ANALYSIS QUALIFYING EXAM, FALL 2001

Directions:

All problems on this test are worth 10 points. There are six Real Analysis problems and five Complex Analysis problems.

Your score will be computed from your best scores on **five** Real Analysis problems and **five** Complex Analysis problems.

In this exam you may use the axiom of choice.

Real Analysis Problems

R1: Consider real numbers $a_{n,m}$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ and assume that inner and outer sums in the expressions

$$A := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{n,m} \right]$$

$$B := \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{n,m} \right]$$

are absolutely convergent.

a) Give an example that shows that we may have $A \neq B$.

b) Under what reasonable additional assumption on $a_{n,m}$ can we conclude $A = B$?

R2: Let $n \geq 1$. Let $O(n)$ denote the set of all real $n \times n$ matrices G which satisfy

$$G^T G = I$$

where I is the identity matrix (O stands for "orthogonal group").

a) Prove that $O(n)$ is compact.

b) Prove that $O(n)$ is not connected.

R3: Prove the mean value theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. For every $a < b$ there exists $a < \xi < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi)$$

R4: The Fourier transform $\widehat{f}$ of a function f in $L^1(\mathbb{R})$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx$$

a) State how the Fourier transform of a function in $L^2(\mathbb{R})$ is defined. (You do not need to prove claims which you use to state this. Do not use the fact b) below.)

b) Let f be in $L^2(\mathbb{R})$. Define

$$Mf(\xi) = \sup_{r>0} \left| \int_{-r}^{r} f(x) e^{-2\pi i x \xi}\, dx \right|$$

A deep fact proved by Carleson and Hunt states that

$$\|Mf\|_2 \leq C\|f\|_2$$

for some universal constant C. Use this theorem to prove

$$\widehat{f}(\xi) = \lim_{r\to\infty} \int_{-r}^{r} e^{-2\pi i x\xi} f(x)\, dx$$

for almost every $\xi \in \mathbb{R}$.

R5: For $n \geq 1$ let $l^1(n)$, $l^2(n)$, $l^\infty(n)$ be the Banach space $\mathbb{R}^n$ equipped with the norm

$$\sum_{i=1}^{n} |x_i|, \quad (\sum_{i=1}^{n} |x_i|^2)^{1/2}, \quad \sup_{i=1,\dots,n} |x_i|$$

respectively. Answer the following questions and prove your assertions:

a) Which of the Banach spaces $l^1(2)$, $l^2(2)$, $l^\infty(2)$ are isometrically isomorphic?

b) Which of the Banach spaces $l^1(3)$, $l^2(3)$, $l^\infty(3)$ are isometrically isomorphic?

R6: Let V be the complex Banach space $l^\infty(\mathbb{N})$, i.e. the space of all sequences $x = (x_n)_{n=1,2,\dots}$ with the norm $\|x\| = \sup_n |x_n|$. Every sequence f in $l^1(\mathbb{N})$ gives rise to a linear functional $\phi_f : V \to \mathbb{C}$ by the formula $\phi_f(x) = \sum_{n=1}^{\infty} f_n x_n$

a) Prove that ϕ_f is continuous for each $f \in l^1(\mathbb{N})$.

b) Prove that there are elements in the dual space of V which are not of the form ϕ_f for any $f \in l^1(\mathbb{N})$. Hint: consider the subspace of V consisting of convergent sequences.

Complex Analysis Problems

C1: Find an explicit conformal mapping from the region

$$\{|z| < 1\} \setminus [0, 1)$$

onto the upper half plane $\{Imz > 0\}$.

C2:
Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{9 + 10x^2 + x^4} \quad .$$

C3:
Define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

a) For m and n positive integers, calculate

$$\frac{\partial}{\partial \overline{z}} z^n \overline{z}^m \quad .$$

b) Let $P(x, y)$ be a polynomial in the two real variables x and y. Then P has the form

$$P(x, y) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_{n,m} z^n \overline{z}^m$$

but you do not need to prove it. Prove however that if

$$\frac{\partial}{\partial \overline{z}} P(x, y) = 0 \quad ,$$

then $a_{n,m} = 0$ or all $m > 0$.

C4:
Let $u(z)$ be a harmonic function on the entire plane $\mathbb{C}$ such that

$$\int\int_{\mathbb{C}} |u(z)|^2 \, dx dy < \infty \quad .$$

Prove $u(z) = 0$ for all z.

