Basic Exam F01

Instructions: Do all of the following:

1. Let K be a compact set of real numbers and let $f(x)$ be a continuous function on K. Prove there exists $x_0 \in K$ such that $f(x) \leq f(x_0)$ for all $x \in K$.

2. Let N denote the positive integers, let $a_n = (-1)^n \frac{1}{n}$, and let α be any real number. Prove there is a one-to-one and onto mapping $\sigma : N \to N$ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \alpha.$$

3. Let E be a set of real numbers and let $\{f_n\}$ be a sequence of continuous real-valued functions on E. Prove that if $f_n(x)$ converges to $f(x)$ *uniformly* on E, then $f(x)$ is continuous on E. (Recall that $f_n(x)$ converges to $f(x)$ uniformly on E means that for every $\epsilon > 0$ there is N such that whenever $n > N$ and $x \in E$, $|f_n(x) - f(x)| < \epsilon$.)

4. Let $\mathcal{S}$ be the set of all sequences $(x_1, x_2, \ldots)$ such that for all n,

$$x_n \in \{0, 1\}.$$

Prove there does not exist a one-to-one mapping from the set $N = \{1, 2, \ldots\}$ *onto* the set $\mathcal{S}$.

5. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function such that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of f exist everywhere and are continuous everywhere, and $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$ and $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ also exist and are continuous everywhere. Prove that

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$$

at every point of $\mathbf{R}^2$.

6. Suppose that $F : \mathbf{R}^2 \to \mathbf{R}^2$ is a continuously differentiable function with $F((0,0)) = (0,0)$ and with the Jacobian of F at $(0,0)$ equal to the identity matrix (i.e., if $F = (f_1, f_2)$ then $\frac{\partial f_i}{\partial x_j}\Big|_{(0,0)} = 1$ if $i = j$ and $= 0$ if $i \neq j$). Outline a proof that there exists $\delta > 0$ such that if $a^2 + b^2 < \delta$ then there is a point (x, y) in $\mathbf{R}^2$ with $F(x, y) = (a, b)$. (Prove this directly: do not just restate the Inverse Function Theorem. Your argument will be part of the proof of the Inverse Function Theorem. You may use any basic

estimation you need about the change in F being approximated by the differential of F without proof.)

7. If V is a real vector space and X is a subspace, let $V^* = \{f : V \to \mathbf{R} \mid f \text{ is linear }\}$ be the dual space of V and $X^o = \{f \in V^* \mid f(x) = 0 \text{ for all } x \in X\}$ be the annihilator of X. Let $T : V \to W$ be a linear transformation of finite dimensional real vector spaces. Recall that the *transpose* of T is the linear map $T^t : W^* \to V^*$ defined by $T^t(f) = f \circ T$. Prove the following:
   a. $\text{im}(T)^o = \ker(T^t)$. [Here $\text{im}(T)$ is the image or range of T and $\ker(T)$ is the kernel or null space of T.]
   b. $\dim \text{im}(T) = \dim \text{im}(T^t)$.

8. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the rotation by 60^o counterclockwise about the plane perpendicular to the vector $(1,1,1)$ and $S : \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection about the plane perpendicular to the vector $(1,0,1)$. Determine the matrix representation of $S \circ T$ in the standard basis $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$. You do not have to multiply the resulting matrices but you must determine any inverses that arise.

9. Let A be a real symmetric matrix. Prove that there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.
   [You cannot just quote a theorem, but must prove it from scratch.]

10. Let V be a complex vector space and $T : V \to V$ a linear transformation. Let $v_1, \ldots, v_n$ be non-zero vectors in V, each an eigenvector of a different eigenvalue. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent.

Basic Examination F02

1. Let K be a compact subset and F be a closed subset in the metric space X. Suppose $K \cap F = \phi$. Prove that

$$0 < \inf\{d(x,y) : x \in K, y \in F\}.$$

2. Show why the Least Upper Bound Property (every set bounded above has a least upper bound) implies the Cauchy Completeness Property (every Cauchy sequence has a limit) of the real numbers.

3. Show that there is a subset of the real numbers which is not the countable intersection of open subsets.

4. By integrating the series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$

prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$. Justify carefully all the steps (especially taking the limit as $x \to 1$ from below).

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives at every point bounded by $A > 0$.

   (a) Show that there is an $M > 0$ such that

$$|f((x,y)) - f((x_1,y_1))| \le M((x-x_1)^2 + (y-y_1)^2)^{1/2}.$$

   (b) What is the smallest value of M (in terms of A) for which this always works?

   (c) Give an example where that value of M makes the inequality an equality.

6. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is continuously differentiable. Suppose for some $v_0 \in \mathbb{R}^3$ and $x_0 \in \mathbb{R}^2$ that $F(v_0) = x_0$ and $F'(v_0) : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. Show that there is a continuously differentiable function γ, $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ for some $\varepsilon > 0$, such that

   (i) $\gamma'(0) \neq \vec{0} \in \mathbb{R}^3$, and

   (ii) $F(\gamma(t)) = x_0$ for all $t \in (-\varepsilon, \varepsilon)$.

7. Let $T : V \to W$ be a linear transformation of finite dimensional real vector spaces. Define the transpose of T and then prove both of the following:

   i. $(\{\text{im}\}(T))^0 = \{\ker(T^t)\}$ where $(\{\text{im}\}(T))^0$ is the annihilator of $\{\text{im}\}(T)$, the image (range) of T, and $\{\ker(T^t)\}$ is the kernel (null space) of T^t.

   ii. $\{\text{rank}(T)\} = \{\text{rank}\}(T^t)$, where the rank of a linear transformation is the dimension of its image.

8. Let T be the rotation of an angle 60^0 counterclockwise about the origin in the plane perpendicular to $(1,1,2)$ in $\{\mathbf{R}\}^3$.

   i. Find the matrix representation of T in the standard basis. Find all eigenvalues and eigenspaces of T.

   ii. What are the eigenvalues and eigenspaces of T if $\{\mathbf{R}\}^3$ is replaced by $\{\mathbf{C}\}^3$.

   [You do not have to multiply any matrices out but must compute any inverses.]

9. Let V be a complex inner product space. State and prove the Cauchy-Schwarz inequality.

10. Let A be an $n \times n$ complex matrix satisfying $A^*A = AA^*$ where A^* is the adjoint of A. Let $V = \{\mathbf{C}\}^{\{n \times 1\}}$, the $n \times 1$ complex column matrices, be an inner product space under the dot product. View $A : V \to V$ as a linear map. Prove that there exists an orthonormal basis of V consisting of eigenvectors of A, i.e., prove this form of the Spectral Theorem for normal operators.

Basic Exam S02

1. Prove that the closed interval $[0, 1]$ is connected.

2. Show that the set Q of rational numbers in $\mathbb{R}$ is not expressible as the intersection of a countable collection of open subsets of $\mathbb{R}$.

3. Suppose that X is a compact metric space (in the covering sense of the word compact). Prove that every sequence $\{x_n : x_n \in X, n = 1, 2, 3 \ldots\}$ has a convergent subsequence. [Prove this directly. Do not just quote a theorem.]

4. (a) Define *uniform continuity* of a function $F : X \to \mathbb{R}$, X a metric space.

   (b) Prove that a function $f : (0, 1) \to \mathbb{R}$ is the restriction to $(0, 1)$ of a continuous function $F : [0, 1] \to \mathbb{R}$ if and only if f is uniformly continuous on $(0, 1)$.

5. State some reasonable conditions under which a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

everywhere on $\mathbb{R}^2$ and prove this equality under the conditions you give.

6. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function with grad $f \neq \vec{0}$ at $\vec{0}$ ($\vec{0} = (0, 0, 0)$ in $\mathbb{R}^3$). Show that there are two other continuously differentiable functions $g : \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R}^3 \to \mathbb{R}$ such that the function

$$(x, y, z) \to (f(x, y, z), g(x, y, z), h(x, y, z))$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one on some neighborhood of $\vec{0}$.

7. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian matrix of F is everywhere nonsingular. Suppose also that $F(\vec{0}) = \vec{0}$ and that $||F((x,y))|| \geq 1$ for all (x,y) with $||(x,y)|| = 1$.

   Prove that $F(\{(x,y) : ||(x,y)|| < 1\}) \supset \{(x,y) : ||(x,y)|| < 1\}$.

   (Hint: Show, with $U = \{(x,y) : ||(x,y)|| < 1\}$, that $F(U) \cap U$ is both open and closed in U).

8. Let V be a finite dimensional real vector space. Let $W \subset V$ be a subspace and $W^o := \{f : V \to F \text{ linear} \,|\, f = 0 \text{ on } W\}$. Prove that

$$\dim(V) = \dim(W) + \dim(W^o).$$

9. Find the matrix representation in the standard basis for either rotation by an angle θ in the plane perpendicular to the subspace spanned by the vectors (1,1,1,1) and (1,1,1,0) in $\mathbf{R}^4$.

   [You do not have to multiply the matrices out but must compute any inverses.]

10. Let V be a complex inner product space and W a finite dimensional subspace. Let $v \in V$. Prove that there exists a unique vector $v_W \in W$ such that

$$||v - v_W|| \leq ||v - w||$$

    for all $w \in W$. Deduce that equality holds if and only if $w = v_W$.

