Qualifying Exam
GEOMETRY/TOPOLOGY
Spring 2001

For the PhD level, do four problems from each part; where Problem 1 counts as two problems.

For the MA level, do five problems in all, with at least two problems from each part. Again, Problem 1 counts as two problems.

Part I: Manifold Theory

1. (20 points) Suppose M is a compact connected 3-manifold and ω is a nowhere zero 1-form defined on M. Suppose that the distribution $\ker \omega$ is integrable, and $\ker \omega = T\mathcal{F}$ for a foliation $\mathcal{F}$.

(4 points) (a) Show that $\omega \wedge d\omega = 0$.

(4 points) (b) Use a partition of unity to show that there is a 1-form α such that $d\omega = \alpha \wedge \omega$.

(2 points) (c) Show $d\alpha \wedge \omega = 0$.

(6 points) (d) Suppose that α' is some other 1-form satsifying $d\omega = \alpha' \wedge \omega$. Show that $\alpha' = \alpha + g\omega$ for some function g, and that $\alpha \wedge d\alpha = \alpha' \wedge d\alpha'$.

(4 points) (e) Suppose that ω' is a nowhere zero 1-form and $\ker \omega = \ker \omega'$. If $d\omega' = \gamma \wedge \omega$, show that $\alpha \wedge d\alpha - \gamma \wedge d\gamma$ is exact.

2. (10 points) On the compact connected manifold M, suppose α is a p-form and β is an $(n-p-1)$-form. Suppose ∂M has two components: $\partial_0 M$ and $\partial_1 M$. Let i_0 and i_1 be the inclusions of $\partial_0 M$ and $\partial_1 M$ into M. Given that $i_0^* \alpha = 0$ and $i_1^* \beta = 0$, show that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

3. (10 points) Suppose $f\colon S^1 \to R^2$ and $g\colon S^1 \to R^2$ are smooth embeddings. Let

$$M = \{(a, b, \vec{v}) \in S^1 \times S^1 \times R^2 \colon f(a) - g(b) = \vec{v}\}.$$

Show that M is a compact submanifold of $S^1 \times S^1 \times R^2$. Let $\pi\colon M \to R^2$ be the projection $\pi(a, b, \vec{v}) = \vec{v}$. Apply Sard's Theorem to π and deduce that for almost every $\vec{v} \in R^2$, $f(S^1)$ is transverse to $g(S^1) + \vec{v}$.

4. (10 points) Suppose that $f\colon M \to N$ is a C^∞ map, M and N are compact connected n-manifolds, and $rank(df) = n$. Show that f is a covering map.

Part II: Algebraic Topology

5. (10 points) Let X be a polyhedron, A a subpolyhedon, $p: \widetilde{X} \to X$ the universal covering space of X and $\bar{A}$ the path component of $p^{-1}(A)$ containing the equivalence class of the constant path at $x_0 \in A$.

(2 points) (a) Give an example in which $\bar{p}: \bar{A} \to A$ (where $\bar{p}$ is the restriction of p) is not the universal covering space of A.

(8 points) (b) Prove that $\bar{p}: \bar{A} \to A$ is the covering space of the kernel of $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ where i is inclusion.

6. (10 points) Let D^n be the unit ball in R^n, S^{n-1} its boundary, and $\mathbf{0} \in R^n$ the origin.

(5 points) (a) Prove that the inclusion $i: (D^n, S^{n-1}) \to (D^n, D^n - \mathbf{0})$ induces an isomorphism $i_*: H_n(D^n, S^{n-1}) \to H_n(D^n, D^n - \mathbf{0})$.

(5 points) (b) Prove that i is not a homotopy equivalence of pairs, that is, there is no map $g: (D^n, D^n - \mathbf{0}) \to (D^n, S^{n-1})$ such that gi and ig are homotopic, as maps of pairs, to identity maps.

7. (10 points) Given a map $f: X \to X$ of a polyhedron, there is an exact sequence

$$\to H_k(X) \xrightarrow{1-f_*} H_k(X) \longrightarrow H_k(T_f) \longrightarrow H_{k-1}(X) \to$$

where T_f is the mapping torus of f. Use the sequence to calculate the homology of the 3-manifold M obtained from $S^2 \times I$ by identifying $(x, 0)$ to $(-x, 1)$ for all $x \in S^2$.

8. (10 points) Let $A \subseteq X \subseteq Q$ and consider

$$H_k(Q, A) \xrightarrow{i_*} H_k(Q, X) \xrightarrow{\partial} H_{k-1}(X, A)$$

where i is inclusion and $\partial[z] = [\partial_k[z]]$ for $\partial_k: H_k(Q, X) \to H_{k-1}(X)$ that comes from the exact sequence of (Q, X).

(4 points) (a) Prove that ∂ is well-defined.

(6 points) (b) Prove that the image of i_* equals the kernel of ∂.

9. (10 points) Let $J(X, x_0) \subseteq \pi_1(X, x_0)$ be the subgroup of cyclic classes, where a class α is *cyclic* if there is a homotopy $\{h_t: X \to X\}$ with $h_0 = h_1 =$ identity such that $[h_t(x_0)] = \alpha$.

(7 points) (a) Prove that $J(X, x_0)$ is contained in the center of $\pi_1(X, x_0)$.

(3 points) (b) Prove that if X is a topological group, then $J(X, x_0) = \pi_1(X, x_0)$.

GEOMETRY-TOPOLOGY QUALIFYING EXAMINATION
September 25, 2002

1. Suppose $P(x,y,z)$, $Q(x,y,z)$, and $R(x,y,z)$ are C^∞ functions on $\mathbb{R}^3$ which vanish identically if $|x| \geq 5$, $|y| \geq 5$, or $|z| \geq 5$. Prove that the volume integral

$$\int_{-6}^{+6} \int_{-6}^{+6} \int_{-6}^{+6} d(Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy) = 0.$$

(Do this directly, not by quoting Stokes' Theorem: this is a special case of the proof of Stokes' Theorem!)

2. Suppose that $V = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}$ is a C^∞ vector field on $\mathbb{R}^3$ with $V \neq \vec{0}$ at the origin. Find a necessary and sufficient condition for there to exist a C^∞ function $\lambda(x,y,z)$ in some neighborhood of the origin such that λV is the gradient of a C^∞ function on the neighborhood.

3. Let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand rule rotation around the positive z-axis by t degrees and $S_s : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand-rule rotation around the positive x-axis by t degrees.

(a) Find the infinitesimal generators of the flows T_t and S_t, i.e., the vector fields X and Y, respectively, on $\mathbb{R}^3$ whose flows are $\{T_t\}$ and $\{S_t\}$.

(b) Compute the commutator

$$T_{-t} \circ S_{-t} \circ T_t \circ S_t.$$

(c) Compare the result of (b) (lowest order non-identically zero term) with the Lie bracket $[X,Y]$.

4. Take as given that a C^∞ 2-form ω on S^2 is of the form $d\theta$ for some C^∞ 1-form θ if and only if $\int_{X^2} \omega = 0$. Use this to show that every C^∞ 2-form Ω on $\mathbb{R}P^2$ has the form $d\Lambda$ for some C^∞ 1-form Λ. (Do not just quote DeRham's Theorem here.)

5. (a) Suppose $F : S^1 \to \mathbb{R}^3$ is a C^∞ function such that dF is nowhere zero (on S^1). Prove that there is a two-dimensional subspace P of $\mathbb{R}^3$ such that $\pi_P \circ F : S^1 \to \mathbb{R}^3$ has nowhere vanishing differential, where π_P = orthogonal projection on P.

(b) Show by example (a picture with explanation is all right) that there is such an F that is also 1 to 1 (injective) but is such that, for all P, $\pi_P \circ F$ fails to be injective.

(c) Show that if $F : S^1 \to \mathbb{R}^4$ is C^∞ and injective then there is a three-dimensional subspace H of $\mathbb{R}^4$ such that $\pi_H \circ F$ is injective, where π_H = orthogonal projection on H.

6. (a) Suppose $F : S^n \to S^n$ is fixed-point free (i.e., for all $p \in S^n$, $p \neq F(p)$). Show that F is homotopic to the antipodal map $p \to -p$, $p \in S^n$.

(b) Use part (a) to show that every vector field on (tangent to) S^{2n}, $n = 1, 2, 3 \ldots$, vanishes somewhere on S^{2n} (i.e., has a zero).

7. (a) Discuss carefully how to obtain the long exact sequence in homology from a short exact sequence of chain complexes. (Include definitions of the maps in the long exact sequence.)

(b) If the short exact sequence is

$$0 \to C_1 \to C_2 \to C_3 \to 0,$$

prove exactness of the long exact sequence at $H_k(C_3)$ [in $\ldots H_k(C_2) \to H_k(C_3) \to H_{k-1}(C_1) \ldots$].

8. (a) Suppose $F : T^2 \to T^2$ (where $T^2 = S^1 \times S^1$) is a continuous function such that $F(p) = p$ for some $p \in T^2$ and

$$F_* : \pi_1(T^2, p) \to \pi_1(T^2, p)$$

is the identity map. Is F necessarily homotopic to the identity map from T^2 to itself?

(b) Is a C^∞ map $F : T^2 \to T^2$ of degree 1 necessarily homotopic to the identity map of T^2 to itself? Explain/prove your answer.

9. (a) Discuss the (a) representation of $\mathbb{C}P^n$ as a cell complex.

(b) Use part (a) to find the homology of $\mathbb{C}P^n$: prove carefully that your calculation is correct.

