(A)$ converges to a limit $f(A)$, as $L \to \infty$, with respect to the given norm.

(ii) Show that there is $\delta > 0$ such that the equation $f(A) = I + B$ has a solution $A$, provided that $\|B\| < \delta$.

2. Determine all complex-valued functions $f$, which are continuous in $[0, 2]$ and satisfy the condition $\int_0^2 f(x)x^n dx = 0$ for $n = 0, 1, 2, 3, \ldots$.

3. Let $a$ be a decreasing $C^1$-function in $[0, \infty)$ such that $\lim_{t \to \infty} a(t) = 0$.

(i) Show that $\lim_{N \to \infty} \int_0^N a(t) \sin(tx) dt$ exists for all $x > 0$.

(ii) For $\epsilon > 0$ show that $\lim_{N \to \infty} \int_0^N a(t) \sin(tx) dt$ converges uniformly for $x \in [\epsilon, \infty)$.

(iii) Show that uniform convergence fails in $(0, \infty)$, for a suitable choice of $a$.

4. Let $f$ be an increasing continuous function in the interval $[a, b]$. Then it is known that the pointwise derivative $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} := f'(x)$ exists for almost all $x \in [a, b]$. Show that

$$\int_a^b f'(x) dx \leq f(b) - f(a).$$

5. Let $f$, $g$ be two measurable functions on the interval $(0, \infty)$ and assume that

$$\int_0^\infty |f(x)| \frac{dx}{x} < \infty, \quad \int_0^\infty |g(x)| \frac{dx}{x} < \infty.$$  

Show that for almost all $x \in (0, \infty)$

$$\int_0^\infty |f(\frac{t}{x})g(t)| \frac{dt}{t}$$

is finite.
6. For the following pairs of domains \((D, \Omega)\) either find a biholomorphic mapping \(F : D \to \Omega\) or show that such a biholomorphic mapping does not exist.
(i) \(D = \{z = x + iy : x < -4\}, \Omega = \{z = x + iy : x > 0, |y| < \sqrt{3}/2\}\),
(ii) \(D = \{z = 0 < |z| < 1\}, \Omega = \{z = x + iy : -1 < x < -1\}\),
(iii) \(D = \{z = x + iy : -1 < x < -1\}, \Omega = \{z : |z| < 1\}\).

7. Let \(f\) be an analytic function on \(A = \{z \in \mathbb{C} : 0 < |z| < 1\}\) satisfying
\[
\int_A |f(x + iy)|^2 dx dy < \infty.
\]
Show that \(f\) has a removable singularity at \(z = 0\).

*Hint:* Estimate the coefficients in the Laurent expansion of \(f\).

8. Let \(p > 0\) and let \(f\) be an entire function satisfying
\[
|f(z)| \leq \exp(|z|^p), \quad z \in \mathbb{C}.
\]
Show that for \(n = 0, 1, 2, \ldots\)
\[
|f^{(n)}(0)| \leq \left( \frac{e^p}{n^{1/p}} \right)^n n!.
\]

9. For \(f \in L^p(\mathbb{R})\) define
\[
B_t f(x) = \int_0^\infty te^{-ty} f(x - y) dy.
\]
(i) Suppose that \(1 \leq p < \infty\). Show that
\[
\lim_{t \to \infty} \left( \int_{-\infty}^{\infty} |B_t f(x) - f(x)|^p dx \right)^{1/p} = 0.
\]
(ii) Suppose that \(f\) is bounded and is supported in \([-1, 1]\). Is it true that \(B_t f\) converges uniformly to \(f^p\)?
Qualifying Exam

ANALYSIS

August 29, 1995

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

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1. Does \( p_N = \prod_{n=2}^{N} \left( 1 + \left( \frac{-1}{n} \right)^2 \right) \) tend to a nonzero limit as \( N \to \infty \)?
(ii) Does \( q_N = \prod_{n=2}^{N} \left( 1 + \frac{(-1)^n}{\sqrt{n}} \right) \) tend to a nonzero limit as \( N \to \infty \)?

Explain.

2. Let \( \{ f_n \} \) be a sequence of measurable functions defined in \([0, 1]\). Show that \( f_n \) converges to 0 in measure if and only if
\[
\lim_{n \to \infty} \int_{0}^{1} \frac{|f_n(x)|}{1 + |f_n(x)|} dx = 0.
\]

3. Calculate
\[
\oint_{C} \frac{-y^2 dx + xy^2 dy}{(x^2 + y^2)^2}
\]
where \( C \) is the plane curve given by the equation \( 10x^{12} + 22y^{8} = 240 \), with the positive orientation.

4. Suppose \( \alpha \geq 0 \), and let \( f \) be a bounded function on the real line with the property that
\[
|f(x + h) - f(x)| \leq A|h|^\alpha
\]
for all \( h \in \mathbb{R} \) and almost all \( x \in \mathbb{R} \).

Show that there is a constant \( C \) and for each \( t > 0 \) a \( C^1 \)-function \( g_t \) such that
\[
\|f - g_t\|_{\infty} \leq C t^\alpha
\]
and
\[
\|g_t'\|_{\infty} \leq C t^{\alpha - 1}.
\]

Hint: Use an approximation of the identity.

5. Let \( f \) be a function in \( C^1(\mathbb{R}) \) with compact support and let \( b > 0 \). Show that the limit
\[
A_b(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R} \setminus [-\epsilon, \epsilon]} \frac{f(x - y)}{y} dy
\]
exists for all \( x \in \mathbb{R} \).

How do \( A_b(x) \) and \( A_c(x) \) differ for \( b \neq c \)?

6. Let \( D = \{ z = x + iy \in \mathbb{C} : y > |x|^{1/m} \} \) and let \( F \) be analytic and bounded in \( D \). What can you say about the growth of \( |F^{(n)}(iy)| \) as \( y \to 0^+ \)?
7. For each of the following compact subsets $K_i$ of $\mathbb{C}$ let $C(K_i)$ be the space of complex-valued continuous functions on $K_i$ (the topology on $K_i$ is the relative topology for subsets of $\mathbb{C}$).

In each case we shall define a subclass $\mathcal{A}_i$ of $C(K_i)$. Prove or disprove that $\mathcal{A}_i$ is dense in $C(K_i)$.

(i) $K_1 = [-1, 1] \times \{0\}$ and $\mathcal{A}_1$ is the set of all (finite) linear combinations of the form $a_0 \cos x + \sum_{j=1}^{n} a_j x^j$ with complex coefficients $a_j$.

(ii) $K_2 = \{z \in \mathbb{C} : |z| = 2\}$ and $\mathcal{A}_2$ is the set of restrictions to $K_2$ of functions which are holomorphic in $\{1 < |z| < 4\}$.

(iii) $K_3 = \{z \in \mathbb{C} : |z| \leq 2\}$ and $\mathcal{A}_3$ is the set of all continuous functions $f$ with the property that $\int_{|z|=1} f(z) z^3 dz = 0$.

8. Calculate

$$\oint_{C} \frac{z^3 + 1}{z^4 + 2z^2 + z + 1} \, dz$$

where $C$ is the circle with center 0 and radius 3, with the positive orientation.

9. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions in a domain $\Omega \subset \mathbb{C}$ which converges to a function $f$, as $n \to \infty$, uniformly on every compact subset of $\Omega$. Suppose that each function $f_n$ has at most $m$ zeros in $\Omega$ for some fixed $m$. Prove that either $f(z) = 0$ for all $z \in \Omega$ or else it has at most $m$ zeros in $\Omega$. 
Qualifying Exam

ANALYSIS

January 17, 1996

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

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1. Let $M(n, \mathbb{R})$ be the vector space of $n \times n$ matrices with real entries. Denote by $\| \cdot \|$ a norm on $M(n, \mathbb{R})$. For $A \in M(n, \mathbb{R})$ let $\text{tr}(A)$ be the trace of $A$ (that is the sum over all entries on the diagonal). Show that there are neighborhoods $U, V$ of the identity matrix $I$ such that for every $A \in V$ there is a unique $B \in U$ with $n^{-1} \text{tr}(B)B^3 = A$.

2. Prove or disprove the following statement:

$$
\lim_{\varepsilon \to 0} \iint_{x^2 + y^2 \geq \varepsilon^2} \frac{f(x, y)}{(x + iy)^3} \, dx \, dy
$$

exists for every function $f \in C^2(\mathbb{R}^2)$ with compact support.

*Hint:* For $0 < a < b$, what are the values of

$$
\iint_{a^2 < x^2 + y^2 < b^2} \frac{x}{(x + iy)^3} \, dx \, dy \quad \text{and} \quad \iint_{a^2 < x^2 + y^2 < b^2} \frac{y}{(x + iy)^3} \, dx \, dy?
$$

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions defined on $[0, 1]$ such that

(i) $\sup_n \sup_{x \in [0, 1]} |f_n(x)| \leq 1$

(ii) $\int_0^1 f_n(x)f_m(x) \, dx = 0$ if $m \neq n$.

Let $A$ be a measurable subset of $[0, 1]$ with positive Lebesgue measure and let $\varepsilon > 0$. Show that there are at most finitely many integers $n$ with the property that $f_n(x) > \varepsilon$ for all $x \in A$.

4. For a Borel set $E \subset \mathbb{R}^n$ let $T(E) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x + y \in E\}$. Given two finite Borel measures $\mu$, $\nu$ with compact support in $\mathbb{R}^n$, the convolution $\lambda = \mu * \nu$ is defined as the Borel measure $\lambda$ such that $\lambda(E) = \iint_{T(E)} d\mu(x) d\nu(y)$ for Borel sets $E$.

(i) Find and prove a formula for $\int f \, d\lambda$, for any continuous function with compact support.

(ii) Let $C_1$ be the line segment in $\mathbb{R}^2$ connecting the origin to $(0, 1, 1)$ and let $C_2$ be the line segment in $\mathbb{R}^3$ connecting the origin to $(1, 0, 2)$. Let $\mu_1, \mu_2$ be arclength measures on $C_1, C_2$, respectively, and let $\lambda = \mu_1 * \mu_2$. Define $F(x, y, z) = x + y + z$. Compute

$$
\int F \, d\lambda.
$$

5. Let $\Omega \subset \mathbb{C}$ be a convex domain and let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function satisfying $\text{Re}(f'(z)) \geq 0$ for all $z \in \Omega$. Prove that $f$ is injective on $\Omega$.

6. For $a \in \mathbb{R}$, using complex integration, evaluate the integral:

$$
\int_{-\infty}^{+\infty} \frac{e^{-iax}}{1 + x + x^2} \, dx
$$

(i.e., evaluate the Fourier transform of the function $(1 + x + x^2)^{-1}$ at the point $a$.)
7. Assume that \( \{a_n\} \in \ell^2, \{b_n\} \in \ell^2. \)
(a) Show that
\[
\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{a_n b_m}{m} < \infty
\]

*Hint:* Estimate the sum
\[
\sum_{2^k n \leq m < 2^{k+1} n} \frac{b_m}{m}
\]
by the Cauchy-Schwarz inequality.

(b) Show that
\[
\sum_{n,m=1}^{\infty} \frac{a_n b_m}{n + m} < \infty.
\]

(*)

(c) If instead of \( \{a_n\} \in \ell^2 \) we assume \( \{a_n\} \in \ell^4 \) (that is \( \sum a_n^4 < \infty \)), what is an appropriate hypothesis on \( \{b_n\} \) so that (*) holds?

8. Let \( \Delta \) be the open unit disk in \( \mathbb{C} \), and for \( \rho > 0 \), let \( \Delta_\rho = \{ z \in \mathbb{C}, |z| < \rho \} \). Let \( M > 0 \).
   Give an explicit value of \( \rho > 0 \) so that \( f(\Delta) \supset \Delta_\rho \) for all holomorphic functions \( f \) defined on \( \Delta \), satisfying
   (1) \( f(0) = 0 \)
   (2) \( f'(0) = 1 \)
   (3) \( |f(z)| \leq M \) for all \( z \in \Delta \).

*Hint:* First find explicitly \( r \) and \( A \) depending only on \( M \) such that \( |f(z) - z| \leq A |z|^2 \) for \( |z| \leq r \). Note that you are asked for an explicit value of \( \rho \) but not for the largest possible value.

9. Let \( F \) be an integrable function defined on \( \mathbb{R} \) and let \( G \) be defined by
\[
G(x) = \int_{0}^{1} tF(x + t) \, dt.
\]
(a) Show that \( G \) is a continuous function on \( \mathbb{R} \).
(b) Show that if \( F \) is continuous, then \( G' \) is differentiable and \( G' \) is continuous.
(c) Show that the converse holds: if \( G' \) is continuous, then \( F \) is equal almost everywhere to a continuous function.
Qualifying Exam

ANALYSIS

August 27, 1996

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case whether or not you feel there is a misstatement do not interpret any problem in such a way that it becomes trivial. If you have any doubts about the interpretation of a problem ask the proctor.
1. Let \( f \) be a real valued function defined in the interval \([-2, 2]\).
   (i) Assume that \( f \) is of class \( C^2 \). Show that there is \( C \geq 0 \) such that for \(|h| \leq 1\)
   \[
   \int_{-1}^{1} \left| f(x + h) + f(x - h) - 2f(x) \right| \, dx \leq C h^2.
   \]
   (ii) Assume that \( f \) is a convex function, no longer assumed to be of class \( C^2 \). Show that the same holds.
   
   **Hint:** Show first that if \( x_0 < x_1 < \ldots < x_N \) and \( x_i - x_{i-1} = h \) then
   \[
   \sum_{i=1}^{N-1} \left| f(x_{i+1}) - 2f(x_i) + f(x_{i-1}) \right| \leq C' h
   \]
   
   **NOTE:** You can consider that it is well known that the restriction of \( f \) to any closed interval included in \((-2, 2)\) is Lipschitzian, and as usual convex means that for any \( x \) and \( y \in [-2, 2] \), \( f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \).

   (iii) Would the above hold for any continuous function \( f \)?

2. (i) Do the following indefinite integrals exist
   \[
   \int_0^\infty t^i \sin t \, dt, \quad \int_0^\infty t^{i-1} \sin t \, dt, \quad \int_0^\infty t^{i-1} \cos t \, dt
   \]
   (ii) For \( \eta \in \mathbb{R} \), set \( I(\eta) = \int_0^1 t^\eta \cos t \, dt \). Is \( I(\eta) \) a continuous function of \( \eta \)? What is the limit of \( I(\eta) \) as \( \eta \to \pm \infty \)?

3. Let \( D \) be a bounded domain in \( \mathbb{R}^2 \), bounded by a smooth Jordan curve \( C \).
   (i) Green's theorem (i.e., Stokes theorem in dimension 2) and the divergence theorem both relate integration along the curve \( C \) and integration on the domain \( D \). State both theorems.
   
   (ii) To a pair of (smooth) functions \( a(x, y) \) and \( b(x, y) \) defined on \( C \), associate a vector field \( \vec{V}(x, y) \) defined on \( C \) such that the flux of \( \vec{V} \) across \( C \) is equal to
   \[
   \int_C a \, dx + b \, dy.
   \]
   And show how Green's theorem follows from the divergence theorem.
   
   (iii) Let \( f \) be a holomorphic function defined on a neighborhood of the closure of \( D \) (i.e., a continuously differentiable, complex valued function satisfying \( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0 \)). Deduce from Green's theorem that \( \int_C f(z) \, dz = 0 \), \( (z = x + iy) \). (Cauchy theorem).
4. (i) Suppose \( f_n, h_n, \) and \( h \) are real valued \( L^1 \) functions defined on the real line, and suppose \( f_n \to 0 \) a.e. and \( h_n \to h \) a.e. as \( n \to +\infty \). Suppose also that \( |f_n(x)| \leq h_n(x) \), for every \( n \in \mathbb{N} \), and \( \int h_n(x)dx \to \int h(x)dx \), as \( n \to +\infty \). Prove that \( \int f_n(x)dx \to 0 \).

Hint: One can apply Fatou’s Lemma to \( h_n - |f_n| \). But this is not the only possible approach.

(ii) Suppose \( f_n, f \in L^1 \) and \( f_n \to f \) a.e. Prove that \( \int |f_n(x) - f(x)|dx \to 0 \) if and only if \( \int |f_n(x)|dx \to \int |f(x)|dx \).

NOTE:

a) One implication above (if or only if) is absolutely immediate, make this clear.

b) You may have seen the results of this problem stated as theorems. However you are asked to give proofs (based on basic results such as Fatou’s Lemma, or the standard version of the Lebesgue dominated convergence Theorem).

5. Let \( 1 \leq p < q < \infty \).

Which of the following statements (i)-(vi) are true, and which are false? Justify all the negative answers by a counterexample, but you do not have to justify the positive answers.

(i) \( L^p(\mathbb{R}) \subset L^q(\mathbb{R}) \).

(ii) \( L^q(\mathbb{R}) \subset L^p(\mathbb{R}) \).

(iii) \( L^p([0, 1]) \subset L^q([0, 1]) \).

(iv) \( L^q([0, 1]) \subset L^p([0, 1]) \).

(v) \( \ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z}) \).

(vi) \( \ell^q(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \).

Justify your answer to the following question:

(vii) For which \( s \geq 1 \), \( L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^s(\mathbb{R}) \)?

6. Let \( \Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \). Let \( h \) be a continuous function defined on \( \mathbb{R} \), and let \( \Phi \) be the map from \( \mathbb{R}^2 \) into itself defined by:

\[
\Phi(x, y) = (x + h(x + y), y - h(x + y)).
\]

Determine the area of \( \Phi(\Omega) \).

Note: Although you can consider first the case when \( h \) is continuously differentiable, you are asked to treat (with justifications) the case when \( h \) is merely continuous.
7. (i) By using complex integration, evaluate the limit, as \( \varepsilon \to 0^+ \) and \( R \to +\infty \), of the integral:

\[
\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \frac{e^{-ix}}{x} \, dx.
\]

(ii) For \( a \in \mathbb{R} \), evaluate the limit, as \( \varepsilon \to 0^+ \) and \( R \to +\infty \), of the integral:

\[
\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \frac{e^{-iax}}{x} \, dx.
\]

(You can gain time, by deducing it from (i), for \( a \neq 0 \)).

(iii) Set

\[
I_\varepsilon(a) = \int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{e^{-iax}}{x} \, dx \quad \text{and} \quad J_R(a) = \int_{-R}^{-1} + \int_{1}^{R} \frac{e^{-iax}}{x} \, dx.
\]

For \( a \in [0, 1] \), does \( I_\varepsilon(a) \) (resp \( J_R(a) \)) have a uniform limit as \( \varepsilon \to 0^+ \) (resp \( R \to +\infty \))? 

8. Let \( \Delta \) be the unit disk in \( \mathbb{C} \), and let \( f \) be a holomorphic function defined on \( \Delta \), assume that \( \int_{\Delta} |f|^2 \, dx \, dy \leq 1 \).

(i) Show that \( |f(z)| \leq \frac{1}{1-|z|} \). (Or show a sharper estimate).

(ii) Conversely, if \( |f(z)| \leq \frac{1}{1-|z|} \), is \( \int_{\Delta} |f|^2 \) finite?

(iii) What can you say about the coefficients in the series expansion of \( f \) (\( f = \sum a_j z^j \))?

(An estimate of \( |a_j| \), or of \( \sum |a_j|^2 \), or of some similar quantity).

9. It is known that if \( g \) is a \( \mathcal{C}^1 \) function defined on \( \mathbb{C} \), there exists \( u \), a \( \mathcal{C}^1 \) function defined on \( \mathbb{C} \), such that \( \frac{\partial u}{\partial \overline{z}} = g \). As usual, \( \frac{\partial}{\partial \overline{z}} = \frac{1}{2} (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) \).

(i) Find \( u \) satisfying \( \frac{\partial u}{\partial \overline{z}} = g \) for

a) \( g(z) = |z|^2 \).

b) \( g(z) = xy \quad (z = x + iy) \).

(ii) If \( g \) has compact support, does there always exist \( u \) a function with compact support satisfying:

\[
\frac{\partial u}{\partial \overline{z}} = g.
\]

Hint: you can start by setting \( g = \frac{\partial v}{\partial z} \), with \( v \) conveniently chosen.
Qualifying Exam

ANALYSIS

January 14, 1997

Instructions: Do 6 of the 9 questions. To facilitate grading, please use a separate packet of paper for each question.

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case whether or not you feel there is a misstatement do not interpret any problem in such a way that it becomes trivial. If you have any doubts about the interpretation of a problem ask the proctor.
Problem 1. In $\mathbb{R}^3$, for which real $p > 0$ is $(x_1^2 + x_2^4 + x_3^6)^{-p}$

(i) integrable in a neighborhood of 0?

(ii) integrable at infinity?

Hint: Decompose $\mathbb{R}^3$ in 'annuli'

$$\Gamma_k = \{(x_1, x_2, x_3) \in \mathbb{R}: 2^k \leq (x_1^2 + x_2^4 + x_3^6) \leq 2^{k+1}\}$$

and estimate the volume of each $\Gamma_k$.

Problem 2. Suppose that $a_n > 0$ for each $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n = +\infty$. Discuss the convergence resp. divergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n^p a_n}$$

for each real $p \geq 0$. (If for some $p$ the series can either converge or diverge, depending on the choice of $a_n$, give an example of both.)