11. Let V be a finite dimensional real inner product space and $T, S : V \to V$ two commuting hermitian linear operators. Show that there exists an orthonormal basis for V consisting of vectors that are simultaneously eigenvectors of T and S.

Basic Examination W 02

Instructions: Do any ten of the following eleven problems.

1. (a) State some reasonably general conditions under which this "differentiation under the integral sign" formula is valid:

$$\frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{\partial f}{\partial x}\, dy.$$

(b) Prove that the formula is valid under the conditions you gave in part (a).

2. Prove that the unit interval $[0,1]$ is sequentially compact, i.e., that every infinite sequence has a convergent

[Prove this directly. Do not just quote general theorems like Heine-Borel.]

3. Prove that the open unit ball in $\mathbb{R}^2$

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

is connected.

[You may assume that intervals in $\mathbb{R}$ are connected. You should not just quote other general results, but give a direct proof.]

4. Prove that the set of irrational numbers in $\mathbb{R}$ is not a countable union of closed sets.

5. (a) Let $f : U \to \mathbb{R}^k$ be a function on an open set U in $\mathbb{R}^n$. Define what it means for f to be differentiable at a point $x \in U$.

(b) State carefully the Chain Rule for the composition of differentiable functions of several variables.

(c) Prove the Chain Rule you stated in part (b).

6. (a) State some reasonably general conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ under which

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

(b) Prove the formula under the conditions you stated.

7. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is everywhere differentiable and that its first derivative (Jacobian) matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

is continuous everywhere and nonsingular everywhere.

[Here we use the notation $F((x,y)) = (F_1(x,y),\ F_2(x,y)) \in \mathbb{R}^2$.]

Suppose also that

$$\|F((x,y))\| \geq 1 \quad \text{if} \quad \|(x,y)\| = 1 \ \text{ and that } \ F((0,0)) = (0,0).$$

Prove that

$$F\left(\{(x,y) : x^2 + y^2 < 1\}\right) \supset \{(x,y) : x^2 + y^2 < 1\}.$$

(Hint: With $U = \{(x,y) : x^2 + y^2 < 1\}$, prove that $F(U) \cap U$ is open and is closed in U.)

8. Let $T : V \to W$ and $S : W \to X$ be linear transformations of finite dimensional real vector spaces. Prove that

$$\text{rank}(T) + \text{rank}(S) - \dim(W) \leq \text{rank}(S \circ T) \leq \max\{\text{rank}(T),\ \text{rank}(S)\}.$$

[The rank of an linear transformation is the dimension of its image.]

9. Let V be a real vector space and $T : V \to V$ be a linear transformation. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of T. Let $0 \neq v_i$ be an eigenvector of T with eigenvalue λ_i for $1 \leq i \leq m$. Show that $\{v_1, \ldots, v_m\}$ is linearly independent.

10. Let V be a finite dimensional complex inner product space and $f : V \to \mathbf{C}$ a linear functional. Show that there exists a vector $w \in V$ such that $f(v) = \langle v, w \rangle$ for all $v \in V$.

11. Let V be a finite dimensional complex inner product space and $T : V \to V$ a linear transformation. Prove that there exists an orthonormal ordered basis for V such that the matrix representation A in this basis is upper triangular, i.e., $A_{ij} = 0$ if $i < j$.

[Hint: First show if $S : V \to V$ is a linear transformation and W is a subspace then W is S-invariant if and only if $W^{\perp}$ is S^*-invariant where S^* is the adjoint of S.]

BASIC EXAM, FALL 2003

Instructions: Do all of the following:

1:

Prove that $\mathbb{R}$ is uncountable. If you like to use the Baire category theorem, you have to prove it.

2:

Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function. Assume that for each element $x \in [0,1]$ there is a positive integer m, such that the m-th derivative of f at x is not zero.

Prove that there exists an integer M such that the following stronger statement holds: For each element $x \in [0,1]$ there is a positive integer m with $m \leq M$ such that the m-th derivative of f at x is not zero.

3:

Prove that the sequence $a_1, a_2, \ldots$ with

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

converges as $n \to \infty$.

4:

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. State the definition of the Riemann integral

$$\int_0^1 f(x)\, dx$$

and prove that it exists.

5:

Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of f) a quadratic polynomial $P(x,y)$ in x and y such that

$$|f(x,y) - P(x,y)| \leq C(x^2 + y^2)^{3/2}$$

for all (x,y) in some small neighborhood of $(0,0)$, where C is a number that may depend on f but not on x and y. Then prove the above estimate.

6:

Let $U = \{(x,y) : x^2 + y^2 < 1\}$ be the standard unit ball in $\mathbb{R}^2$ and let ∂U denote its boundary.

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian determinant of F is everywhere non-zero. Suppose also that $F(x, y) \in U$ for some $(x, y) \in U$ and $F(x, y) \notin U \cup \partial U$ for all $(x, y) \in \partial U$. Prove that $U \subset F(U)$.

7:

Prove that the space of continuous functions on the closed interval $[0, 1]$ with the metric

$$\mathrm{dist(f,g)} := \sup_{x\in[0,1]} |f(x) - g(x)| = \|f - g\|_\infty$$

is complete. You do not need to show that this is a metric space.

8:

Prove the following three statements. You certainly may choose an order of these statements and then use the earlier statements to prove the later statements.

a) If $T : V \to W$ is a linear tranformation between two finite dimensional real vector spaces V, W, then

$$\dim(im(T)) = \dim(V) - \dim(ker(T))$$

b) If $T : V \to V$ is a linear transformation on a finite dimensional real inner product space and T^* denotes its adjoint, then $im(T^*)$ is the orthogonal complement of $ker(T)$ in V.

c) Let A be a n by n real matrix, then the maximal number of linearly independent rows (row rank) in the matrix equals the maximal number of linearly independent columns (column rank).

9:

Consider a 3 by 3 real symmetric matrix with determinant 6. Assume $(1, 2, 3)$ and $(0, 3, -2)$ are eigenvectors with eigenvalues 1 and 2. Give answers to a) and b) below and justify the answers.

a)

Give an eigenvector of the form $(1, x, y)$ for some real x, y which is linearly independent of the two vectors above.

b)

What is the eigenvalue of this eigenvector.

10:

a) Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix

$$A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

prove that there does not exist a real valued matrix B such that BAB^{-1} is a diagonal matrix.

b) Do the same for the matrix

$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$.

Basic Examination May 2003

1. (a) Suppose $f : (0,1) \to \mathbb{R}$ is a continuous function. Define what it means for f to be **uniformly continuous**.

   (b) Show that if $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then there is a continuous function $F : [0,1] \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in (0,1)$.

2. Prove: If $a_1, a_2, a_3, \ldots$ is a sequence of real numbers with

$$\sum_{j=1}^{+\infty} |a_j| < +\infty,$$

then $\lim\limits_{N \to +\infty} \sum\limits_{j=1}^{N} a_j$ exists.

3. Find a subset S of the real numbers $\mathbb{R}$ such that both (i) and (ii) hold for S:

   (i) S is not the countable union of closed sets

   (ii) S is not the countable intersection of open sets.

4. Consider the following equation for a function $F(x,y)$ on $\mathbb{R}^2$:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} \qquad (*)$$

   (a) Show that if a function F has the form $F(x,y) = f(x+y) + g(x-y)$ where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are twice differentiable, then F satisfies the equation (*).

   (b) Show that if $F(x,y) = ax^2 + bxy + cy^2,\ a, b, c \in \mathbb{R}$, satisfies (*) then $F(x,y) = f(x,y) + g(x-y)$ for some polynomials f and g in one variable.

5. Consider the function $F(x,y) = ax^2 + 2bxy + cy^2$ on the set $A = \{(x,y) : s^2 + y^2 = 1\}$.

   (a) Show that F has a maximum and minimum on A.

   (b) Use Lagrange multipliers to show that if the maximum of F on A occurs at a point (x_0, y_0), then the vector (x_0, y_0) is an eigenvector of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

6. Formulate some reasonably general conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ which guarantee that
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
and prove that your conditions do in fact guarantee that this equality holds.

7. Let V be a finite dimensional real vector space. If $W \subset V$ be a subspace let $W^o := \{f : V \to F \text{ linear}, | \, f = 0 \text{ on } W\}$. Let $W_i \subset V$ be subspaces for $i = 1, 2$. Prove that
$$W_1{}^o \cap W_2{}^o = (W_1 + W_2)^o.$$

8. Let V be an n-dimensional complex vector space and $T : V \to V$ a linear operator. Suppose that the characteristic polynomial of T has n distinct roots. Show that there is a basis B for V such that the matrix representation of T is the basis B is diagonal. (Make sure that you prove that your choice of B is in fact a basis.)

9. Let $A \in \mathbf{M}_3(\mathbf{R})$ satisfy $\det(A) = 1$ and $A^t A = I = AA^t$ where I is the 3×3 identity matrix. Prove that the characteristic polynomial of A has 1 as a root.

10. Let V be a finite dimensional real inner product space and $T : V \to V$ a hermitian linear operator. Suppose the matrix representation of T^2 in the standard basis has trace zero. Prove that T is the zero operator.