10. (a) Let X = the space obtained by attaching two discs to S^1, the first disc being attached by $S^1 = \partial D_1 \to S^1$ being the 7 times around (counterclockwise) map, e.g., $z \to z^7$, $|z| = 1$, $z \in C$ and the second being attached by $S^1 = \partial D_2 \to S^1$ being the 5 times around map $z \to z^5$. Find the homology of X.

(b) Can X be made a C^∞ manifold? Why or why not?

M.A. or Ph.D. Qualifying Exam
GEOMETRY-TOPOLOGY
Spring 2002

Instructions:

For a Ph.D. pass do 4 problems from each section to a total of 8 problems. For a M.A. pass do 2 from one and 3 from the other section to a total of 5 problems.

Geometry

1. Let M be a closed (compact, without boundary) manifold. Show that any smooth function

$$f : M \to \mathbb{R}$$

has a critical point.

2. (a) Show that every closed 1-form on S^n, $n > 1$, is exact.

(b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$, is exact.

3. Let M^d be a d-dimensional manifold and $\omega_1, \dots, \omega_p$ be pointwise linearly independent 1-forms. If $\theta_1, \dots, \theta_p$ are 1-forms so that

$$\sum_{i=1}^{p} \omega_i \wedge \theta_i = 0,$$

then there exist smooth functions f_{ij} so that

$$\theta_i = \sum_{i=1}^{p} f_{ij}\omega_j, \quad i = 1, \dots, p.$$

(**Hint:** try $p = 1$)

4. Let M be the set of all straight lines in $\mathbb{R}^2$ (not just those which pass through the origin). Show that M is a smooth manifold and identify it with a well-known manifold.

(**Hint:** Lines not through the origin have a unique closest point to the origin and that point determines the line uniquely. What happens at the origin?)

5. Let $f : M^m \to N^n$ be a smooth bijection so that $Df : T_pM \to T_{f(p)}N$ is injective for all p. Show that f is a diffeomorphism.

Topology

6. (a) Show that if $f : S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.

   (b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

   (c) Show that if X has S^{2n+1} as universal covering space then X is orientable.

7. (a) Outline the construction of the universal covering of a path connected locally simply connected space X.

   (b) Give an example of a path connected space which does not have a universal covering space.

8. Let X be a finite cell complex constructed inductively by gluing all p-cells onto cells of dimension $< p$. Assume no $p-1$ and $p+1$ cells are used to construct X. Show that

$$H_p(X, \mathbb{Z}) \simeq \mathbb{Z}^{n_p}$$

when n_p is the number of p-cells used in the construction.

9. Let $(M, \partial M)$ be a compact oriented n-manifold with connected boundary ∂M. Show that there is no retract $r : M \to \partial M$, i.e., a map $r : M \to \partial M$ such that $r(x) = x$ if $x \in \partial M$.
(**Hint:** Prove that $H_{n-1}(\partial M) \to H_{n-1}(M)$ is trivial.)

10. Let $X = T^2 - \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.

   (a) Compute the homology groups $H_*(X, \mathbb{Z})$.

   (b) Compute the fundamental group of X.

Qualifying Exam
Geometry/Topology
Winter 2002

Instructions:

For the PhD level, do four problems from each part.

For the MA level, do five problems in all, with at least two problems from each part.

Part I: Differentiable Manifolds

1. Let M be a smooth three dimensional manifold and α is a 1-form on M s.t. $\alpha \wedge d\alpha \neq 0$ at every point of M. (10 points)

(i) Let $H = \ker\ \alpha \subseteq TM$. Show that H is a two-dimensional plane field of TM which is not integrable.

Hint: Use the formula $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$, where X, Y are two arbitrary vector fields.

(ii) Show that there exists a unique vector field V s.t.

(a) $\alpha(V) = 1$, (b) $\langle V \rangle \oplus H = TM$, (c) $d\alpha(V, W) = 0$

for any vector field W. Here $\langle V \rangle$ is the line field generated by V.

2. Let M be a closed smooth manifold and X be a vector field on M. Denote the flow generated by X by $\varphi_t : M \to M$, i.e., φ_t is defined by: (10 points)

$$\frac{d\varphi_t}{dt}(x) = X(\varphi_t(x)) \quad \text{for any} \qquad x \in M.$$

Given a function f, prove that:

$$f \circ \varphi_1 - f \circ \varphi_0 = \int_0^1 \varphi_t^*(df)(X)dt.$$

3. Let M_n be the space of $n \times n$ real matrices and M_n^k be the subspace of all matrices of rank k in M_n. (10 points)

(i) Show that M_n^k is a submanifold of M_n.

(ii) Find the dimension of M_n^k.

4. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ as usual. (10 points)

(a) Show that, for each C^∞ 1-form ω on S^2 with $d\omega = 0$, there is a C^∞ function $f : S^2 \to \mathbb{R}$ such that $df = \omega$.

(b) Show that, for each 2-form Ω on S^2 such that $\Omega = d\theta$ for some 1-form θ,

$$\int_{S^2} \Omega = 0.$$

(c) Is the converse of (b) true, i.e., is it true that if Ω is a 2-form on S^2 with $\int_{S^2} \Omega = 0$ then there is always a 1-form θ on S^2 such that $\Omega = d\theta$? Prove your answer.

5. Let S^2 be as in Problem 4. Consider the 2-form on $\mathbb{R}^3 - \{(0, 0, 0)\}$ (10 points)

$$\sigma = (x^2 + y^2 + z^2)^{-3/2}(x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy).$$

(a) Show that σ is closed on $\mathbb{R}^3 - \{(0, 0, 0)\}$.

(b) Show that the 2-form

$$\omega = x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy$$

is closed but not exact on S^2.

(c) Find $\int_{S^2} \omega$.

(d) Suppose M is compact, 2-dimensional, oriented embedded submanifold of $\mathbb{R}^3 - \{(0, 0, 0\}$. What are the possible values of $\int_M \sigma$? Prove your answer.

Part II: Algebraic Topology

6. (a) Define: chain complex, chain map, chain homotopy. (10 points)

(b) Prove that if $f_1, f_2 : C \to C'$ and $g_1, g_2 : C' \to C''$ are chain homotopic chain maps then $g_1 \circ g_1,\ g_2 \circ f_2 : C \to C''$ are also chain homotopic.

7. Let $p : \widetilde{X} \to X$ be a covering space and let $f : X \to X$ be a map such that $f(x_0) = x_0$. A map $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ such that $f(\widetilde{x}_0) = \widetilde{x}_0$ for some $\widetilde{x}_0 \in p^{-1}(x_0)$ is a *lift* of f if $p\widetilde{f} = fp$. (10 points)

(a) Prove that f has a *lift* if and only if $f_*(H) \subseteq H$ where $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$.

(b) Give an example of a space X, a map $f : X \to X$ and a covering space $p = \widetilde{X} \to X$ such that f has *no lifts* to $\widetilde{X}$.

8. The following diagram of groups and homomorphisms is commutative and both horizontal sequences are exact. The symbol "id" denotes the identity. Prove that if $c \in C$ such that $\gamma(c) = 1$ then there exists $b \in B$ such that $\beta(b) = 1$ and $\varphi(b) = c$, and thus that $\varphi(\ker \beta) = \ker \gamma$. (10 points)

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\varphi} & C & \xrightarrow{\delta} & D \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow[\alpha']{} & B' & \xrightarrow[\varphi']{} & C' & \xrightarrow[\delta']{} & D
\end{array}$$

9. Let (X_1, A_1) and (X_2, A_2) be pairs of finite polyhedra and subpolyhedra. (10 points)

(a) Write the *relative* Mayer-Vetoris sequence for the pair $(X_1 \cup X_2,\ A_1 \cup A_2)$. You do not have to define the homomorphism or prove anything about it.

(b) Use part (a) to prove that if X is a finite polyhedra, S^r is the r-sphere, $p_0 \in S^r$ and $k > r$ then

$$H_k(X \times S^r,\ X \times p_0) \simeq H_{k-r}(X).$$

10. Let $p : E \to B$ be a covering space and $f : X \to B$ a map. Define (10 points)

$$E^* = \{(x, e) \in X \times B : f(x) = p(e)\}.$$

Prove that $q = E^* \to X$ defined by $q(x, e) = x$ is a covering space.

M.A. or Ph.D. Qualifying Exam
GEOMETRY/TOPOLOGY
Fall 2003

Instructions: All problems are worth ten points.

1. Explain carefully how the classical "divergence theorem"

$$\iint_S \vec{V} \cdot \vec{n}\ d(\text{area}) = \iiint_V \text{div}\ \vec{V}\ d(\text{volume})$$

(V a bounded volume in $\mathbb{R}^3, S =$ boundary of V) follows from the Stokes' Theorem for differential forms.

2. Without using deRham's Theorem, prove:

(a) every closed 1-form on S^2 is exact.

(b) a two-form Ω is exact on S^2 if and only if

$$\int_{S^2} \Omega = 0.$$

3. Show that the set of all lines in $\mathbb{R}^2$ has a natural structure as a differentiable manifold. What (already familiar) manifold is it?

4. Show that S^3 is the union of two solid tori ($S^1\times$ 2-disc) with a embedded torus ($S^1 \times S^1$) as their common boundary. (**Hint:** Express $\mathbb{R}^3$ with a solid torus removed as a union of circles and a single straight line and then add a point at infinity.)

5. Suppose M is a compact manifold (with empty boundary).

(a) Prove that, if $f : M \to \mathbb{R}$ is a C^∞ function, then f has at least two critical points.

(b) A C^∞ function on $S^1 \times S^1$ cannot have only two critical points. Prove this (e.g.) by deforming a homotopically nontrivial S' along the gradient flow of $f : M \to \mathbb{R}$.