C5:
Let $F(z)$ be continuous on the closed unit disc $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ and analytic on the open disc $\mathbb{D} = \{z : |z| < 1\}$.

a) Prove

$$\lim_{\lambda \uparrow 1} F(\lambda z) = F(z)$$

uniformly on $\overline{\mathbb{D}}$

b) If also

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

on $\mathbb{D}$, prove that

$$\lim_{n \to \infty} a_n = 0 \quad .$$

New/revised

ANALYSIS QUALIFYING EXAMINATION

SEPTEMBER 18, WEDNESDAY, 2002

2:00 - 6:00 PM

ROOM: MS 5200

Instructions

Work any 10 problems, but must include 2 problems each from Part I, Part II and Part III respectively. All problems are worth 10 points.

Part I

1. Let f, g be two absolutely continuous functions on the interval $[0, 1]$ which are everywhere positive. Show that the pointwise quotient f/g is also absolutely continuous on $[0, 1]$. [**Hint**: Is g bounded from below?]

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a real-valued function in $L^1(X, \mathcal{M}, \mu)$. Show that there exists a non-negative function $f^* \in L^1([0, \infty))$ which is monotone non-increasing (i.e. $f^*(x) \leq f^*(y)$ for all $x \geq y \geq 0$), and such that

$$\int_X |f|^p d\mu = \int_0^\infty f^*(x)^p \, dx$$

for all $0 < p < \infty$. [**Hint**: Choose f^* so that $\mu(\{x \in X : |f(x)| \geq \lambda\}) = m(\{y \in [0, \infty) : f^*(y) \geq \lambda\})$ for all $\lambda > 0$, where m is Lebesgue measure.]

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $1 \leq p < \infty$. Let $f_1, f_2, \dots$ be a sequence of functions in $L^p(X, \mathcal{M}, \mu)$ which converge pointwise μ-a.e. to a function $f \in L^p(X, \mathcal{M}, \mu)$. Show that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

4. Let E and F be two Lebesgue measurable sets of the real line of finite measure, and let χ_E and χ_F be their respective characteristic functions.

(a) Show that the convolution $\chi_E * \chi_F$, defined by

$$\chi_E * \chi_F(x) = \int_{\mathbf{R}} \chi_E(y)\chi_F(x - y) \, dy$$

is a continuous function.

(b) Show that $\chi_E * \chi_F$ lies in L^p for every $1 \leq p \leq \infty$.

5. Let $0 < \alpha < 1$. A function $f \in C([0, 1])$ is said to be *Hölder continuous of order* α if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in [0,1]$.

Show that for every $0 < \alpha < 1$, the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n x)$$

is Hölder continuous of order α but is nowhere differentiable on $[0,1]$.

6. Let $n > 1$ be an integer, and let $B = \{x \in \mathbf{R}^n : |x| < 1\}$ be the open unit ball in $\mathbf{R}^n$. Show that there exists a constant $0 < C < \infty$ depending only on n, such that

$$\int_B |u(x)|^2 \, dx \le C \int_B |\nabla u(x)|^2 dx$$

for all smooth, compactly supported real-valued functions $u : B \to \mathbf{R}$.

Part II

7. Show that for every Hilbert space H and every closed convex subset $\Omega \subset H$, there exists a unique element $x \in \Omega$ of minimal norm, i.e., $\|x\| \le \|y\|$ for all $y \in \Omega$.

8. Show that the set

$$\Omega = \left\{ f \in C([-1,1]) : \int_{-1}^{1} f(x) signum(x) \, dx = 1 \right\}$$

is a close convex set Ω in the Banach space $C([-1,1])$ which does not contain any element of minimal norm. Here $signum(x)$ is the function which equals $+1$ for positive x, -1 for negative x, and 0 for $x = 0$.

[**Hint**: If $C([-1,1])$ were replaced by the larger space $L^\infty([-1,1])$, what would be the element of minimal norm?]

9. Let $T : L^2([0,1]) \to L^2([0,1])$ be the operator

$$Tf(x) = \int_0^x f(y) \, dy.$$

(a) Show that the Fourier coefficients

$$\widehat{Tf}(n) = \int_0^1 e^{-2\pi i n x} Tf(x) \, dx$$

of Tf obey the bound

$$|\widehat{Tf}(n)| \le \frac{C\|f\|_{L^2([0,1])}}{n}$$

for all non-zero n, all $f \in L^2([0,1])$, and some constant $0 < C < \infty$.