Basic Exam (S04)

In several problems you will need the usual "norm" terminology. If V is a real vector space, then a norm on V is a map $\| \ \| : V \to [0, \infty)$ such that $\|v + w\| \leq \|v\| + \|w\|$, $\|cv\| = |c|\|v\|$, and $\|v\| = 0$ if and only if $v = 0$. Each norm determines a metric d on V via the relation $d(v, w) = \|v - w\|$. The Euclidean norm (also called the "inner product" norm) on $\mathbb{R}^n$ is given by

$$\left\| \sum_{k=1}^{n} x_k e_k \right\|_2 = [\sum_{k=1}^{n} |x_k|^2]^{1/2}.$$

where e_k is the usual vector basis. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we define

$$\|T\| = \sup\{\|T(x)\|_2 : \|x\|_2 \leq 1\}.$$

For all x, $\|T(x)\| \leq \|T\|\|x\|$.

1. Let $\mathcal{S}$ denote the set of sequences $a = (a_1, a_2, \ldots)$, with $a_k = 0$ or 1. Show that the mapping $\theta : \mathcal{S} \to \mathbb{R}$ defined by

$$\theta((a_1, a_2, \ldots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots$$

is an injection. Include an explanation of why the infinite series converges. Hint: if $a \neq b$, you may assume that

$$\begin{aligned} a &= (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots). \\ b &= (a_1, \ldots, a_{n-1}, 1, b_{n+1}, \ldots) \end{aligned}$$

2. Is $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$? Prove your assertion.

3. a) Carefully define when a function f on $[0, 1]$ is Riemann integrable.
b) Show that if f_n are Riemann integrable functions on $[0, 1]$ and f_n converges to f uniformly, then f is Riemann integrable.

4. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

5. Suppose that G is an open set in $\mathbb{R}^n$, $f : G \to \mathbb{R}^m$ is a function, and that $x_0 \in G$.
a) Carefully define what is meant by $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$.
b) Suppose that I is a line segment in G such that $f'(x)$ is defined for all $x \in I$. Show that if f is differentiable at all the points of I, then for some point c in I

$$\|f(q) - f(p)\|_2 \leq \|f'(c)\| \, \|q - p\|_2 .$$

Hint: let w be a unit vector with $\|f(q) - f(p)\|_2 = (f(q) - f(p)) \cdot w$.

6. Let $\| \; \|$ be any norm on $\mathbb{R}^n$.

a) Prove that there exists a constant d with $\|x\| \leq d \|x\|_2$ for all $x \in \mathbb{R}^n$, and use this to show that $N(x) = \|x\|$ is continuous in the usual topology on $\mathbb{R}^n$.

b) Prove that there exists a constant c with $\|x\| \geq c \|x\|_2$ (Hint: use the fact that N is continuous on the sphere $\{x : \|x\|_2 = 1\}$).

c) Show that if L is an n-dimensional subspace of an arbitrary normed vector space V, then L is closed.

7. Let V be a finite dimensional real vector space. Let $W_1, W_2 \subset V$ be subspaces. Show both of the following:

a) $W_1^0 \cap W_2^0 = (W_1 + W_2)^0$

b) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

[Note: W_i^0 is the annihilator of W_i.]

8. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a rotation about the axis $(1, 0 - 1)$ by an angle of 30^o (you can use either orientation).

a) Find the matrix representation $A \in \mathbf{M}_3(\mathbf{R})$ of T in the standard basis. (You do not have to multiply out matrices but must evaluate inverses.)

b) Find all the eigenvalues of $A \in \mathbf{M}_3(\mathbf{R})$.

c) Find all the eigenvalues of $A \in \mathbf{M}_3(\mathbf{C})$.

9. Let V be a finite dimensional real inner product space under $(\; , \;)$ and $T : V \to V$ a linear operator. Show the following are equivalent:

a) $(Tx, Ty) = (x, y)$ for all $x, y \in V$.

b) $||T(x)|| = ||x||$ for all $x \in V$.

c) $T^*T = Id_V$, where T^* is the adjoint of T.

d) $TT^* = Id_V$.

10. Let T be a real symmetric matrix. Show that T is similar to a diagonal matrix.

[You cannot use the Spectral Theorem.]

Basic Examination September, 2005

Do all problems

1. A real number α is said to be *algebraic* if for some finite set of integers $a_0, \ldots, a_n$, not all 0,
$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0\,.$$
Prove that the set of algebraic real numbers is countable.

2. State some reasonable conditions on a real-valued function $f(x,y)$ on $\mathbb{R}^2$ which guarantee that $\dfrac{\partial^2 f}{\partial x dy} = \dfrac{\partial^2 f}{\partial y \partial x}$ at every point of $\mathbb{R}^2$. Then prove that your conditions do in fact guarantee this equality.

3. (a) Prove that if $f_j : [0,1] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly on $[0,1]$ to a (necessarily continuous) function $F : [0,1] \to \mathbb{R}$ then
$$\int_0^1 F^2(x)dx = \lim \int_0^1 f_j^2(x)dx\,.$$
(b) Give an example of a sequence $f_j : [0,1] \to \mathbb{R}$ of continuous functions which converges to a continuous function $F : [0,1] \to \mathbb{R}$ pointwise and for which
$$\lim \int_0^1 f_j^2(x)dx \quad \text{exists but}$$
$$\lim \int_0^1 f_j^2(x)dx \neq \int_0^1 F^2(x)dx$$
(f_j converges to F "pointwise" means that for each $x \in [0,1]$, $F(x) = \lim f_j(x)$).

4. Suppose $F : [0,1] \to [0,1]$ is a C^2 function with $F(0) = 0$, $F(1) = 0$, and $F''(x) < 0$ for all $x \in [0,1]$. Prove that the arc length of the curve $\{(x, F(x)) : x \in [0,1]\}$ is less than 3. (Suggestion: Remember that $\sqrt{a^2+b^2} < |a| + |b|$ when you are looking at the arc length formula - and at picture of what $\{(x, f(x)\}$ could look like.)

5. Prove carefully that $\mathbb{R}^2$ is not a (countable) union of sets $S_i, i = 1, 2\ldots$ with each S_i being a subset of some straight line L_i in $\mathbb{R}^2$.

6. (a) Prove that if P is a real-coefficient polynomial and if A is a real symmetric matrix, then the eigenvalues of $P(A)$ are exactly the numbers $P(\lambda)$, where λ is an eigenvalue of A.
(b) Use part (a) to prove that if A is a real symmetric matrix, then A^2 is nonnegative definite.
(c) Check part (b) by verifying directly that det A^2 and trace(A^2) are nonnegative when A is real symmetric.

7. Let A be a real $n \times m$ matrix. Prove that the maximum number of linearly independent rows of A = the maximum number of linearly independent columns. ("Row rank = column rank").

8. For a real $n \times n$ matrix A, let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear mapping. Set $||A|| = \sup_{\vec{x} \in \mathbb{R}^n} ||, ||\vec{x}|| = 1$ (here $||\vec{x}||$ = usual euclidean norm, i.e.

$$||(x_1, \ldots, x_n)|| = (x_1^2 + \cdots + x_n^2)^{1/2}).$$

   (a) Prove that $||A + B|| \leq A|| + ||B||$

   (b) Use part (a) to check that the set M of all $n \times n$ matrices is a metric space if the distance function d is defined by

$$d(A, B) = ||B - A||.$$

   (c) Prove that M is a complete metric space with this "distance function". (Suggestion: The ijth element of $A = \langle T_A e_j, e_i \rangle$ where $e_i = (0, \ldots, 1 \ldots 0)$, 1 in ith position.)

9. Suppose V_1 and V_2 are subspaces of a finite-dimensional vector space V.

   (a) Show that

$$\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(\text{span}(V_1, V_2))$$

   where $\text{span}(V_1, V_2)$ is by definition the smallest subspace that contains both V_1 and

   (b) Let $n = \dim V$. Use part (a) to show that, if $k < n$, then an intersection of k subspaces of dimension $n - 1$ always has dimension at least $n - k$. (Suggestion: Do induction on k

10. (a) For each $n = 2, 3, 4, \ldots$, is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ but $A^n = 0$? (Give example or proof of nonexistence.)

   (b) Is there an $n \times n$ upper triangular matrix A with $A^n \neq 0$ but $A^{n+1} = 0$? (Give example or proof of nonexistence.)
   [Note: A square matrix is *upper triangular* if all the entries below the main diagonal are 0.]

Note: Throughout this exam, $M_n(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex entries.

Linear Algebra.

1. Given $n \geq 1$, let $\text{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ denote the trace of a matrix:

$$\text{tr}(A) = \sum_{k=1}^{n} A_{k,k}.$$

   (a) Determine a basis for the kernel (or null-space) of tr.