6. (a) Prove carefully that a group of homeomorphisms of S^{2n}, each of which has no fixed points (unless it is the identity map), contains at most two elements.

(b) Give a counterexample for some S^{2n+1}, $n \geq 1$.

7. Find the homology groups with $\mathbb{Z}$ coefficients, of $\mathbb{R}P^n$, $n = 2, 3, 4 \ldots$ by some systmatic rigorous method.

8. Find the homology and the fundamental group of $S^1 \times S^1$ with two points removed.

9. Prove that if a compact (empty boundary) manifold X has S^{2n+1}, $n \geq 1$, as a covering space, then X is orientable.

10. Suppose M is a compact orientable manifold (empty boundary). Prove that

$$H_n(M,\ \mathbb{Z}) \simeq \mathbb{Z}.$$

(You may assume M is triangulated.)

GEOMETRY/TOPOLOGY QUALIFYING EXAMINATION

January 18, 2003

Manifold Theory

1. Let M be a smooth compact manifold of dimension n. Show that there is no immersion of M into $\mathbb{R}^n$.

2. The n-dimensional torus T^n is defined to be $\mathbb{R}^n/\mathbb{Z}^n$, i. e. for any x and y in $\mathbb{R}^n$, $x \sim y$ iff $x - y \in \mathbb{Z}^n$. Let α and β be two functions on $\mathbb{R}^n$ such that (i) $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$ iff $x - y \in \mathbb{Z}^n$ and (ii) α/β is an irrational constant. Then

$$v = \alpha(x)\frac{\partial}{\partial x^1} + \beta(x)\frac{\partial}{\partial x^2}$$

is a vector field on $\mathbb{R}^n$ descending to T^n, where

$$\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\}$$

are coordinate vector fields. Find all functions f on T^n such that $vf = 0$.

3. Let M and N be smooth compact connected manifolds and $f: M \to N$ be a smooth map such that, for any point $m \in M$, $rank(df_m) = \dim(N)$. Show that (i) for any $n \in N$, $f^{-1}(n)$ is a submanifold of M and (ii) for any n_1 and n_2 in N, the submanifolds $f^{-1}(n_1)$ and $f^{-1}(n_2)$ of M are diffeomorphic to each other.

4. Let

$$\tilde{\theta} = \frac{1}{2}\{(x^2dx^1 - x^1dx^2) + (x^4dx^3 - x^3dx^4) + \cdots + (x^{2n}dx^{2n-1} - x^{2n-1}dx^{2n})\}$$

be a 1-form on $\mathbb{R}^{2n}$ and θ be its restriction to the unit sphere

$$S^{2n-1} = \{x = (x^1, \dots, x^{2n}) | (x^1)^2 + \cdots + (x^{2n})^2 = 1\}.$$

The kernel K of θ is a distribution on S^{2n-1}:

$$K = \{v | v \in TS^{2n-1}, \theta(v) = 0\}.$$

Decide whether or not K is integrable.

5. Let $T^{2m} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be torus of dimension $2n$. Consider the 2-form

$$\omega = dx^1 \wedge dx^{n+1} + dx^2 \wedge dx^{n+2} + \cdots + dx^n \wedge dx^{2n}$$

defined on $\mathbb{R}^{2n}$ descending to T^{2n}.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

(i) Show that ω is closed but not exact on T^{2n}.

(ii) Let $i: T^n \to T^{2n}$ be the subtorus defined by the equation

$$x^{n+1} = x^{n+2} = \cdots = x^{2n} = 0.$$

What is $i^*\omega$?

(iii) Let $\Sigma = S^2 \setminus \{\cup_{i=1}^m D_i\}$, where D_i, $i = 1, \ldots, m$ are m open disks in S^2 with disjoint closures. Show that

$$\int_\Sigma f_1^*\omega = \int_\Sigma f_2^*\omega$$

if $f_1, f_2: (\Sigma, \partial\Sigma) \to (T^{2n}, T^n)$ are homotopic to each other, where $\partial\Sigma$ is the boundary of Σ.

Algebraic Topology

1. Let X be a path connected space and let $x_0, x_1 \in X$. Prove carefully that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

2. (i) Define what is meant by a "chain homotopy" P between chain maps $f_\#, g_\#: C \to D$ and prove that chain homotopic chain maps induce the same homomorphism of homology. (ii) Let X and Y be spaces and let $F: X \times I \to Y$ be a homotopy between maps f and g. Define a chaim homotopy P between the induced chain maps $f_\#, g_\#: C(X) \to C(Y)$ of singular chains. (iii) Verify that P satisfies the definition of a chain homotopy ONLY for the restriction of P to $C_1(X)$.

3. Let X be a locally contractible space and H a subgroup of $\pi_1(X, x_0)$. Describe carefully how to construct a topological space X_H and a map $p: X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ and show that it has the required property of p_*. (Note: Although X_H will be a covering space, you don't have to verify this unless you want to use some general properties of covering spaces.)

4. Prove that the real even-dimensional projective spaces $\mathbb{R}P^{2n}$ have the fixed point property, that is, every for every map $f: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ there is a solution to $f(x) = x$. (Hint: Consider maps on the covering space S^{2n}.)

5. Use the Mayer-Vietoris sequence to calculate the homology of $S^1 \times S^2$. You may assume the homology calculations for $S^1, S^1 \times S^1$ and S^2.

M.A. or Ph.D

Qualifying Exam

Geometry/Topology

Fall 2004

Instruction: All problems are worth ten points.

1. Let M be a connected smooth manifold. Construct the orientation cover M_0.

   a) Show that M_0 is a smooth manifold.

   b) Show that M_0 is a 2:1 covering of M.

   c) Show that M is orientable iff M_0 is the union of two disconnected components.

2. Let ω be a smooth nowhere vanishing 1-form on a smooth connected manifold M.

   a) Show that ker ω is a smooth co-dimension 1 distribution on M.

   b) Show that ker ω is integrable iff $d\omega$ vanishes on ker ω.

   c) Find a co-dimension 1 distribution on $\mathbb{R}^3$ that is not integrable.

3. Show that $S^1 \times S^n$ is parallelizable, i.e., one can find $(n+1)$ vector fields that are everywhere linearly independent.
   ($S^k \subset \mathbb{R}^{k+1}$ is the unit sphere)

4. Let $\omega = \frac{-ydx+xdy}{(x^2+y^2)^\alpha}$ and consider $\int_\gamma \omega$, where $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$.

   a) For which α is $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$, whenever γ_1 and γ_2 are smoothly homotopic, i.e., then exists $F : S^1 \times [0,1] \to \mathbb{R}^2 - \{0\}$ such that $\gamma_0(t) = F(t,0)$, $\gamma_1(t) = F(t,1)$?

   b) What are the possible values for $\int_\gamma \omega$ when α is chosen as in part a)?

5. Show that a closed (compact without boundary) n-manifold cannot be immersed in $\mathbb{R}^n$.

6. Let $\mathbb{C}^*$ be the set of all non-zero complex numbers with the induced topology from $\mathbb{C}$. It is a topological group with respect to the usual multiplication. Let f be a continuous homomorphism from C^* to itself.

   (i) Find all possible $f|_{S^1}$, where $S^1 = \{z \mid |z| = 1, z \in \mathbb{C}^*\}$.

   (ii) Classify such $f|_{S^1}$ up to homotopy.

7. Let $X_1 = S^1 \vee_{x_1=x_2} S^2$ be the space obtained from the disjoint union of the circle S^1 and the S^2 by identifying a point $x_1 \in S^1$ with a point $x_2 \in S^2$. Define $X_2 = S^1 \vee_{x_1=x_2} S^1$ similarly.

   (i) Find $\pi_1(X_1)$ and $\pi_1(X_2)$.

   (ii) Find their universal coverings.

8. Let $f : S^2 \to T^2$ be a continuous map from 2-sphere to 2-torus T^2.
   What is the induced map

$$f_* : H_*(S^2) \to H_*(T^2)$$

   on the homology groups?

9. Let X be a topological space, and define $S(X)$ to be the quotient space of $X \times I$ by contracting $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. Here $I = [0, 1]$. What is the relationship between $H_*(S(x))$ and $H_*(x)$?

10. Let K be a finite simplicial complex and K^n be the subcomplex consisting of all simplices in K of dimension less than or equal to n. Denote the underlying topological spaces of K and K^n by $|K|$ and $|K^n|$.

    (i) What is the relative singular homology $H_*(|K^n|, |K^{n-1}|)$?

    (ii) Write down the long exact sequence for the triple $(|K^n|, |K^{n-1}|, |K^{n-2}|)$, i.e. , the long exact sequence relating the singular homology groups $H_*(|K^n|, |K^{n-1}|)$, $H_*(|K^{n-1}|, |K^{n-2}|)$ and $H_*(|K^n|, |K^{n-2}|)$.

    (iii) Use (i) and (ii) to show that singular homology of $|K|$ is same as the simplicial hmology of $|K|$. (Hint: identify the connecting boundary map in (ii)).

GEOMETRY/TOPOLOGY QUALIFYING EXAMINATION

Winter, 2004

Manifold Theory

1. (a) Let $M = SL(2,\mathbb{R}) = \{A \in M_2\mathbb{R}; \det A = 1\}$. Show that M is a submanifold of $M_2(\mathbb{R})$ (the space of two-by-two matrices). Given $A \in M$, regard T_AM as a subspace of $M_2\mathbb{R}$. Consider three vector fields H, X, Y on M defined by

$$H(A) = A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X(A) = A \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y(A) = A \cdot \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \in T_AM.$$

Find the flows of H, X and Y.