Problem 3. Let $S \subset \mathbb{R}^3$ be the plane through the origin with the unit normal vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$, and let $S$ be oriented by $n$. For $r > 0$ we denote by $C_r$ the circle of radius $r$ in $S$ centered at the origin. Let $T$ be the unit tangent vector field to $C_r$ in the positive (counterclockwise) direction.

(i) Prove that for each continuous vector field $F$ on $\mathbb{R}^3$ we have

$$\lim_{r \to 0} \frac{1}{r} \int_{C_r} F \cdot T \, ds = 0.$$  

(Here $ds$ denotes the arc length on $C_r$.)

(ii) Suppose in addition that $F$ is differentiable at the origin. Prove that the following limit exists:

$$\lim_{r \to 0} \frac{1}{r^2} \int_{C_r} F \cdot T \, ds.$$ 

(iii) Calculate the limit in (ii) for $n = (1, 0, 0)$ and

$$F(x) = (e^{x_1}, x_2 \sin x_3, x_3 \cos x_2).$$

Problem 4. Let $1 < p < +\infty$. Let $f_j$ be a sequence in $L^p(\mathbb{R})$ such that for all $j = 1, 2, 3, \ldots$.  

1
(i) \( \|f_j\|_p = 1 \) (the \( L_p(\mathbb{R}) \) norm), and

(ii) for almost every \( x \in \mathbb{R} \) we have \( f_j(x) = 0 \) or \( |f_j(x)| \geq j \).

Show that for any \( g \in L^q(\mathbb{R}) \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \int_{\mathbb{R}} f_jg \) tends to 0 as \( j \) tends to \( +\infty \).

**Problem 5.** Let \( f \in L^1([0,1]) \). For \( k \in \mathbb{N} \), let \( f_k \) be the step function defined on \([0,1]\) by:

\[
f_k(x) = k \int_{\frac{j}{k}}^{\frac{j+1}{k}} f(t) \, dt \quad \text{for} \quad \frac{j}{k} \leq x < \frac{j+1}{k}.
\]

Show that \( f_k \) tends to \( f \) in \( L^1 \) norm as \( k \) tends to \( +\infty \).

Hint: Treat first the case when \( f \) is continuous, and use approximation.

**Problem 6.** (This problem can be done even if you have not done the problem 5 above.) In the problem 5 above, is the convergence of \( f_k \) to \( f \) a dominated convergence in the sense of the Lebesgue dominated convergence theorem?

Hint: Try \( f(x) = 1/(x \log^2 x) \).

**Problem 7.** Let \( f \) be a holomorphic function, defined on the unit disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), such that:

\[
f(0) = 0 \text{ and } |f'(z)| \geq 1 \text{ for every } z \in D.
\]

Show that \( f(D) \supseteq D \).

Hint: If a line segment \([0, \zeta]\) is included in \( f(D) \), show that there exists a \( z \in D \) such that \( f(z) = \zeta \) and \( |z| \leq |\zeta| \).

**Problem 8.** Let \( D \) be the unit disc as in problem 7 above. Suppose that \( f \) is a nonconstant holomorphic function on \( D \) satisfying \( \text{Re}(f(z)) \geq 0 \) for all \( z \in D \) and \( f(0) = 1 \). Prove

\[
|f(z)| \leq \frac{1 + |z|}{1 - |z|}.
\]

**Problem 9.** Let \( 0 < a < 1 \). Evaluate the integral

\[
I(a) = \int_{-\infty}^{+\infty} \frac{e^{ax}}{1 + e^x} \, dx.
\]

Hint: Compare \( I(a) \) with \( \int_{-\infty}^{+\infty} \frac{e^{a(x+2\pi i)}}{1 + e^x} \, dx \) by a contour integration in \( \mathbb{C} \).
Problem I.

(a) Find an explicit value of $\epsilon > 0$ so that for every real number $x \in [0, 1]$,

$$|\sqrt{x} - \sqrt{x + \epsilon}| \leq \frac{1}{200}.$$

(b) Find an explicit integer $N$ such that there exists a polynomial $P$ of degree at most $N$ such that for every real number $x \in [0, 1]$

$$|\sqrt{x} - P(x)| \leq \frac{1}{100}.$$

(Hint: You could use the Taylor expansion of the function $\sqrt{x}$ in powers of $(x - 1)$.)

Problem II.

Let $y(t)$ be a continuously differentiable solution to the following initial value problem on the interval $[0, T]$:

$$\frac{dy}{dt} = y^2 \left[ 2 + \sin(e^t + y) \right]$$

$$y(0) = 1.$$

Show that $T < 1$.

Problem III.

Let $f$ and $g$ be twice continuously differentiable real functions defined on $\mathbb{R}^2$. Assume that they vanish at $(0, 0)$ and that they do not have any other common zero.

Under which conditions on the partial derivatives of $f$ and $g$ at $(0, 0)$ is the improper integral

$$\iint_{|x|^2 + |y|^2 \leq 1} \frac{dx \, dy}{|f|^{\frac{1}{2}} + |g|^{\frac{1}{2}}}$$

convergent?

You are asked a complete discussion.
**Problem IV.**

On the interval $[-1, +1]$ consider the standard Banach spaces $L^1$ and $L^2$ with the usual norms

$$
\|f\|_1 = \int_{-1}^{+1} |f(x)| \, dx; \quad \|f\|_2 = \left( \int_{-1}^{+1} |f(x)|^2 \, dx \right)^{\frac{1}{2}}. \quad \|f\|_2 \leq \sqrt{2} \|f\|_2
$$

(a) How are these two norms related? (Give inequalities, state sharp results, and give justifications.)

(b) Let $\{f_j\}$ be a sequence of functions in $L^2$. Assume that $f_j \geq 0$, that $\|f_j\|_1 = 2$, and that

$$
\|f_j\|_2 - \sqrt{2} \leq 2^{-j}.
$$

Show that $f_j$ tends almost everywhere to the constant 1 on the interval $[-1, +1]$. (Hint: Write $f_j = 1 + h_j$.)

(c) If we drop the hypothesis that $f_j \geq 0$, what should replace the above conclusion?

**Problem V.**

Let $f$ be a locally integrable function defined on $\mathbb{R}$. Let

$$
L_f = \{ x \in \mathbb{R} \mid f(x) = \lim_{h \to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, dt \}
$$

be the set at which small averages of $f$ converge to $f$.

(a) Is the set $L_f$ always non-empty? Explain what you know about this. Give a non-trivial example of a function $f$ and a point $x \in L_f$ at which $f$ is not continuous.

(b) For $t > 0$ define a function $g_t$ on $\mathbb{R}$ by $g_t(x) = \frac{1}{t} \left( 1 - \frac{|x|}{t} \right)$ if $|x| < t$, and $g_t(x) = 0$ if $|x| \geq t$.

Graph the function $g_t$. Suppose $x_0 \in L_f$. Prove that

$$
\lim_{t \to 0^+} f * g_t(x_0) = f(x_0), \quad (\ast \text{ denotes convolution}).
$$

Hint: For $a > 0$, let $\chi_{[-a, +a]}$ denote the characteristic function of the interval $[-a, +a]$. Show the following identity, explain its meaning:

$$
g_t(x) = \int_0^1 \left( \frac{\chi_{[-ht, +ht]}(x)}{2th} \right)(x) \, 2hdh.
$$

**Problem VI.**

For each $t > 0$, let $f_t$ be a measurable function defined on the real line $\mathbb{R}$. Let

$$
E = \{ x \in \mathbb{R} \mid \lim_{t \to \infty} f_t(x) \text{ exists.} \}
$$

(a) Show by an example that the set $E$ need not be measurable.

(b) If each of the functions $f_t$ is continuous, show that $E$ is a Borel set and hence measurable.
Problem VIIc.

Let $f$ be a holomorphic function, defined in a neighborhood of the disk
\[ \Delta = \{ \zeta \in \mathbb{C}, \; |\zeta| \leq 4 \}. \]
Suppose that $f$ does not vanish on the circle $\{ \zeta \in \mathbb{C}, \; |\zeta| = 4 \}$, and satisfies
\[ \int_{|\zeta|=4} \frac{\zeta^k f'(\zeta) \, d\zeta}{f(\zeta)} = 2^{k+2} \pi i \]
for $k = 0, 1, 2$. Find all the zeros of $f$ in $\Delta$.

Problem VIIIc.

Let $D^* = \{ z \in \mathbb{C}, \; 0 < |z| < 1 \}$. Let $f$ be a non constant holomorphic function on $D^*$. Assume that $\text{Im} \ f(z) \geq 0$ if $\text{Im} \ z \geq 0$ and that
\[ \text{Im} \ f(z) \leq 0 \text{ if } \text{Im} \ z \leq 0. \]
(a) Show that if $z \in D^*$ and $z$ is not real, then $f(z)$ is not real. Show that if $z \in (-1, 0) \cup (0, 1)$, then $f'(z) \neq 0$.

(b) Show that either $f$ has a holomorphic extension to the unit disk satisfying $f'(0) \neq 0$, or that $f$ has a meromorphic extension to the unit disk with a simple pole at 0.

Problem IXc.

It is a well known fact that if $u$ is a continuous function defined on $\mathbb{R}$, then
\[ u(0) = \lim_{\tau \to 0^+} \frac{1}{\sqrt{2\pi \tau}} \int_{-2}^{+2} u(t)e^{-\frac{t^2}{2\tau}} \, dt. \]
(This is just approximation by convolution with a Gaussian kernel, and the limits of integration $\pm 2$ could be replaced by others).

Let $f$ be a holomorphic function on $\mathbb{C}$. Show that:
\[ f(i) = \lim_{\tau \to 0^+} \frac{1}{\sqrt{2\pi \tau}} \int_{-2}^{+2} f(t)e^{-\frac{(t-i)^2}{2\tau}} \, dt. \]
(Hint: Change contour of integration. Take the piecewise linear path from $-2$ to $-2+i$, to $2+i$, to $2$.)
Problem VIIr.

Let \( \{f_n\} \) be a sequence of continuous functions defined on the unit interval \([0, 1]\). Suppose that \( \lim_{n \to \infty} \int_0^1 |f_n(t)| \, dt = +\infty \). For each of the following two statements, either give an example of a sequence \( \{f_n\} \) which satisfies the statement, or else prove that the statement is false.

(a) For every continuously differentiable function \( g \) on \([0, 1]\), \( \lim_{n \to \infty} \int_0^1 f_n(t)g(t) \, dt = 0 \).

(b) For every continuous function \( h \) on \([0, 1]\), \( \lim_{n \to \infty} \int_0^1 f_n(t)h(t) \, dt = 0 \)
QUALIFYING EXAM

ANALYSIS

August 24, 1998

Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Notation used in this exam:
1. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
2. $D = \{z \in \mathbb{C} | |z| < 1\}$ denotes the unit disc in the complex plane.
3. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,
   \[ ||f||_p = \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p}. \]
   Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.
4. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then
   \[ ||f||_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}. \]
   $L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $||f||_{\infty} < +\infty$.
5. $L^p_{loc}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \subset \mathbb{R}$.
6. If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f * g$ is defined to be the function
   \[ f * g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt \]
   whenever the integral converges.

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Problem I  Define an infinite sequence of real numbers \( \{a_1, a_2, \ldots, a_n, \ldots\} \) by setting \( a_1 = 1 \), \( a_2 = 2 \), and \( a_{n+1} = 2a_n + 3a_{n-1} \) for \( n \geq 2 \).

(a) Let \( b_n = \frac{a_{n+1}}{a_n} \) for \( n \geq 1 \). Prove that \( \lim_{n \to \infty} b_n \) exists and evaluate the limit.

(b) What is the radius of convergence \( \rho \) of the infinite series \( \sum_{n=1}^{\infty} a_n x^n \)?

(c) For \( |x| < \rho \), evaluate \( \sum_{n=1}^{\infty} a_n x^n \). Does this infinite series converge when \( x = \rho \), the radius of convergence found in part (b)?

Problem II  In this problem suppose that \( A \) and \( B \) are strictly positive real numbers.

(a) Show that there is a constant \( K > 0 \) so that for all \( A \) as above
\[
\int_0^{+\infty} \frac{dx}{A^3 + x^3} = K A^{-2}.
\]
Without computing \( K \) explicitly, show that \( K < 3/2 \).

(b) Show that there is a universal constant \( K \) so that for all \( A \) and \( B \) as above,
\[
\int_0^{+\infty} \frac{dx}{(A^3 + x^3)(B^3 + x^3)} \leq K (A + B)^{-3} \left[ \min\{A, B\} \right]^{-2}.
\]

(c) Find an estimate for
\[
\left| \int_0^{\infty} \frac{\sin(x)}{(A + x)^3} \, dx \right|
\]
which is better as \( A \to 0 \) than can be obtained by observing that
\[
\left| \int_0^{\infty} \frac{\sin(x)}{(A + x)^3} \, dx \right| \leq \int_0^{\infty} \frac{|\sin(x)|}{(A + x)^3} \, dx \leq \int_0^{\infty} \frac{dx}{A^3 + x^3} = K A^{-2}.
\]

Problem III

(a) Let \( \Omega \) be a convex set in \( \mathbb{R}^2 \) with smooth boundary. Using only the Fundamental Theorem of Calculus for functions of one variable, prove that if \( f \) and \( g \) are continuously differentiable functions in a neighborhood of the closure of \( \Omega \), then
\[
\oint_C f(x, y) \, dx + g(x, y) \, dy = \iint_{\Omega} \left[ \frac{\partial g}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right] \, dx \, dy
\]
where \( C \) is the simple closed curve bounding \( \Omega \) taken in the counter-clockwise direction.

(b) Evaluate
\[
\oint_C \frac{xy^2 \, dx - x^2 y \, dy}{(x^2 + y^2)^2}
\]
where \( C \) is the ellipse \( 25(x - 1)^2 + 16(y - 2)^2 = 400 \), taken in the counter-clockwise sense.
Problem IV
(a) Give an example of a sequence of functions \( f_n \in L^1([0,1]), \ n = 1, 2, \ldots \) and a function \( g \in L^1([0,1]) \) with the following properties:
1. \( f_n(x) \to g(x) \) for almost all \( x \in [0,1] \);
2. \( \int_0^1 |f_n(x)| \, dx = 2 \) for every \( n = 1, 2, \ldots \);
3. \( \int_0^1 |g(x)| \, dx = 1 \).

(b) Show that for any sequence \( \{f_n\} \) and function \( g \) as in part (a) it follows that
\[
\lim_{n \to \infty} \int_0^1 |f_n(x) - g(x)| \, dx = 1.
\]

Problem V
Let \( f \in L^4([0,1]) \).

(a) Show that \( f \in L^2([0,1]) \) and that \( \|f\|_2 \leq \|f\|_4 \).

(b) Does there exist a constant \( C \) so that for all \( f \in L^4([0,1]) \), \( \|f\|_4 \leq C \|f\|_2^2 \)?

(c) For a given function \( f_0 \in L^4([0,1]) \), let \( C \) be a constant such that
\[
\int_0^1 |f_0(x)|^4 \, dx \leq C \left( \int_0^1 |f_0(x)|^2 \, dx \right)^2.
\]
Find a constant \( A \) depending only on \( C \) so that
\[
\left( \int_0^1 |f_0(x)|^2 \, dx \right)^{\frac{1}{2}} \leq A \int_0^1 |f_0(x)| \, dx.
\]
(HINT: Estimate \( \int_0^1 |f_0(x)|^2 \, dx = \int_0^1 |f_0(x)|^\alpha |f_0(x)|^{2-\alpha} \, dx \) by using Hölder’s inequality with appropriate exponents and an appropriate \( \alpha < 1 \).)

Problem VI
Define a function \( \phi \) on \( \mathbb{R} \) by setting
\[
\phi(x) = \begin{cases} 
1 - |x| & \text{if } |x| < 1, \\
0 & \text{if } |x| \geq 1.
\end{cases}
\]

(a) Show that if \( g \) is a continuously differentiable function on \( \mathbb{R} \), then the convolution of \( g \) with \( \phi \) is also continuously differentiable.

(b) Find a function \( \chi \in L^\infty(\mathbb{R}) \) so that if \( g \) is continuously differentiable on \( \mathbb{R} \), then
\[
\frac{d(g * \phi)}{dx}(x) = g * \chi(x).
\]

(c) Show that if \( f \in L^1(\mathbb{R}) \), then the convolution of \( f \) with \( \phi \) is continuously differentiable.
(HINT: Approximate \( f \) by a sequence of continuously differentiable functions \( g_n \), and then use (a) and (b) and results about limits of continuously differentiable functions.)
Problem VII Let \( 0 < \alpha < 1 \). Evaluate the improper integral
\[
\int_0^{+\infty} \frac{dx}{x^\alpha (1 + x)}.
\]
(HINT: Consider the complex plane slit along the positive real axis, and consider a closed contour in this slit plane consisting of a part of a large circle of radius \( R \) taken in the counterclockwise sense, a part of a small circle of radius \( \epsilon \) taken in the clockwise sense, and two lines parallel to the positive axis joining these circles, one above the positive \( x \) axis and one below.)

Problem VIII Let \( f \) be a holomorphic function defined in the unit disc \( \mathbb{D} \). Show that the following two assertions are equivalent:

1. There exists a constant \( C > 0 \) and a positive integer \( n \) such that
\[
|f(z)| \leq \frac{C}{(1 - |z|^2)^n}.
\]

2. There exists a positive constant \( A \) and a positive integer \( k \) such that the coefficients \( \{a_n\} \) in the power series expansion \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) satisfy the inequality
\[
|a_m| \leq A m^k.
\]

Try to give sharp results relating the constants \( C \) and \( n \) to the constants \( A \) and \( k \).

Problem IX

(a) Let \( f \) be a holomorphic function defined in the unit disc \( \mathbb{D} \) and suppose that \( f(0) = 1 \) and \( \Re[f(z)] > 0 \) for all \( z \in \mathbb{D} \). Show that for \( -1 < x < +1 \)
\[
|f(x)| \leq \frac{1 + |x|}{1 - |x|}.
\]

What can you say if there is equality at some point \( x \neq 0 \)?

(b) Let \( V = \{z = re^{i\theta} \in \mathbb{C} \mid r > 0 \text{ and } |\theta| < \frac{\pi}{2}\} \). Let \( \mathbb{D} \) denote the unit disc, and suppose \( f : \mathbb{D} \to V \) is holomorphic with \( f(0) = 1 \). Prove that if \( -1 < x < +1 \) then
\[
|f(x)| \leq \left( \frac{1 + |x|}{1 - |x|} \right)^{\frac{1}{2}}
\]
**QUALIFYING EXAM**

**ANALYSIS**

January 19, 2000

Version for Math 722

**Instructions:** Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

**Notation used in this exam:**

1. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
2. $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ denotes the unit disc in the complex plane.
3. For points $x$ and $y$ in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
4. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,
   \[
   \|f\|_p = \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p}.
   \]

Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $\|f\|_p < +\infty$.

5. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then
   \[
   \|f\|_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
   \]

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $\|f\|_{\infty} < +\infty$.

6. $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \subset \mathbb{R}$.

7. If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f \ast g$ is defined to be the function
   \[
   f \ast g(x) = \int_{\mathbb{R}} f(x-t) g(t) \, dt
   \]

whenever the integral converges.

*The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, do not interpret any problem in such a way that it becomes trivial.*
Problem I  Let \( \{a_n\}, \ n=1, 2, \ldots, \) be a sequence of strictly positive real numbers.

a) Prove that
\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]

b) Give an example of a sequence \( \{a_n\} \) of strictly positive real numbers such that all of the above inequalities are strict inequalities.

Problem II  Let
\[
Q = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < 1, \ 0 < y < 1 \right\}.
\]
For positive real numbers \( a \) and \( b \) define a function \( F_{a,b} : Q \to \mathbb{R} \) by the equation
\[
F_{a,b}(x, y) = x^a y^b \int_0^\infty \frac{ds}{(x+s)(y^3+s^3)}.
\]
For which \( a \) and \( b \) is it true that \( F_{a,b} \) is a bounded function on \( Q \)?

Problem III  Let \( U = \{ x \in \mathbb{R}^n \mid |x| < 1 \} \) be the open unit ball in \( \mathbb{R}^n \) and let \( \rho : U \to \mathbb{R} \) be a \( C^\infty \) function such that \( \rho(0) = 0 \) and \( \nabla \rho(0) \neq 0 \). Let \( \Sigma = \{ x \in U \mid \rho(x) = 0 \} \). For \( x \in U \) let
\[
d(x) = \inf_{y \in \Sigma} |x - y|.
\]

a) For \( x \in V = \{ x \in \mathbb{R}^n \mid |x| < \frac{1}{2} \} \), prove that there is a point \( y \in \Sigma \) such that \( d(x) = |x - y| \).

b) For \( x \in V - \Sigma \) and for any \( y \in \Sigma \) such that \( d(x) = |x - y| \), prove that vector \( \nabla \rho(y) \) is a scalar multiple of the vector \( x - y \).

c) Prove that there is an open set \( W \) with \( 0 \in W \subset V \) and a \( C^\infty \) function \( \varphi : W \to \mathbb{R} \) so that for \( x \in W \), \( |\varphi(x)| = d(x) \).