(b) Show that T is a continuous, compact operator (i.e. the image of the closed unit ball is compact). [**Hint**: use (a)].

(c) Show that for any complex non-zero λ, the operator $T - \lambda$ has no kernel (i.e. there is no non-zero $f \in L^2([0,1])$ such that $(T - \lambda)f = 0$).

(d) Show that the operator $T - \lambda$ is invertible on $L^2([0,1])$ for all complex non-zero λ. [**Hint**: Use (b) and (c)].

Part III

10. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_1, f_2, \ldots$ be sequence of complex-valued functions in $L^1(X, \mathcal{M}, \mu)$ such that

$$\|f_n\|_{L^1(X,\mathcal{M},\mu)} \leq 2^{-n}$$

and

$$\|f_n\|_{L^\infty(X,\mathcal{M},\mu)} \leq 1/2$$

for all $n = 1, 2, \ldots$. Show that the infinite product

$$\prod_{n=1}^{\infty} (1 + f_n(x))$$

is convergent for μ-a.e. x, and that the product is a measurable function. [**Hint**: You may need to compare $|1 + z|$ and $\exp(|z|)$ in the disk $|z| \leq 1/2$.]

11. Find the linear fractional tranformation $f : \mathbf{C} \to \mathbf{C}$ such that $f(0) = 1$, $f(1) = 0$, and $f(\infty) = i$. What is the image of the line $\{z : \mathrm{Re}(z) = 1\}$ under f?

12. Suppose the function $f(z)$ is continuous on the closed unit disk $\{|z| \leq 1\}$ and analytic on the open disk $\{|z| < 1\}$. Assume $f(z) = 0$ for all z in the semi-circle $\{z : |z| = 1, \mathrm{Im}(z) > 0\}$. Prove that $f(z) = 0$ on the closed disk.

13. Let $U_n(z)$ be a sequence of *positive* harmonic functions on a *connected* open set Ω containing the origin. Show that if

$$\lim_{n \to \infty} U_n(0) = 0,$$

then

$$\lim_{n \to \infty} \|U_n\|_{L^\infty(K)} = 0$$

for all compact subsets $K \subset \Omega$.

14. Evaluate the integral

$$\lim_{N\to\infty}\int_0^N \cos(x^2)\mathrm{d}x;$$

justify your reasoning.

15. Let $1 \leq p < \infty$ and let $U(z)$ be a harmonic function on the entire plane $\mathbf{C}$ such that

$$\int\int_{\mathbf{C}} |U(z)|^p \mathrm{d}x\mathrm{d}y < \infty.$$

Prove $U(z) = 0$ for all z.

Analysis Qualifying Examination - May 18, 2002

Instructions:

Work any 12 problems, and especially three from Problems 9 to 14. All problems are worth ten points.

1. Let V be a finite dimensional real vector space, and let $|| \ ||_V$ be a norm on V. Let $\mathcal{P}$ be the set of one-dimensional linear subspaces of V ($\mathcal{P}$ is called a real projective space.) For $W_1, W_2 \in \mathcal{P}$ define

$$d(W_1, W_2) = \inf\{||v_1 - v_2||_V : v_j \in W_j \text{ and } ||v_j||_V = 1\}.$$

Prove that d is a metric on $\mathcal{P}$ and that $\mathcal{P}$ is compact with respect to this metric.

2. Let $\{a_n\}_n$ be a sequence of real numbers such that $\lim_n a_n = 0$ but such that $\sum_n a_n$ is divergent. Show that for any real number r there is a sequence $\sigma_n \in \{-1, 1\}$ such that

$$\sum_n \sigma_n a_n = r.$$

3. Prove or disprove that there exists an infinite dimensional real Banach space X containing a countable subset S such that every $x \in X$ is a (finite) linear combination of elements of S.

4. Let $f_n(x)$ be a sequence of Borel measurable real-valued functions on the interval $[0,1]$ such that $f_n(x) \to 0$ for all $x \in [0,1]$. Let $\epsilon > 0$. Prove there is a Borel set $A \subset [0,1]$ such that

(i) $m([0,1] \cap A') < \epsilon$;

where m is Lebesgue measure, and

(ii) $f_n(x) \to 0$ uniformly on A.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $f \in L^1(X, \mathcal{M}, \mu)$. Prove

$$\lim_{p \to 0} \Big(\int |f|^p d\mu \Big)^{1/p} = \exp\Big(\int \log |f| d\mu \Big)$$

where $e^{-\infty}$ is defined to be 0. Hint: Jensen's inequality.