(b) Show that $[H, X] = 2X$.

2. State the general Stokes Theorem, and explain how the classical version

$$\int\int_S (\nabla \times \vec{v}) \cdot \vec{n}\, dA = \int_{\partial S} \vec{v} \cdot d\vec{r}$$

follows. Here S is a compact surface in $\mathbb{R}^3$ with normal vector $\vec{n}$ and boundary ∂S, and $\vec{r}$ is the position vector.

3. Describe diffeomorphisms between $SO(3), \mathbb{R}P^3$ and $UT(S^2)$, the unit tangent bundle of S^2. You need not check that the maps are smooth. ($SO(3)$ is the special orthogonal group and $UT(S^2)$ is the set of tangent vectors of length one.)

4. Let X be the space of symmetric n-by-n real matrices and let X_k be the subspace of matrices of rank k in X. Show that X_k is a submanifold and find its dimension.

5. Suppose that $f: M \to N$ is C^∞, M and N are compact connected n-manifolds, and $rank(df) = n$. Show that f is a covering map.

Algebraic Topology

6. Consider the exact sequence of abelian groups and homomorphisms

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

Prove that if there is a homomorphism $\gamma: B \to A$ such that $\gamma\alpha: A \to A$ is the identity, then B is isomorphic to $A \oplus C$.

7. Prove that the n-sphere S^n admits a continuous field of nonzero tangent vectors if and only if n is odd.

8. Let $p: \widetilde{X} \to X$ be the universal covering space of a space X and let $f: X \to X$ be a map.

(a) Prove that there exist *lifts* of f to $\widetilde{X}$, that is, maps $\tilde{f}: \widetilde{X} \to \widetilde{X}$ such that $p\tilde{f} = fp$.

(b) Suppose $\tilde{f}_1, \tilde{f}_2$ are lifts of f and there exist $\tilde{x}_1, \tilde{x}_2 \in \widetilde{X}$ such that $\tilde{f}(\tilde{x}_1) = \tilde{x}_1, \tilde{f}(\tilde{x}_2) = \tilde{x}_2$ and $p(\tilde{x}_1) = p(\tilde{x}_2)$. Prove that there exists a covering transformation $\sigma: \widetilde{X} \to \widetilde{X}$ such that $\tilde{f}_2 = \sigma \tilde{f}_1 \sigma^{-1}$.

9. Let $X_k = S^1 \times D^2 - \{p_1, p_2, \ldots, p_k\}$ be the solid torus (circle cross disc) with $k \geq 1$ points deleted from its interior. Calculate the homology of X_k.

10. Let $\Omega(X)$ denote the *loop space* of a metric space X with metric d. That means $\Omega(X)$ is the set of all maps $a: [0,1] \to X$ such that $a(0) = a(1)$, with the topology given by the metric $d(a,b) = \max_{0 \leq t \leq 1} d(a(t), b(t))$. Suppose $a, b \in \Omega(X)$ such that $a(0) = b(0) = x_0$. Prove that the classes $[a], [b] \in \pi_1(X, x_0)$ are conjugate in $\pi_1(X, x_0)$ if and only if a and b lie in the same path component of $\Omega(X)$.

Geometry/Topology

QUALIFYING EXAM

September 2005

Do all problems

Manifold Theory

1. (a) Define complex projective space , P^n, $n \geq 1$, and prove that it is a compact differentiable manifold.
   (b) Show that P^n is orientable for all n.
   (c) Prove that P^1 is diffeomorphic to S^2.

2. Suppose N is an embedded submanifold of a (C^∞) manifold M. A vector field X on M is *tangent to* N if $X(p) \in T_pN \subset T_pM$ for all $p \in N$.
   (a) Prove that if X and Y are vector fields on M that are each tangent to N, then $[X, Y]$ is also tangent to N.
   (b) Illustrate this principle for two vector fields (your choice) tangent to $S^2 \subset {}^3$ (with $[X, Y] \not\equiv 0$), computing $[X, Y]$ and checking that $[X, Y]$ is tangent to S^2

3. DeRham's Theorem says that (closed p-forms)/(exact p-forms) $\cong H^p(M, \ )$ for a C^∞ manifold M. Verify this in the special case $M = S^2$, where $H^1 = 0$, $H^2 = 1$ (given) by proving, by direct constructions:
   (a) Every closed 1-form $\dot{\omega}$ (i.e., $dw = 0$) is exact (i.e., $\omega = df$, some function f).
   (b) A (closed, necessarily) 2-form Ω on S^2 is $d\Theta$, for some 1-form Θ, if and only if $\int_{S^2} \Omega = 0$.
   (c) There is a 2-form Ω on S^2 such that $\int_{S^2} \Omega \neq 0$.

4. A vector field V on ^{3} is said to be *gradient-like* at a point $(x, y, z) \in {}^3$ if there is a neighborhood U of (x, y, z) and a nowhere-vanishing -valued function λ on U with the property that $\text{curl}(\lambda V) \equiv \vec{0}$ on U. (So λV is the gradient of a function in a neighborhood of (x, y, z).)
   Use the Frobenius Theorem to find a condition under which a nowhere vanishing vector field V on ^{3} is gradient-like at each $(x, y, z) \in {}^3$. Demonstrate that your condition works

5. Prove that if M is a compact C^∞ manifold, then for some positive integer k there is a C^∞ mapping $F : M \to {}^k$ such that $dF|_q$ is injective for all $q \in M$.

Algebraic Topology

6. (a) Write down the Mayer-Vietoris sequence associated to a pair of open sets U and V with $U \cup V =$ a topological space X.
   (b) Describe explicitly how the dimension-lowering map(s) in this long exact sequence arise and prove that this map is well-defined.

7. Let X be a space that has a simply-connected covering space $p : \tilde{X} \to X$.
   (a) Prove that X is semi-locally simply connected. (i.e. each $x \in X$ has a neighborhood such that every loop at x that is in the neighborhood is contractible in X.)
   (b) Prove that $p : \tilde{X} \to X$ is "universal" in the sense that, given any covering space $p' : X' \to X$, there is a covering space $q : \tilde{X} \to X'$ with $p = p'q$.

8. (a) Prove that if X is a topological manifold of dimension n then, for each $x_0 \in X$, the relative homology $H_n(X, X - \{x_0\})$ is isomorphic to .
   (b) Explain how if X is a C^∞ n-manifold that is orientable, then an orientation of X picks out a particular generator of $H_n(X, X - \{x_0\})$ for each $x_0 \in X$.

9. Let X be the space obtained by deleting from the closed ball of radius 2 in ^{3} the unit circle in the (x, y) plane, i.e.,

$$X = \{(x, y, z) \in \ ^3 : x^2 + y^2 + z^2 \leq 2\} \backslash \{(x, y, 0) \in \ ^3 : x^2 + y^2 = 1\}.$$

   Compute the homology groups of X.

10. Viewing the unit circle S^1 in the plane as the complex numbers of norm one, let $\mu : S^1 \times S^1 \to S^1$ be complex multiplication. Given maps $f, g : S^1 \to S^1$, **define** their "product" $h : S^1 \to S^1$ by $h(z) = \mu(f(z), g(z))$. Prove that the degrees are related by $\deg(h) = \deg(f) + \deg(g)$. (Hint: First show that, for maps of the circle, degree can be defined in terms of the fundamental group.)

Geometry/Topology
QUALIFYING EXAM
Winter 2005

Manifold Problems

1. Let M^2 be a smooth 2-manifold and $f : M^2 \to \mathbf{R}$ be a smooth surjective map without critical points. Assume that for any finite closed interval $[a,b] \hookrightarrow \mathbf{R}$, $f^{-1}([a,b])$ is compact. What is M^2?

2. Show that $T^2 \times S^2$ is parallelizable, i.e., there are 4 vector fields that are everywhere linearly independent.

3. Let $V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ be a nowhere zero C^∞ vector field on $\mathbf{R}^3$. Show that the following three statements are equivalent.

   a) The orthogonal-to-V plane field is integrable on some neighbourhood of $\mathbf{0} \in \mathbf{R}^3$.

   b) There exists a nowhere-zero C^∞ function $f : \mathbf{R}^3 \to \mathbf{R}$ such that $\operatorname{curl}(fV) \equiv \mathbf{0}$ on some neighbourhood of $\mathbf{0} \in \mathbf{R}^3$.

   c) $V \cdot \operatorname{curl}(V) \equiv 0$ on some neighbourhood of $\mathbf{0} \in \mathbf{R}^3$.

4. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function and $x \in \mathbf{R}^n$ be a critical point of f. The Hessian $H(t)_x$ at x be a bilinear form: $T_x\mathbf{R}^n \times T_x\mathbf{R}^n \to \mathbf{R}$ defined as follows. For any two vectors V_1 and V_2 in $T_x\mathbf{R}^n$, extend V_2 to a vector field $\tilde{V}_2$ near x, and define $H(f)_x(V_1,V_2) =: D_{v_1}(D_{\tilde{v}_2} f)$.
   Show that:

   (1) $H(f)_x(V_1,V_2) = H(f)_x(V_2,V_1)$.

   (2) $H(f)_x(V_1,V_2)$ is independent of the choice of the extension $\tilde{V}_2$.

5. (1) State Stokes' Theorem in its most general form.
   (2) Use the Stokes' Theorem to prove that for any vector field X defined on $\mathbf{R}^n$, $\int_\Omega (\operatorname{div} X) dx^1 \cdots dx^n = \pm \int_{\partial\Omega} (X \cdot N) ds$ where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and a unit normal field N on $\partial\Omega$. Here ds is the "area" form.