Problem IV  Let \( B \) denote the set of all Borel subsets of \( \mathbb{R} \). Let \( \mu : B \to [0, \infty) \) (the finite, non-negative real numbers) be a set function with the property that if \( E_j \in B \) are mutually disjoint, then
\[
\mu \left( \bigcup_{j=1}^\infty E_j \right) = \sum_{j=1}^\infty \mu(E_j).
\]

a) Prove that if \( \{E_j\} \) is a sequence of Borel sets with \( E_j \subset E_{j+1} \), then
\[
\mu \left( \bigcup_{j=1}^\infty E_j \right) = \lim_{j \to \infty} \mu(E_j).
\]

b) Prove that if \( \{E_j\} \) is a sequence of Borel sets with \( E_j \supset E_{j+1} \), then
\[
\mu \left( \bigcap_{j=1}^\infty E_j \right) = \lim_{j \to \infty} \mu(E_j).
\]

c) Suppose that for every Borel set \( E \) with Lebesgue measure \( |E| = 0 \), it follows that \( \mu(E) = 0 \). Prove that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that if \( E \in B \) and \( |E| < \delta \), then \( \mu(E) < \varepsilon \).
Problem V  Let $f \in L^1(\mathbb{R})$. For $x \in \mathbb{R}$ and $y > 0$ set

$$F(x, y) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(t) e^{-x \sqrt{\frac{t-1}{y}}} \, dt.$$ 

a)  Show that $\lim_{y \to \infty} F(x, y)$ exists for all $x \in \mathbb{R}$. What is the limit?

b)  Show that there is a constant $C$ independent of $f$ so that for all $y > 0$

$$|F(x, y)| \leq C \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \, dt.$$ 

c)  Show that $\lim_{y \to 0} F(x, y)$ exists for almost all $x \in \mathbb{R}$. What is the limit?

Problem VI  

a)  Prove Hölder’s inequality for measurable functions $f$ and $g$ defined on $\mathbb{R}$ if $1 < p, q < +\infty$ and if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\int_{\mathbb{R}} |f(x)g(x)| \, dx \leq \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} |g(x)|^q \, dx \right)^{\frac{1}{q}}.$$ 

b)  Suppose that $f$ is a measurable function on $\mathbb{R}$ and that $|f| \neq 0$ on a set of positive measure. For $0 < p < +\infty$, let

$$A_f(p) = \log \left[ \int_{\mathbb{R}} |f(x)|^p \, dx \right],$$ 

and let

$$E_f = \left\{ p \in (0, \infty) \mid A_f(p) < +\infty \right\}.$$ 

Using part a), prove that $E_f$ is connected and prove that if $E_f$ has non-empty interior, then $A_f$ is a convex function on the interior of $E_f$.

c)  Give an example of a function $f$ for which $E_f$ in part b) consists of exactly one point.

Problem VII  Let $f$ be a holomorphic function defined in an open set $\Omega$ containing the origin, and suppose there is an integer $m \geq 1$ so that $f^{(j)}(0) = 0$ for $0 \leq j \leq m-1$ and $f^{(m)}(0) \neq 0$. Prove that there are constants $A$ and $B$ so that for $0 < |w| < A$, the equation $f(z) = w$ has exactly $m$ distinct roots in the set $\{z \in \mathbb{C} \mid |z| < B\}$.

Problem VIII For each of the following, prove that there is no holomorphic function $f$ defined on the unit disk $\mathbb{D}$ with the stated properties:

a)  $\lim_{|z| \to 1} |f(z)| = +\infty$.

b)  For all $z \in \mathbb{D}$, $|f(z)| < 1$ and

$$f \left( \frac{1}{2} \right) = 0, \quad f \left( \frac{1}{3} \right) = 0, \quad f \left( \frac{1}{5} \right) = \frac{1}{5}.$$ 

Problem IX  Let $\alpha$ be a real number. Evaluate

$$\int_{-\infty}^{+\infty} \frac{\cos(\alpha x)}{e^x + e^{-x}} \, dx.$$
QUALIFYING EXAM

ANALYSIS

January 19, 2000

Version for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Notation used in this exam:
1. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
2. $D = \{ z \in \mathbb{C} | |z| < 1 \}$ denotes the unit disc in the complex plane.
3. For points $x$ and $y$ in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.
4. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,

$$||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.$$ 

Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

5. If $\mu$ is a positive measure on a set $X$, and $f$ is a complex valued measurable function on $X$, then

$$||f||_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.$$ 

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $||f||_{\infty} < +\infty$.

6. $L^p_{loc}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset \subset \mathbb{R}$.

7. If $f$ and $g$ are measurable functions on $\mathbb{R}$, the convolution $f \ast g$ is defined to be the function

$$f \ast g(x) = \int_{\mathbb{R}} f(x-t)g(t) \, dt$$

whenever the integral converges.

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Problem I  Let \( \{a_n\}, n = 1, 2, \ldots, \) be a sequence of strictly positive real numbers.

a) Prove that
\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]

b) Give an example of a sequence \( \{a_n\} \) of strictly positive real numbers such that all of the above inequalities are strict inequalities.

Problem II  Let
\[
Q = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < 1, \quad 0 < y < 1 \right\}.
\]
For positive real numbers \( a \) and \( b \) define a function \( F_{a,b} : Q \to \mathbb{R} \) by the equation
\[
F_{a,b}(x, y) = x^a y^b \int_0^\infty ds \frac{ds}{(x+s)(y^2+s^3)}.
\]
For which \( a \) and \( b \) is it true that \( F_{a,b} \) is a bounded function on \( Q \)?

Problem III  Let \( U = \{ x \in \mathbb{R}^n \mid |x| < 1 \} \) be the open unit ball in \( \mathbb{R}^n \) and let \( \rho : U \to \mathbb{R} \) be a \( C^\infty \) function such that \( \rho(0) = 0 \) and \( \nabla \rho(0) \neq 0 \). Let \( \Sigma = \{ x \in U \mid \rho(x) = 0 \} \). For \( x \in U \) let
\[
d(x) = \inf_{y \in \Sigma} |x - y|.
\]

a) For \( x \in V = \{ x \in \mathbb{R}^n \mid |x| < \frac{1}{2} \} \), prove that there is a point \( y \in \Sigma \) such that \( d(x) = |x - y| \).

b) For \( x \in V - \Sigma \) and for any \( y \in \Sigma \) such that \( d(x) = |x - y| \), prove that vector \( \nabla \rho(y) \) is a scalar multiple of the vector \( x - y \).

c) Prove that there is an open set \( W \) with \( 0 \in W \subset V \) and a \( C^\infty \) function \( \varphi : W \to \mathbb{R} \) so that for \( x \in W \), \( |\varphi(x)| = d(x) \).

Problem IV  Let \( B \) denote the set of all Borel subsets of \( \mathbb{R} \). Let \( \mu : B \to [0, \infty) \) (the finite, non-negative real numbers) be a set function with the property that if \( E_j \in B \) are mutually disjoint, then
\[
\mu \left( \bigcup_{j=1}^\infty E_j \right) = \sum_{j=1}^\infty \mu(E_j).
\]

a) Prove that if \( \{E_j\} \) is a sequence of Borel sets with \( E_j \subset E_{j+1} \), then
\[
\mu \left( \bigcup_{j=1}^\infty E_j \right) = \lim_{j \to \infty} \mu(E_j).
\]

b) Prove that if \( \{E_j\} \) is a sequence of Borel sets with \( E_j \supset E_{j+1} \), then
\[
\mu \left( \bigcap_{j=1}^\infty E_j \right) = \lim_{j \to \infty} \mu(E_j).
\]

C) Suppose that for every Borel set \( E \) with Lebesgue measure \( |E| = 0 \), it follows that \( \mu(E) = 0 \). Prove that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) so that if \( E \in B \) and \( |E| < \delta \), then \( \mu(E) < \epsilon \).
Problem V  Let \( f \in L^1(\mathbb{R}) \). For \( x \in \mathbb{R} \) and \( y > 0 \) set
\[
F(x, y) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(t) e^{-\pi \frac{(x-t)^2}{y}} \, dt.
\]
a) Show that \( \lim_{y \to \infty} F(x, y) \) exists for all \( x \in \mathbb{R} \). What is the limit?
b) Show that there is a constant \( C \) independent of \( f \) so that for all \( y > 0 \)
\[
|F(x, y)| \leq C \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \, dt.
\]
c) Show that \( \lim_{y \to 0} F(x, y) \) exists for almost all \( x \in \mathbb{R} \). What is the limit?

Problem VI
a) Prove Hölder’s inequality for measurable functions \( f \) and \( g \) defined on \( \mathbb{R} \): if \( 1 < p, q < +\infty \) and if \( \frac{1}{p} + \frac{1}{q} = 1 \) then
\[
\int_{\mathbb{R}} |f(x)g(x)| \, dx \leq \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} |g(x)|^q \, dx \right)^{\frac{1}{q}}.
\]
b) Suppose that \( f \) is a measurable function on \( \mathbb{R} \) and that \( |f| \neq 0 \) on a set of positive measure. For \( 0 < p < +\infty \), let
\[
A_f(p) = \log \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right),
\]
and let
\[
E_f = \left\{ p \in (0, +\infty) \mid A_f(p) < +\infty \right\}.
\]
Using part a), prove that \( E_f \) is connected and prove that if \( E_f \) has non-empty interior, then \( A_f \) is a convex function on the interior of \( E_f \).
c) Give an example of a function \( f \) for which \( E_f \) in part b) consists of exactly one point.

Problem VII  State carefully and give a complete proof of the Baire Category Theorem. Then
describe one significant application of this result.

Problem VIII
a) If \( T \) is a distribution on \( \mathbb{R}^2 \), give a precise definition of the distribution \( \Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \).
b) For \( (x, y) \in \mathbb{R}^2 \), let \( F(x, y) = \log(x^2 + y^2) \). Derive an explicit formula for \( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \) in the sense of distributions.

Problem IX  For \( f \in L^1(\mathbb{R}^n) \), the Fourier transform of \( f \) is the function
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) \, dx.
\]
a) Prove that if \( f \in L^1(\mathbb{R}^n) \) then \( \hat{f} \) is a continuous function on \( \mathbb{R}^n \), and that \( \lim_{|\xi| \to \infty} \hat{f}(\xi) = 0 \).
b) Suppose that \( f \) and \( g \) belong to \( L^1(\mathbb{R}^n) \) and that there is a real number \( \alpha > \frac{n}{2} \) such that
\[
\hat{f}(\xi) = (1 + |\xi|^2)^{\alpha} \hat{g}(\xi).
\]
Prove that, after redefining \( g \) on a set of measure zero if necessary, \( g \) is continuous on \( \mathbb{R}^n \).
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison

Wednesday, January 17, 2001

Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( ||f||_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then

\[
||f||_\infty = \inf \{ t > 0 \mid \mu(\{x \in X : |f(x)| > t\}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( ||f||_\infty < +\infty \).

(7) \( L^p_{loc}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \mathbb{R} \).

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function

\[
f \ast g(x) = \int_{\mathbb{R}} f(x - t) \, g(t) \, dt
\]

whenever the integral converges.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.
Problem I

(a) Show that \( n^n e^{-n} \leq n! \leq n^n \) for all positive integers \( n \).

(b) Let \( \{c_n\}, n = 1, 2, \ldots \), be a sequence of positive real numbers. Suppose there is a real number \( \alpha \in \mathbb{R} \) and a constant \( C > 0 \) so that for all \( n \geq 1 \)

\[
\frac{c_{n+1}}{c_n} = 1 + \frac{\alpha}{n} + R(n) \quad \text{where} \quad |R(n)| \leq \frac{C}{n^2}.
\]

Show that, depending on \( \alpha \), the sequence \( \{c_n\} \) has a limit which is either zero, positive, or infinite.

(c) Using the results of part (b), show that the sequence \( \left\{ \frac{n!}{n^n e^{-n} \sqrt{n}} \right\}, \quad n = 1, 2, \ldots \), has a finite non-zero limit.

Problem II (NOTE: This is an advanced calculus problem. Do not quote theorems from the theory of Lebesgue integration for its solution). For each positive integer \( n \) let \( a^{(n)} \) be an infinite sequence of complex numbers so that

\[
a^{(n)} = \left( a_0^{(n)}, a_1^{(n)}, \ldots, a_j^{(n)}, \ldots \right).
\]

(a) Suppose there is a positive real number \( M \) such that for all \( n \)

\[
\sum_{j=0}^{+\infty} |a_j^{(n)}|^2 \leq M.
\]

Prove that

\[
\sum_{j=0}^{+\infty} \liminf_{n \to \infty} |a_j^{(n)}|^2 \leq M.
\]

Is it also true that

\[
\sum_{j=0}^{+\infty} \limsup_{n \to \infty} |a_j^{(n)}|^2 \leq M?
\]

Either prove this or show that it is false by giving a counter-example.

(b) Assume that for each \( j, \lim_{n \to \infty} a_j^{(n)} = \alpha_j \) exists. If

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)}|^2 = \sum_{j=0}^{\infty} |\alpha_j|^2,
\]

prove that

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)} - \alpha_j|^2 = 0.
\]

Problem III For \( x > 0 \), let \( F(x) = \int_0^\infty \frac{1 - e^{-xt^2}}{t^2} \, dt \).

(a) Show that this improper integral converge.

(b) Show that the function \( F \) is differentiable, and find an explicit formula for \( F'(x) \) in terms of elementary functions.

(c) Use the results of part (b) to find an explicit expression for \( F(x) \) in terms of elementary functions.
Problem IV Let $E \subset \mathbb{R}$ be a proper non-empty measurable set, so that $\emptyset \neq E \neq \mathbb{R}$. Assume that $E$ is invariant under translation by rational numbers. Explicitly, this means that if $x \in E$ and if $r$ is any rational number, then $x + r \in E$. Show that either $E$ has Lebesgue measure zero or that $\mathbb{R} - E$ has Lebesgue measure zero. Give examples to show that both conclusions are possible.

Problem V Let $\{E_j\}_{j \in \mathbb{N}}$ be a countable collection of measurable subsets of $\mathbb{R}^d$.
(a) Show that the set $A$ of points $x \in \mathbb{R}^d$ that belong to all but finitely many of the sets $E_j$ is measurable.
(b) Show that if $\lim_{j \to \infty} |E_j| = 0$, the set $A$ defined in part (a) has measure zero.
(c) Show that the set $B$ of points $x \in \mathbb{R}^d$ that belong to infinitely many of the set $E_j$ is measurable.
(d) If $\lim_{j \to \infty} |E_j| = 0$, must the set $B$ defined in part (c) have measure zero? Either prove that this is true, or show that it is false by giving a counter-example.

Problem VI For $t \in \mathbb{R}$ let $g(t) = (1 + |t|)^{-1}$. Fix $x \in \mathbb{R}$ and, for each non-zero $h \in \mathbb{R}$ set
$$G_h(t) = \frac{g(x + h - t) - g(x - t)}{h}.$$ 
(a) Prove that each $G_h \in L^2(\mathbb{R})$.
(b) Prove that $\lim_{h \to 0} G_h(t)$ exists for almost every $t \in \mathbb{R}$.
(c) Prove that if $G_0$ is the limit function found in part (b), then
$$\lim_{h \to 0} \int_{\mathbb{R}} |G_h(t) - G_0(t)|^2 \, dt = 0.$$ 
(d) Let $f \in L^2(\mathbb{R})$, and define
$$f \ast g(x) = \int_{\mathbb{R}} f(t) g(x - t) \, dt.$$ 
Prove that $f \ast g$ is a continuously differentiable function on $\mathbb{R}$. Be sure to justify all your steps including the existence of the integral defining $f \ast g$, the continuity of this function, and the continuous differentiability of this function.

Problem VII Assume that $0 < \alpha < 2$. Evaluate the definite integral $\int_0^\infty \frac{\log(1 + x^2)}{x^{1+\alpha}} \, dx$. 
(Hint: First integrate by parts).
Problem VIII  Let $L^2(\mathbb{D})$ denote the space of square integrable complex valued measurable functions on the open unit disk $\mathbb{D}$ in the complex plane with norm $||f||_2 = \left( \int \int_{\mathbb{D}} |f(z)|^2 \, dm(z) \right)^{\frac{1}{2}}$, where $dm(z)$ denotes Lebesgue measure on $\mathbb{D}$. Let

$$B^2(\mathbb{D}) = \left\{ f \in L^2(\mathbb{D}) \mid f \text{ is holomorphic on } \mathbb{D} \right\}.$$

(a) Prove that there is a constant $C$ so that for all $f \in B^2(\mathbb{D})$ and all $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{C}{1-|z|} ||f||_2.$$

(b) Show that the inequality in part (a) is sharp in the sense that there does not exist a constant $\alpha$ with $\alpha < 1$ so that

$$|f(z)| \leq \frac{C}{(1-|z|)^{\alpha}} ||f||_2.$$

(c) Let $B_N(z, w) = \frac{1}{\pi} \sum_{j=0}^{N} (j+1)z^j w^j$, and for $f \in L^2(\mathbb{D})$, set

$$B_N[f](z) = \int \int_{\mathbb{D}} B_N(z, w) f(w) \, dm(w).$$

Show that if $f \in L^2(\mathbb{D})$, then the sequence $\{B_N[f]\}$ belongs to $B^2(\mathbb{D})$ and converges both in the norm $|| \cdot ||_2$ and uniformly on compact subsets of $\mathbb{D}$.

(d) If $f \in B^2(\mathbb{D})$, prove that

$$\lim_{N \to \infty} B_N[f] = f.$$

Be sure to indicate in what sense you are considering the limit.

Problem IX

(a) Prove that the series

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2}$$

converges to a meromorphic function on $\mathbb{C}$.

(b) Prove that there is an entire function $f(z)$ so that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2} + f(z).$$

(c) Prove that the function $f$ found in part (b) is identically zero.

(d) Evaluate

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$
QUALIFYING EXAM  

in  

ANALYSIS  

Department of Mathematics  

University of Wisconsin-Madison  

Wednesday, January 17, 2001  

Version for Math 725  

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:  

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.  

(2) \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.  

(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.  

(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.  

(5) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),  

\[
\| f \|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \| f \|_p < +\infty \).  

(6) If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then  

\[
\| f \|_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]

\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \| f \|_\infty < +\infty \).  

(7) \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).  

(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function  

\[
f \ast g(x) = \int_{\mathbb{R}} f(x-t) \, g(t) \, dt
\]

whenever the integral converges.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.
Problem I
(a) Show that \( n^n e^{-n} \leq n! \leq n^n \) for all positive integers \( n \).
(b) Let \( \{c_n\}, n = 1, 2, \ldots \), be a sequence of positive real numbers. Suppose there is a real number \( \alpha \in \mathbb{R} \) and a constant \( C > 0 \) so that for all \( n \geq 1 \)
\[
\frac{c_{n+1}}{c_n} = 1 + \frac{\alpha}{n} + R(n) \quad \text{where} \quad |R(n)| \leq \frac{C}{n^2}.
\]
Show that, depending on \( \alpha \), the sequence \( \{c_n\} \) has a limit which is either zero, positive, or infinite.
(c) Using the results of part (b), show that the sequence \( \left\{ \frac{n!}{n^n e^{-n} \sqrt{n}} \right\}, \ n = 1, 2, \ldots \), has a finite non-zero limit.

Problem II (NOTE: This is an advanced calculus problem. Do not quote theorems from the theory of Lebesgue integration for its solution). For each positive integer \( n \) let \( a^{(n)} \) be an infinite sequence of complex numbers so that
\[
a^{(n)} = \left( a_0^{(n)}, a_1^{(n)}, \ldots, a_j^{(n)}, \ldots \right).
\]
(a) Suppose there is a positive real number \( M \) such that for all \( n \)
\[
\sum_{j=0}^{+\infty} |a_j^{(n)}|^2 \leq M.
\]
Prove that
\[
\sum_{j=0}^{+\infty} \liminf_{n \to \infty} |a_j^{(n)}|^2 \leq M.
\]
Is it also true that
\[
\sum_{j=0}^{+\infty} \limsup_{n \to \infty} |a_j^{(n)}|^2 \leq M?
\]
Either prove this or show that it is false by giving a counter-example.
(b) Assume that for each \( j, \lim_{n \to \infty} a_j^{(n)} = a_j \) exists. If
\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)}|^2 = \sum_{j=0}^{\infty} |a_j|^2,
\]
prove that
\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} |a_j^{(n)} - a_j|^2 = 0.
\]

Problem III For \( x > 0 \), let \( F(x) = \int_0^x \frac{1 - e^{-xt^2}}{t^2} \, dt \).
(a) Show that this improper integral converge.
(b) Show that the function \( F \) is differentiable, and find an explicit formula for \( F'(x) \) in terms of elementary functions.
(c) Use the results of part (b) to find an explicit expression for \( F(x) \) in terms of elementary functions.
Problem IV Let $E \subset \mathbb{R}$ be a proper non-empty measurable set, so that $\emptyset \neq E \neq \mathbb{R}$. Assume that $E$ is invariant under translation by rational numbers. Explicitly, this means that if $x \in E$ and if $r$ is any rational number, then $x + r \in E$. Show that either $E$ has Lebesgue measure zero or that $\mathbb{R} - E$ has Lebesgue measure zero. Give examples to show that both conclusions are possible.