   (b) For $X \in M_n(C)$, show that $\text{tr}(X) = 0$ if and only if there exists an integer m and matrices $A_1, \ldots, A_m, B_1, \ldots, B_m \in M_n(\mathbb{C})$ so that

$$X = \sum_{i=j}^{m} A_j B_j - B_j A_j$$

2. Let V be a finite-dimensional vector space, and let V^* denote the dual space; that is, the space of linear maps $\phi : V \to \mathbb{C}$. For a set $W \subset V$, let

$$W^{\perp} = \{\phi \in V^* : \phi(w) = 0 \ \forall w \in W\}.$$

   For a subset $U \subset V^*$, let

$${}^{\perp}U = \{v \in V : \phi(v) = 0 \ \forall \phi \in U\}.$$

   (a) Show that for any subset $W \subset V$, ${}^{\perp}(W^{\perp}) = \text{span}(W)$.
   Recall that the span of a set of vectors is the smallest vector sub-space that contains these vectors.

   (b) Let $W \subset V$ be a linear subspace. Give an explicit isomorphism between $(V/W)^*$ and $W^{\perp}$. Show that it is an isomorphism.

3. Let A be a Hermitian-symmetric $n \times n$ complex matrix. Show that if $\langle Av, v\rangle \geq 0$ for all $v \in \mathbb{C}^n$, then there exists an $n \times n$ matrix T so that $A = T^*T$.

4. Let $\mathcal{A} = M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with complex entries. We say that $\mathcal{I} \subseteq \mathcal{A}$ is a *two-sided ideal* in $\mathcal{A}$ if

   (i) for all $A, B \in \mathcal{I}$, $A + B \in \mathcal{I}$
   (ii) for all $A \in \mathcal{I}$ and $B \in \mathcal{A}$, AB and BA belong to $\mathcal{I}$

   Show that the only two-sided ideals in $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$ itself.

Analysis.

1. For a subset $X \subset \mathbb{R}$, we say that X is *algebraic*, if there exists a family $\mathcal{F}$ of polynomials with rational coefficients, so that $x \in X$ if and only if $p(x) = 0$ for some $p \in \mathcal{F}$.

   (a) Show that the set $\mathbb{Q}$ of rational numbers is algebraic.

   (b) Show that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not algebraic.

2. Let X be the set of all infinite sequences $\{\sigma_n\}_{n=1}^{\infty}$ of 1's and 0's endowed with the metric

$$\text{dist}\left(\{\sigma_n\}_{n=1}^{\infty}, \{\sigma_n'\}_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\sigma_n - \sigma_n'|.$$

   Give a direct proof that every infinite subset of X has an accumulation point.

3. Let X, Y be two topological spaces. We say that a continuous function $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for any compact set $K \subset Y$.

   (a) Give an example of a function that is proper but not a homeomorphism.

   (b) Give an example of a function that is continuous but not proper.

   (c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (that is, has a continuous derivative) and

$$|f'(x)| \geq 1 \qquad \text{for all } x \in \mathbb{R}.$$

   Show that f is proper.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (i.e., continuously differentiable). Show that

$$\lim_{n \to \infty} \sum_{j=1}^{n} |f(\tfrac{j-1}{n}) - f(\tfrac{j}{n})|$$

   is equal to

$$\int_0^1 |f'(t)| \, dt.$$

5. (a) Suppose

$$\lim_{n \to \infty} a_n = A$$

   Show that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n = A$$

   (b) Show by example that the converse is false.

6. Consider the set of $f : [0,1] \to \mathbb{R}$ that obey

$$|f(x) - f(y)| \leq |x - y| \qquad \text{and} \qquad \int_0^1 f(x)\,dx = 1.$$

Show that this is a compact subset of $C([0,1])$.

7. Let us make $M_n(\mathbb{C})$ into a metric space in the following fashion:

$$\operatorname{dist}(A, B) = \left\{ \sum_{i,j} |A_{i,j} - B_{i,j}|^2 \right\}^{1/2}$$

(which is just the usual metric on $\mathbb{R}^{n^2}$).

(a) Suppose $F : \mathbb{R} \to M_n(\mathbb{C})$ is continuous. Show that the set

$$\{x \in \mathbb{R} : F(x) \text{ is invertible}\}$$

is open (in the usual topology on $\mathbb{R}$).

(b) Show that on the set given above, $x \mapsto [F(x)]^{-1}$ is continuous.

8. Let (X, d) be a metric space. Prove that the following are equivalent:

(a) There is a countable dense set.

(b) There is a countable basis for the topology.

Recall that a collection of open sets $\mathcal{U}$ is called a basis if every open set can be written as a union of elements of $\mathcal{U}$.

Basic Exam Spring 2006

PROBLEM 1

(A) Define precisely the notion of Riemann integrability for a function $f(x)$ on $[0,1]$.

(B) Suppose that $f_n(x)$ is a sequence of Riemann integrable functions on $[0,1]$ such that $\{f_n(x)\}$ converges uniformly to $f(x)$. Prove that $f(x)$ is Riemann integrable.

PROBLEM 2

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \in \mathbf{R}$. Show that there exists a unique number $\rho \geq 0$ such that $F(x)$ converges if $|x| < \rho$ and $F(x)$ diverges if $|x| > \rho$.

PROBLEM 3

Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{5/2}}$$

converges for all $x \in \mathbf{R}$ and that $f(x)$ is a continuous function on $\mathbf{R}$ with a continuous derivative. State clearly any facts you assume.

PROBLEM 4

The point $P = (1,1,1)$ lies on the surface S in $\mathbf{R}^3$ defined by

$$x^2y^3 + x^3z + 2yz^4 = 4$$

Prove that there exists a differentiable function $f(x,y)$ defined in an open neighborhood $\mathcal{N}$ of $(1,1)$ in $\mathbf{R}^2$ such that $f(1,1) = 1$ and $(x, y, f(x,y))$ lies in S for all $(x,y) \in \mathcal{N}$.

PROBLEM 5

(A) Define uniform continuity for a function f defined on a metric space X with distance function $\rho(x,y)$.

(B) Prove that if $0 < \alpha < 1$, then $F(x) = x^\alpha$ is uniformly continuous on $[0, \infty)$.

PROBLEM 6

Let W be the subset of the space $C[0,1]$ of real-valued, continuous functions on $[0,1]$ satisfying the conditions:

$$|f(x) - f(y)| < |x - y| \qquad \int_0^1 f(x)^2\,dx = 1$$

(A) Prove that W is uniformly bounded, i.e., there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [0,1]$.
Hint: Show first that $|f(0)| \le 2$ for all $f \in W$.

(B) Prove that W is a compact subset of $C[0,1]$ under the sup norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$.

PROBLEM 7

A matrix T (with entries, say, in the field $\mathbf{C}$ of complex numbers) is diagonalizable if there exists a non-singular matrix S such that STS^{-1} is diagonal. Prove that if $a, \lambda \in \mathbf{C}$ with $a \neq 0$, then the following matrix is not diagonalizable:

$$T = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & \lambda \end{pmatrix}$$

PROBLEM 8

A linear transformation T is called *orthogonal* if it is non-singular and ${}^tT = T^{-1}$. Prove that if $T : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}$ is orthogonal, then there exists a vector $v \in \mathbf{R}^{2n+1}$ such that $Tv = \pm v$.

PROBLEM 9

Let S be a real, $n \times n$-symmetric matrix S, i.e., ${}^tS = S$.

(A) Prove that the eigenvalues of S are real.
(B) State and prove the Spectral Theorem for S.

PROBLEM 10

Let Y is an arbitrary set of commuting matrices in $M_n(\mathbf{C})$ (i.e., $AB = BA$ for all $A, B \in Y$). Prove that there exists a non-zero vector $v \in \mathbf{C}^n$ which is a common eigenvector of all elements of Y.

BASIC QUAL WINTER 2006

(February 18, 2006)

Problem 1. Show that for each $\epsilon > 0$ there exists a sequence of intervals (I_n) with the properties

$$\bigcup_{n=1}^{\infty} I_n \supset \mathbb{Q} \quad \text{and} \quad \sum_{n=1}^{\infty} |I_n| < \epsilon.$$

Problem 2. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = \infty$. Under what condition(s) is the function

$$f(x) = \sum_{n=1}^{\infty} (-1)^n a_n \, x^n$$

well-defined and left-continuous at $x = 1$? Carefully prove your assertion.

Problem 3. Consider a function $f\colon [a,b] \to \mathbb{R}$ which is twice continuously differentiable (including the endpoints). Let $a = x_0 < x_1 < \cdots < x_n = b$ be the uniform partition of $[a,b]$, i.e., $x_{i+1} - x_i = (b-a)/n$ for all $0 \leq i < n$. Show that there exists M such that for all $n \geq 1$,

$$\left| \frac{1}{n}\Big(\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\Big) - \int_a^b f(x)\mathrm{d}x \right| \leq \frac{M}{n^2}.$$

[Recall that the sum is an approximation of the integral in the Trapezoid Rule. It may be instructive to first solve the problem for $n = 1$ and then address the general case.]

Problem 4. Consider a decreasing sequence of continuous functions $f_n\colon [0,1] \to \mathbb{R}$ obeying the uniform bound $|f_n| \leq M$ for some $M \in (0,1)$. Suppose the point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous on $[0,1]$. Prove that $f_n \to f$ uniformly on $[0,1]$.
[You may use without proof that $[0,1]$ is compact as well as sequentially compact.]