Geometry/Topology
QUALIFYING EXAM
Winter 2005

Topology Problems

1. Sketch the proof of:

   THEOREM. *If D is a subspace of S^n homeomorphic to I^k for some $k \geq 0$ then the reduced homology groups $\tilde{H}_i(S^n - D, \mathbf{Z})$ are trivial for all i.*

   (Hint: Induction on k.) (This is a special case of Alexander duality. No credit for saying "Applying Alexander duality ...".)

2. Show that $\mathbf{R}P^3$ is not homotopy equivalent to $\mathbf{R}P^2 \vee S^3$. (You could use cup products, degree, or covering spaces.)

3. Suppose $F : X \times I \to Y$ is a homotopy between $f : X \to Y$ and $g : X \to Y$.
   (6 pts) a) Indicate how to construct prism operators $P : C_n(X) \to C_{n+1}(Y)$ satisfying $g_* - f_* = \partial P + P\partial$ where $f_* : C_n(X) \to C_n(Y)$, $g_* : C_n(X) \to C_n(Y)$ are the chain maps.
   (4 pts) b) Show that the induced homomorphisms $H_n(f)$, $H_n(g)$ are equal.

4. Give examples of a) two nonhomeomorphic connected regular 3-sheeted covering spaces of the bouquet of two circles and b) an irregular connected 3-sheeted cover of the bouquet of two circles.

5. (5 pts) a) Find the Euler characteristic of X_4^2, the 2-skeleton of the 4-simplex.
   (5 pts) b) Give a reason why $H_2(X_4^2)$ is free abelian and find its rank.

Qualifying Exam
GEOMETRY-TOPOLOGY
Spring 2006

All ten problems have equal value.

Part I: Differentiable Manifolds

1. (i) Suppose that M is a closed (that is, compact and without boundary) smooth m-manifold. Show that there is a smooth embedding $f\colon M \hookrightarrow \mathbb{R}^n$ for sufficiently large n.
(ii) Adapt/extend your argument to show that if $g\colon M \to \mathbb{R}^n$ is a given *continuous* map, then the smooth embedding (of part (i))$f\colon M \hookrightarrow \mathbb{R}^n$ can be chosen to be arbitrarily (pointwise) close to g (again for sufficiently-large-but-fixed n).

2. Let $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ be a 2-form defined on $\mathbb{R}^{2n}$, where $(x_1, y_1, \cdots, x_n, y_n)$ are the coordinates of $\mathbb{R}^{2n}$.
(i) Show that as a bilinear form defined on $\mathbb{R}^{2n}$, ω is non-degenerate.
(ii) Let $f\colon \mathbb{R}^{2n} \to \mathbb{R}^1$ be smooth. Show that there is a unique vector field X_f on $\mathbb{R}^{2n}$ such that for any vector field Y on $\mathbb{R}^{2n}$, $df(Y) = \omega(X_f, Y)$.
(iii) Use the formula $\mathcal{L}_X = i_X \circ d + d \circ i_X$ to compute the Lie derivative $\mathcal{L}_{X_f}\omega$. Here $i_X\colon \Omega^k(\mathbb{R}^{2n}) \to \Omega^{k-1}(\mathbb{R}^{2n})$ denotes the *interior product* (or *contraction*) defined by

$$i_X(\eta)(Y_1, \ldots, Y_{k-1}) := \eta(X, Y_1, \ldots, Y_{k-1}).$$

3. Let $\omega = (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} \sum_{i=1}^n (-1)^{i-1} x_i \, dx_1 \wedge dx_2 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$
be an $(n-1)$-form defined on $\mathbb{R}^n - \{\mathbf{0}\}$.
(i) Suppose f is a smooth map from a closed oriented manifold M of dimension $n-1$ to $\mathbb{R}^n - \{\mathbf{0}\}$. Show that $\int_M f^*\omega$ only depends on the homotopy class of f.
(ii) Find the possible values of the integrals in (i) in the case that $n = 3$ and $M = S^2$.

4. Suppose M and N are two smooth manifolds of positive dimensions m and n respectively, and f is a smooth map from M to N.
(i) If $m < n$, is it possible that f is surjective? Justify appropriately your answer.
(ii) If $m \geq n$, must some point-inverse $f^{-1}(y)$ be a smooth $(m-n)$-dimensional submanifold of M? Justify appropriately your answer.
(iii) Show that some point-inverse $f^{-1}(y)$ can be homeomorphic to a Cantor set.
Hint: The model case is where $f\colon \mathbb{R}^1 \to [0, \infty) \subset \mathbb{R}^1$, with $f^{-1}(0) =$ Cantor set. The desired f can be constructed as a suitable limit of sums of C^∞ bump functions. Supply details, extending your argument to the general case (arbitrary M and N) of the question.

5. (i) Let $f\colon \mathbb{R}^n \to \mathbb{R}^1$ be a smooth function. Show that there are smooth functions $g_1, \cdots, g_n$ from $\mathbb{R}^n$ to $\mathbb{R}^1$ such that $f(\boldsymbol{x}) = f(\mathbf{0}) + \sum_{j=1}^n g_j(\boldsymbol{x})x_j$ and $g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$, where $\boldsymbol{x} = (x_1, \ldots, x_j, \ldots, x_n)$. (Hint: Recall that such g_j's can be defined using integrals.)

(ii) Let F be a diffeomorphism of $\mathbb{R}^n$ to itself. Use (i) to find a smooth isotopy (= a smooth homotopy which is a diffeomorphism at each fixed time of the homotopy) between F and $DF(\mathbf{0})$. (Hint: To find the isotopy F_t, $0 \le t \le 1$, you may assume that $F(\mathbf{0}) = \mathbf{0}$ (justify this) and then define $F_t(\boldsymbol{x}) = F(t\boldsymbol{x})/t$ for $0 < t \le 1$.)

Part II: Algebraic Topology

6. (i) Define what it means for two spaces X and Y to be *homotopically equivalent* (equivalently, to have the *same homotopy type*).

(ii) Define what it means for a space W to be *contractible.* (If you wish, you may reference your definition in part (i).)

(iii) Suppose that X is a manifold and W is an arbitrary contractible space. Show that X and the wedge $Y := X \underset{x_0 \sim w_0}{\vee} W$ are homotopically equivalent. (Here $x_0 \in X$ and $w_0 \in W$ are (arbitrary) points, and $X \underset{x_0 \sim w_0}{\vee} W$ is the one-point union of X and W at these points.)

7. (i) Define what it means for a map $p\colon X \to Y$ to have the (*unique*) *homotopy lifting property* (= HLP here), equivalently known as the (*unique*) *covering homotopy property* (= CHP). Recall (partly to establish some notation) that the definition begins:
$p\colon X \to Y$ has the HLP if, given any space W and homotopy $F\colon W \times [0,1] \to Y$ such that ....

(ii) Show that a covering map $p\colon X \to Y$ has the HLP for the special case where W is a point.

8. (i) Suppose that $X = U \cup V$ is the union of two open subsets U and V whose intersection is path-connected, and let $x_0 \in U \cap V$. State (carefully) the (Seifert -)van Kampen Theorem for these data, relating $\pi_1(X, x_0)$ to $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$.

(ii) Prove the special case of this theorem which asserts that the natural homomorphism $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is an epimorphism.

9. Suppose that $\mathcal{A} = \{\partial_n^A : A_n \to A_{n-1} \mid n \geq 0\}$ is a chain complex (with it understood that $A_{-1} = 0$).
(i) Define the nth homology group $H_n(\mathcal{A})$.
(ii) Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are chain complexes, with (connecting) homomorphisms $\alpha = \{\alpha_n \colon A_n \to B_n\}$ and $\beta = \{\beta_n \colon B_n \to C_n\}$ such that

$$
\begin{array}{ccccccccc}
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow[\partial_{n+1}^A]{} & A_n & \xrightarrow[\partial_n^A]{} & A_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow[\partial_{n+1}^B]{} & B_n & \xrightarrow[\partial_n^B]{} & B_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow[\partial_{n+1}^C]{} & C_n & \xrightarrow[\partial_n^C]{} & C_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}
$$

all squares commute and all (vertical) columns are exact. Show how to define the *boundary homomorphism* $\partial_n \colon H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$, and justify that it is well-defined.
(iii) Define and prove *exactness* at $H_{n-1}(\mathcal{A})$ (for the long exact sequence $\cdots \to H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A}) \to H_{n-1}(\mathcal{B}) \to \cdots$).

10. Let S^p and S^q be (standard) spheres of (arbitrary) dimensions $p \geq 0$ and $q \geq 0$. Compute the homology groups $H_n(S^p \times S^q)$ for all $n \geq 0$. You may use any reasonable method (e.g. Mayer-Vietoris, or cellular homology), as long as you present your argument with suitable completeness and clarity.

Qualifying Exam
GEOMETRY-TOPOLOGY
September 2007

Each problem is worth 10 points. In order to pass this examination, you must demonstrate proficiency in both parts, differentiable manifolds and algebraic topology. In particular, if you score fewer than 20 points on one part, you will not pass no matter how well you do on the other part.

Part I: Differentiable Manifolds

1. Let M_n be the linear space of all $n \times n$ of real matrices and S_n be the subspace of all $n \times n$ symmetric matrices. Consider the smooth map $\psi: M_n \to S_n$ defined by $\psi(A) = A^t A - I_n$ for $A \in M_n$, where A^t is the transpose of A and I_n is the identity matrix. (a) (5 points) Show that $0 \in S_n$ is a regular value of ψ. (b) (5 points) Use (a) to show that the group $O(n)$ of all orthogonal $n \times n$-matrices is a compact Lie group.