Problem V Let $\{E_j\}_{j \in \mathbb{N}}$ be a countable collection of measurable subsets of $\mathbb{R}^d$.
(a) Show that the set $A$ of points $x \in \mathbb{R}^d$ that belong to all but finitely many of the sets $E_j$ is measurable.
(b) Show that if $\lim_{j \to \infty} |E_j| = 0$, the set $A$ defined in part (a) has measure zero.
(c) Show that the set $B$ of points $x \in \mathbb{R}^d$ that belong to infinitely many of the set $E_j$ is measurable.
(d) If $\lim_{j \to \infty} |E_j| = 0$, must the set $B$ defined in part (c) have measure zero? Either prove that this is true, or show that it is false by giving a counter-example.

Problem VI For $t \in \mathbb{R}$ let $g(t) = (1 + |t|)^{-1}$. Fix $x \in \mathbb{R}$ and, for each non-zero $h \in \mathbb{R}$ set
$$G_h(t) = \frac{g(x + h - t) - g(x - t)}{h}.$$ 
(a) Prove that each $G_h \in L^2(\mathbb{R})$.
(b) Prove that $\lim_{h \to 0} G_h(t)$ exists for almost every $t \in \mathbb{R}$.
(c) Prove that if $G_0$ is the limit function found in part (b), then
$$\lim_{h \to 0} \int_{\mathbb{R}} \left| G_h(t) - G_0(t) \right|^2 \, dt = 0.$$
(d) Let $f \in L^2(\mathbb{R})$, and define
$$f \ast g(x) = \int_{\mathbb{R}} f(t) \, g(x - t) \, dt.$$ 
Prove that $f \ast g$ is a continuously differentiable function on $\mathbb{R}$. Be sure to justify all your steps including the existence of the integral defining $f \ast g$, the continuity of this function, and the continuous differentiability of this function.

Problem VII Let $C_0^\infty(\mathbb{R})$ denote the space of infinitely differentiable complex valued functions with compact support on $\mathbb{R}$.
(a) Show that for any $\varphi \in C_0^\infty(\mathbb{R})$,
$$\lim_{t \to 0} \int_{|t| > \epsilon} \frac{\varphi(t)}{t} \, dt$$
exists. Denote this limit by $T[\varphi]$.
(b) Show that the linear functional $T$ defined in part (a) is a distribution on $\mathbb{R}$. In particular, check carefully that all the hypotheses in the definition of a distribution are satisfied.
(c) If $S$ is a distribution on $\mathbb{R}$, define carefully what is meant by the support of $S$.
(d) Find two distributions $T_1$ and $T_2$ on $\mathbb{R}$ whose supports are, respectively, $(-\infty, 0]$ and $[0, +\infty)$, such that if $T$ is the distribution defined in part (a), then $T = T_1 + T_2$. 
Problem VIII  
In this problem $L^2$ stands for $L^2(0, 1)$ and its norm is denoted simply by $|| \cdot ||$; also $W^{1, 2} = \{ f \in L^2 \mid f' \in L^2 \}$, with norm $||f||_W = (||f||^2 + ||f'||^2)^{1/2}$; finally $C^1([0, 1])$, with norm $||f||_1 = \text{Sup} |f| + \text{Sup}|f'|$.

(a) Let $B$ be the closed unit ball of $W^{1, 2}$; show that it is a compact subset of $L^2$.

(b) Let $B'$ be the closed unit ball of $C^1$; show that it is a relatively compact subset of $L^2$. Is it a compact subset of $L^2$?

(c) Let $E$ be a closed subspace of $L^2$, such that $E \cap W^{1, 2} = \{0\}$ (for example the linear span of a function in $L^2$ and not in $W^{1, 2}$). Let $\varphi$ be a continuous linear form on $E$, continuous with respect to the $L^2$ norm. Show that for every $\epsilon > 0$, there exists a continuous linear form $\tilde{\varphi}$ on $L^2$, whose restriction to $E$ is $\varphi$, and such that

$$\sup_{f \in B} |\tilde{\varphi}(f)| \leq \epsilon$$

with $B$ as in part (a). Can one take $\epsilon = 0$?

You can use the following geometric form of the Hahn-Banach Theorem: If $K$ is a convex compact subset of a normed vector space $E$, and $L$ is a closed affine subspace of $E$ that does not intersect $K$, there exists a close affine hyperplane containing $L$ and still not intersecting $K$.

---

Problem IX  
Let $H$ be a separable Hilbert space with norm $|| \cdot ||_H$ and inner product $(\cdot, \cdot)_H$. Let $\{\varphi_n\}$, $n = 1, 2, \ldots$, be a complete orthonormal basis for $H$. Let $0 \leq \delta < 1$ and let $\{f_n\}$ be a sequence of elements of $H$ such that for every finite set of complex numbers $\{a_n\}$ we have

$$\left|\sum a_n (\varphi_n - f_n)\right|_H^2 \leq \delta^2 \sum |a_n|^2.$$

(a) Prove that the series $K[x] = \sum_{n=1}^{\infty} (x, \varphi_n)_H (\varphi_n - f_n)$ converges in norm for every $x \in H$.

(b) Prove that $K$ defines a bounded linear transformation from $H$ to $H$, and show that if $K^*$ is the adjoint operator, then $||K^*|| \leq \delta$.

(c) Prove that for each $n \geq 1$, $(I - K)[\varphi_n] = f_n$, and there exists a unique element $g_n \in H$ such that $(I - K^*)[g_n] = \varphi_n$. Hint: Prove that if an operator $T$ has operator norm less than 1, then $I - T$ is invertible.

(d) Prove that for $m, n \geq 1$, $(f_n, g_m)_H = \begin{cases} 1 & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$

(e) Prove that for every $x \in H$, $x = \sum_{n=1}^{\infty} (x, g_n)_H f_n$ where the series converges in the norm in $H$. 
QUALIFYING EXAM
in
ANALYSIS
Department of Mathematics
University of Wisconsin-Madison
Tuesday January 10, 2006
Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
2. \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
3. For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
4. If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
5. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
\[ \|f\|_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p} \]
Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
6. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then
\[ \|f\|_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \} \]
\( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_{\infty} < +\infty \).
7. \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).
8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function
\[ f \ast g(x) = \int_{\mathbb{R}} f(x-t) \, g(t) \, dt \]
whenever the integral converges.
9. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I
(a) Determine all values of $\alpha, \beta$ such that the (possibly) improper integral
$$\int_0^1 x^\alpha \sin(x^\beta) \, dx$$
converges.
(b) Determine all values of $\alpha, \beta$ such that the improper integral
$$\int_0^1 x^\alpha |\sin(x^\beta)| \, dx$$
converges.

Problem II
(a) Suppose $f \in C^2((-\epsilon, \epsilon))$. Define
$$F(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y}, & x \neq y, \\ f'(x), & x = y. \end{cases}$$
Show that $F \in C^1((-\epsilon, \epsilon) \times (-\epsilon, \epsilon))$.
(b) Suppose $f \in C^2((-\epsilon, \epsilon))$ and $f(0) = f'(0) = 0$ and $f''(0) = 1$. Show that there exist $\delta, \eta > 0$ and $\varphi \in C^1((-\delta, \delta))$ such that $|\varphi(x)| < \eta$ for $|x| < \delta$ and $\varphi'(0) = -1$ and $f(\varphi(x)) = f(x)$.
(c) Show that the above $\varphi$ satisfies $\varphi(\varphi(x)) = x$ for $|x| < \eta'$ and for some $\eta' \in (0, \eta]$.

Problem III
Let $a_k$ be a sequence of non-negative numbers satisfying $0 < \sum_{k=1}^{\infty} a_k < \infty$. Show that
$$\lim_{x \to +\infty} \sum_{k=1}^{\infty} a_k \sin \frac{x}{k}$$
does not exist.

Problem IV
Show that if $f \in L^1(\mathbb{R})$ then
$$\lim_{\lambda \in \mathbb{R}, \lambda \to \infty} \int_{\mathbb{R}} f(x) e^{-i\lambda x} \, dx = 0.$$

Problem V
Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $1 \leq p \leq q \leq r \leq \infty$, and let $L^p = L^p(\Omega)$, $L^q = L^q(\Omega)$, and $L^r = L^r(\Omega)$.
(a) Show that
$$L^p \cap L^r \subset L^q \subset L^p + L^r.$$ By definition, $L^p + L^r = \{g + h : g \in L^p, h \in L^r\}$.
(b) Define canonical norms on $L^p \cap L^r$, $L^p + L^r$ by
$$\|f\|_{L^p \cap L^r} = \max\{\|f\|_{L^p}, \|f\|_{L^r}\},$$
$$\|f\|_{L^p + L^r} = \inf\{\|g\|_{L^p} + \|h\|_{L^r} : f = g + h, g \in L^p, h \in L^r\}.$$ (You don’t need to verify the above two are norms.)
Prove that two inclusions in (a) are continuous maps.
Problem VI  Assume \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}^+), \) i.e. \( a_1^2 + a_2^2 + \cdots < \infty \) and \( b_1^2 + b_2^2 + \cdots < \infty. \)

(a) Using the Cauchy-Schwarz inequality show that
\[
\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_n b_m}{m}
\]
is convergent.

(b) Show that
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n + m}
\]
is convergent.

(c) If instead of assuming \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}^+) \) we assume \( a_n \in l^p(\mathbb{Z}^+) \) and \( b_n \in l^{p'}(\mathbb{Z}^+) \), with \( p, p' \in (1, \infty), 1/p + 1/p' = 1 \), prove that (b) still holds.

Problem VII  Suppose that \( f \) is holomorphic in \( \{z = x + iy : y > 0\} \) and that \( \lim_{z \to 0} f(z) = L \) exists (and is finite). Let \( S = \{z = x + iy : y > |x|\} \). Show that
\[
\lim_{S \ni z \to 0} zf'(z) = 0.
\]

Problem VIII

(a) Show that
\[
\frac{\pi^2}{\sin^2(\pi x)} = \sum_{n=\infty}^{\infty} \frac{1}{(z-n)^2}.
\]

(b) Evaluate
\[
\sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

Problem IX  Show that there is no entire function \( f(z) \) satisfying
\[
|f(z) - e^z| \leq 3|z|, \quad z \in \mathbb{C}.
\]
QUALIFYING EXAM
in
ANALYSIS
Department of Mathematics
University of Wisconsin-Madison
Tuesday January 10, 2006
Versions for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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5. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < \infty \),
   \[ \|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}. \]
   Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < \infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < \infty \).
6. If \( \mu \) is a positive measure on a set \( X \), and \( f \) is a complex valued measurable function on \( X \), then
   \[ \|f\|_{\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}. \]
   \( L^\infty(X) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_{\infty} < \infty \).
7. \( L^p_{\text{loc}}(\mathbb{R}) \) is the space of measurable, complex valued functions on \( \mathbb{R} \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R} \).
8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R} \), the convolution \( f \ast g \) is defined to be the function
   \[ f \ast g(x) = \int_{\mathbb{R}} f(x - t) g(t) \, dt \]
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9. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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(b) Determine all values of $\alpha, \beta$ such that the improper integral
\[ \int_0^1 x^\alpha |\sin(x^\beta)| \, dx \]
converges.

Problem II
(a) Suppose $f \in C^2((-\epsilon, \epsilon))$. Define
\[ F(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y}, & x \neq y, \\ f'(x), & x = y. \end{cases} \]
Show that $F \in C^1((-\epsilon, \epsilon) \times (-\epsilon, \epsilon))$.
(b) Suppose $f \in C^2((-\epsilon, \epsilon))$ and $f(0) = f'(0) = 0$ and $f''(0) = 1$. Show that there exist $\delta, \eta > 0$ and $\varphi \in C^1((-\delta, \delta))$ such that $|\varphi(x)| < \eta$ for $|x| < \delta$ and $\varphi'(0) = -1$ and $f(\varphi(x)) = f(x)$.
(c) Show that the above $\varphi$ satisfies $\varphi(\varphi(x)) = x$ for $|x| < \eta'$ and for some $\eta' \in (0, \eta]$.

Problem III Let $a_k$ be a sequence of non-negative numbers satisfying $0 < \sum_{k=1}^\infty a_k < \infty$. Show that
\[ \lim_{x \to +\infty} \sum_{k=1}^\infty a_k \sin \frac{x}{k} \]
does not exist.

Problem IV Show that if $f \in L^1(\mathbb{R})$ then
\[ \lim_{\lambda \to +\infty} \int_{\mathbb{R}} f(x)e^{-\lambda x} \, dx = 0. \]

Problem V Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $1 \leq p \leq q \leq r \leq \infty$, and let $L^p = L^p(\Omega)$, $L^q = L^q(\Omega)$, and $L^r = L^r(\Omega)$.

(a) Show that
\[ L^p \cap L^r \subset L^q \subset L^p + L^r. \]
By definition, $L^p + L^r = \{g + h : g \in L^p, h \in L^r\}$.
(b) Define canonical norms on $L^p \cap L^r$, $L^p + L^r$ by
\[ \|f\|_{L^p \cap L^r} = \max\{\|f\|_{L^p}, \|f\|_{L^r}\}, \]
\[ \|f\|_{L^p + L^r} = \inf\{\|g\|_{L^p} + \|h\|_{L^r} : f = g + h, g \in L^p, h \in L^r\}. \]
(You don't need to verify the above two are norms.)
Prove that two inclusions in (a) are continuous maps.
Problem VI  Assume \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}^+) \), i.e. \( a_1^2 + a_2^2 + \cdots < \infty \) and \( b_1^2 + b_2^2 + \cdots < \infty \).

(a) Using the Cauchy-Schwarz inequality show that
\[
\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_n b_m}{m}
\]
is convergent.

(b) Show that
\[
\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_n b_m}{n + m}
\]
is convergent.

(c) If instead of assuming \( \{a_n\}, \{b_n\} \in l^2(\mathbb{Z}^+) \) we assume \( a_n \in l^p(\mathbb{Z}^+) \) and \( b_n \in l^{p'}(\mathbb{Z}^+) \), with \( p, p' \in (1, \infty) \), \( 1/p + 1/p' = 1 \), prove that (b) still holds.

Problem VII  Find all functions \( f \in L^1(\mathbb{R}^d) \) with the property that \( f * f = f \). Find all functions \( f \in L^2(\mathbb{R}^d) \) with the property that \( f * f = f \).

Problem VIII  One says that a distribution \( T \) on \( \mathbb{R} \) has order \( s \) at \( p \in \mathbb{R} \), if \( s \) is the smallest non-negative integer such that there is a neighborhood \( U \) of \( p \) such that
\[
|<T, \varphi>| \leq C \max_{x \in U, k \leq s} |\varphi^{(k)}(x)|
\]
for some constant \( C > 0 \) and for all smooth functions \( \varphi \) whose supports are contained in \( U \).

Let \( 0 < b_{n+1} < a_n < b_n \) for \( n = 1, 2, 3, \ldots \). Let \( \chi_{[a_n, b_n]} \) be the characteristic function of \( [a_n, b_n] \) and
\[
f = \sum_{n=1}^{\infty} c_n \chi_{[a_n, b_n]}, \quad c_n \in \mathbb{R}.
\]
Assume that \( f \in L^1(-\infty, \infty) \).

(a) Prove that the distribution derivative \( f' \) has order 0 at 0 if \( \sum_{n=1}^{\infty} |c_n| < \infty \).

(b) Prove that \( f' \) has order 1 at 0 if \( \sum_{n=1}^{\infty} |c_n| = \infty \).

Problem IX  Let \( X \) be a Banach space. Assume that \( \{v_1, v_2, \ldots\} \) is a dense subset of the unit ball of \( X \). Define the map \( A : L^1(\mathbb{N}) \to X \) by
\[
A((\alpha_1, \alpha_2, \ldots)) = \sum_{n=1}^{\infty} \alpha_n v_n.
\]

(a) Show that the map \( A \) is well defined, linear, and continuous.

(b) Show that \( A(L^1(\mathbb{N})) = X \).
Instructions: Do six of the nine problems. To receive credit on a problem, you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).

1. Let $X$ be a metric space with metric $d$.
   (i) Define $\rho : X \times X \to \mathbb{R}$ by
   $$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$ 
   Prove that $\rho$ is a metric on $X$.
   (ii) Show that a subset $U$ of $X$ is open with respect to the metric $d$ if and only if it is open with respect to the metric $\rho$.

2. Let $u : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function and let $\Delta u = \partial^2_x u + \partial^2_y u + \partial^2_z u$ be the Laplacian of $u$.
   Suppose that $\Delta u = 1$ on $\mathbb{R}^3$ and $u(x, y, z) = x^3 y^3$ on the sphere of radius $R$ centered at the origin. Find $u(0, 0, 0)$.

3. Let $I$ be a compact subset of $(0, 2\pi)$. Show that the series
   $$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$
   converges uniformly on $I$.

4. Let $F$ be a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta_F(x)$ denote the distance of $x$ to $F$, i.e. $\delta_F(x) = \inf\{|x - y| : y \in F\}$.
   (i) Prove that $\delta_F$ is Lipschitz continuous, in fact
   $$|\delta_F(x) - \delta_F(y)| \leq |x - y|.$$
   (ii) Let
   $$M(x) = \int \frac{\delta_F(y)}{|x - y|^2} dy.$$ 
   Show that $M(x) < \infty$ for almost every $x \in F$.
   **Hint:** For part (ii) consider the integral $\int_F M(x) dx$. 


5. On the interval $[-1,1]$ consider the standard Banach spaces $L^1$ and $L^2$ with the norms
\[ \|f\|_{L^1} = \int_{-1}^{1} |f(x)| \, dx; \quad \|f\|_{L^2} = \left( \int_{-1}^{1} |f(x)|^2 \, dx \right)^{1/2}. \]
Let \( \{f_j\}_{j=1}^{\infty} \) denote a sequence of functions in $L^2$. Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and
\[ |\|f_j\|_{L^2} - \sqrt{2}| \leq 2^{-j}. \]
Show that $\lim_{j \to \infty} f_j(x) = 1$ for almost every $x \in [-1,1]$.

Hint: Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that $f_n$ converges weakly to $f \in L^2$ if
\[ \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) g(x) \, dx = \int_{\mathbb{R}} f(x) g(x) \, dx \quad \text{for all} \quad g \in L^2(\mathbb{R}). \]
Find a sequence of bounded, (Borel) measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

7. For $z \in \mathbb{C}$ evaluate
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \log |e^{i\theta} - z| \, d\theta. \]

Suggestion: Treat the easier case $|z| > 1$ first.

8. Let $S = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, -1 < y < 1\}$, and let $f : S \to \mathbb{C}$ be a holomorphic function which satisfies the inequality
\[ |f(z)| \leq 1 + |z|^2 \quad \text{for all} \quad z \in S. \]
Show that for any $n = 0, 1, \ldots$ there is a constant $C_n$ such that
\[ |f^{(n)}(x)| \leq C_n (1 + |x|^2) \quad \text{for all} \quad x \in \mathbb{R}. \]
What can you say about the constant $C_n$?

9. Let $E$ denote a compact subset of $\mathbb{R}$ of measure 0 (here measure refers to Lebesgue measure on the real line). Let $f : \mathbb{C} \setminus E \to \mathbb{C}$ be a holomorphic function. Show that if $f$ is bounded on any bounded subset of $\mathbb{C} \setminus E$, then $f$ extends to a holomorphic function on $\mathbb{C}$. 

**Qualifying Exam in Analysis**  
**Real Analysis (Math 721-725) Version**  
**Wednesday, January 17, 2007**

**Instructions:** Do six of the nine problems. To receive credit on a problem, you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).

1. Let $X$ be a metric space with metric $d$.
   (i) Define $\rho : X \times X \to \mathbb{R}$ by
   \[ \rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}. \]
   Prove that $\rho$ is a metric on $X$.
   (ii) Show that a subset $U$ of $X$ is open with respect to the metric $d$ if and only if it is open with respect to the metric $\rho$.

2. Let $u : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function and let $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ be the Laplacian of $u$.
   Suppose that $\Delta u = 1$ on $\mathbb{R}^3$ and $u(x, y, z) = x^3y^3$ on the sphere of radius $R$ centered at the origin. Find $u(0, 0, 0)$.

3. Let $I$ be a compact subset of $(0, 2\pi)$. Show that the series
   \[ \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \]
   converges uniformly on $I$.

4. Let $F$ be a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta_F(x)$ denote the distance of $x$ to $F$, i.e. $\delta_F(x) = \inf\{|x - y| : y \in F\}$.
   (i) Prove that $\delta_F$ is Lipschitz continuous, in fact
   \[ |\delta_F(x) - \delta_F(y)| \leq |x - y|. \]
   (ii) Let
   \[ M(x) = \int \frac{\delta_F(y)}{|x - y|^2} \, dy. \]
   Show that $M(x) < \infty$ for almost every $x \in F$.
   **Hint:** For part (ii) consider the integral $\int_{F} M(x) \, dx$. 
5. On the interval $[-1, 1]$ consider the standard Banach spaces $L^1$ and $L^2$ with the norms
\[
\|f\|_{L^1} = \int_{-1}^{1} |f(x)| \, dx; \quad \|f\|_{L^2} = \left( \int_{-1}^{1} |f(x)|^2 \, dx \right)^{1/2}.
\]
Let $\{f_j\}_{j=1}^{\infty}$ denote a sequence of functions in $L^2$. Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and
\[
\left| \|f_j\|_{L^2} - \sqrt{2} \right| \leq 2^{-j}.
\]
Show that $\lim_{j \to \infty} f_j(x) = 1$ for almost every $x \in [-1, 1]$.