Problem 5. Consider a function $f(x,y)$ which is twice continuously differentiable. Suppose that f has its unique minimum at $(x,y) = (0,0)$. Carefully prove that then at $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} \geq \Big(\frac{\partial^2 f}{\partial x \partial y}\Big)^2$$

[You may use without proof that the mixed partials are equal for C^2 functions.]

Problem 6. Let $-\infty < a < b < \infty$. Prove that a continuous function $f\colon [a,b] \to \mathbb{R}$ attains all values in $[f(a), f(b)]$.

Problem 7. Let V be a complex inner product space and $v, w \in V$. Prove the Cauchy-Schwarz inequality

$$|(v, w)| \leq |v||w|.$$

Problem 8. Let $T\colon V \to W$ be a linear transformation of finite dimensional real inner product spaces. Show that there exists a unique linear transformation $T^t\colon W \to V$ such that

$$\langle T(v), w\rangle_W = \langle v, T^t(w)\rangle_V \text{ for all } v \in V \text{ and } w \in W$$

where $\langle\ ,\ \rangle_X$ is the inner product on $X = V$ or W.

Problem 9. Let $A \in \mathbb{M}_3(\mathbb{R})$ be invertible and satisfy $A = A^t$ and $\det A = 1$. Prove that A has one as an eigenvalue.

Problem 10. Let $T\colon V \to V$ be a linear operator on a finite dimensional complex inner product space. Show that there exists an ordered orthonormal basis for V such that the matrix representation A of T in this basis is upper triangular, i.e, $A = (a_{ij})$ with $a_{ij} = 0$ if $j < i$.
[You cannot use canonical form theorems without proof.]

BASIC 2001 FALL

1. Let S be a subset of $\mathbb{R}^n$ with the distance function $d(x,y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$ so that $(S, d|_{S\times S})$ is a metric space.

   a) Given $y \in S$, is $E = \{x \in S : d(x,y) \geq r\}$ a closed set in S?
   b) Is the set E in part a) contained in the closure of $\{x \in S : d(x,y) > r\}$ in S?

Prove your answers.

2. Let $f : (a,b) \to \mathbb{R}$ be continuous and differentiable in $(a,b)\backslash\{c\}$. If $\lim_{x\to c} f'(x) = d \in \mathbb{R}$, show that f is differentiable at c, and $f'(c) = d$.

3. Let T be a linear transformation of the vector space V into itself. If Tv and v are linearly dependent for each $v \in V$, show that T must be a scalar multiple of the identity.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and its second derivative, f'', satisfies $|f''(x)| \leq B$.

   a) Prove that

$$|2Af(0) - \int_{-A}^{A} f(x)dx| \leq \frac{A^3}{3}B$$

   b) Use the result of part a) to justify the following estimate:

$$|\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f(a + \frac{2k-1}{2n}(b-a))| \leq Cn^{-2},$$

where C is a constant that does not depend on n.

5. a) Show that, given a continuous function, $f : [0,1] \to \mathbb{R}$, which vanishes at $x = 1$, there is a sequence of polynomials vanishing at $x = 1$ which converges uniformly to f on $[0,1]$.
   b) If f is continuous on $[0,1]$, and

$$\int_0^1 f(x)(x-1)^k dx = 0 \text{ for each } k = 1, 2, ...,$$

   show that $f(x) \equiv 0$.

6. Let T be a linear transformation from a finite dimensional vector space V into a finite dimensional vector space W. Compute (with proof)

$$\dim\ (\text{Null } T) + \dim\ (\text{Range } T)$$

and

$$\dim\ (\text{Null } T^*) + \dim\ (\text{Range } T)$$

in terms of the dimensions of V and W. Here T^* denotes the adjoint of T.

7. Let $A(x)$ be a function on $\mathbb{R}$ whose values are $n \times n$ matrices. Starting from the definition that the derivative $A'(x)$ is the matrix you get by differentiating the entries in $A(x)$, show that when $A(x)$ is invertible and differentiable for all x, $A^{-1}(x)$ is differentiable, and

$$(A^{-1})'(x) = -A^{-1}(x)A'(x)A^{-1}(x).$$

8. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Does it follow that

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = \infty?$$

Prove your answer.

9. Suppose $u_n : \mathbb{R} \to \mathbb{R}$ is differentiable and solves

$$u_n'(x) = F(u_n(x), x),$$

where F is continuous and bounded.

a) Suppose $u_n \to u$ uniformly. Show that u is differentiable and solves

$$u'(x) = F(u(x), x).$$

b) Suppose

$$u'(x) = F(u(x), x), u(x_0) = y_0$$

has a unique solution $u : \mathbb{R} \to \mathbb{R}$ and $u_n(x_0)$ converges to y_0 as $n \to \infty$. Show that u_n uniformly converges to u.

10. Suppose that $\{\vec{v}_j\}_{j=1}^n$ is a basis for the complex vector space $\mathbb{C}^n$.

a) Show that there is a basis $\{\vec{w}_j\}_{j=1}^n$ such that $(\vec{w}_j, \vec{v}_k) = \delta_{jk}$. Here $(\cdot, \cdot)$ is the standard inner product, $(\vec{w}, \vec{v}) = \overline{w}_1 v_1 + \overline{w}_2 v_2 + \cdots + \overline{w}_n v_n$, and $\delta_{jk} = 1$ when $j = k$ and 0 otherwise.

b) If the $\vec{v}_j$'s are eigenvectors for a linear transformation T of $\mathbb{C}^n$, show that the $\vec{w}_j$'s are eigenvectors for T^*, the adjoint of T with respect to $(\ ,)$.

11. Let f be bounded real function on [0,1]. Show that f is Riemann integrable if and only if f^3 is Riemann integrable.

12. a) Suppose that $x_0 < x_1 < \cdots < x_n$ are points in [a,b]. Define linear functions on $\mathbb{P}^n$, the vector space of polynomials of degree less than or equal n, by setting

$$l_j(p) = p(x_j) \quad j = 0, \ldots, n$$

Show that the set $\{l_j\}_{j=0}^n$ is linearly independent.

b) Show that there are unique coefficients c_j such that

$$\int_a^b p(x)dx = \sum_{j=0}^{n} c_j l_j(p)$$

for all $p \in \mathbb{P}^n$.

Basic Exam, Spring 2007

1. Let A be a real $m \times n$ matrix, $m > n$, whose columns are linearly independent and $\mathbf{b} \in \mathrm{R}^m$. Show that the vector $\mathbf{x}^* \in \mathrm{R}^n$ that minimizes the functional

$$g(\mathbf{x}) = ||A\mathbf{x} - \mathbf{b}||_2^2$$

is given by the solution of the normal equations

$$A^t A\mathbf{x} = A^t\mathbf{b}.$$

Here $||\mathbf{z}||_2^2 = \langle \mathbf{z}, \mathbf{z} \rangle = \sum_i z_i^2$.

2. Let V, W, Z be n-dimensional vector spaces and $T : V \to W$ and $U : W \to Z$ be linear transformations. Prove that if the composite transformation $UT : V \to Z$ is invertible, then both T and U are invertible. (Do not use determinants in your proof!)

3. Consider the space of infinite sequences of real numbers

$$\mathcal{S} = \{(a_0, a_1, a_2, ...) : a_n \in \mathrm{R}, n = 0, 1, 2, ...\}$$

endowed with the standard operations of addition and scalar multiplication:

$$(a_0, a_1, ...) + (b_0, b_1, ...) = (a_0 + b_0, a_1 + b_1, ...); \quad c(a_0, a_1, ...) = (ca_0, ca_1, ...), \quad c \in \mathrm{R}.$$

For each pair of real numbers A and B, prove that the set of solutions $(x_0, x_1, x_2, ...)$ of the linear recursion

$$x_{n+2} = Ax_{n+1} + Bx_n, \quad n = 0, 1, 2, ...$$

is a linear subspace of $\mathcal{S}$ of dimension 2.

4. Suppose that A is a symmetric $n \times n$ real matrix with distinct eigenvalues $\lambda_1, ..., \lambda_l$, $(l \le n)$. Find the sets

$$X = \left\{ \mathbf{x} \in \mathrm{R}^n : \lim_{k \to \infty} (\mathbf{x}^t A^{2k} \mathbf{x})^{1/k} \text{ exists} \right\}$$

and

$$L = \left\{ \lim_{k \to \infty} (\mathbf{x}^t A^{2k} \mathbf{x})^{1/k} : x \in X \right\},$$

where R^n is identified with the set of real column vectors, and $\mathbf{x}^t$ denotes the transpose of $\mathbf{x}$.

5. Let T be a normal linear operator on a finite dimensional complex inner product linear space V. Prove that if $\mathbf{v}$ is an eigenvector of T, then $\mathbf{v}$ is also an eigenvector of its adjoint T^*.

6. Consider the integral equation

$$(*) \qquad y(t) = y_0 + \int_0^t f(s, y(s))ds$$

where $f(t, y)$ is continuous on $[0, T] \times \mathrm{R}$ and is Lipschitz in y with Lipschitz constant K. Assume that you have shown that the iterates defined by

$$y^n(t) = y_0 + \int_0^t f(s, y^{n-1}(s))ds, \quad y^0(t) \equiv y_0$$

converge uniformly to a solution $y(t)$ of (*). Show that if $Y(t)$ is a solution of (*) and satisfies $|Y(t) - y_0| \leq C$ for some constant C and all $t \in [0, T]$, then $Y(t) \equiv y(t)$ on $[0, T]$.