2. Let $f: \mathcal{R}^{n+1} \to \mathcal{R}$ be a smooth function defined on $\mathcal{R}^{n+1}$. Assume that $0 \in \mathcal{R}$ is a regular value of f. (a) (3 points) Show that $M = f^{-1}(0)$ is a smooth submanifold of $\mathcal{R}^{n+1}$. (b) (3 points) Show that M has a non-vanishing normal field, i.e. there is a smooth map $N: M \to \mathcal{R}^{n+1}$ with $N(x) \neq 0$ for any $x \in M$ such that $< N(x), v >$ $= 0$ for any tangent vector V of M at any point x in M. (c) (4 points) Use (b) to show that $M \times S^1$ is parallelizable, i.e, there exist $n+1$ linearly independent vector fields defined on $M \times S^1$.

3. Let $S^{2n-1} = \{z = (z_1, \cdots, z_n) | \, |z_1|^2 + \cdots |z_n|^2 = 1\}$ be the unit sphere in $\mathcal{R}^{2n}$ identified with $\mathcal{C}^n$, where $z_i, i = 1, \cdots, n$, are the complex coordinates of a point z in $\mathcal{C}^n$. The cyclic group $\mathcal{Z}_p = \mathcal{Z}/(p)$ of order p acts on S^{2n-1} freely by the formula: $\phi(z) = e^{\frac{2\pi i}{p}} z$, where ϕ is the generator of $\mathcal{Z}_p$. Let M be the quotient space of S^{2n-1} under this action. (a) (5 points) Show that any closed 1-form on S^n is exact if $n > 1$. (b) (5 points) Show that the same conclusion as (a) is true for $M = S^{2n-1}/\mathcal{Z}_p$ if $n > 1$.

4. Let M be the space defined in Problem 3 with $n > 1$. (a) (5 points) Let $f: M \to T^m$ be a continuous map, where $T^m =$

$S^1 \times \cdots \times S^1$ is the standard m-dimensional torus. Show that f is homotopically trivial. (b) (5 points) Construct explicitly a homotopically non-trivial map $f: M \to S^{2n-1}$.

5. Let D be a bounded domain in $\mathcal{R}^n$ with a smooth boundary S and let X be a smooth vector field defined on $\mathcal{R}^n$. (a) (5 points) Let $\omega = dx_1 \cdots dx_n$. Show that the Lie derivative $\mathcal{L}_X \omega = div(X)\omega$. (b) (5 points) Use Stokes' theorem to show that

$$\int_D div(X)\,\omega = \pm \int_S < X, N > dS.$$

Here N is the outer unit normal vector field along S, $< X, N >$ is the Euclidean inner product of X and N, and "dS" is the "area" form on S. Explain carefully the geometrical meaning of the term "dS".

Part II: Algebraic Topology

6. Let $f, g: X \to Y$ be homotopic maps of not necessarily connected spaces. Prove the following for the specific dimensions given; do not refer to any general result. (a) (3 points) Prove that $f_{*,0} = g_{*,0}: H_0(X) \to H_0(Y)$. (a) (7 points) Prove that $f_{*,1} = g_{*,1}: H_1(X) \to H_1(Y)$.

7. Let $x_1, x_2 \in S^1$, the circle, and define subsets of the torus $T = S^1 \times S^1$ as follows: $A = (S^1 \times \{x_1\}) \cup (\{x_1\} \times S^1)$ (a figure-eight) and $B = (S^1 \times \{x_1\}) \cup (S^1 \times \{x_2\})$ (two disjoint circles). Assume the homology of T is known. (a) (4 points) Calculate the homology of the pair (T, A). (b) (6 points) Calculate the homology of the pair (T, B).

8. (a) (3 points) Prove that the 2-sphere S^2 is simply-connected. (b) (7 points) Each compact orientable surface S_g can be obtained by adding g handles to S^2, that is, g pairs of open discs are removed from S^2 and g copies of $S^1 \times [0, 1]$ are attached to pairs of bounding circles. Prove that S^2 is the only compact orientable surface that is simply-connected.

9. (a) (2 points) Define the *degree* $deg(f)$ of a map $f: S^2 \to S^2$ and prove that this is well-defined, that is, independent of any choices required for the definition. (b) (8 points) Give a detailed proof

that, given an integer k, there exists a map $f_k: S^2 \to S^2$ such that $deg(f) = k$.

10. Let X be a path-connected space and let $p: \widetilde{X} \to X$ be a normal n-sheeted covering space, that is, each $p^{-1}(x)$ consists of n points and, for each pair of such points, there is a deck transformation taking one to the other. (a) (5 points) Prove that if $\widetilde{X}$ is path-connected, then the order of the fundamental group $\pi_1(X, x_0)$ is at least n. (b) (5 points) Prove that if X is simply-connected, then $\widetilde{X}$ has n path components.

Qualifying Exam
GEOMETRY-TOPOLOGY
March 2007

Each problem is worth 10 points. In order to pass this examination, you must demonstrate proficiency in both parts, differential manifolds and algebraic topology. In particular, if you score fewer than 20 points on one part, you will not pass no matter how well you do on the other part.

Part I: Differentiable Manifolds

1, Let M be a smooth three-dimensional manifold and let α be a 1-form on M such that $\alpha \wedge d\alpha \neq 0$ at every point of M.

(i) Let $H = ker\, \alpha \subseteq TM$. Show that H is a two-dimensional subbundle of TM that is not integrable. (Hint: Use the formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

where X, Y are two arbitrary vector fields.)

(ii) Show that there is a unique vector field V such that

$$(a)\, \alpha(V) = 1 \quad (b) < V > \oplus H = TM \quad (c)\, d\alpha(V, W) = 0$$

for any vector field W. Here $< V >$ is the line field generated by V.

2. Suppose $f: S^1 \to \mathbf{R}^2$ and $g: S^1 \to \mathbf{R}^2$ are smooth embeddings. Let

$$M = \{(a, b, \vec{v}) \in S^1 \times S^1 \times \mathbf{R}^2 : f(a) - g(b) = \vec{v}\}.$$

Show that M is a compact submanifold of $S^1 \times S^1 \times \mathbf{R}^2$. Let $\pi: M \to \mathbf{R}^2$ be the projection $\pi(a, b, \vec{v}) = \vec{v}$. Apply Sard's Theorem to π and deduce that for almost every $\vec{v} \in \mathbf{R}^2$, $f(S^1)$ is transverse to $g(S^1) + \vec{v}$.

3. Let $f: \mathbf{R}^n \to \mathbf{R}$ be a smooth function and let $x \in \mathbf{R}^n$ be a critical point of f. The Hessian $H(f)_x$ at x is a bilinear function $H(f)_x: T_x\mathbf{R}^n \times T_x\mathbf{R}^n \to \mathbf{R}$ defined as follows: given vectors V_1 and V_2 in $T_x\mathbf{R}^n$, extend V_2 to a vector field $\widetilde{V}_2$ near x and define

$$H(f)_x(V_1, V_2) = D_{V_1} D_{\widetilde{V}_2} f.$$

(i) Show that $H(f)_x(V_1, V_2)$ is independent of the extension $\widetilde{V}_2$.

(ii) Show that $H(f)_x(V_1, V_2) = H(f)_x(V_2, V_1)$.

4. Suppose M is a compact, connected n-manifold, α is a p-form and β is an $n-p-1$-form. The boundary ∂M has two components, $\partial_0 M$ and $\partial_1 M$. Let i_0 and i_1 be the inclusions of $\partial_0 M$ and $\partial_1 M$ into M. Given that $i_0^* \alpha = 0$ and $i_1^* \beta = 0$, show that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta.$$

5. Show that $T^2 \times S^2$ is parallelizable.

Part II: Algebraic Topology

6. The *cone* CA of a space A is obtained from $A \times I$ by identifying $A \times \{1\}$ to a point p. Prove that, if (X, A) a topological pair, then $H_k(X, A)$ is isomorphic to $\widetilde{H}_k(X \cup CA)$ for all k, where $\widetilde{H}$ denotes reduced homology.

7. (a) Let X be a path-connected, locally path-connected and simply-connected space. Prove that if $f, g: X \to S^1$ are maps, then they are homotopic. (b) Represent S^1 as the complex numbers of norm one and define $p_k: S^1 \to S^1$ by $p_k(z) = z^k$. Prove that a map $f: S^1 \to S^1$ such that $f(1) = 1$ is homotopic to p_k, for some k, by a homotopy $F: S^1 \times I \to S^1$ such that $F(1, t) = 1$ for all t.

8. Calculate the homology of the complement of a finite set of $n \geq 1$ points in $\mathbf{R}^3$ and give a convincing argument that your calculation is correct.

9. Let $A \subseteq X$ and let $i: A \to X$ be inclusion inducing $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$. Given a covering space $p: \widetilde{X} \to X$, prove that the kernel of i_* is contained in the image of $p_{A*}: \pi_1(p^{-1}(A), \tilde{x}_0) \to \pi_1(A, x_0)$, where $p_A: p^{-1}(A) \to A$ denotes the restriction of p.

10. Suppose $B \subseteq A \subseteq X$ and $j: (X, B) \to (X, A)$ is inclusion. Define $\partial: H_k(X, A) \to H_{k-1}(A, B)$ so that the kernel of ∂ is equal to the image of $j_*: H_k(X, B) \to H_k(X, A)$ and prove that it has this property.