6. On the two-point space $\{0, 1\}$ let μ be the measure such that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let X be the infinite product space

$$\prod_{j=1}^{\infty} X_j$$

where each $X_j = \{0, 1\}$, and let σ be the product measure on X (defined by $\sigma\left(\bigcap_{j=1}^{n}\{x : x_j = a_j\}\right) = 2^{-n}$). Find an explicit mapping $f : X \to [0, 1]$ such that there exists $X' \subset X$, with $\mu(X') = 0$ and $Y' \subset [0, 1]$ with Lebesgue measure zero such that

$$f : X \setminus X' \to [0, 1] \setminus Y'$$

is one-to-one and measure preserving, i. e. $m(F(E)) = \mu(E)$ where m denotes Lebesque measure.

7. a). For x and ξ real, show that every partial sum of the series

$$e^{ix\xi} = \sum_{n=0}^{\infty} \frac{i^n x^n \xi^n}{n!}$$

is bounded by $e^{|x||\xi|}$.

(b). Let $\mathcal{H}$ be the Hilbert space of Lebesque measureable functions on the real line with inner product

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}e^{-x^2}\,dx.$$

Prove that the set of polynomials $a_0 + a_1 x + \ldots + a_n x^n$ is dense in $\mathcal{H}$. Hint: Suppose $g \in \mathcal{H}$ is orthogonal to the polynomials. Show that $G(x) = g(x)e^{-x^2} \in L^1(\mathbb{R}, dx)$ and that

$$\hat{G}(\xi) = \int e^{ix\xi} G(x)dx = 0$$

for all $\xi \in \mathbb{R}$. Use this information to show $g = 0$ almost everywhere.

8. Let H be the Hilbert space $L^2(\mathbb{R})$ of square (Lebesgue) integrable functions on the line $\mathbb{R}$ and define $U : H \to H$ by

$$Uf(x) = f(x - 1).$$

Show that U has no (non-zero) eigenvectors.

9. Let f be any conformal mapping from the strip $S = \{z \in \mathbb{C} : -1 < \Im z < 1\}$ onto the unit disc $\mathbb{D} = \{z : |z| < 1\}$ such that uniformly in $y \in (-1, 1)$,

$$\lim_{x\to\infty} f(x + iy) = 1, \text{ and } \lim_{x\to-\infty} f(x + iy) = -1.$$

Find the of images in $\mathbb{D}$ of the set of horozontal lines in S and the set of vertical line segments in S. Hint: In each case the set of images does not depend on the choice of f.

10. Let $I = [0, 1]$ be the closed unit interval in $\mathbb{R}$ and let $U = \mathbb{C} \setminus I$.

(a) Prove there exists a non-constant bounded analytic function on U.

(b) If $f(z)$ is bounded and analytic on U and if f has a continuous extension to $\mathbb{C}$, prove f is constant.

11. Let $u(z)$ be an harmonic function on the complex plane $\mathbb{C}$ such that

$$\int\int_{\mathbb{C}} |u(z)|^2 dxdy < \infty.$$

Prove $u(z) = 0$ for all $z \in \mathbb{C}$.

12. Evaluate

$$\int_0^\infty \frac{x^2 dx}{x^4 + 6x^2 + 13}.$$

13. Let U be a domain in the complex plane such that $0 \in U$ and let $f : U \to U$ be an analytic function from U into U such that $f(0) = 0$ and $|f'(0)| < 1$. Let

$$f^{(n)}(z) = f \circ f \circ \cdots \circ f(z)$$

be the function obtained by composing f n-times. Prove that $f^{(n)}(z) \to 0 (n \to \infty)$ uniforml on compact subsets of U. Hint: First find a disc $B = \{|z| < a\} \subset U$ such that $f(B) \subset B$.

14. Let $S = [0, 1] \times [0, 1]$ be the unit square in $\mathbb{C}$ and let $f : S \to \mathbb{C}$ be a *continuous* map such that $f(z) \neq 0$ for all $z \in S$. Prove there exists continuous $g : S \to \mathbb{C}$ such that

$$f(z) = e^{g(z)}$$

for all $z \in S$.

Analysis Qualifying Examination - January 2002

Instructions:

Work any 12 problems. All problems are worth ten points.

1. Suppose f_n is a sequence of continuous functions on $[0,1]$ which converges to a continuous function f on $[0,1]$. Does it follow that f_n converges uniformly? Give a proof or provide a counterexample.