7. Let $f : \mathrm{R} \to \mathrm{R}$ be a twice continuously differentiable function with f'' uniformly bounded, and with a simple root at x^* (i.e., $f(x^*) = 0, f'(x^*) \neq 0$). Consider the fixed point iteration

$$x_n = F(x_{n-1}) \qquad \text{where} \qquad F(x) = x - \frac{f(x)}{f'(x)}.$$

Show that if x_0 is sufficiently close to x^*, then there exists a constant C so that for all n,

$$|x_n - x^*| \leq C|x_{n-1} - x^*|^2.$$

8. Suppose the functions f_n are twice continuously differentiable on $[0, 1]$ and satisfy

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in [0, 1], \text{ and}$$

$$|f_n'(x)| \leq 1, \quad |f_n''(x)| \leq 1 \quad \text{for all } x \in [0, 1], \ n \geq 1.$$

Prove that $f(x)$ is continuously differentiable on $[0, 1]$.

9. (a) Define "f is Riemann integrable on $[0, 1]$".
(b) Prove that every continuous function on $[0, 1]$ is Riemann integrable.

10. Suppose the functions $f_n(x)$ on R satisfy:
(i) $0 \leq f_n(x) \leq 1$ for all $x \in \mathrm{R}$ and $n \geq 1$.
(ii) $f_n(x)$ is increasing in x for every $n \geq 1$.
(iii) $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathrm{R}$, where f is continuous on R.
(iv) $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$.
Show that $f_n(x) \to f(x)$ uniformly on R.

11. (a) Consider the equations

$$u^3 + xv - y = 0, \quad v^3 + yu - x = 0.$$

Can these equations be solved uniquely for u, v in terms of x, y in a neighborhood of $x = 0, y = 1, u = 1, v = -1$? Explain your answer.
(b) Give an example in which the conclusion of the implicit function theorem is true but the hypothesis is not.

12. Let c_0 be the normed space of real sequences $x = (x_1, x_2, ...)$ such that $\lim_{k\to 0} x_k = 0$ with the supremum norm $||x|| = \sup_k |x_k|$.
(a) Show that c_0 is complete.
(b) Is the unit ball $\{x \in c_0 : ||x|| \leq 1\}$ compact? Prove your answer.
(c) Is the set $\{x \in c_0 : \sum_k k|x_k| \leq 1\}$ compact? Prove your answer.

Basic Exam Fall 08

Instructions

Solve any 10 of the following 12 problems. You will not receive credit for more than 10 problems. Indicate which problems you wish to be graded by circling the corresponding numbers.

(1) For which of the values $a = 0, 1, 2$ is the function $f(t) = t^a$ uniformly continuous on $[0, \infty)$? Prove your assertions.

(2) Suppose that A is a non-empty connected subset of $\mathbb{R}^2$.
   (a) Prove that if A is open, then it is path connected.
   (b) Is (a) true if A is closed? Prove your assertion.

(3) Give an example of a sequence of continuous real-valued functions f_n on $[0, 1]$ such that $f(t) = \lim f_n(t)$ is continuous, but $\int_0^1 f_n(t)dt$ does not converge to $\int_0^1 f(t)dt$.

(4) (a) Suppose that K and F are subsets of $\mathbb{R}^2$ with K closed and bounded and F closed. Prove that if $K \cap F = \emptyset$, then $d(K, F) > 0$. Recall that

$$d(K, F) = \inf\{d(x, y) : x \in K, y \in F\}.$$

   (b) Is (a) true if K is just closed? Prove your assertion.

(5) A rearrangement of a series $\sum_{n=1}^{\infty} a_n$ is a series of the form $\sum_{k=1}^{\infty} a_{n(k)}$, were $n : \mathbb{N} \to \mathbb{N}$ is a bijection (i.e. one-to-one and onto). Show that there is a rearrangement of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges to π.

(6) Suppose that V is an n-dimensional vector space ($n \in \mathbb{N}$) and that $T : V \to V$ is a linear mapping. *Prove* that

$$\dim \ker T + \dim \operatorname{range} T = n$$

Note: Do not just quote a standard theorem.

(7) Suppose that $T = [t_{ij}]$ is a complex $n \times n$ matrix, and that $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues of T, with corresponding non-zero eigenvectors $v_1, \ldots, v_r$. Show that $v_1, \ldots, v_r$ are linearly independent.

(8) Must the eigenvectors of a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ span $\mathbb{C}^n$? Prove your assertion.

(9) (a) Prove that any linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ must have an eigenvector.
   (b) Is (a) true for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$?

(10) Given $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we let $\|v\| = (\sum |v_j|^2)^{1/2}$. If $f = (f_1, \dots, f_n) : [a, b] \to \mathbb{R}^n$ is a continuous function, we define

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \dots, \int_a^b f_n(t)dt \right).$$

Prove that

$$\| \int_a^b f(t)dt \| \leq \int_a^b \|f(t)\| dt.$$

(11) Consider the Poisson equation with periodic boundary conditions on $[0, 1]$

$$\frac{\partial^2 u}{\partial x^2} = f, \quad x \in (0, 1)$$
$$u(0) = u(1).$$

A second order accurate approximation to the problem is given by the solution to the following system of equations

$$\mathbf{Au} = \Delta x^2 \mathbf{f}$$

where

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & \dots & 0 & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots \\ & & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}$$

$\mathbf{u} = [u_0, u_1, ..., u_{n-1}]$, $\mathbf{f} = [f_0, f_1, ..., f_{n-1}]$ and $u_i \approx u(x_i)$ with $x_i = i\Delta x$, $\Delta x = 1/n$ and $f_i = f(x_i)$ for $i = 0, \dots, n-1$.

a. Show that the matrix $\mathbf{A}$ is singular.

b. What condition must $\mathbf{f}$ satisfy so that a solution exists?

(12) Consider the least squares problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|_2$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \geq n$. Prove that if $\mathbf{x}$ and $\mathbf{x} + \alpha \mathbf{z}$ $(\alpha \neq 0)$ are minimizers then $\mathbf{z} \in \text{null}(A)$.

Basic Exam, March 2008

1. Let $g \in C([a,b])$, with $a \le g(x) \le b$ for all $x \in [a,b]$. Prove the following:

   (i) g has at least one fixed point p in the interval $[a,b]$.

   (ii) If there is a value $\gamma < 1$ such that

$$|g(x) - g(y)| \le \gamma |x - y|$$

for all $x, y \in [a,b]$, then the fixed point p is unique, and the iteration

$$x_{n+1} = g(x_n)$$

converges to p for any initial guess $x_0 \in [a,b]$.

2. Let $\{f_n(x)\}$ be a sequence of continuous functions on the unit interval $[0,1]$ such that $f_n(x) \ge 0$ for all n and x and such that for all $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = 0.$$

Prove or give a counterexample to the assertion:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

3. Assuming that $f \in C^4[a,b]$ is real, derive a formula for the error of approximation $E(h)$ when the second derivative is replaced by the finite-difference formula

$$f''(x) \sim \frac{f(x+h) - 2f(x) + f(x-h)}{h^2},$$

and h is the mesh size. (Assume that x, $x+h$, $x-h \in (a,b)$).

4. Let X be a compact subset of $\mathbb{R}^N$ and let $\{f_n(x)\}$ be a sequence of continuous real functions on X such that

   $0 \le f_{n+1}(x) \le f_n(x)$

and

   $\lim f_n(x) = 0$ for all $x \in X$.

Prove Dini's Theorem that $f_n(x)$ converges to 0 *uniformly* on X.

5. (a) Let $F(x,y)$ be a continuous function on the plane such that for every square S having its sides parallel to the axes,

$$\int\int_S F(x,y)dxdy = 0.$$

Prove $F(x,y) = 0$ for all (x,y).

(b) Assume $f(x,y)$, $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$ are all continuous in the plane. Use part (a) to prove that

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right).$$

Hint: You may assume the double integral in (a) equals the iterated integral $\int(\int F(x,y)dx)dy$ and equals the iterated integral $\int(\int F(x,y)dy)dx$.

6. Let Y be a complete *countable* metric space. Prove there is $y \in Y$ such that $\{y\}$ is open.

7. Let $a(x)$ be a function on $\mathbb{R}$ such that

(i) $a(x) \geq 0$ for all x, and

(ii) There exists $M < \infty$ such that for all *finite* $F \subset \mathbb{R}$,

$$\sum_F a(x) \leq M.$$

Prove $\{x : f(x) > 0\}$ is countable.

8. Assume V is an n-dimensional vector space over the rationals $\mathbb{Q}$, and T is a $\mathbb{Q}$-linear tranformation $T : V \to V$ such that $T^2 = T$. Prove that every vector $v \in V$ can be written uniquely as $v = v_1 + v_2$ such that $T(v_1) = v_1$ and $T(v_2) = 0$.