Qualifying Exam
Geometry/Topology
September 2008

Each problem is worth 10 points. In order to pass this examination, you must demonstrate proficiency in both parts, manifold theory and algebraic topology. In particular, if your score is fewer than 20 points on one part, you will not pass no matter how well you do on the other part.

Part I: Manifold Theory

1. Let $G(k,n)$ be the collection of all k-dimensional linear subspaces in $\mathbf{R}^n$.

(i) Define a natural topological and smooth structure on $G(k,n)$, and show that with respect the structures you defined, $G(k,n)$ is a compact smooth manifold.

(ii) Show that $G(k,n)$ is diffeomorphic to $G(n-k,n)$.

2. Let M and N be two smooth manifolds, and $f : M \to N$ be a smooth map. Assume that $df_x : T_xM \to T_{f(x)}N$ is surjective for all x in M and that the inverse image $f^{-1}(y)$ is compact for all y in N.

(i) Show that for any x in M there is an open neighborhood U of x such that $f^{-1}(U)$ is diffeomorphic to $U \times f^{-1}(x)$.

(ii) Assume further that N is connected, can you take U to be N in (i) ? (Justify your answer)

3. Let M be a connected smooth manifold. Show that for any two points x and y in M there is a diffeomorphism f of M such that $f(x) = y$.

4. Let $\theta = \Sigma_{i=1}^{n}(x_i dy_i - y_i dx_i)$ be a 1-form defined on $\mathbf{R}^{2n}$, where $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ are the coordinates of $\mathbf{R}^{2n}$. Consider the $2n-1$ dimensional distribution $D = ker\theta$. Is D integrable? (Justify your answer)

5. Let D be a bounded domain in $\mathbf{R}^n$ with a smooth boundary S, $j : S \to \mathbf{R}^n$ be the inclusion map and X be a smooth vector field defined on $\mathbf{R}^n$.

(i) Denote the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ by ω. Show that $j^*(i_X\omega) = <X, N> dS$, where N is the outer unit normal vector field along S, $<X, N>$ is the Euclidean inner product of X and N. Here $i_X\omega$ is the contraction of ω along X, dS is the "area" form on S. Explain carefully the definition and geometrical meaning of the term "dS".

(ii) Use (i) and Stokes theorem to show that

$$\int_D \mathcal{L}_X \omega = \int_S < X, N > dS.$$

Here $\mathcal{L}_X \omega$ is the Lie derivative of ω along X.

Part II: Algebraic Topology

6. Find $\pi_1(T^2 \setminus \{k\ pts\})$, where T^2 is the two-dimensional torus.

7. Find the homology groups $H_i(\Delta_n^{(k)}), i = 0, 1, \cdots, k$. Here $\Delta_n^{(k)}$ is the k-skeleton of the n-simplex Δ_n with $k \leq n$.

8. Let G be a topological group with the identity element e. For any two continuous loops γ_1 and $\gamma_2 : S^1 \to G$ sending $1 \in S^1$ to $e \in G$, define $\gamma_1 * \gamma_2 : S^1 \to G$ by $\gamma_1 * \gamma_2(t) = \gamma_1(t) \circ \gamma_2(t)$ for $t \in S^1$. Here $\circ$ is the product operation in G.

(i) Show that the product $*$ so defined induces a product structure on $\pi_1(G, e)$ and that this new product on $\pi_1(G, e)$ is the same as the usual one.

(ii) Is $\pi_1(G, e)$ commutative? (Justify your answer)

9. (i) Show that any continuous map $f : S^2 \to T^2$ is null-homotopic.

(ii) Show that there exists a continuous map $f : T^2 \to S^2$ which is not null-homotopic.

10. Let A and B be two chain complexes with boundary operators ∂_A and ∂_B respectively, and $f : A \to B$ be a chain map. Define a new chain complex C whose ith chain group is $C_i = A_i \oplus B_{i+1}$ and whose boundary operator ∂_C is defined by $\partial_C(a, b) = (\partial_A(a), \partial_B(b) + (-1)^{deg(a)} f(a))$ for any $(a, b) \in C_i$. Here A_i and B_i are the ith chain groups of A and B respectively.

(i) Show that C so defined is indeed a chain complex and that there is a short exact sequence of chain complexes:

$$0 \to B \to C \to A \to 0$$

sending B_{i+1} to C_i and C_i to A_i.

(ii) Write down the long exact sequence of the homology groups associated to the short exact sequence in (i). What is the connecting boundary map in the long exact sequence?

(iii) Let $(f_*)_i : H_i(A) \to H_i(B)$ be the induced map of f on the ith homology groups. Show that $(f_*)_i : H_i(A) \to H_i(B)$ is an isomorphism for all i if and only if $H_i(C) = 0$ for all i.

1. Let M and N be smooth (C^∞) manifolds, not necessarily of the same dimension, and $F: M \to N$ be a smooth map.
   (a) Define the map F^* of p-forms on N to p-forms on M (p=0,1,2,...).
   (b) Prove that, if ω is a p-form on N, then $F^*(d_N\omega)=d_M(F^*\omega)$.

2. Let M be a C^∞ manifold and X a C^∞ vector field on M.
   (a) Suppose $X(p)\neq 0$ for some particular $p\in M$. Show , using the flow of X, that there is a neighborhood U of p and a coordinate system$(x_1,\ldots,x_n)$ on U with $X=\partial/\partial x_1$ on U.
   (b) Use part (a) to prove that if Y is another C^∞ vector field on M with [X, Y]=0 everywhere on M, then $\varphi_s(\psi_t(p))=\psi_t(\varphi_s(p))$ for all s, t with $|t|$ and $|s|$ sufficiently small, where φ, ψ are the flows of X and Y respectively. [Suggestion: Write Y near p in the coordinate system of part (a)].

3. Gauss's Divergence Theorem asserts that if U is a bounded open set in R^3 with smooth boundary and if X is a smooth vector field defined in a neighborhood of the closure of U, then $$\iiint_U \text{divergence}(X)\, d(\text{vol}) = \iint_{\partial U} X\cdot N\, d(\text{area})$$
where N is the exterior unit normal to ∂U. Show how the Divergence Theorem follows from Stokes Theorem for differential forms on manifolds with boundary.

4. (a)Let θ be a 1-form on S^2 with $d\theta=0$. Construct a function f on S^2 with $df = \theta$.

   (b)Let θ be a 1-form on S^1 x (0, 1) with $d\theta=0$. Show that there is a function f: S^1 x $(0, 1)\to R$ with $df = \theta$ if and only if $\int_{S^1\times 1/2} \theta = 0$.

   (c) Use part (b) to show that if ω is a 2-form on S^2 with $\int_{S^2}\omega = 0$ then there is a 1-form θ on S^2 with $d\theta = \omega$. [Suggestion: You may assume the Poincaré Lemma so that $\omega = d\theta_1$ on S^2 - {South pole} and $\omega = d\theta_2$ on S^2 - {North pole} . Use Stokes theorem to show $\theta_1 - \theta_2$ satisfies the integral condition of part (b)].

5. Let SO(3) = the set of all 3x3 matrices A with AA^t = identity (orthogonal matrices) and determinant of A = 1. Also, for each 3x3 matrix B, let
$$\exp(B) = I + B + \left(\frac{B^2}{2!}\right) + \left(\frac{B^3}{3!}\right) + \ldots .$$
   (a) Prove that the infinite series for exp(B) converges for each 3x3 matrix B, so that exp is a map from the space of 3x3 matrices to itself.

You may *assume* from here on that this map is smooth and that the series can be differentiated term by term to give the differential of the mapping.

   (b) Show that the map exp is injective on some neighborhood of the 0 matrix in the space of all 3x3 matrices. [Suggestion: Inverse function theorem].

   (c) Prove that exp(B) is in SO(3) if B satisfies B^t = -B (B is "anti-symmetric") .

   (d) Show that the mapping exp restricted to the vector space of 3x3 anti-symmetric matrices is a surjective (onto) map from some neighborhood of the 0 matrix to a neighborhood of the identity matrix in SO(3). [Suggestion: Note that every element of SO(3) is a rotation around an axis, so check this case.]

(e) Discuss how to combine parts (b), (c), and (d) to give coordinate charts on SO(3) and thus to make SO(3) a differentiable manifold.

6. Let M and N be two compact, oriented manifolds of the same dimension. And let ω be a nowhere vanishing n-form on N with $\int_N \omega = 1$. Let F: M → N be a smooth map.

(a) Set $\deg_\omega F = \int_M F^* \omega$. Show that $\deg_\omega F$ is independent of the choice of ω.

[You may assume deRham's Theorem]. We shall call the common value the degree of F.

(b) Show that there is a smooth map from $S^2 \times S^2$ to S^4 of degree 1.

(c) Show that no map from S^4 to $S^2 \times S^2$ has degree 1.

7. Describe carefully the basic algebraic construction of algebraic topology, namely, how to go from a short exact sequence of chain complexes to a long exact sequence in homology. Give explicitly, in particular, the construction of the "connecting homomorphism", the map where the dimension drops, and prove exactness at its image, that is, prove that the image of the connecting homorphism = the kernel of the map that follows it. [You need not prove exactness of the long exact sequence elsewhere].

8(a)Prove that S^n is simply connected if $n > 1$.

(b)Prove that $\pi_1(RP^n) = Z_2$, $n>1$.

(c)Prove that RP^n is orientable if n is odd ($n >1$).

9. Find by any method the homology groups of RP^n with integer coefficients.

10 (a)Define complex projective space CP^n .

(b)Show that CP^n is compact.