*Hint:* Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that $f_n$ converges weakly to $f \in L^2$ if
\[
\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)g(x) \, dx = \int_{\mathbb{R}} f(x)g(x) \, dx \quad \text{for all } g \in L^2(\mathbb{R}).
\]
Find a sequence of bounded, (Borel) measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

7. Let $g : \mathbb{R}^2 \to \mathbb{R}$,
\[
g(x, y) = \begin{cases} 
x^2 + y^2 & \text{if } x^2 + y^2 \leq 1; \\
1 & \text{if } x^2 + y^2 \geq 1.
\end{cases}
\]
Find the distribution $(\partial^2_x + \partial^2_y)g$.

8. For any $m \in \{1, 2, \ldots\}$ let $f_m(x) = |x|^{-m}$ on $\mathbb{R} \setminus \{0\}$.
Find a distribution $T_m$ on $\mathbb{R}$ which agrees with $f_m$ on $\mathbb{R} \setminus \{0\}$, i.e.
\[
T_m(\phi) = \int_{\mathbb{R}} f_m(x)\phi(x) \, dx \text{ for any } \phi \in C_0^\infty(\mathbb{R} \setminus \{0\}).
\]
Give complete justifications (in particular show that your choice of $T_m$ really defines a distribution).

9. Let $H$ be an infinite dimensional Hilbert space.
   (i) Prove that there is no relatively compact neighborhood of the origin.
   (ii) Let $T : H \to H$ be linear, bounded and surjective, and let $B$ be the closed unit ball centered at the origin. Show that $T(B)$ is not compact.
Instructions: Do six of the nine problems. To receive credit on a problem, you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).
1. Let \( f \) be a continuous function on \( \mathbb{R} \) and let for \( n = 1, 2, \ldots \),
\[
F_n(x) = \int_0^x (x - t)^{n-1} f(t) dt.
\]
Prove that \( F_n \) is \( n \) times differentiable, and prove a simple formula for its \( n \)-th derivative.

2. Given a sequence \( \{c_n\}_{n=0}^\infty \) of complex numbers, we let \( s_n = \sum_{k=0}^n c_k \) denote the partial sums and \( \sigma_N = \frac{s_0 + \ldots + s_N}{N+1} \) their arithmetic means. We say that the series \( \sum_{n=0}^\infty c_n \) is Cesàro summable to \( \sigma \) if \( \lim_{N \to \infty} \sigma_N = \sigma \).

Show that if \( \sum_{n=0}^\infty c_n \) is Cesàro summable to \( \sigma \) and \( \lim_{n \to \infty} nc_n = 0 \) then the series \( \sum_{n=0}^\infty c_n \) converges and \( \sum_{n=0}^\infty c_n = \sigma \). \textit{Hint:} First write \( \sigma_N - s_N = \sum \alpha_k c_k \) with suitable coefficients \( \alpha_k \).

3. Recall the following definition: a function \( g : (0, 1) \to \mathbb{R} \) has \textit{bounded variation} if
\[
\sup_{N \geq 2} \sup_{0 < x_N < \ldots < x_1 < 1} |g(x_1) - g(x_2)| + \ldots + |g(x_{N-1}) - g(x_N)| < \infty,
\]
where the second sup is taken over all strictly decreasing sequences \( x_N < \ldots < x_1 \) with \( x_i \in (0, 1) \).

Find the exponents \( p \) for which the function \( f : (0, 1) \to \mathbb{R} \),
\[
f(x) = x^p \sin(1/x)
\]
has bounded variation.

4. Consider the inequality
\[
\left| \int_\mathbb{R} \prod_{i=1}^n f_i(x) dx \right| \leq \prod_{i=1}^n \left( \int_\mathbb{R} |f_i(x)|^{p_i} dx \right)^{1/p_i} \quad (\ast)
\]
for measurable functions on \( \mathbb{R} \) and \( p_i \in (1, \infty), i = 1, \ldots, n \).

(a) Assume that (\ast) holds for all measurable \( f_1, \ldots, f_n \). Prove that necessarily \( \sum_{i=1}^n 1/p_i = 1 \).

(b) Conversely, show that if \( \sum_{i=1}^n 1/p_i = 1 \) holds then (\ast) holds for all measurable \( f_1, \ldots, f_n \). \textit{(Note: The familiar case \( n = 2 \) can be assumed).}

\textit{Continued on next page.}
5. (i) Prove the identity
\[
\{ y : y \in E_k \text{ for infinitely many } k \} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k.
\]
(ii) Let \( A \) be the set of points \( x \in (0, 1) \) with the property that there are infinitely many fractions \( p/q \) with integers \( p, q \) such that
\[|x - p/q| < 1/q^3.\]
Show that \( A \) is a set of measure 0.

6. Assume that \( I = [a, b] \) is a compact interval and \( f \in L^2(I) \) (with the usual Lebesgue measure). Show that if
\[
\int_{[a,b]} f(x)x^n dx = 0 \text{ for } n = 0, 1, 2, \ldots,
\]
then \( f(x) = 0 \) almost everywhere in \( I \).

7C. Evaluate the following improper integrals which involve a parameter \( a > 0 \).
(a) \[
\int_{-\infty}^{+\infty} \frac{x \sin(x)}{x^2 + a^2} dx;
\]
(b) \[
\int_{0}^{\infty} \frac{1}{x^2 + a^2} \sqrt{x} dx.
\]

8C. For each of the following, either construct a holomorphic function in the unit disk \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) with the stated properties, or show that no such function exists.
(a) For every integer \( n \geq 2 \) we have \( f \left( \frac{1}{n} \right) = \frac{1}{n-1} \) and \( f \left( \frac{i}{n} \right) = \frac{1}{in-1} \).
(b) For every integer \( n \geq 1 \) we have \( |f^{(n)}(0)| \geq (n/3)^n \).
(c) The function \( f \) extends to a continuous function on the closure of \( D \), \( |f(e^{i\theta})| = 1 \) for \( 0 \leq \theta \leq 2\pi \), and \( f \left( \frac{i}{2} \right) = f \left( \frac{1}{2} \right) = 0 \).

9C. Let \( \Omega \subset \mathbb{C} \) be an open set containing the point 0. Suppose that \( f : \Omega \to \Omega \) is a holomorphic mapping, with \( f(0) = 0 \) and \( f'(0) = 1 \). Suppose that \( f \) has the Taylor expansion \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) at 0. Define the iterates of \( f_k : \Omega \to \Omega \) by setting \( f_1(z) = f(z) \), \( f_2(z) = f(f_1(z)) \), and then by induction, \( f_k(z) = f(f_{k-1}(z)) \).
(a) Show that if \( f(z) = z + a_n z^n + O(z^{n+1}) \), then \( f_k(z) = z + k a_n z^n + O(z^{n+1}) \).
(b) Show that if \( \Omega \) is bounded and connected, then \( f(z) \equiv z \) for all \( z \in \Omega \).
Instructions: Do six of the nine problems. To receive credit on a problem, you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).
1. Let \( f \) be a continuous function on \( \mathbb{R} \) and let for \( n = 1, 2, \ldots \)

\[ F_n(x) = \int_0^x (x-t)^{n-1} f(t) \, dt. \]

Prove that \( F_n \) is \( n \) times differentiable, and prove a simple formula for its \( n \)-th derivative.

2. Given a sequence \( \{c_n\}_{n=0}^{\infty} \) of complex numbers, we let \( s_n = \sum_{k=0}^{n} c_k \) denote the partial sums and \( \sigma_N = \frac{s_0 + \cdots + s_N}{N+1} \) their arithmetic means. We say that the series \( \sum_{n=0}^{\infty} c_n \) is Cesáro summable to \( \sigma \) if \( \lim_{N \to \infty} \sigma_N = \sigma \).

Show that if \( \sum_{n=0}^{\infty} c_n \) is Cesáro summable to \( \sigma \) and \( \lim_{n \to \infty} nc_n = 0 \) then the series \( \sum_{n=0}^{\infty} c_n \) converges and \( \sum_{n=0}^{\infty} c_n = \sigma \).

Hint: First write \( \sigma_N - s_N = \sum \alpha_k c_k \) with suitable coefficients \( \alpha_k \).

3. Recall the following definition: a function \( g : (0,1) \to \mathbb{R} \) has bounded variation if

\[
\sup_{N \geq 2} \sup_{0 < x_N < \cdots < x_1 < 1} |g(x_1) - g(x_2)| + \cdots + |g(x_{N-1}) - g(x_N)| < \infty,
\]

where the second supremum is taken over all strictly decreasing sequences \( x_N < \cdots < x_1 \) with \( x_i \in (0,1) \).

Find the exponents \( p \) for which the function \( f : (0,1) \to \mathbb{R} \),

\[ f(x) = x^p \sin(1/x) \]

has bounded variation.

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\left| \int_{\mathbb{R}} \prod_{i=1}^{n} f_i(x) \, dx \right| \leq \prod_{i=1}^{n} \left( \int_{\mathbb{R}} |f_i(x)|^{p_i} \, dx \right)^{1/p_i} \quad (\ast)
\]

for measurable functions on \( \mathbb{R} \) and \( p_i \in (1, \infty) \), \( i = 1, \ldots, n \).

(a) Assume that \( (\ast) \) holds for all measurable \( f_1, \ldots, f_n \). Prove that necessarily \( \sum_{i=1}^{n} 1/p_i = 1 \).

(b) Conversely, show that if \( \sum_{i=1}^{n} 1/p_i = 1 \) holds then \( (\ast) \) holds for all measurable \( f_1, \ldots, f_n \). (Note: The familiar case \( n = 2 \) can be assumed).

Continued on next page.
5. (i) Prove the identity

\[ \{ y : y \in E_k \text{ for infinitely many } k \} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k. \]

(ii) Let \( A \) be the set of points \( x \in (0,1) \) with the property that there are infinitely many fractions \( p/q \) with integers \( p, q \) such that

\[ |x - p/q| < 1/q^3. \]

Show that \( A \) is a set of measure 0.

6. Assume that \( I = [a, b] \) is a compact interval and \( f \in L^2(I) \) (with the usual Lebesgue measure). Show that if

\[ \int_{[a,b]} f(x)x^n dx = 0 \text{ for } n = 0, 1, 2, \ldots, \]

then \( f(x) = 0 \) almost everywhere in \( I \).

7R. Determine all tempered distributions \( T \in \mathcal{S}'(\mathbb{R}^2) \) with the property that

\[ x^2T = y^2T \equiv 0 \text{ (this means } T(x^2\phi) = T(y^2\phi) = 0 \text{ for any } \phi \in \mathcal{S}(\mathbb{R}^2)). \]

8R. (i) Let \( X \) be a metric space. Give the definition of a nowhere dense set in \( X \) and state the Baire category theorem.

(ii) Let \( \{ f_n \}_{n=1}^{\infty} \) be a sequence of continuous functions on \( X = [0,1] \) and assume that \( \sup_n |f_n(x)| < \infty \) for every \( x \in [0,1] \). Show that there exists an interval \( (a, b) \subset [0,1] \) and an \( M \in \mathbb{R} \) so that \( |f_n(x)| \leq M \) for all \( x \in (a, b) \) and all \( n = 1, 2, \ldots \).

9R. Assume that \( H \) is a Hilbert space and \( T \in \mathcal{L}(H) \) is a bounded linear operator on \( H \). Assume that \( T \) is self-adjoint, i.e. \( \langle Tx, y \rangle = \langle x, Ty \rangle \) for any \( x, y \in H \).

(a) Show that for any \( x, y \in H \)

\[ \text{Re} \langle x, Ty \rangle = \frac{1}{4} [\langle x + y, T(x + y) \rangle - \langle x - y, T(x - y) \rangle]. \]

(b) Use part (a) and the parallelogram identity to prove that

\[ \|T\|_{\mathcal{L}(H)} = \sup_{\|x\|=1} |\langle x, Tx \rangle|. \]
QUALIFYING EXAM

in
ANALYSIS
Department of Mathematics
University of Wisconsin-Madison
Wednesday January 16, 2008
Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \(\mathbb{R}\) and \(\mathbb{C}\) denote the fields of real and complex numbers respectively.
(2) \(D = \{z \in \mathbb{C} : |z| < 1\}\) denotes the unit disc in the complex plane.
(3) For points \(x\) and \(y\) in \(\mathbb{R}^n\), \(|x - y|\) denotes the Euclidean distance between the points.
(4) If \(E \subseteq \mathbb{R}^n\) is a Lebesgue measurable set, then \(|E|\) denotes its Lebesgue measure.
(5) If \(\mu\) is a positive measure on a set \(X\), and if \(f\) is a complex valued measurable function on \(X\), then for \(1 \leq p < +\infty\),

\[\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}.\]

Two functions on \(X\) are said to be equivalent if they are equal except on a set of \(\mu\) measure zero. For \(1 \leq p < +\infty\), \(L^p(X) = L^p(X, d\mu)\) is the space of equivalence classes of complex valued measurable functions such that \(\|f\|_p < +\infty\).

(6) If \(\mu\) is a positive measure on a set \(X\), and if \(f\) is a complex valued measurable function on \(X\), then

\[\|f\|_\infty = \inf\{t > 0 : \mu\{x \in X : |f(x)| > t\} = 0\}.\]

\(L^\infty(X) = L^\infty(X, d\mu)\) is the space of equivalence classes of measurable, complex valued functions on \(X\) such that \(\|f\|_\infty < +\infty\).

(7) \(L^p(\mathbb{R}^n)\) and \(L^\infty(\mathbb{R}^n)\) denote the spaces of equivalence classes of functions as defined in (5) and (6) where the measure \(d\mu\) is Lebesgue measure.

(8) \(L^p_{loc}(\mathbb{R}^n)\) is the space of equivalence classes of measurable, complex valued functions on \(\mathbb{R}^n\) which belong to \(L^p(K)\) for every compact set \(K \subseteq \mathbb{R}^n\).

(9) If \(f\) and \(g\) are measurable functions on \(\mathbb{R}^n\), the convolution \(f \ast g\) is defined to be the function

\[f \ast g(x) = \int_{\mathbb{R}^n} f(x-t) g(t) \, dt\]

whenever the integral converges. Here \(dt\) denotes Lebesgue measure.

(10) If \(T\) is a distribution and \(\varphi\) is a test function, then \(\langle T, \varphi \rangle\) denotes the value of the distribution applied to the test function.

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Problem I  Suppose that the function $f : [0, 1] \to \mathbb{R}$ satisfies the following properties:
- $f \in C^1([0, 1]);$
- $f(0) = f(1) = 0;$
- $f(x) \leq 1$ for all $x \in [0, 1];$
- $f$ is concave, i.e. $f'$ is nonincreasing on $[0, 1].$

(a) Prove that the arc length of graph of $f$ does not exceed 3.

(b) Does there exist a constant $\mu < 3$ such that the arc length of graph of $f$ is less than $\mu$ for all above $f$?

Hint: Use the inequality $\sqrt{a^2 + b^2} \leq |a| + |b|.$

Problem II  Let $\{c_n\}$ be a sequence of complex numbers satisfying
\[ |c_{n+1}| \leq 2|c_n| \]
for $n \geq 0.$ Let $f_0$ be a continuously differentiable function on $[0, 1],$ and for $n \geq 1,$ define continuously differentiable functions $\{f_n\}$ on $[0, 1]$ by setting
\[ f_n(x) = c_{n-1}f_{n-1}^2(x). \]
Show that there exists an $\epsilon > 0$ such that if $|f_0(x)| < \epsilon$ for all $x \in [0, 1],$ then the infinite series
\[ \sum_{n=0}^{\infty} f_n(x), \quad 0 \leq x \leq 1 \]
converges to a continuously differentiable function $f$ on $[0, 1].$

Hint: Show that $|c_n| \leq 2^n|c_0|$ and $|f_n(x)| \leq \frac{1}{4}|f_{n-1}(x)|$ for suitable $c.$

Problem III  Let $I(\lambda) = \int_0^1 e^{-\lambda(x^5-x^6)} \, dx.$ Show that there are positive constants $C_1$ and $C_2$ so that for $\lambda \geq 10,$
\[ I(\lambda) = C_1 \lambda^{-\frac{1}{2}} + E(\lambda) \]
where $|E(\lambda)| \leq C_2 \lambda^{-\frac{2}{7}}.$

Problem IV  Let $A$ and $B$ be real numbers. Show that there is a constant $C$ independent of $A,$ $B,$ and $N$ so that
\[ \left| \int_{-N}^{+N} \left[ e^{i(Ax+Bx^2)} - 1 \right] \frac{dx}{x} \right| \leq C. \]

Problem V  Let $\mu$ be a positive measure on $X$ with $\mu(X) < \infty.$

(a) Let $f \in L^\infty(X, d\mu).$ Prove that $f \in L^p(X, d\mu)$ for $1 \leq p < +\infty$ and that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$

(b) Suppose that $f \in L^p(X, d\mu)$ for $1 \leq p < +\infty.$ Suppose also that there is a constant $L > 0$ such that $\|f\|_p \leq L$ for all $1 \leq p < +\infty.$ Prove that $f \in L^\infty(X, d\mu)$ and $\|f\|_\infty \leq L.$

Problem VI  Let $f_n \in L^\infty(\mathbb{R})$ for $n = 1, 2, \ldots,$ and suppose that $\lim_{n \to \infty} \|f_n\|_{L^\infty(\mathbb{R})} = \infty.$ Show that there exists $g \in L^1(\mathbb{R})$ such that the sequence $\{\|f_ng\|_{L^1(\mathbb{R})}\}$ is not bounded.
Problem VII
(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function which is one-to-one. Prove that there exist complex numbers $a$ and $b$ with $a \neq 0$ such that $f(z) = az + b$.
(b) Let $\mathbb{C}^* = \{ \zeta \in \mathbb{C} \mid \zeta \neq 0 \}$. If $g : \mathbb{C}^* \to \mathbb{C}^*$ is holomorphic and one-to-one, what can you say about $g$?

Problem VIII
Prove that the improper integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$ exist, and that they both equal $\frac{\sqrt{2\pi}}{4}$.

Problem IX
For each of the following, either construct a holomorphic function $f$ in the unit disk $\mathbb{D} = \{ \zeta \in \mathbb{C} \mid |\zeta| \leq 1 \}$ with the stated properties, or show that no such function exists.
(a) For every sequence $\{a_n\} \subset \mathbb{D}$ with $\lim_{n \to \infty} |a_n| = 1$, it follows that $\lim_{n \to \infty} |f(z_n)| = +\infty$.
(b) $|f''(0)| = 2$, $|f(z)| \leq 1$ for all $z$ such that $|z| = \frac{1}{2}$, and $|f\left(\frac{3}{4}\right)| = \frac{5}{3}$.
(c) $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, and $f\left(1 - \frac{1}{n^2}\right) = 0$ for $n = 1, 2, 3, \ldots$, and $f$ is not identically zero.
Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
2. \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) denotes the unit disc in the complex plane.
3. For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.
4. If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
5. If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
   \[ \|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p} . \]
   Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
6. If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then
   \[ \|f\|_{L^\infty} = \inf \{ t > 0 \mid \mu(\{ x \in X : |f(x)| > t \}) = 0 \} . \]
   \( L^\infty(X) = L^\infty(X, d\mu) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_{L^\infty} < +\infty \).
7. \( L^p(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \) denote the spaces of equivalence classes of functions as defined in (5) and (6) where the measure \( d\mu \) is Lebesgue measure.
8. \( L^p_{\text{loc}}(\mathbb{R}^n) \) is the space of equivalence classes of measurable, complex valued functions on \( \mathbb{R}^n \) which belong to \( L^p(K) \) for every compact set \( K \subset \mathbb{R}^n \).
9. If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), the convolution \( f \ast g \) is defined to be the function
   \[ f \ast g(x) = \int_{\mathbb{R}^n} f(x - t) \, g(t) \, dt \]
   whenever the integral converges. Here \( dt \) denotes Lebesgue measure.
10. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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**Problem I** Suppose that the function \( f : [0, 1] \to \mathbb{R} \) satisfies the following properties:

- \( f \in C^1([0, 1]) \);
- \( f(0) = f(1) = 0 \);
- \( f(x) \leq 1 \) for all \( x \in [0, 1] \);
- \( f \) is concave, i.e. \( f' \) is nonincreasing on \([0, 1]\).

(a) Prove that the arc length of graph of \( f \) does not exceed 3.

(b) Does there exist a constant \( \mu < 3 \) such that the arc length of graph of \( f \) is less than \( \mu \) for all above \( f \)?