2. Let $\{a_n\}_n$ be a sequence of positive numbers converging to 0. Show that given any $x > 0$ there exist non-negative integers $k_1, k_2, \ldots$ such that $\Sigma_n k_n a_n = x$.

3. Let f and g be Lebesgue integrable functions on the interval $[0,1]$. Set

$$F(x) = \int_0^x f(y)dy, \; G(y) = \int_0^y g(x)dx$$

where dy and dx denote Lebeqgue measure. Assume $h(x,y) = f(y)g(x)$ is Lebesgue measureable on $[0,1] \times [0,1]$. Prove

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(y)G(y)dy.$$

4. Let $F(x)$ be a bounded real valued function on $\mathbb{R}$ such that

$$\lim_{x \to -\infty} F(x) = 0$$

and such that for all $\epsilon > 0$ there is $\delta > 0$ such that whenever $(a_j, b_j), 1 \leq j \leq n$ is a finite family of pairwise disjoint open intervals,

$$\sum (b_j - a_j) < \delta \Rightarrow \sum |F(b_j) - F(a_j)| < \epsilon.$$

Prove there is $f \in L^1(\mathbf{R})$ such that for all $x \in \mathbb{R}$,

$$F(x) = \int_{-\infty}^x f(t)dt.$$

5. Let $f : [0,1] \to \mathbb{C}$ be Lebesgue measurable. Assume $fg \in L^2([0,1], \mu)$ for every $g \in L^2([0,1], \mu)$, where μ is the Lebesgue measure on $[0,1]$. Prove that $f \in L^\infty([0,1], \mu)$.

6. Let $x \in \mathbb{R}$ have decimal expansion

$$x = a_0.a_1a_2a_3\ldots$$

and assume this expansion is unique (it is unique for almost all x). Define

$$A_n(x) = \begin{cases} 1, & \text{if } a_n \text{ is even} \\ -1, & \text{if } a_n \text{ is odd.} \end{cases}$$

Let $f \in L^1(\mathbb{R})$. Prove

$$\lim_{n\to\infty} \int_{\mathbf{R}} f(x)A_n(x)dx = 0.$$

7. Let $0 < p^* < \infty$. Let $n \geq 2$ and let μ be Lebesgue measure on $\mathbb{R}^n$. Prove there exists a Lebesgue measurable f on $\mathbb{R}^n$ such that for $0 < p < \infty$,

$$f \in L^p(\mathbb{R}^n, \mu) \Leftrightarrow p = p^*.$$

8. Let $\mathcal{H}$ be a Hilbert space.

a). Show that if $T : \mathcal{H} \to \mathcal{H}$ is a linear transformation such that $id_{\mathcal{H}} - T$ is a bounded operator on $\mathcal{H}$ with $\|id_{\mathcal{H}} - T\| < 1$, then T is a bounded, invertible operator on $\mathcal{H}$.

b). Assume $\{e_n\}_n \subset \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$; that is $\{e_n\}$ is an orthonormal system, i.e. $< e_n, e_m >= \delta_{n,m}$ for all n, m, and $\overline{\text{span}}\{e_n\}_n = \mathcal{H}$. Show that if $\{f_n\}_n \subset \mathcal{H}$ is an orthonormal system such that $\Sigma_n \|e_n - f_n\|^2 < 1$ then $\{f_n\}_n$ is a basis for $\mathcal{H}$.

9. Find the Laurent series expansion for

$$\frac{1}{z(z+1)}$$

valid in $\{1 < |z-1| < 2\}$.

10. Prove, by using the residue calculus, that

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \pi\sqrt{2}$$

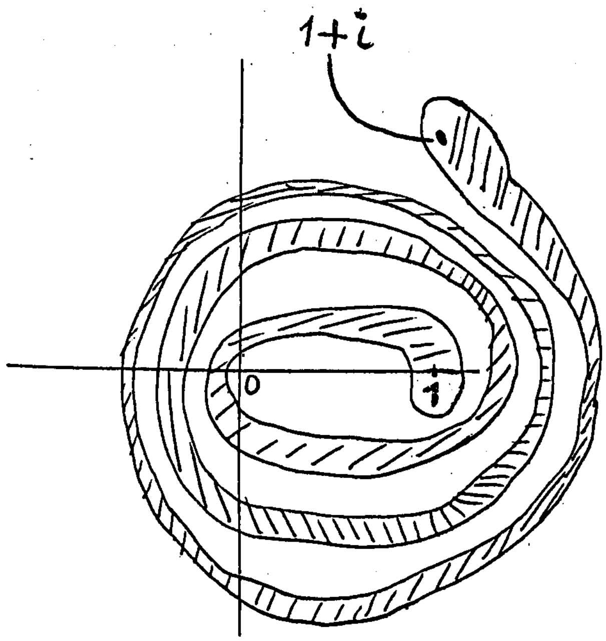
11. Let f and g be continuous functions on the real line related by Fourier transform, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)e^{-ikx}dk, \qquad g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

Prove that *both* f and g cannot be compactly supported (i.e., vanish outside some finite interval of the real line).