(c)Show that $CP^1 = S^2$ (homeomorphic is enough).

(d) Show that CP^n is simply connected.

(e) Find the homology of CP^n (integer coefficients). [Any method will do. But cell complex decomposition is the easiest].

Qualifying Exam
GEOMETRY-TOPOLOGY
March 2009

Instructions: Do any ten of the following twelve problems. Please do not turn in work on more than ten problems and label each problem carefully by its number. Start each problem on a new page.

1. (a) Show that a closed 1-form θ on $S^1 \times (-1,1)$ is dF for some function $F: S^1 \times (-1,1) \to \mathbf{R}$ if and only if $\int_{S^1} i^*\theta = 0$ where $i: S^1 \to S^1 \times (-1,1)$ is defined by $i(p) = (p,0)$ for $p \in S^1$. (b) Show that a 2-form ω on S^2 is $d\theta$ for some 1-form θ on S^1 if and only if $\int_{S^2} \omega = 0$.

2. Suppose that M, N are connected C^∞ manifolds of the same dimension $n \geq 1$ and $F: M \to N$ is a C^∞ map such that $dF: T_pM \to T_{F(p)}N$ is surjective for each $p \in M$. (a) Prove that if M is compact, then F is onto and F is a covering map. (b) Find an example of such an everywhere nonsingular equidimensional map where N is compact, F is onto, $F^{-1}(p)$ is finite for each $p \in N$, but F is not a covering map. [A clearly explained pictorial version of F will be acceptable; you do not need to have a "formula" for F.]

3. (a) Suppose that M is a C^∞ connected manifold. Prove that, given an open subset U of M and a finite set of points $p_1, p_2, \ldots, p_k$ in M, there is a diffeomorphism $F: M \to M$ such that $f(\{p_1, p_2, \ldots, p_k\}) \subset U$. [Suggestion: Construct F one point at a time.] (b) Use part (a) to show that if M is compact and the Euler characteristic $\chi(M) = 0$, then there is a vector field on M which vanishes nowhere. You may assume that if a vector field has isolated zeros, then the sum of the indices at the zero points equals $\chi(M)$.

4. A smooth vector field V on $\mathbf{R^3}$ is said to be "gradient-like" if, for each $p \in \mathbf{R^3}$, there is a neighborhood U_p of p and a function $\lambda_p: U_p \to \mathbf{R} - \{\mathbf{0}\}$ such that $\lambda_p V$ on U_p is the gradient of some C^∞ function on U_p. Suppose V is nowhere zero on $\mathbf{R^3}$. Then show that V is gradient-like if and only if $curl\, V$ is perpendicular to V at each point of $\mathbf{R^3}$

5. Suppose that M is a compact C^∞ manifold of dimension n. (a) Show that there is a positive integer k such that there is an immersion $F: M \to \mathbf{R^k}$. (b) Show that if $k > 2n$, there is a $(k-1)$-dimensional subspace H of $\mathbf{R^k}$ such that $P \circ F$ is an immersion, where $P: \mathbf{R^k} \to H$ is orthogonal projection.

6. Let $Gl^+(n, \mathbf{R})$ be the set of $n \times n$ matrices with determinant > 0. Note that $Gl^+(n, \mathbf{R})$ can be considered to be a subset of $\mathbf{R^{n^2}}$ and this subset is open. (a) Prove that $Sl^+(n, \mathbf{R}) = \{A \in Gl^+(n, \mathbf{R}) : \det A = 1\}$ is a submanifold. (b) Identify the tangent space of $Sl^+(n, \mathbf{R})$ at the identity matrix I_n. (c) Prove that, for every $n \times n$ matrix B, the series $I_n + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots + \frac{1}{n!}B^n \cdots$ converges to some $n \times n$ matrix. Notation: this sum $= e^B$. (d) Prove that if $e^{tB} \in Sl^+(n, \mathbf{R})$ for all $t \in \mathbf{R}$,

then $trace\, B = 0$. (e) Prove that if $trace\, B = 0$, then $e^B \in Sl^+(n, \mathbf{R})$. [Suggestion: Use one-parameter subgroups or note that it suffices to treat complex-diagonable B since such are dense.]

7. (a) Define complex projective space $\mathbf{CP^n}$. (b) Calculate the homology of $\mathbf{CP^n}$. Any systematic method such as Mayer-Vietoris or cellular homology is acceptable.

8. Let $p: E \to B$ be a covering space and $f: X \to B$ a map. Define $E^* = \{(x, e) \in X \times B: f(x) = p(e)\}$. Prove that $q: E^* \to X$ defined by $q(x, e) = x$ is a covering space.

9. (a) Explain carefully and concretely what it means for two (smooth) maps of S^1 into $\mathbf{R^2}$ to be transversal. (b) Do the same for maps of S^1 into $\mathbf{R^3}$. (c) Explain what it means for transversal maps to be "generic" and prove that they are indeed generic in the cases of 9(a) and 9(b).

10. Let M be the 3-manifold with boundary obtained as the union of the two-holed torus in 3-space and the bounded component of its complement. Let X be the space obtained from M by deleting k points from the interior of M. (a) Calculate the fundamental group of X. (b) Calculate the homology of X.

11. Let P be a finite polyhedron. (a) Define the Euler characteristic $\chi(P)$ of P. (b) Prove that if P_1, P_2 are subpolyhedra of P such that $P_1 \cap P_2$ is a point and $P_1 \cup P_2 = P$, then $\chi(P) = \chi(P_1) + \chi(P_2) - 1$. (c) Suppose that $p: E \to P$ is an n-sheeted covering space of P, that is $p^{-1}(x)$ is n points for each $x \in P$. Prove that $\chi(E) = n\chi(P)$

12. Let $f: T \to T = S^1 \times S^1$ be a map of the torus inducing $f_\pi: \pi_1(T) \to \pi_1(T) = \mathbf{Z} \oplus \mathbf{Z}$ and let F be a matrix prepresenting f_π. Prove that the determinant of F equals the degree of the map of the map f.

QUALIFYING EXAM

Geometry and Topology, March 23, 2010

1. Let M_n be the space of all $n \times n$ matrices with real entries and let S_n be the subset consisting of all symmetric matrices. Consider the map $F : M_n \longrightarrow S_n$ defined by $F(A) = AA^t - I$, where I is the identity matrix and A^t is the transpose of A.

   (a) Show that $0_{n \times n}$ (the $n \times n$ matrix with all entries 0) is a regular value of F.

   (b) Deduce that $O(n)$, the set of all $n \times n$ matrices such that $A^{-1} = A^t$ is a submanifold of M_n.

   (c) Find the dimension of $O(n)$ and determine the tangent space of $O(n)$ at the identity matrix as a subspace of the tangent space of M_n which is M_n itself.

2. Show that $T^2 \times S^n$, $n \geq 1$ is parallelizable, where S^n is the n sphere, $T^2 = S^1 \times S^1$ is the two torus, and a manifold of dimension k is said to be parallelizable if there are k vector fields $V_1, \ldots, V_k$ on it with $V_1(p), \ldots, V_k(p)$ linearly independent for all points p of the manifold.

3. Suppose $\pi : M_1 \longrightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \longrightarrow T_{\pi(p)}M_2$ is a vector space isomorphism.

   (a) Show that if M_1 is connected, then π is a covering space projection.

   (b) Given an example where M_2 is compact but $\pi : M_1 \longrightarrow M_2$ is not a covering space (but has the π_* isomorphism property).

4. Let $\mathcal{F}^k(M)$ denote the differentiable (C^∞) k-forms on a manifold M. Suppose U and V are open subsets of a differentiable manifold.

   (a) Explain carefully how the usual exact sequence

   $$0 \longrightarrow \mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \longrightarrow \mathcal{F}(U \cap V) \longrightarrow 0$$

   arises.

   (b) Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part (a) and describe exaplicitly how the map

   $$H^k_{deR}(U \cap V) \longrightarrow H^{k+1}_{deR}(U \cup V)$$

   arises.

5. Explain carefully why the following holds: if $\pi : S^N \longrightarrow M$, $N > 1$ is a covering space with M orientable, then every closed k-form on M, $1 \leq k < N$ is exact.

(Suggestion: Recall that the covering transformations in this situation form a group G with $S^N/G \cong M$).

6. Calculate the singular homology of $\mathbb{R}^n$, $n > 1$, with k points removed, $k \geq 1$. (Your answer will depend on k and n).

7. (a) Explain what is meant by adding a handle to a 2-sphere for a two dimensional orientable surface in general.

   (b) Show that a 2 sphere with a positive number of handles attached can not be simply connected.

8. (a) Define the degree $\deg f$ of a C^∞ map $f: S^2 \longrightarrow S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.

   (b) Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f: S^2 \longrightarrow S^2$ of degree k.

9. Explain how Stokes Theorem for manifolds with boundary gives, as a special case, the classical divergence theorem (about $\iiint_U \operatorname{div} V d(\text{vol})$, where U is a bounded open set in $\mathbb{R}^3$ with smooth boundary and V is a C^∞ vector field on $\mathbb{R}^3$).

10. (a) Show that every map $F: S^n \longrightarrow S^1 \times \cdots \times S^1$ (k copies of S^1) is null-homotopic (homotopic to a constant map).

    (b) Show that there is a map $F: S^1 \times \cdots S^1$ (n copies) $\longrightarrow S^n$ such that F is not null-homotopic.

    (c) Show that every map $F: S^n \longrightarrow S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_1 + \cdots + n_k = n$, $n_j > 0$, $k \geq 2$, has degree 0. (You may use any definition of degree you like, and you may assume F is C^∞).