*Hint: Use the inequality \( \sqrt{a^2 + b^2} \leq |a| + |b| \).*

**Problem II** Let \( \{c_n\} \) be a sequence of complex numbers satisfying

\[
|c_{n+1}| \leq 2|c_n|
\]

for \( n \geq 0 \). Let \( f_0 \) be a continuously differentiable function on \([0, 1]\), and for \( n \geq 1 \), define continuously differentiable functions \( \{f_n\} \) on \([0, 1]\) by setting

\[
f_n(x) = c_{n-1} f_{n-1}^2(x).
\]

Show that there exists an \( \epsilon > 0 \) such that if \( |f_0(x)| < \epsilon \) for all \( x \in [0, 1] \), then the infinite series

\[
\sum_{n=0}^{\infty} f_n(x), \quad 0 \leq x \leq 1
\]

converges to a continuously differentiable function \( f \) on \([0, 1]\).

*Hint: Show that \( |c_n| \leq 2^n |c_0| \) and \( |f_n(x)| \leq \frac{1}{4} |f_{n-1}(x)| \) for suitable \( \epsilon \).*

**Problem III** Let \( I(\lambda) = \int_0^1 e^{-\lambda(x^5-x^6)} \, dx \). Show that there are positive constants \( C_1 \) and \( C_2 \) so that for \( \lambda \geq 10 \),

\[
I(\lambda) = C_1 \lambda^{-\frac{1}{2}} + E(\lambda)
\]

where \( |E(\lambda)| \leq C_2 \lambda^{-\frac{3}{2}} \).

**Problem IV** Let \( A \) and \( B \) be real numbers. Show that there is a constant \( C \) independent of \( A \), \( B \), and \( N \) so that

\[
\left| \int_{-N}^{+N} e^{i(Ax^2+Bx^2)} - 1 \, \frac{dx}{x} \right| \leq C.
\]

**Problem V** Let \( \mu \) be a positive measure on \( X \) with \( \mu(X) < \infty \).

(a) Let \( f \in L^\infty(X, d\mu) \). Prove that \( f \in L^p(X, d\mu) \) for \( 1 \leq p < +\infty \) and that \( \lim_{p \to \infty} \|f\|_p = \|f\|_\infty \).

(b) Suppose that \( f \in L^p(X, d\mu) \) for \( 1 \leq p < +\infty \). Suppose also that there is a constant \( L > 0 \) such that \( \|f\|_p \leq L \) for all \( 1 \leq p < +\infty \). Prove that \( f \in L^\infty(X, d\mu) \) and \( \|f\|_\infty \leq L \).

**Problem VI** Let \( f_n \in L^\infty(\mathbb{R}) \) for \( n = 1, 2, \ldots \), and suppose that \( \lim_{n \to \infty} \|f_n\|_{L^\infty(\mathbb{R})} = \infty \). Show that there exists \( g \in L^1(\mathbb{R}) \) such that the sequence \( \{\|f_n g\|_{L^1(\mathbb{R})}\} \) is not bounded.
Problem VII  Let \( \varphi > 0 \) be a continuous function on \( \mathbb{R} \) such that \( \lim_{|x| \to \infty} \varphi(x) = 0 \). Let

\[
Sf(x) = \int_x^{x+1} f(x) \varphi(x) \, dx.
\]

Prove or disprove that \( S : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \) is compact.

Problem VIII
(a) Show that there exists an infinitely differentiable function \( f \) defined on \( \mathbb{R} \) such that

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} x^n f(x) \, dx = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots .
\]

(b) Let \( f \) be a continuous function on \( \mathbb{R} \) with compact support in \([0, 1]\). Prove the following version of the Weierstrass approximation theorem: For each \( \epsilon > 0 \) there exists a polynomial \( P(x) \) such that

\[
|f(x) - P(x)| < \epsilon, \quad 0 \leq x \leq 1.
\]

\[\text{Hint: Consider } f_t(x) = \frac{t}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(y) e^{-|x-y|^2} \, dy, \quad t > 0.\]

(c) Either prove or disprove the statement that there exists an infinitely differentiable function \( f \) defined on \( \mathbb{R} \) that satisfies (a) and has compact support.

Problem IX  Let \( H \) be a Hilbert space. Prove that every non-empty convex closed set \( E \subset H \) contains a unique element \( x \) of minimal norm.
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison

Wednesday August 20, 2008

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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(3) For points \( x \) and \( y \) in \( \mathbb{R}^n \), \( |x - y| \) denotes the Euclidean distance between the points.

(4) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.

(5) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[ ||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}. \]

Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( ||f||_p < +\infty \).

(6) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then

\[ ||f||_\infty = \inf \left\{ t > 0 \mid \mu \left( \{ x \in X \mid |f(x)| > t \} \right) = 0 \right\}. \]

\( L^\infty(X) = L^\infty(X, d\mu) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( ||f||_\infty < +\infty \).

(7) \( L^p(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \) denote the spaces of equivalence classes of functions as defined in (5) and (6) where the measure \( d\mu \) is Lebesgue measure.

(8) \( L^p_{\text{loc}}(\mathbb{R}^n) \) is the space of equivalence classes of measurable, complex valued functions on \( \mathbb{R}^n \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R}^n \).

(9) If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), the convolution \( f \ast g \) is defined to be the function

\[ f \ast g(x) = \int_{\mathbb{R}^n} f(x - t) \, g(t) \, dt \]

whenever the integral converges. Here \( dt \) denotes Lebesgue measure.

(10) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Let $f_n$ be a continuous non-negative function on $[0, 1]$ for $n = 1, 2, \ldots$. Suppose that 
\[(*) \quad f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots \quad \text{for all } x \in [0, 1].\]

Let $f(x) = \lim_{n \to \infty} f_n(x)$ and $M = \sup_{x \in [0, 1]} f(x)$.

(a) Prove that there exists $y \in [0, 1]$ with $f(y) = M$.

(b) Show by example that the conclusion of (a) needs not hold if instead of $(*)$ we merely assume that for every $x \in [0, 1]$ there exists $n_x$ such that for all $n \geq n_x$ one has $f_n(x) \geq f_{n+1}(x)$.

Problem II  Let
\[Q(x) = \sum_{j,k=1}^{n} a_{j,k}x_jx_k\]
be a real symmetric quadratic form, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\{a_{j,k}\}$ are real numbers with $a_{j,k} = a_{k,j}$. Let $\Sigma_{n-1} = \{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j^2 = 1\}$ denote the unit sphere in $\mathbb{R}^n$.

(a) Prove that there is a point $x_0 \in \Sigma_{n-1}$ so that $Q(x_0) \geq Q(x)$ for all $x \in \Sigma_{n-1}$.

(b) Prove that there is a real number $\lambda$ so that $\nabla Q(x_0) = \lambda x_0$. (Here $\nabla Q$ denotes the gradient of $Q$).

(c) Prove that if $A = (a_{j,k})$ is a symmetric $n \times n$ matrix, and if $A \cdot x$ denotes matrix multiplication, then there is a non-zero vector $x_0 \in \mathbb{R}^n$ and a real number $\lambda$ such that $A \cdot x_0 = \lambda x_0$.

Problem III  Let $f_n$ be a sequence of real-valued $C^1$ functions on $[0, 1]$, such that, for all $n$,
\[|f_n'(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1),\]
\[\int_0^1 f_n(x) \, dx = 0.\]

Prove that the sequence $f_n$ has a subsequence that converges uniformly on $[0, 1]$.

Problem IV  Let $E \subset [0, 1]$ be a measurable set with positive Lebesgue measure. Let $\chi$ be the characteristic function of $E$.

(a) Let $F(x) = \int_{\mathbb{R}} \chi(x-t)\chi(t) \, dt$. Prove that $F$ is a continuous function.

(b) Let $E + E = \{e_1 + e_2 \in \mathbb{R} \mid e_1 \in E, e_2 \in E\}$. Prove that $E + E$ contains a non-empty open subset of $\mathbb{R}$.
Problem V  Let $p > 1$ and let

$$\varphi(x) = \begin{cases} 
  x^{-\frac{1}{p}} & \text{if } x \in (0, 1], \\
  0 & \text{if } x \notin (0, 1]. 
\end{cases}$$

Let $\{r_n\}$ be a countable dense set in $\mathbb{R}$, and put $F(x) = \sum_{n=1}^{\infty} 2^{-n} \varphi(x - r_n)$.

(a) Prove that $F \in L^1(\mathbb{R})$, and so in particular, $F(x) < \infty$ for almost every $x \in \mathbb{R}$.

(b) Prove that $F^p$ is not integrable on any non-empty interval $[a, b] \subset \mathbb{R}$.

Problem VI  Construct a (non-measurable) function $f : \mathbb{R} \to \mathbb{R}$ with the following property: for any $g : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - g(x)| < 1$ for all $x \in \mathbb{R}$, $g$ is not measurable.

Problem VII  Let $f$ be holomorphic on the open unit disk $D$, and suppose that

$$\int\int_D |f(z)|^2 \, dx \, dy < +\infty.$$ 

(a) If $K \subset D$ is a compact set, prove that there is a constant $C$, independent of the function $f$ such that

$$\sup \{ |f''(z)| \mid z \in K\} \leq C \left( \int\int_D |f(z)|^2 \, dx \, dy \right)^{\frac{1}{2}}.$$ 

(b) Prove that if the Taylor expansion of $f$ is $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$ 

Problem VIII  Let $f$ be a holomorphic function in the disk centered at the origin of radius 2 except at the point $z = 1$, where it has a simple pole. Suppose that the Taylor series expansion of $f$ for $|z| < 1$ is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1.$$

Problem IX  Evaluate

$$\int_{x=1}^{2} \frac{x^{2k+1}}{\sqrt{(x^2 - 1)(4 - x^2)}} \, dx, \quad \text{for } k = 0, 1, 2, \ldots.$$
QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison

Wednesday August 20, 2008
Versions for Math 725

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(5) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),

\[
||f||_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}.
\]

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\[
||f||_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]

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whenever the integral converges. Here \( dt \) denotes Lebesgue measure.

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\[ f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots \quad \text{for all } x \in [0, 1]. \]
Let \( f(x) = \lim_{n \to \infty} f_n(x) \) and \( M = \sup_{x \in [0, 1]} f(x) \).

(a) Prove that there exists \( y \in [0, 1] \) with \( f(y) = M \).
(b) Show by example that the conclusion of (a) needs not hold if instead of (*) we merely assume that for every \( x \in [0, 1] \) there exists \( n_x \) such that for all \( n \geq n_x \) one has \( f_n(x) \geq f_{n+1}(x) \).

Problem II  Let
\[ Q(x) = \sum_{j,k=1}^{n} a_{j,k} x_j x_k \]
be a real symmetric quadratic form, where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( \{a_{j,k}\} \) are real numbers with \( a_{j,k} = a_{k,j} \). Let \( \Sigma_{n-1} = \{ x \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j^2 = 1 \} \) denote the unit sphere in \( \mathbb{R}^n \).

(a) Prove that there is a point \( x_0 \in \Sigma_{n-1} \) so that \( Q(x_0) \geq Q(x) \) for all \( x \in \Sigma_{n-1} \).
(b) Prove that there is a real number \( \lambda \) so that \( \nabla Q(x_0) = \lambda x_0 \). (Here \( \nabla Q \) denotes the gradient of \( Q \).)
(c) Prove that if \( A = (a_{j,k}) \) is a symmetric \( n \times n \) matrix, and if \( A x \) denotes matrix multiplication, then there is a non-zero vector \( x_0 \in \mathbb{R}^n \) and a real number \( \lambda \) such that \( A x_0 = \lambda x_0 \).

Problem III  Let \( f_n \) be a sequence of real-valued \( C^1 \) functions on \([0, 1]\), such that, for all \( n \),
\[ |f'_n(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1), \]
\[ \int_0^1 f_n(x) \, dx = 0. \]
Prove that the sequence \( f_n \) has a subsequence that converges uniformly on \([0, 1]\).

Problem IV  Let \( E \subset [0, 1] \) be a measurable set with positive Lebesgue measure. Let \( \chi \) be the characteristic function of \( E \).

(a) Let \( F(x) = \int_{\mathbb{R}} \chi(x-t) \chi(t) \, dt \). Prove that \( F \) is a continuous function.
(b) Let \( E + E = \{ e_1 + e_2 \in \mathbb{R} \mid e_1 \in E, e_2 \in E \} \). Prove that \( E + E \) contains a non-empty open subset of \( \mathbb{R} \).
Problem V  Let $p > 1$ and let

$$\varphi(x) = \begin{cases} x^{-\frac{1}{p}} & \text{if } x \in (0, 1], \\ 0 & \text{if } x \notin (0, 1]. \end{cases}$$

Let $\{r_n\}$ be a countable dense set in $\mathbb{R}$, and put $F(x) = \sum_{n=1}^{\infty} 2^{-n} \varphi(x - r_n)$.

(a) Prove that $F \in L^1(\mathbb{R})$, and so in particular, $F(x) < \infty$ for almost every $x \in \mathbb{R}$.
(b) Prove that $F^p$ is not integrable on any non-empty interval $[a, b] \subset \mathbb{R}$.

Problem VI  Construct a (non-measurable) function $f : \mathbb{R} \to \mathbb{R}$ with the following property: for any $g : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - g(x)| < 1$ for all $x \in \mathbb{R}$, $g$ is not measurable.

Problem VII  Let $H$ be a Hilbert space. Let a sequence $f_n \in H$ converge weakly to $f$ (i.e. $\lim_{n \to \infty} (f_n, g) = (f, g)$ for any $g \in H$).

(a) Show that $\|f\| \leq \limsup_{n \to \infty} \|f_n\|$.
(b) Give an example for which the inequality in (a) is strict.
(c) Show that if $\|f\| = \lim_{n \to \infty} \|f_n\|$, then $\lim_{n \to \infty} \|f - f_n\| = 0$ (i.e. $f_n$ converges to $f$ strongly).

Problem VIII  Let $T \in \mathcal{D}'(\mathbb{R})$ be a distribution on $\mathbb{R}$.

(a) Prove that if $\frac{dT}{dx} = 0$, then there is a constant $c$ so that for all $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\langle T, \varphi \rangle = c \int_{\mathbb{R}} \varphi(x) \, dx.$$

(b) What can you conclude if there is a positive integer $N$ such that $\frac{d^N T}{dx^N} = 0$?

Problem IX  Let $\eta$ be a Schwartz function on $\mathbb{R}^n$. Show that for any complex number $a$ with $\text{Re} \, a > 0$ the function $g_a$ defined by

$$g_a(\xi) = \eta(\xi) |\xi|^a$$

is the Fourier transform of a function in $L^1(\mathbb{R}^n)$. 
QUALIFYING EXAM in ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday January 14, 2009
Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(4) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
\[
\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
\]
Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
(5) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then
\[
\|f\|_\infty = \inf \{ t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0 \}.
\]
\( L^\infty(X) = L^\infty(X, d\mu) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_\infty < +\infty \).
(6) \( L^p(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \) denote the spaces of equivalence classes of functions as defined in (4) and (5) where the measure \( d\mu \) is Lebesgue measure.
(7) \( L^p_{loc}(\mathbb{R}^n) \) is the space of equivalence classes of measurable, complex valued functions on \( \mathbb{R}^n \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R}^n \).
(8) If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), the convolution \( f \ast g \) is defined to be the function
\[
f \ast g(x) = \int_{\mathbb{R}^n} f(x-t) g(t) \, dt
\]
whenever the integral converges. Here \( dt \) denotes Lebesgue measure.
(9) \( \mathcal{S}(\mathbb{R}) \) is the Schwartz space of infinitely differentiable functions \( \varphi \) on \( \mathbb{R} \) such that \( \sup_{x \in \mathbb{R}} |x|^m |\varphi^{(n)}(x)| < +\infty \) for all non-negative integers \( m \) and \( n \).
(10) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Prove or disprove the statement that the infinite series \( \sum_{n=1}^{\infty} \frac{(-1)^n e^{-x/n}}{n} \) converges uniformly on the interval \([0, +\infty)\).

Problem II  If \( p = (x_1, x_2) \) and \( q = (y_1, y_2) \) are two points in \( \mathbb{R}^2 \), let \( d(p, q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \) denote the Euclidean distance from \( p \) to \( q \).

(a) Let \( A > 0 \) and let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a continuously differentiable function whose partial derivatives satisfy \( \left| \frac{\partial \varphi}{\partial x_j}(x_1, x_2) \right| \leq A \) for \( j = 1, 2 \) and for all \((x_1, x_2) \in \mathbb{R}^2 \). Show that for any points \( p, q \in \mathbb{R}^2 \) we have
\[
|\varphi(p) - \varphi(q)| \leq \sqrt{2A} d(p, q).
\]
Also, show that the constant \( \sqrt{2A} \) cannot be replaced by any smaller number.

(b) Let \( A > 0 \) and suppose that a mapping \( \Psi : \mathbb{R}^2 \to \mathbb{R} \) is given by \( \Psi = (\Psi_1, \Psi_2) \), where each \( \Psi_j : \mathbb{R}^2 \to \mathbb{R} \) is continuously differentiable and satisfies \( \left| \frac{\partial \Psi_k}{\partial x_j}(x_1, x_2) \right| \leq A \) for \( j = 1, 2, k = 1, 2 \), and for all \((x_1, x_2) \in \mathbb{R}^2 \). Find the smallest constant \( C \) such that for all \( p, q \in \mathbb{R}^2 \)
\[
d(\Psi(p), \Psi(q)) \leq C d(p, q).
\]
(c) Show that there exists \( A > 0 \) so that if \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) is a mapping as in part (b) given by \( \Psi = (\Psi_1, \Psi_2) \) with \( \left| \frac{\partial \Psi_j}{\partial x_k}(x_1, x_2) \right| \leq A \) for \( j = 1, 2, k = 1, 2 \), and for all \((x_1, x_2) \in \mathbb{R}^2 \), then for any point \((a_1, a_2) \in \mathbb{R}^2 \), there exists a unique point \((x_1, x_2) \in \mathbb{R}^2 \) which is a solution to the equation:
\[
\Psi(x_1, x_2) = (a_1, a_2) + (x_1, x_2).
\]

Problem III  Let \( \{a_k\} \) and \( \{b_k\} \) be two infinite sequences of real numbers. Suppose that \( a_k > 0 \) and \( 0 \leq b_k \leq 1 \) for all \( k \geq 1 \), and suppose that \( \sum_{k=1}^{\infty} a_k = +\infty \). Prove that there is an increasing sequence \( \{k_n\} \) of positive integers such that:
(i) \( \sum_{n=1}^{\infty} a_{k_n} = +\infty \);
(ii) \( \lim_{n \to \infty} b_{k_n} \) exists.

Problem IV  For \( \lambda > 1 \), define \( H(\lambda) = \int_{0}^{\infty} e^{-\lambda(x^3 + x^4)} \, dx \). Prove that there are positive constants \( A \) and \( C \) so that \( |H(\lambda) - AA^{-\frac{1}{2}}| \leq CA^{-1} \) for \( \lambda > 1 \).

Problem V  Let \( f \in L^1(\mathbb{R}) \). Prove that \( \lim_{\lambda \to +\infty} \int_{\mathbb{R}} e^{i\lambda t^2} f(t) \, dt = 0. \)

Problem VI  Let \( f_n \in L^1(\mathbb{R}) \) for \( n = 1, 2, \ldots \), and suppose \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = 0 \). For any \( \alpha > 0 \), let \( E_n(\alpha) = \left\{ x \in \mathbb{R} \mid |f_n(x)| > \alpha \right\} \). For each of the following statements, either prove the statement is true, or show that it is false by giving a counterexample:
(a) There is a set \( E \subset \mathbb{R} \) with \( |E| = 0 \) so that if \( x \notin E \), then \( \lim_{n \to \infty} f_n(x) = 0 \).
(b) Let \( \alpha > 0 \). Then \( \lim_{n \to \infty} \sqrt{\alpha} |E_n(\alpha)| = 0. \)
(c) \( \lim_{n \to \infty} \frac{1}{n} \left| E_n \left( \frac{1}{n} \right) \right| = 0. \)
(d) \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left| E_n \left( \frac{1}{n} \right) \right| = 0. \)
**Problem VII**  Let $f$ be holomorphic in the open unit disk $D$ and suppose $|f(z)| \leq 1$ for all $z \in D$.

(a) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the Taylor series expansion of $f$, prove that $\sum_{n=0}^{\infty} |a_n|^2 \leq 1$.

(b) If $f \left(\frac{1}{2}\right) = f \left(-\frac{1}{2}\right) = 0$, prove that $|f(z)| \leq \left|\frac{4z^2 - 1}{4 - z^2}\right|$ for all $z \in D$.

**Problem VIII**  Let

\[
L_1 = \{ z \in \mathbb{C} \mid \text{Re}[z] = +1, \text{Im}[z] \leq 0 \} \\
L_{-1} = \{ z \in \mathbb{C} \mid \text{Re}[z] = -1, \text{Im}[z] \leq 0 \}.
\]

(a) Show that there exists a unique holomorphic function $f$ defined on $\mathbb{C} \setminus (L_1 \cup L_{-1})$ such that $(f(z))^2 = 1 - z^2$ and $f(i) > 0$.