9. Let V be a vector space over $\mathbb{R}$.

(a) Prove that if V is odd dimension, and if T is an $\mathbb{R}$-linear transformation $T : V \to V$ of V, then T has a non-zero eigenvector $v \in V$.

(b) Show that for every even positive integer n, there is a a vector space V over $\mathbb{R}$ of dimension n, and an $\mathbb{R}$-linear transformation $T : V \to V$ of V, such that there is no non-zero $v \in V$ satisfying $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$.

10. Suppose A is an $n \times n$ complex matrix such that A has n distinct eigenvalues. Prove that if B is an $n \times n$ complex matrix such that $AB = BA$, then B is diagonalizable.

11. Assume A is an $n \times n$ complex matrix such that for some positive integer m the power $A^m = I_n$, where I_n is the $n \times n$ identity matrix. Prove that A is diagonalizable.

12. Let A be an $n \times n$ real symmetric ($a_{i,j} = a_{j,i}$) matrix, and let $S = \{x \in \mathbb{R}^n : \sum x_j^2 = 1\}$ be the unit sphere of $\mathbb{R}^n$. Let $x \in S$ be such that

$$(Ax, x) = \sup_S (Ay, y)$$

where $(z, y) = \sum z_j y_j$ is the usual inner product on $\mathbb{R}^n$. (By compactness such x exists.)

(a) Prove that $(x, y) = 0 \Longrightarrow (Ax, y) = 0$. Hint: Expand

$$(A(x + \epsilon y), x + \epsilon y).$$

(b) Use (a) to prove x is an eigenvector for A.

(c) Use induction to prove $\mathbb{R}^n$ has an orthonormal basis of eigenvectors for A.

Note: If you use part (c) to prove part (a) or part (b), then your solution should include a proof of part (c) that does not use part (a) or part (b).

Basic Exam Fall 2009

Instructions for qualifying exams

Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!

Complete this sheet. Read the directions of the exam very carefully.

STUDENT ID NUMBER: ______________________

DATE: ______________________

HOME DEPARTMENT: ______________________

INSTRUCTIONS: Do any 10 of the following questions. If you attempt more than 10 questions, indicate below which ones you would like to be considered for credit (otherwise the first 10 will be taken). Each question counts for 10 points. Little or no credit will be given for answers without adequate justification. You have 4 hours. Good luck.

NOTATION: We denote by $\mathbb{N} = 1, 2\dots$ the natural numbers, by $\mathbb{R}$ and $\mathbb{C}$ the real and complex numbers respectively, and by $M_n(\mathbb{R}), M_n(\mathbb{C})$ the $n \times n$ matrices with real and complex coefficients respectively.

#	Score	Counts in 10?
1	______	____
2	______	____
3	______	____
4	______	____
5	______	____
6	______	____
7	______	____
8	______	____
9	______	____
10	______	____
11	______	____
12	______	____
Total	______	10

1. (i). For each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to \mathbb{R}$ be a function with $|f_n(m)| \leq 1$ for all $m, n \in \mathbb{N}$. Prove that there is an infinite subsequence of distinct positive integers n_i, such that for each $m \in \mathbb{N}$, $f_{n_i}(m)$ converges.

(ii). For n_i as in (i), assume that in addition $\lim_{m\to\infty} \lim_{i\to\infty} f_{n_i}(m)$ exists and equals 0. Prove or disprove: The same holds for the reverse double limit $\lim_{i\to\infty} \lim_{m\to\infty} f_{n_i}(m)$.

2. (i). Let X be a complete metric space with respect to a distance function d. We say that a map $T : X \to X$ is a *contraction* if for some $0 < \lambda < 1$ and all $x, y \in X$: $d(f(x), f(y)) \leq \lambda d(x, y)$. Prove that if T is a contraction then it has a fixed point, i.e., there is an $x \in X$ such that $T(x) = x$.

(ii). Using (i) show that given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose first derivative satisfies $f'(x) = e^{-x^2} - e^{-x^4}$, there exists $\alpha \in \mathbb{R}$ with $f(\alpha) = \alpha$.

3. The purpose of this problem is to give a multi variable calculus proof of the geometric and arithmetic means inequality along the concrete steps below. The inequality has numerous other proofs and naturally you are not allowed to use it (or them) below.

(i). Let $\mathbb{R}^n_+ \subset \mathbb{R}^n$ be the (open) subset of vectors all whose coordinates are *positive*, and $f : \mathbb{R}^n_+ \to \mathbb{R}$ be defined by:

$$f(x_1, \ldots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdot x_2 \cdots x_n}$$

(i). Explain *carefully* why f attains a global (not necessarily unique) minimum at some $p \in \mathbb{R}^n_+$. (Hint: what happens when $x_i \to 0, \infty$?)

(ii). Find p.

(iii). Deduce that if all $x_i \in \mathbb{R}$ are positive and $\Pi x_i = 1$ then $\Sigma x_i \geq n$, with equality iff $x_i = 1$ for all i. (This is a special case of the geometric and arithmetic means inequality, from which the general statement can be immediately deduced – no need to write down this part here).

4. Let V be a finite dimensional dimensional $\mathbb{R}$-vector space, whose dimension we denote by $\dim(V)$, equipped with an inner product

$$<,>: V \times V \to \mathbb{R}.$$

For a vector subspace $U \subseteq V$, denote by $U^\perp$ its orthogonal complement, i.e., the set of $v \in V$ such that $< v, u >= 0$ for all $u \in U$. Show that $\dim(U) + \dim(U^\perp) = \dim(V)$.

5. Show that if $\alpha_1, \ldots \alpha_n \in \mathbb{R}$ are all *different*, and some $a_1, \ldots a_n \in \mathbb{R}$ satisfy:

$$\Sigma\, a_i e^{\alpha_i t} = 0 \qquad \forall t \in (-1, 1),$$

then necessarily $a_i = 0$ for all $1 \leq i \leq n$. (Hint: you may use the differentiation operator and a theorem in Linear Algebra on distinct eigenvalues.)

6. Consider the function $f(x,y) = \sin^3(xy) + y^2|x|$ defined on the region $S \subset \mathbb{R}^2$ given by

$$S = \{(x,y) \in \mathbb{R}^2; \quad x^{2010} + y^{2010} \leq 1\}.$$

Define what it means for f to be uniformly continuous on S and prove that f is indeed uniformly continuous. (You can use any theorem you wish in the proof, as long as it is stated correctly and you justify properly why it can be applied, e.g., if you are using a general theorem on continuous functions, show that the function in question is indeed continuous, and if you are using a metric property of a set explain why it has it.)

7. Let $V \simeq \mathbb{R}^n$ be an n -dimensional vector space over $\mathbb{R}$, and denote by $\mathrm{End}(V)$ the vector space of $\mathbb{R}$-linear transformations of V. (Note that $\dim(\mathrm{End}(V)) = \dim(V)^2 = n^2$.) Then for $T \in \mathrm{End}(V)$ show that the dimension of the subspace W of $\mathrm{End}(V)$ spanned by T^k, for k running through non-negative integers, satisfies the inequality $\dim(W) \leq \dim(V) = n$.

8. For a matrix $A \in M_n(\mathbb{R})$, define $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Let $v_0 \in \mathbb{R}^n$. Prove that the function $v : \mathbb{R} \to \mathbb{R}^n$ given by $v(t) = e^{At}v_0$ solves the linear differential equation $v'(t) = Av(t)$ with the initial condition $v(0) = v_0$. Explain precisely which theorems in calculus you are using in your proof and why they are applicable.

9. If $A \in M_{2n+1}(\mathbb{R})$ is such that $AA^t = \mathrm{Id}_{2n+1}$ the identity matrix, then prove that one of 1 or -1 is an eigenvalue of A.

10. (i). Let $I = [0,2]$. If $f : I \to \mathbb{R}$ is a continuous function such that $\int_I f(x)dx = 36$, prove that there is an $x \in I$ such that $f(x) = 18$.

(ii) Consider $I^2 \subset \mathbb{R}^2$, and let $g : I^2 \to \mathbb{R}$ be a continuous function such that $\int_{I^2} g(x,y)dxdy = 36$. Prove that there is $(x,y) \in I^2$ such that $g(x,y) = 9$.

11. (i). State the Cayley-Hamilton theorem for matrices $A \in M_n(\mathbb{C})$.

(ii). Prove it directly for diagonalisable matrices.

(iii). Identify $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ through some (say, the natural) linear isomorphism. Through this identification $M_n(\mathbb{C})$ becomes a metric space with the Euclidean metric. Fact: The subset of diagonalisable matrices in $M_n(\mathbb{C})(\simeq \mathbb{C}^{n^2})$ is dense. Use this fact, together with part (ii), to prove the Cayley-Hamilton theorem.

12. Let V be an $n(\geq 2)$-dimensional vector space over $\mathbb{C}$ with a set of basis vectors $e_1, \cdots, e_n$. Let T be a linear transformation of V satisfying $T(e_1) = e_2, \cdots, T(e_{n-1}) = e_n, T(e_n) = e_1$.

(i). Show that T has 1 as an eigenvalue and write down an eigenvector with eigenvalue 1. Show that up to scaling it is unique.

(ii). Is T diagonalisable? (Hint: calculate the characteristic polynomial.)