12. Let $f(z)$ be an entire function which is not a constant. Prove that $e^{f(z)}$ has an essential singularity at $z = \infty$.

13. Explain why there is a unique analytic function g in the domain D roughly sketched below (shaded region) with the property that $e^{g(z)} = z$ for all $z \in D$ and $g(1) = 2\pi i$. Compute $g(1+i)$.



ANALYSIS QUALIFYING EXAM
FALL 2003

Directions. Each problem is worth 10 points. A complete solution to one problem is worth more than two half-solutions to two problems. You are to solve any 10 problems.

Problem 1. Let μ be a finite Borel measure on $[0,1]$. Suppose that

$$f : [0,1] \times [0,1] \to \mathbb{R}$$

is a Borel function, such that for each x, the map $t \mapsto f(x,t)$ is differentiable, and that $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ for some Borel function $g(x)$ satisfying $\int_0^1 g(x)d\mu(x) < \infty$. Carefully prove that $F(t) = \int_0^1 f(x,t)d\mu(x)$ satisfies

$$F'(t) = \int_0^1 \frac{\partial f}{\partial t}(x,t)d\mu(x).$$

Problem 2. Let μ be a positive Borel measure on the unit interval $I = [0,1]$, such that $\mu(I) = 1$. Let $\xi_n(x) = x^n$, $n = 0,1,2,\ldots$.

(a) Let H be the Hilbert space $L^2([0,1],\mu)$. Show that $\xi_n \to 0$ in norm if and only if $\mu(\{1\}) = 0$.

(b) Let H be as in part (a). Show that if $f \perp \xi_n$ for all n, then f must be a.e. zero.

(c) Let V be the Banach space $L^\infty([0,1],\mu)$. Show that $\xi_n \to 0$ in norm if and only if for some $\varepsilon > 0$, $\mu([1-\varepsilon,1]) = 0$.

Problem 3. Let $f \in L^1(\mathbb{R})$. Define its Fourier transform by the formula

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{2\pi i(xt)}dx.$$

Assume that both $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Show that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{-2\pi i(xt)}dt \qquad x \text{ a.e.}$$

Problem 4. Let H be a Hilbert space. Show that the unit sphere $S = \{\xi \in H : ||\xi|| = 1\}$ is compact (in the norm topology) if and only if H is finite-dimensional.

Problem 5. Let H be a Hilbert space. A sequence $\xi_n \in H$ is said to converge weakly to ξ if for all $\zeta \in H$, $\langle \xi_n, \zeta \rangle \to \langle \xi, \zeta \rangle$.

(a) Show that $||\xi|| \leq \limsup_n ||\xi_n||$;

(b) Give an example in which the inequality in (a) is strict;

(c) Show that if $||\xi|| = \lim_n ||\xi_n||$, then $||\xi - \xi_n|| \to 0$.

Problem 6. We let $\ell^p = L^p(\mathbb{N})$, where $\mathbb{N}$ has the usual counting measure, and we let $L^p = L^p([0,1])$ where $[0,1]$ has the usual Lebesgue measure. Show that $\ell^1 \subsetneq \ell^2 . \subsetneq \ell^\infty$, and that $L^1 \supsetneq L^2 \supsetneq L^\infty$.

Problem 7. Define the Haar functions on $[0,1]$ by

$$e_0(x) = 1,$$
$$e_{n,k}(x) = \begin{cases} 2^{\frac{n}{2}}, & \text{if } \frac{k-1}{2^n} \le x < \frac{k-\frac{1}{2}}{2^n} \\ -2^{\frac{n}{2}}, & \text{if } \frac{k-\frac{1}{2}}{2^n} \le x < \frac{k}{2^n} \\ 0, & \text{otherwise.} \end{cases}$$

Show that these form an orthonormal basis for $L^2[0,1]$.