(b) If $f$ is the function from part (a), find an explicit formula for $f(x)$ when $x \in \mathbb{R} \setminus \{-1, +1\}$.

(c) Evaluate $\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{1+2x^2+x^4} \, dx$ by integrating $f(z)(1+2z^2+z^4)^{-1}$ along appropriate contours.

**Problem IX**

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function. Suppose that

\[
\lim_{|z| \to \infty} |f(z)| = +\infty.
\]

Prove that $f$ is a polynomial.

(b) Is the conclusion in part (a) still valid if we only assume that for each $\theta \in [0, 2\pi]$,

\[
\lim_{r \to \infty} |f(re^{i\theta})| = +\infty?
\]

[HINT: Consider the function $g(z) = z + e^z$.]
QUALIFYING EXAM in ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday January 14, 2009
Version for Math 725

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

1. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
2. \( \mathcal{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
3. If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
4. If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
   \[
   \|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.
   \]
   Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( \|f\|_p < +\infty \).
5. If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then
   \[
   \|f\|_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
   \]
   \( L^\infty(X) = L^\infty(X, d\mu) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( \|f\|_\infty < +\infty \).
6. \( L^p(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \) denote the spaces of equivalence classes of functions as defined in (4) and (5) where the measure \( d\mu \) is Lebesgue measure.
7. \( L^p_{\text{loc}}(\mathbb{R}^n) \) is the space of equivalence classes of measurable, complex valued functions on \( \mathbb{R}^n \) which belong to \( L^p(K) \) for every compact set \( K \subset \subset \mathbb{R}^n \).
8. If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), the convolution \( f \ast g \) is defined to be the function
   \[
   f \ast g(x) = \int_{\mathbb{R}^n} f(x-t) \, g(t) \, dt
   \]
   whenever the integral converges. Here \( dt \) denotes Lebesgue measure.
9. \( S(\mathbb{R}) \) is the Schwartz space of infinitely differentiable functions \( \varphi \) on \( \mathbb{R} \) such that \( \sup_{x \in \mathbb{R}} |x|^m |\varphi^{(n)}(x)| < +\infty \) for all non-negative integers \( m \) and \( n \).
10. If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.

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Problem I  Prove or disprove the statement that the infinite series \( \sum_{n=1}^{\infty} \frac{(-1)^n e^{-x/n}}{n} \) converges uniformly on the interval \([0, +\infty)\).

Problem II  If \( p = (x_1, x_2) \) and \( q = (y_1, y_2) \) are two points in \( \mathbb{R}^2 \), let \( d(p, q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \) denote the Euclidean distance from \( p \) to \( q \).

(a) Let \( \alpha > 0 \) and \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a continuously differentiable function whose partial derivatives satisfy \( \left| \frac{\partial \varphi}{\partial x_j}(x_1, x_2) \right| \leq A \) for \( j = 1, 2 \) and for all \( (x_1, x_2) \in \mathbb{R}^2 \). Show that for any points \( p, q \in \mathbb{R}^2 \) we have

\[
|\varphi(p) - \varphi(q)| \leq \sqrt{2} A d(p, q).
\]

Also, show that the constant \( \sqrt{2} A \) cannot be replaced by any smaller number.

(b) Let \( A > 0 \) and suppose that a mapping \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by \( \Psi = (\Psi_1, \Psi_2) \), where each \( \Psi_j : \mathbb{R}^2 \to \mathbb{R} \) is continuously differentiable and satisfies \( \left| \frac{\partial \Psi_k}{\partial x_j}(x_1, x_2) \right| \leq A \) for \( j = 1, 2, k = 1, 2 \), and for all \( (x_1, x_2) \in \mathbb{R}^2 \). Find the smallest constant \( C \) such that for all \( p, q \in \mathbb{R}^2 \)

\[
d(\Psi(p), \Psi(q)) \leq C d(p, q).
\]

(c) Show that there exists \( A > 0 \) so that if \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) is a mapping as in part (b) given by \( \Psi = (\Psi_1, \Psi_2) \) with \( \left| \frac{\partial \Psi_j}{\partial x_k}(x_1, x_2) \right| \leq A \) for \( j = 1, 2, k = 1, 2 \), and for all \( (x_1, x_2) \in \mathbb{R}^2 \), then for any point \( (a_1, a_2) \in \mathbb{R}^2 \), there exists a unique point \( (x_1, x_2) \in \mathbb{R}^2 \) which is a solution to the equation:

\[
\Psi(x_1, x_2) = (a_1, a_2) + (x_1, x_2).
\]

Problem III  Let \( \{a_k\} \) and \( \{b_k\} \) be two infinite sequences of real numbers. Suppose that \( a_k > 0 \) and \( 0 \leq b_k \leq 1 \) for all \( k \geq 1 \), and suppose that \( \sum_{k=1}^{\infty} a_k = +\infty \). Prove that there is an increasing sequence \( \{k_n\} \) of positive integers such that:

(i) \( \sum_{n=1}^{\infty} a_{k_n} = +\infty \);  

(ii) \( \lim_{n \to \infty} b_{k_n} \) exists.

Problem IV  For \( \lambda > 1 \), define \( H(\lambda) = \int_{0}^{\infty} e^{-\lambda(x^3 + x^4)} \, dx \). Prove that there are positive constants \( A \) and \( C \) so that \( \left| H(\lambda) - A\lambda^{-\frac{3}{4}} \right| \leq C\lambda^{-1} \) for \( \lambda > 1 \).

Problem V  Let \( f \in L^1(\mathbb{R}) \). Prove that \( \lim_{\lambda \to +\infty} \int_{\mathbb{R}} e^{i\lambda t^2} f(t) \, dt = 0 \).

Problem VI  Let \( f_n \in L^1(\mathbb{R}) \) for \( n = 1, 2, \ldots \), and suppose \( \lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)| \, dx = 0 \). For any \( \alpha > 0 \), let \( E_n(\alpha) = \{ x \in \mathbb{R} : |f_n(x)| > \alpha \} \). For each of the following statements, either prove the statement is true, or show that it is false by giving a counterexample:

(a) There is a set \( E \subset \mathbb{R} \) with \( |E| = 0 \) so that if \( x \notin E \), then \( \lim_{n \to \infty} f_n(x) = 0 \).

(b) Let \( \alpha > 0 \). Then \( \lim_{n \to \infty} \sqrt{n} |E_n(\alpha)| = 0 \).

(c) \( \lim_{n \to \infty} \frac{1}{n} \left| E_n \left( \frac{1}{n} \right) \right| = 0 \).

(d) \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left| E_n \left( \frac{1}{n} \right) \right| = 0 \).
Problem VII  Let \( \mathcal{S}(\mathbb{R}) \) denote the Schwartz space, let \( \chi \in \mathcal{S}(\mathbb{R}) \), and let \( a > 0 \). Show that the function 
\[
g(\xi) = \chi(\xi)|\xi|^a
\]
is the Fourier transform of a function \( f \in L^1(\mathbb{R}) \).

Problem VIII  Determine the limits in the sense of distributions on \( \mathbb{R} \) of each of the following:

(a) \( \lim_{t \to \infty} \frac{\cos(tx)}{1 + x^2} \);

(b) \( \lim_{t \to 0} \frac{3t^2|x|}{(t^2 + x^2)^2} \);

(c) \( \lim_{R \to \infty} G_R(x) \) where \( G_R(x) = \int_{\mathbb{R}} \chi(R^{-1}\xi) e^{i\xi x} \, dx \) and \( \chi \in \mathcal{S}(\mathbb{R}) \) satisfies \( \chi(0) = 1 \).

Problem IX  For \( f \in L^2(0,1) \), define \( T[f](x) = xf(x) \).

(a) Show that \( T \) is a continuous self adjoint operator on \( L^2([0,1]) \).

(b) Show that \( T \) does not have any eigenvalues.

(c) For which \( \lambda \in \mathbb{C} \) is the operator \( T - \lambda I \) not invertible?
QUALIFYING EXAM in ANALYSIS
Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 19, 2009
Version for Math 722

Instructions:
Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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Notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.
(2) $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ denotes the unit disc in the complex plane.
(3) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.
(4) If $\mu$ is a positive measure on a set $X$, and if $f$ is a complex valued measurable function on $X$, then for $1 \leq p < +\infty$,

$$||f||_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}.$$

Two functions on $X$ are said to be equivalent if they are equal except on a set of $\mu$ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.
(5) If $\mu$ is a positive measure on a set $X$, and if $f$ is a complex valued measurable function on $X$, then

$$||f||_{\infty} = \inf \{ t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0 \}.$$

$L^\infty(X) = L^\infty(X, d\mu)$ is the space of equivalence classes of measurable, complex valued functions on $X$ such that $||f||_{\infty} < +\infty$.
(6) $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ denote the spaces of equivalence classes of functions as defined in (4) and (5) where the measure $d\mu$ is Lebesgue measure.
(7) If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, the convolution $f \ast g$ is defined to be the function

$$f \ast g(x) = \int_{\mathbb{R}^n} f(x-t) g(t) \, dt$$

whenever the integral converges. Here $dt$ denotes Lebesgue measure.
(8) $S(\mathbb{R})$ is the Schwartz space of infinitely differentiable functions $\varphi$ on $\mathbb{R}$ such that $\sup_{x \in \mathbb{R}} |x|^m |\varphi^{(n)}(x)| < +\infty$ for all non-negative integers $m$ and $n$.
(9) If $T$ is a distribution and $\varphi$ is a test function, then $\langle T, \varphi \rangle$ denotes the value of the distribution applied to the test function.
Problem 1
Let $b > 1$. A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers is defined inductively by specifying $a_0 > 0$, and then setting
\[ a_{n+1} = \frac{a_n + \frac{b^3}{2a_n^2}}{2} \]
for $n \geq 0$.
(a) Show that if $L = \lim_{n \to \infty} a_n$ exists, then $L = b$.
(b) Show that there is an open interval $I_b \subset \mathbb{R}$ containing $b$ so if $a_0 \in I_b$, then $L = \lim_{n \to \infty} a_n$ exists.
(c) What can you say about the length of $I_b$?

Problem 2
Let $f(x) = \sum_{n=1}^{\infty} (1 + n^4x^2)^{-1}$.
(a) Show that $f$ is continuously differentiable on $(0, \infty)$.
(b) Show that there is a constant $C > 0$ so that $f(x) \leq C x^{-\frac{1}{2}}$ for all $0 < x \leq 1$, and $f(x) \leq C x^{-2}$ for $x \geq 1$.
(c) Show that the improper Riemann integral $\int_0^\infty f(x) \, dx = \lim_{N \to \infty} \int_0^N f(x) \, dx$ exists.

Problem 3
Let $X$ and $Y$ be normed vector spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. Let $\Omega \subset X$ be an open set and let $x \in \Omega$. Recall that a function $F : \Omega \to Y$ is differentiable at $x \in \Omega$ if there is a continuous linear transformation $S_x : X \to Y$ such that
\[ \lim_{||h||_x \to 0} \frac{||F(x + h) - F(x) - S_x(h)||_Y}{||h||_X} = 0. \]
We then say that $S_x$ is the derivative of $F$ at $x$.
(a) Let $X$, $Y$, and $Z$ be normed vector spaces with norms $\| \cdot \|_X$, $\| \cdot \|_Y$, and $\| \cdot \|_Z$. Let $F : X \to Y$ be differentiable at a point $x_0 \in X$ with derivative $S_{x_0}$, and let $G : Y \to Z$ be differentiable at the point $F(x_0)$ with derivative $T_{F(x_0)}$. Prove that the composition $G \circ F : X \to Z$ defined by $G \circ F(x) = G(F(x))$ is differentiable at $x_0$, and compute its derivative.
(b) Let $M_n$ denote the space of all real $n \times n$ matrices $m$, and define $F : M_n \to M_n$ by $F(m) = m^3$. Prove that $F$ is differentiable at every matrix $m \in M_n$, and compute the derivative of $F$.
(c) Let $\Omega \subset M_n$ denote the set of invertible $n \times n$ matrices, and define $G : \Omega \to M_n$ by setting $G(m) = m^{-1}$. Prove that $\Omega$ is an open subset of $M_n$, and that $G$ is differentiable at every point $m \in \Omega$, and compute the derivative of $G$. 
Problem 4
Let \( \{f_0, f_1, \ldots, f_n, \ldots\} \) be a sequence of Lebesgue measurable functions on the interval \([0, 1]\).

(a) Suppose that
\[
\begin{align*}
(i) & \quad \lim_{n \to \infty} f_n(x) = f_0(x) \text{ for almost every } x \in [0, 1]; \\
(ii) & \quad \int_0^1 |f_n(x)| \, dx < +\infty \text{ for every } n \geq 0; \\
(iii) & \quad \lim_{n \to \infty} \int_0^1 |f_n(x)| \, dx = \int_0^1 |f_0(x)| \, dx.
\end{align*}
\]
Prove that \( \lim_{n \to \infty} \int_0^1 |f_n(x) - f_0(x)| \, dx = 0. \)

(b) Suppose that \( \lim_{n \to \infty} \int_0^1 |f_n(x) - f_0(x)| \, dx = 0. \)
Prove that there is a subsequence \( n_k \to \infty \) such that \( \lim_{k \to \infty} f_{n_k}(x) = f_0(x) \) for almost every \( x \in \mathbb{R}. \)

Problem 5
If \( f \in L^1(\mathbb{R}) \) and \( y > 0 \), define \( f_y(x) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(x-t)e^{-\frac{t^2}{y}} \, dt. \)

(a) Prove that for each \( y > 0 \), the function \( f_y \in L^1(\mathbb{R}) \); i.e. \( \int_{\mathbb{R}} |f_y(t)| \, dt < +\infty \) for each \( y > 0 \).

(b) Prove that \( \lim_{y \to 0} \int_{\mathbb{R}} |f(x) - f_y(x)| \, dx = 0. \)

(c) There exists a constant \( C > 0 \) such that for every \( f \in L^1(\mathbb{R}), \)
\[
\left\{ x \in \mathbb{R} \mid \sup_{y > 0} |f_y(x)| > \lambda \right\} \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f(t)| \, dt.
\]
Using this inequality, prove that if \( f \in L^1(\mathbb{R}) \), then for almost every \( x \in \mathbb{R}, \) \( \lim_{y \to 0} f_y(x) = f(x). \)

Problem 6
(a) For which real numbers \( a \in \mathbb{R} \) and \( b > 0 \) is it true that \( \left| \int_0^N e^{ixb}(1 + x)^a \, dx \right| \) is bounded independently of the number \( N > 0? \)

(b) For which real numbers \( a \in \mathbb{R} \) and \( b > 0 \) is it true that the improper integral \( \left| \int_0^\infty e^{ixb}(1 + x)^a \, dx \right| \) converges?
Problem 7
Using contour integration, evaluate the convergent improper integral
\[ \int_0^\infty \frac{\sqrt{x} \cos(x)}{1 + x^2} \, dx. \]

Note: There is a problem here. Unfortunately, the examination committee was not careful, and this integral cannot be computed using the standard contour integral technique. If students applied the method correctly or essentially correctly, they received full or high credit for the problem.

Problem 8
Let \( F : \mathbb{C} \to \mathbb{C} \) be an entire holomorphic function.

(a) Prove that if \( \int \int |F(x + iy)|^2 \, dx \, dy < +\infty \), then \( F(z) = 0 \) for all \( z \in \mathbb{C} \).

(b) Suppose that \( F \) is non-constant. Prove that for every \( w \in \mathbb{C} \) and every \( \epsilon > 0 \) there exists \( z \in \mathbb{C} \) so that \( |F(z) - w| < \epsilon \). (Note: Do not just quote Picard's theorem.)

(c) Suppose that for each \( w \in \mathbb{C} \), the equation \( F(z) = w \) has at most 1 solution. Prove that there are complex numbers \( a \neq 0 \) and \( b \) so that \( F(z) = az + b \).

Problem 9
Let \( f \) be a function holomorphic in the open unit disk \( \mathbb{D} \).

(a) Suppose that \( |f(z)| \leq 1 \) for every \( z \in \mathbb{D} \). Prove that \( |f^{(N)}(z)| \leq N!(1 - |z|)^{-N} \) for any integer \( N \geq 1 \) and any \( z \in \mathbb{D} \).

(b) Suppose that \( |f(z)| \leq 1 \) for every \( z \in \mathbb{D} \). Let \( 0 \neq w_j \in \mathbb{D} \), and suppose that \( f(w_j) = 0 \) for \( 1 \leq j \leq N \). Prove that \( |f(0)| \leq \prod_{j=1}^{N} |w_j| \). What can you conclude if there is equality?

(c) Let \( \{x_n\} \) be a sequence of distinct real numbers such that \( |x_n| < \frac{1}{2} \) for every \( n \geq 1 \), and suppose that \( f(x_n) \) is real for every \( n \geq 1 \). Prove that \( f(\bar{z}) = f(z) \) for all \( z \in \mathbb{D} \).
QUALIFYING EXAM in ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 19, 2009
Version for Math 725

Instructions:
Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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Notation used on the Analysis exams:
(1) \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers respectively.
(2) \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) denotes the unit disc in the complex plane.
(3) If \( E \subset \mathbb{R}^n \) is a Lebesgue measurable set, then \( |E| \) denotes its Lebesgue measure.
(4) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then for \( 1 \leq p < +\infty \),
\[
|f|_p = \left[ \int_X |f(x)|^p \, d\mu(x) \right]^{1/p}.
\]
Two functions on \( X \) are said to be equivalent if they are equal except on a set of \( \mu \) measure zero. For \( 1 \leq p < +\infty \), \( L^p(X) = L^p(X, d\mu) \) is the space of equivalence classes of complex valued measurable functions such that \( |f|_p < +\infty \).
(5) If \( \mu \) is a positive measure on a set \( X \), and if \( f \) is a complex valued measurable function on \( X \), then
\[
|f|_\infty = \inf \{ t > 0 \mid \mu(\{ x \in X \mid |f(x)| > t \}) = 0 \}.
\]
\( L^\infty(X) = L^\infty(X, d\mu) \) is the space of equivalence classes of measurable, complex valued functions on \( X \) such that \( |f|_\infty < +\infty \).
(6) \( L^p(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \) denote the spaces of equivalence classes of functions as defined in (4) and (5) where the measure \( d\mu \) is Lebesgue measure.
(7) If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), the convolution \( f \ast g \) is defined to be the function
\[
f \ast g(x) = \int_{\mathbb{R}^n} f(x-t) g(t) \, dt
\]
whenever the integral converges. Here \( dt \) denotes Lebesgue measure.
(8) \( S(\mathbb{R}) \) is the Schwartz space of infinitely differentiable functions \( \varphi \) on \( \mathbb{R} \) such that \( \sup_{x \in \mathbb{R}} |x|^m |\varphi^{(n)}(x)| < +\infty \) for all non-negative integers \( m \) and \( n \).
(9) If \( T \) is a distribution and \( \varphi \) is a test function, then \( \langle T, \varphi \rangle \) denotes the value of the distribution applied to the test function.
Problem 1
Let \( b \geq 1 \). A sequence \( \{a_n\}_{n=0}^{\infty} \) of positive real numbers is defined inductively by specifying \( a_0 > 0 \), and then setting
\[
a_{n+1} = \frac{a_n}{2} + \frac{b^3}{2a_n^2}
\]
for \( n \geq 0 \).
(a) Show that if \( L = \lim_{n \to \infty} a_n \) exists, then \( L = b \).
(b) Show that there is an open interval \( I_b \subset \mathbb{R} \) containing \( b \) so if \( a_0 \in I_b \), then \( L = \lim_{n \to \infty} a_n \) exists.
(c) What can you say about the length of \( I_b \)?

Problem 2
Let \( f(x) = \sum_{n=1}^{\infty} (1 + n^4 x^2)^{-1} \).
(a) Show that \( f \) is continuously differentiable on \((0, \infty)\).
(b) Show that there is a constant \( C > 0 \) so that \( f(x) \leq C x^{-\frac{3}{2}} \) for all \( 0 < x \leq 1 \), and \( f(x) \leq C x^{-2} \) for \( x \geq 1 \).
(c) Show that the improper Riemann integral \( \int_{0}^{\infty} f(x) \, dx = \lim_{N \to \infty} \int_{0}^{N} f(x) \, dx \) exists.

Problem 3
Let \( X \) and \( Y \) be normed vector spaces with norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \). Let \( \Omega \subset X \) be an open set and let \( x \in \Omega \). Recall that a function \( F : \Omega \to Y \) is differentiable at \( x \in \Omega \) if there is a continuous linear transformation \( S_x : X \to Y \) such that
\[
\lim_{||h||_X \to 0} \frac{||F(x+h) - F(x) - S_x(h)||_Y}{||h||_X} = 0.
\]
We then say that \( S_x \) is the derivative of \( F \) at \( x \).
(a) Let \( X, Y, \) and \( Z \) be normed vector spaces with norms \( \| \cdot \|_X, \| \cdot \|_Y, \) and \( \| \cdot \|_Z \). Let \( F : X \to Y \) be differentiable at a point \( x_0 \in X \) with derivative \( S_{x_0} \), and let \( G : Y \to Z \) be differentiable at the point \( F(x_0) \) with derivative \( T_{F(x_0)} \). Prove that the composition \( G \circ F : X \to Z \) defined by \( G \circ F(x) = G(F(x)) \) is differentiable at \( x_0 \), and compute its derivative.
(b) Let \( M_n \) denote the space of all real \( n \times n \) matrices \( m \), and define \( F : M_n \to M_n \) by \( F(m) = m^3 \). Prove that \( F \) is differentiable at every matrix \( m \in M_n \), and compute the derivative of \( F \).
(c) Let \( \Omega \subset M_n \) denote the set of invertible \( n \times n \) matrices, and define \( G : \Omega \to M_n \) by setting \( G(m) = m^{-1} \). Prove that \( \Omega \) is an open subset of \( M_n \), and that \( G \) is differentiable at every point \( m \in \Omega \), and compute the derivative of \( G \).
Problem 4

Let \{f_0, f_1, \ldots, f_n, \ldots\} be a sequence of Lebesgue measurable functions on the interval \([0, 1]\).