Basic Examination, Spring 2009

2–6pm, Saturday, March 28, 2009

Instructions: Work any 10 problems. All problems are worth ten points; parts of a problem do not carry equal weight. You must tell us which 10 problems you want us to grade.

The grading will emphasize your attention to detail.

The problems are listed in no particular order.

Problem 1. Set $a_1 = 0$ and define a sequence $\{a_n\}$ via the recurrence

$$a_{n+1} = \sqrt{6 + a_n} \qquad \text{for all } n \geq 1.$$

Show that this sequence converges and determine the limiting value.

Problem 2. Compute the norm of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$$

That is, determine the maximum value of the length of Ax over all unit vectors x.

Problem 3. We wish to find a quadratic polynomial P obeying

$$P(0) = \alpha, \quad P'(0) = \beta, \quad P(1) = \gamma, \quad \text{and} \quad P'(1) = \delta$$

where $'$ denotes differentiation.

(a) Find a minimal system of linear constraints on $(\alpha, \beta, \gamma, \delta)$ such that this is possible.

(b) When the constraints are met, what is P? Is it unique? Explain your answer.

Problem 4. Let (X, d) be an arbitrary metric space.

(a) Give a definition of *compactness* of X involving open covers.

(b) Define *completeness* of X.

(c) Define *connectedness* of X.

(d) Is the set of rational numbers $\mathbb{Q}$ (with the usual metric) connected? Justify your answer.

(e) Suppose X is complete. Show that X is compact in the sense of part (a) if and only if for every $r > 0$, X can be covered by finitely many balls of radius r.

Problem 5. Compute e^{At} when

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

Recall that e^{At} is defined by the property that a smooth vector function $x(t)$ obeys

$$\frac{dx}{dt}(t) = Ax(t) \quad \Longleftrightarrow \quad x(t) = e^{At}x(0)$$

Problem 6. Show that a continuous function $f : [0, 1) \to \mathbb{R}$ is *uniformly* continuous if and only if there is a continuous function $g : [0, 1] \to \mathbb{R}$ that obeys $f(x) = g(x)$ for all $x \in [0, 1)$.

Problem 7. (a) Define what it means for $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable at a point $a \in \mathbb{R}^n$.
(b) Using this definition, formulate and prove an appropriate form of the chain rule, that is, a theorem describing the derivative of $g(f(x))$ at $x = a$.

Problem 8. Let $M_{n\times n}(\mathbb{R})$ denote the vector space of $n \times n$ matrices with real entries.
(a) Show that

$$\langle A, B\rangle = \operatorname{tr}(AB^T)$$

defines an inner product on $M_{n\times n}(\mathbb{R})$. More precisely, show that it obeys the axioms of an inner product. Note: tr denotes the *trace* of a matrix and ^{T} denotes the *transpose.*
(b) Given $C \in M_{n\times n}(\mathbb{R})$, we define a linear transformation

$$\Phi_C : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}) \quad \text{by} \quad \Phi_C(A) = CA - AC$$

Compute the adjoint of Φ_C. Check that when C is symmetric, then Φ_C is self-adjoint.
(c) Show that whatever the choice of C, the map Φ_C is not onto (i.e. is not surjective).

Problem 9. Let us say that a real symmetric $n \times n$ matrix A is a *reflection* if $A^2 = \mathrm{Id}$ and

$$\operatorname{rank}(A - \mathrm{Id}) = 1,$$

where Id denotes the identity matrix. Given *distinct* unit vectors $x, y \in \mathbb{R}^n$ show that there is a reflection with $Ax = y$ and $Ay = x$. Moreover, show that the reflection A with these properties is unique.

Problem 10. (a) Rigorously justify the following:

$$\int_0^1 \frac{dx}{1+x^2} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n}{2n+1}$$

(b) Deduce the value of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

Problem 11. (a) Explain the following (overly informal) statement:

> Every matrix can be brought to Jordan normal form; moreover the normal form is essentially unique.

No proofs are required; however, all statements must be clear and precise. All required hypotheses must be included. The meaning of the phrases 'brought to', 'Jordan normal form', and 'essentially unique' must be defined explicitly.
(b) Define the *minimal polynomial* of a matrix. How may it be determined for a matrix in Jordan normal form?

Problem 12. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Show that

$$\operatorname{div}(F) = \rho$$

for all points $(x, y) \in \mathbb{R}^3$ if and only if

$$\iint_{\partial\Omega} F \cdot dS = \iiint_{\Omega} \rho \, dx\, dy\, dz$$

for all balls Ω (with all radii $r > 0$ and all possible centers).
[You may use the various standard theorems of vector calculus without proof.]

Basic exam, Spring 2010

Instructions: Do any 10 of the following questions. If you attempt more than 10 questions, indicate which one you would like to be considered for credit by crossing question we should not check (otherwise the first 10 will be taken). All problems worth 10 points. Parts of the problem do not carry equal weight. Little or no credit will be given for answers without adequate justification. Write your university identification number at the top of each sheet of paper.

Good luck!

Problem 1: Let $u_1, \ldots, u_n$ be orthonormal basis of $\mathbb{R}^n$ and let $y_1, \ldots, y_n$ be a collection of vectors in $\mathbb{R}^n$ satisfying $\sum_i ||y_i||^2 < 1$. Prove that vectors $u_1 + y_1, \ldots, u_n + y_n$ are linearly independent.

Problem 2: Let A be $n \times n$ real symmetric matrix and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Prove that

$$\lambda_k = \max_{U, \dim U = k} \min_{x \in U, ||x||=1} < Ax, x >,$$

where $< \cdot, \cdot >$ denotes the usual scalar product in $\mathbb{R}^n$ and the maximum is taken over all k dimensional subspaces of $\mathbb{R}^n$

Problem 3: Let S and T be two normal transformations in the complex finite dimensional vector space V with a positive definite Hermitian inner product such that $ST = TS$. Prove that S and T have joint basis of eigenvectors.

Problem 4: (i). Let $A = (a_{i,j})$ be $n \times n$ real symmetric matrix such that $\sum_{i,j} a_{i,j} x_i x_j \leq 0$ for every vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Prove that if $tr(A) = 0$ then $A = 0$.
(ii). Let T be a linear transformation in the complex finite dimensional vector space V with a positive definite Hermitian inner product. Suppose that $TT^* = 4T - 3I$, where I is identity transformation. Prove that T is positive definite Hermitian and find all possible eigenvalues of T.

Problem 5: Let A, B two $n \times n$ complex matrices which have the same minimal polynomial $M(t)$ and the same characteristic polynomial $P(t) = (t - \lambda_1)^{a_1} \cdots (t - \lambda_k)^{a_k}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$. Prove that if $P(t)/M(t) = (t - \lambda_1) \cdots (t - \lambda_k)$, then these matrices are similar.

Problem 6: Let $A = \begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix}$.
(i). Find Jordan form J of A and a matrix P such that $P^{-1}AP = J$.
(ii). Compute A^{100} and J^{100}.
(iii). Find a formula for a_n, when $a_{n+1} = 4a_n - 4a_{n-1}$ and $a_0 = a, a_1 = b$.

Problem 7: Let $\{f_n\}$ be a sequence of real-valued functions on the line, and assume that there is a $B < \infty$ such that $|f_n(x)| \leq B$ for all n and x. Prove that there is a subsequence $\{f_{n_k}\}$ such that $\lim_{k \to \infty} f_{n_k}(r)$ exists for all rational numbers r.

Problem 8: Assume that K is a closed subset of a complete metric space (χ, d) with the property that, for any $\epsilon > 0$, K can be covered by a finite number of sets $B_\epsilon(x)$, where

$$B_\epsilon(x) = \{y \in \chi : d(x,y) < \epsilon\}.$$

Prove that K is compact.

Problem 9: Assume that $f(x,y,z)$ is a real valued, continuously differentiable function such that $f(x_0,y_0,z_0) = 0$. If $\vec{\nabla} f(x_0,y_0,z_0) \neq \vec{0}$, show that there is a differentiable surface, given parametrically by $(x(s,t), y(s,t), z(s,t))$ with $(x(0,0), y(0,0), z(0,0)) = (x_0,y_0,z_0)$, on which $f = 0$.

Problem 10: Let $f(x,y)$ be the function defined by

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

when $(x,y) \neq (0,0)$ with $f(0,0) = 0$.

(a) Compute the directional derivatives of $f(x,y)$ at (0,0) in all directions where they exist.

(b) Is $f(x,y)$ differentiable at (0,0)? Prove your answer.

Problem 11: Suppose $\sum_{n=1}^\infty |a_n| < \infty$. Let σ be a one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N}$. The series $\sum_{n=1}^\infty a_{\sigma(n)}$ is called a "rearrangement" of $\sum_{n=1}^\infty a_n$. Prove that all rearrangements of $\sum_{n=1}^\infty a_n$ are convergent and have the same sum.

Problem 12: Assume that $\{f_n\}$ is a sequence of nonnegative continuous functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. Is necessarily true that

(a) There is a B such that $f_n(x) \leq B$ for $x \in [0,1]$ for all n?

(b) There are points x_0 in [0,1] such that $\lim_{n\to\infty} f_n(x_0) = 0$?

Prove your answers.