Problem 8. Let $I = [-1,1]$ be the closed unit interval in $\mathbb{R}$. Let $U = \mathbb{C} \setminus I$.

(a) Show that there exists a non-constant bounded analytic function on U.

(b) Prove that if $f(z)$ is bounded and analytic on U and if f has a continuous extension to $\mathbb{C}$, then f must be constant.

Problem 9. Let $A = \{z : \mathrm{Im} z > 0\} \setminus \{z : \mathrm{Re} z = 0 \text{ and } 0 \le \mathrm{Im} z \le 1\}$. Find a conformal map that maps A one-to-one onto the upper half place $\{z : \mathrm{Im} z > 0\}$, or show that no such map exists.

Problem 10. Use a contour integral to evaluate

$$\int_0^\infty \frac{1}{1+x^{2n}} dx, \qquad n \ge 1.$$

Problem 11. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial. Show that there must be at least one point with $|z| = 1$ and $|p(z)| \ge 1$. (Hint: count zeros of $a_{n-1}z^{n-1} + \cdots + a_0$).

Problem 12. Let D be the open unit disk in the complex plane. Endow D with the Lebesgue measure λ. Let $A \subset L^2(D, \lambda)$ be the subspace consisting of those L^2 functions, which are analytic on the disk.

(a) Show that A is infinite-dimensional.

(b) Show that A is a closed subspace of $L^2(D, \lambda)$.

Analysis Qualifying Examination
January 11, 2003

Work any 10 problems, but include at least 3 problems from Part II. All problems s are worth 10 points, and a complete solution to one problem will be valued more highly than two half solutions to two problems.

Part I

1. Let μ be a finite, positive, regular Borel measure on $\mathbf{R}^2$, and let $\mathcal{G}$ be the family of finite unions of squares of the form

$$S = \{j2^n \leq x \leq (j+1)2^n; k2^n \leq y \leq (k+1)2^n\},$$

where j, k, and n are integers. Prove that the set of linear combinations of characteristic functions of elements of $\mathcal{G}$ is dense in $L^1(\mu)$.

2. Prove there is a constant C such that for every closed bounded interval $I = [a, b] \subset \mathbf{R}$ there is a constant α_I such that

$$\int_I \Big|\log|x| - \alpha_I\Big| dx \leq C(b-a).$$

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K(x, y)$ be measureable with respect to the product σ-algebra $\mathcal{M} \times \mathcal{N}$. Assume there is a constant $A > 0$ such that for all $x \in X$

$$\int_Y |K(x,y)| d\nu(y) \leq A,$$

and for all $y \in Y$,

$$\int_X |K(x,y)| d\mu(x) \leq A.$$

Let $1 \leq p \leq \infty$, and for $f \in L^p(X, \mathcal{M}, \mu)$ define

$$Tf(y) = \int_X f(x) K(x,y) d\mu(x).$$

Prove

$$||TF||_{L^p(\nu)} \leq A||f||_{L^p(\mu)}.$$

4. Prove or disprove: If F is a strictly increasing continuous map from the real line $\mathbf{R}$ onto itself and if $A \subset \mathbf{R}$ is Lebesgue measureable, then $f^{-1}(A)$ is Lebesgue measurable.

5. Is the Banach space ℓ^∞ of bounded complex sequences $a = \{a_n\}_{n=1}^\infty$ with the supremum norm $||a||_\infty = sup_n |a_n|$ separable? Prove your answer is correct.

6. Let X be a finite-dimensional real normed linear space with norm $|| \; ||$, and let

$$a_1, a_2, \ldots, a_n$$

be a vector space basis over $\mathbf{R}$ for X. For $x = \sum_{j=1}^{n} x_j a_j \in X$, write $||x||^* = \sum_{j=1}^{n} |x_j|$. Prove there is a constant $C > 0$ such that for all $x \in X$,

$$C^{-1}||x||^* \leq ||x|| \leq C||x||^*.$$

Hint. One inequality is easy; for the other use the Hahn-Banach theorem and induction.

7. Let X be an infinite-dimensional complete normed linear space over $\mathbf{R}$. Prove that every vector space basis for X is uncountable. *Hint.* Use Problem 7 to show finite-dimensional subspaces of X are closed.

8. Let $n \geq 2$, let H be the Hilbert space $L^2(\mathbf{R}^n)$ of square (Lebesgue) integrable function on $\mathbf{R}^n$ and let e be a fixed vector in $\mathbf{R}^n$, $e \neq 0$. Prove that the linear transformation $T : H \to H$ defined by

$$Tf(x) = f(x + e)$$

has no nonzero eigenvector.