(a) Suppose that
   (i) \(\lim_{n \to \infty} f_n(x) = f_0(x)\) for almost every \(x \in [0, 1]\);
   (ii) \(\int_0^1 |f_n(x)| \, dx < +\infty\) for every \(n \geq 0\);
   (iii) \(\lim_{n \to \infty} \int_0^1 |f_n(x)| \, dx = \int_0^1 |f_0(x)| \, dx\).

Prove that \(\lim_{n \to \infty} \int_0^1 |f_n(x) - f_0(x)| \, dx = 0\).

(b) Suppose that \(\lim_{n \to \infty} \int_0^1 |f_n(x) - f_0(x)| \, dx = 0\).

Prove that there is a subsequence \(n_k \to \infty\) such that \(\lim_{k \to \infty} f_{n_k}(x) = f_0(x)\) for almost every \(x \in \mathbb{R}\).

Problem 5

If \(f \in L^1(\mathbb{R})\) and \(y > 0\), define \(f_y(x) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(x - t)e^{-\frac{t^2}{y}} \, dt\).

(a) Prove that for each \(y > 0\), the function \(f_y \in L^1(\mathbb{R})\); i.e. \(\int_{\mathbb{R}} |f_y(t)| \, dt < +\infty\) for each \(y > 0\).

(b) Prove that \(\lim_{y \to 0} \int_{\mathbb{R}} |f(x) - f_y(x)| \, dx = 0\).

(c) There exists a constant \(C > 0\) such that for every \(f \in L^1(\mathbb{R})\),

\[
\left| \left\{ x \in \mathbb{R} \mid \sup_{y > 0} |f_y(x)| > \lambda \right\} \right| \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f(t)| \, dt.
\]

Using this inequality, prove that if \(f \in L^1(\mathbb{R})\), then for almost every \(x \in \mathbb{R}\), \(\lim_{y \to 0} f_y(x) = f(x)\).

Problem 6

(a) For which real numbers \(a \in \mathbb{R}\) and \(b > 0\) is it true that \(\left| \int_0^N e^{ix^b}(1 + x)^a \, dx \right|\) is bounded independently of the number \(N > 0\)?

(b) For which real numbers \(a \in \mathbb{R}\) and \(b > 0\) is it true that the improper integral \(\left| \int_0^\infty e^{ix^b}(1 + x)^a \, dx \right|\) converges?
Problem 7

(a) Let $H_1$ and $H_2$ be Hilbert spaces, and let $T : H_1 \to H_2$ be a continuous linear operator. Give a precise definition of the adjoint operator $T^*$. 

(b) Let $(a, b) \subset \mathbb{R}$ be a (possibly infinite) open interval. If $f \in L^2(a, b)$, explain what it means that the distributional derivative $f'$ is also in $L^2(a, b)$.

(c) Let $R^+$ denote the positive real axis $[0, \infty)$. Let $H_1(R)$ (respectively $H^1(R^+)$) be the space of real-valued functions $f \in L^2(R)$ (respectively $f \in L^2(R^+)$) such that the distributional derivative $f'$ is also in $L^2(R)$ (respectively $L^2(R^+)$). Then $H^1(R)$ and $H^1(R^+)$ are Hilbert spaces with inner product given by

$$\langle f, g \rangle_{H^1(R)} = \int_R f(x)g(x) \, dx + \int_R f'(x)g'(x) \, dx;$$

$$\langle f, g \rangle_{H^1(R^+)} = \int_{R^+} f(x)g(x) \, dx + \int_{R^+} f'(x)g'(x) \, dx.$$ 

Let $T : H^1(R) \to H^1(R^+)$ be the mapping given by restriction. Compute explicitly the adjoint operator $T^*$.

Problem 8

Let $g$ be a positive decreasing function defined on $(0, \infty)$. Show that the following are equivalent:

(a) There exists a distribution $T$ on $\mathbb{R}$ such that $\langle T, \varphi \rangle = \int_0^\infty \varphi(x) g(x) \, dx$ for all test functions $\varphi \in C_0^\infty(\mathbb{R})$ which have compact support in the open interval $(0, \infty)$.

(b) There exists a non-negative integer $k \in \mathbb{N}$ and a constant $C > 0$ such that for all $x \in (0, 1)$, $g(x) \leq C x^{-k}$.

Problem 9

If $I \subset \mathbb{R}$ is a closed interval, a function $f : I \to \mathbb{R}$ is monotonic on $I$ if either it is non-decreasing on $I$ or it is non-increasing on $I$. A function $f$ on the interval $[0, 1]$ is said to be nowhere monotonic if there is no closed subinterval $I \subset [0, 1]$ on which $f$ is monotonic. Prove that there exists a continuous function $f$ on $[0, 1]$ which is nowhere monotonic.

Hint: It may be easier to show more: the “typical” function in the space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$ is nowhere monotonic.
ANALYSIS QUALIFIER - JANUARY 2010. UNIVERSITY OF WISCONSIN-MADISON
Complex Analysis - Math 722.

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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PROBLEMS I-VI. ADVANCED CALCULUS - REAL ANALYSIS 1

I. 1) Evaluate
\[ \int_{\Gamma} (x - y^3)dx + x^3dy \]
where \( \Gamma \) is the unit circle in \( \mathbb{R}^2 \), with counterclockwise orientation.

2) Find a function \( \lambda \) such that for any closed continuously differentiable curve \( C \) (i.e. a curve parameterized by a \( C^1 \) map \( t \to \gamma(t), t \in [0, 1] \), with \( \gamma(0) = \gamma(1) \)).
\[ \int_C (x - y^3)dx + x^3dy = \int_C \lambda(x, y)dy \]

II. 1) Let \( f \) be a continuously differentiable function on \( \mathbb{R}^2 \). Assume that at each point
\[ \frac{\partial f}{\partial x_1} > |\frac{\partial f}{\partial x_2}|. \]
Show that if \( f(x_1, x_2) = f(x_1', x_2') \), then \( |x_1' - x_1| < |x_2' - x_2| \) unless \( (x_1, x_2) = (x_1', x_2') \).

2) Let \( \phi \) be the map from \( \mathbb{R}^2 \) into itself defined by:
\[ \phi(x_1, x_2) = (x_1 + \sin(\frac{x_1}{2} + \frac{x_2}{4}), x_2 + \cos(\frac{x_1}{4} + \frac{x_2}{2})). \]
Show that \( \phi \) is a diffeomorphism from \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \), i.e. a ‘smooth’ map with smooth inverse. Evaluate the partial derivatives of \( \phi^{-1} \) at the point \((0, 1)\).

Hint. Question 1 can be used to prove injectivity. For proving surjectivity, a possibility is to prove that \( \Phi(\mathbb{R}^2) \) is closed and open in \( \mathbb{R}^2 \).
III. For $\lambda > 0$, set

$$ F(\lambda) = \int_0^1 e^{-10\lambda x^4 + \lambda x^6} \, dx. $$

Prove that there exist $A$ and $C > 0$ such that $F(\lambda) = \frac{A}{\lambda^\frac{1}{4}} + E(\lambda)$, where $|E(\lambda)| \leq \frac{C}{\lambda^\frac{1}{2}}$.

IV. Find a sequence of bounded measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to $\frac{1}{2}\chi$, where $\chi$ is the characteristic function of the interval $[0, 1]$.

Does there exist a sequence of bounded measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to $2\psi$, where $\psi$ is the characteristic function of a set of positive measure?

Recall that a sequence $f_n$ in $L^2(\mathbb{R})$ tends weakly to $f \in L^2(\mathbb{R})$, if and only if for every $g \in L^2(\mathbb{R})$, $\int f_n g \to \int fg$.

V. 1) Let $f \in L^1([0, 2\pi])$. Prove that $\int_0^{2\pi} f(x) \cos(nx) \, dx \to 0$, as $n \to \infty$.

You are asked to give a proof, not simply to quote a Theorem. By essentially the same proof, that you are not asked to repeat, one also has $\int_0^{2\pi} f(x) \sin(nx) \, dx \to 0$, as $n \to \infty$.

Prove that, for any sequence $(\alpha_n)$ in $\mathbb{R}$, $\int_0^{2\pi} f(x) \cos^2(nx + \alpha_n) \, dx \to \frac{1}{2} \int_0^{2\pi} f(x) \, dx$.

2) Let $(a_n)$ and $(b_n)$ be sequences in $\mathbb{R}$ such that on a set of positive measure in $[0, 2\pi]$, $a_n \cos nx + b_n \sin nx$ tends pointwise to $0$. Prove that $a_n$ and $b_n \to 0$.

Hint: Write $a_n \cos nx + b_n \sin nx = \rho_n \cos(nx + \alpha_n)$ and use that fact that $\cos^2 \theta \leq |\cos \theta|$.

VI. (a) Give an example of a sequence of functions $f_n \in L^1([0, 1])$, $n = 1, 2, \cdots$, with the following properties:

1. $\lim_{n \to \infty} f_n(x) = 1$ for any $x \in [0, 1]$;
2. $\int_0^1 |f_n(x)| \, dx = 2$ for any $n = 1, 2, \cdots$

(b) Show that if the $f_n$ are as in part (a) then

$$ \lim_{n \to \infty} \int_0^1 |f_n(x) - 1| \, dx = 1. $$
PROBLEMS VII-IX, COMPLEX ANALYSIS.

VII C. The Gamma function is defined for $\text{Re}[z] > 0$ by the convergent improper integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.$$ 

The object of this problem is to prove that for $0 < \text{Re}[z] < 1$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$ 

Either present your own proof of this identity, or complete the following outline:

- Use contour integration to show that if $x$ is real and $0 < x < 1$ then
  $$\int_0^\infty \frac{r^{x-1}}{1+r} dr = \frac{\pi}{\sin(\pi x)}.$$ 

- Show that if $0 < x < 1$, then $\Gamma(x)\Gamma(1-x) = \int_0^\infty \left[ \int_0^\infty e^{-s-t} \left( \frac{t}{s} \right)^x \frac{dt}{t} \right] ds.$

- Show that if $0 < x < 1$, then
  $$\int_0^\infty \left[ \int_0^\infty e^{-s-t} \left( \frac{t}{s} \right)^x \frac{dt}{t} \right] ds = \int_0^\infty \frac{r^{x-1}}{1+r} dr.$$ 

- Explain why the identity $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ for $0 < x < 1$ implies the desired identity.

VIII C. Let $f$ be a function which is holomorphic in the open unit disk $D = \{ \zeta \in \mathbb{C} \mid |\zeta| < 1 \}$, and assume that

$$\iint_D |f'(x+iy)|^2 dx dy = M^2 < +\infty.$$ 

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, prove that $M^2 = \pi \sum_{n=1}^{\infty} n|a_n|^2$.

Prove that $\text{Sup}_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < +\infty$.

Prove that there are real constants $C, \alpha > 0$ such that $|f'(z)| \leq C(1-|z|)^{-\alpha}$ for all $z \in D$, and find the smallest possible value of $\alpha$.

IX C. Let $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ be the open unit disk, and let $S = \{ w \in \mathbb{C} \mid |\text{Im}[w]| < \frac{\pi}{2} \}$ be the horizontal strip centered at zero of width $\pi$.

Find an explicit biholomorphic mapping from $S$ to $D$; i.e. find a holomorphic function $\Phi : S \to D$ which is one-to-one and onto, and whose inverse is also holomorphic.

Let $f : S \to S$ be a holomorphic function with $f(0) = 0$. Show that

$$\left| \frac{e^{f(z)} - 1}{e^{f(z)} + 1} \right| \leq \left| \frac{e^z - 1}{e^z + 1} \right|.$$ 

Let $f : S \to S$ be a holomorphic function with $f(0) = 0$. What can you conclude about $f$ if $f'(0) = 1$? Why?
Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

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I. 1) Evaluate
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\[ \int_C (x - y^3)dx + x^3dy = \int_C \lambda(x, y)dy. \]

II.
1) Let \( f \) be a continuously differentiable function on \( \mathbb{R}^2 \). Assume that at each point
\[ \frac{\partial f}{\partial x_1} > \left| \frac{\partial f}{\partial x_2} \right|. \]
Show that if \( f(x_1, x_2) = f(x'_1, x'_2) \), then \( |x'_1 - x_1| < |x'_2 - x_2| \) unless \( (x_1, x_2) = (x'_1, x'_2) \).

2) Let \( \phi \) be the map from \( \mathbb{R}^2 \) into itself defined by:
\[ \phi(x_1, x_2) = (x_1 + \sin\left(\frac{x_1}{2} + \frac{x_2}{4}\right), \ x_2 + \cos\left(\frac{x_1}{4} + \frac{x_2}{2}\right)). \]
Show that \( \phi \) is a diffeomorphism from \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \), i.e. a ‘smooth’ map with smooth inverse. Evaluate the partial derivatives of \( \phi^{-1} \) at the point \( (0, 1) \).

Hint. Question 1 can be used to prove injectivity. For proving surjectivity, a possibility is to prove that \( \Phi(\mathbb{R}^2) \) is closed and open in \( \mathbb{R}^2 \).
III. For \( \lambda > 0 \), set
\[
F(\lambda) = \int_0^1 e^{-10\lambda x^4 + \lambda x^6} \, dx.
\]
Prove that there exist \( A \) and \( C > 0 \) such that
\[
F(\lambda) = A \frac{\lambda}{\lambda^2} + E(\lambda), \quad \text{where } |E(\lambda)| \leq C \frac{\lambda}{\lambda^2}.
\]

IV. Find a sequence of bounded measurable sets in \( \mathbb{R} \) whose characteristic functions converge weakly in \( L^2(\mathbb{R}) \) to \( \frac{1}{2} \chi \), where \( \chi \) is the characteristic function of the interval \([0, 1]\).

Does there exist a sequence of bounded measurable sets in \( \mathbb{R} \) whose characteristic functions converge weakly in \( L^2(\mathbb{R}) \) to \( 2\psi \), where \( \psi \) is the characteristic function of a set of positive measure?

Recall that a sequence \( f_n \) in \( L^2(\mathbb{R}) \) tends weakly to \( f \in L^2(\mathbb{R}) \), if and only if for every \( g \in L^2(\mathbb{R}) \),
\[
\int f_n g \to \int f g.
\]

V. 1) Let \( f \in L^1([0, 2\pi]) \). Prove that \( \int_0^{2\pi} f(x) \cos(nx) \, dx \to 0 \), as \( n \to \infty \).

You are asked to give a proof, not simply to quote a Theorem. By essentially the same proof, that you are not asked to repeat, one also has \( \int_0^{2\pi} f(x) \sin(nx) \, dx \to 0 \), as \( n \to \infty \).

Prove that, for any sequence \( (\alpha_n) \) in \( \mathbb{R} \),
\[
\int_0^{2\pi} f(x) \cos^2(nx + \alpha_n) \, dx \to \frac{1}{2} \int_0^{2\pi} f(x) \, dx.
\]

2) Let \( (a_n) \) and \( (b_n) \) be sequences in \( \mathbb{R} \) such that on a set of positive measure in \([0, 2\pi]\),
\[a_n \cos nx + b_n \sin nx \to 0.\]

Prove that \( a_n \) and \( b_n \to 0 \).

Hint: Write \( a_n \cos nx + b_n \sin nx = \rho_n \cos(nx + \alpha_n) \) and use that fact that \( \cos^2 \theta \leq |\cos \theta| \).

VI. (a) Give an example of a sequence of functions \( f_n \in L^1([0, 1]), n = 1, 2, \ldots \), with the following properties:

1. \( \lim_{n \to \infty} f_n(x) = 1 \) for any \( x \in [0, 1] \);

2. \( \int_0^1 |f_n(x)| \, dx = 2 \) for any \( n = 1, 2, \ldots \).

(b) Show that if the \( f_n \) are as in part (a) then
\[
\lim_{n \to \infty} \int_0^1 |f_n(x) - 1| \, dx = 1.
\]
PROBLEMS VII-IX, REAL ANALYSIS 2.

VII R. Let \( W \) be the space of continuous functions \( f \) on \([0,1]\), whose distributional derivative on \((0,1)\), is an integrable function. In one variable, this simply means that \( f(x) = f(0) + \int_0^x g(t) \, dt \), for some integrable function \( g \), and then \( f' = g \). On \( W \) one considers the norm defined by

\[
\|f\|_W = |f(0)| + \int_0^1 |f'(t)| \, dt.
\]

Let \( \Lambda \) be the space of continuous functions on \([0,1]\), that are Hölder continuous of order \( \frac{1}{2} \) (i.e. functions \( f \) such that for some constant \( C > 0 \), for every \( x \) and \( y \),
\[
|f(x) - f(y)| \leq C|x - y|^{\frac{1}{2}}.
\]

On \( \Lambda \) one considers the norm

\[
\|f\|_{\Lambda} = |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\frac{1}{2}}}.
\]

Equipped with these norms \( W \) and \( \Lambda \) are Banach spaces. And it is immediate that the diagonal \( \Delta \subset W \times \Lambda \), that is the set of all \((f,f)\) for \( f \in W \cap \Lambda \), is a closed subspace of \( W \times \Lambda \). You are not asked to justify the above.

1) Show that if \( f \in W \) and \( f' \in L^2 \) (not only \( \in L^1 \)), then \( f \in \Lambda \).

2) For each integer \( N > 0 \), set \( u_N(x) = \frac{1}{N} \sin(Nx) \). Find constants \( A_N \) such that, \( A_N \to 0 \) as \( N \to \infty \), and for every \( x \) and \( y \in \mathbb{R} \)

\[
|u_N(x) - u_N(y)| \leq A_N|x - y|^{\frac{1}{2}}.
\]

3) Prove that there is a Hölder continuous function \( f \), of Hölder exponent \( \frac{1}{2} \), defined on \([0,1]\) (i.e. \( f \in \Lambda \)) whose distributional derivative on \((0,1)\), is not an integrable function (i.e. \( f \notin W \)).

Even if you were not able to prove the result of question 2, you can use it for applying the open mapping theorem to the projection of \( \Delta \) on \( \Lambda \), when arguing by contradiction.
VIII R. Let $H$ be a Hilbert space on $\mathbb{R}$, with scalar product denoted by $\langle \cdot , \cdot \rangle$, and associated norm denoted by $\| \cdot \|$.

1) Assume that $(x_n)$ and $(y_n)$ are sequences in $H$ such that $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ and $\langle x_n, y_n \rangle \to 1$ as $n \to \infty$. Show that $(x_n - y_n) \to 0$ as $n \to \infty$.

2) Let $T$ be a continuous linear map from $H$ into itself.

2.1) Recall what is the definition of the adjoint operator $T^*$.

2.2) Assume that $T$ is self adjoint, i.e. that $T^* = T$. And assume that, for some sequence $x_n \in H$, with $\|x_n\| \leq 1$:

$$1 = \text{Sup}_{\|x\| \leq 1} \|T(x)\| = \lim_{n \to \infty} \|T(x_n)\| .$$

Using question 1, show that $T^2(x_n) - x_n$ tends to 0 as $n \to \infty$.

Conclude that at least one of the 2 operators $T + 1$ or $T - 1$ is not invertible. Here 1 denotes the identity map on $H$ (i.e. $1(x) = x$).

IX R. We shall use the following normalization for the Fourier transform on $\mathbb{R}^n$. When it makes (classical) sense:

$$\hat{f}(\xi_1, \cdots, \xi_n) = \int_{\mathbb{R}^n} f(y_1, \cdots, y_n) e^{-i(y_1\xi_1 + \cdots + y_n\xi_n)} dy_1 \cdots dy_n .$$

And the Fourier inversion formula is thus, when it makes sense:

$$f(x_1, \cdots, x_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi_1, \cdots, \xi_n) e^{i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \cdots d\xi_n .$$

1) Let $f$ be a a bounded function on $\mathbb{R}^2$, explain how (in the theory of tempered distributions) one defines its Fourier transform $\hat{f}$, even if $f$ is not integrable. What is the Fourier transform of the constant function 1?

2) Let $\varphi$ be a continuous function with compact support on $\mathbb{R}$. Its Fourier transform is therefore a continuous function $\hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} \varphi(x) e^{-ix\xi} \, dx$. Let $\phi$ be the function of two variables defined by

$$\phi(x_1, x_2) = \varphi(x_1) .$$

Find the Fourier transform of $\phi$, in terms of $\hat{\varphi}$?