Part II

9. Let D be the domain in the complex plane $\mathbf{C}$ that is the intersection of the two open disks centered at ± 1 whose boundary circles pass through $\pm i$. Find a conformal map f of D onto the open unit disk $\Delta = \{|w| < 1\}$ such that $f(i) = 1$ and $f(-i) = -1$. (You may express f as a composition of other specific maps.) What are the images of arcs of circles passing through $\pm i$ under your map f? (Justify your answer.)

10. Let

$$f_m(z) = \sum_{k=-m}^{m} \frac{1}{(z - m - ik)^2}, \qquad g_n(z) = \sum_{m=1}^{n} f_m(z).$$

Show that the sequence $\{g_n(z)\}_{n=1}^{\infty}$ converges normally to ∞ as $n \to \infty$. *Hint.* Look first at $g_n(0)$.

11. Show by contour integration that

$$\int_0^{2\pi} \frac{d\theta}{x + \cos\theta}\, d\theta = \frac{2\pi}{\sqrt{x^2 - 1}}, \qquad x > 1.$$

Determine for which complex values of z the integral

$$\int_0^{2\pi} \frac{d\theta}{z + \cos\theta}\, d\theta$$

exists and evaluate the integral. Justify your reasoning.

12. Let S be a sequence of points in the complex plane that converges to 0. Let $f(z)$ be defined and analytic on some disk centered at 0 except possibly at the points of S and at 0. Show that either $f(z)$ extends to be meromorphic in some disk containing 0, or else for any complex number w there is a sequence $\{\zeta_j\}$ such that $\zeta_j \to 0$ and $f(\zeta_j) \to w$.

Analysis Qualifying Exam, September 2004.

Attempt any ten of the twelve questions.
Each question is worth 10 points.

Q1. For each natural number n, let $f_n : [0,1] \to \mathbf{R}$ be a sequence of absolutely integrable functions, and let $f : [0,1] \to \mathbf{R}$ be another absolutely integrable function such that

$$\int_0^1 |f_n(x) - f(x)| \, dx \to 0$$

as $n \to 0$.

(a) Show that there exists a subsequence f_{n_j} of f_n which converges to f pointwise almost everwhere.

(b) Give a counterexample to show that the assertion fails if "pointwise almost everywhere" is replaced by "uniformly".

Q2. Let $f : [0,1] \to \mathbf{R}^+$ be a non-negative, absolutely integrable function. Prove that the following two statements are equivalent.

(i) There exists a constant $0 < C < \infty$ such that

$$\left(\int_{[0,1]} f(x)^p \, dx \right)^{1/p} \leq Cp \quad \text{for all } 1 \leq p < \infty.$$

(ii) There exists a constant $0 < c < \infty$ such that

$$\int_{[0,1]} e^{cf(x)} \, dx < \infty.$$

Q3. Let E be a measurable subset of the real line.

(a) Let $\chi_E : \mathbf{R} \to \mathbf{R}$ be the characteristic function of E (i.e. $\chi_E(x) = 1$ when $x \in E$ and $\chi_E(x) = 0$ when $x \notin E$. If E has finite Lebesgue measure, show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) := \int_{\mathbf{R}} \chi_E(y)\chi_E(y-x)\, dy$$

is continuous.

(b) Suppose instead that E has positive Lebesgue measure $0 < m(E) \leq \infty$. Using (a), show that the set $E - E := \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon) := \{x \in \mathbf{R} : -\varepsilon < x < \varepsilon\}$ for some $\varepsilon > 0$.

Q4. Let $f : \mathbf{R} \to \mathbf{R}$ be a measurable bijective function such that

$$f(x) + f(y) = f(x+y)$$

for all $x, y \in \mathbf{R}$.

(a) Show that f is continuous.
Hint: Show that the set $E := f^{-1}((-\varepsilon, \varepsilon)) := \{x \in \mathbf{R} : -\varepsilon < f(x) < \varepsilon\}$ obeys the hypotheses of Q3(b) for each $\varepsilon > 0$.

(b) Show that there exists a non-zero real number c such that $f(x) = cx$ for all real numbers $x \in \mathbf{R}$.
Hint: work on the rational numbers $\mathbf{Q}$ first.

Q5. Let X be a Banach space endowed with the norm $\|\ \|_X$, let V be a closed subspace of X, and let $T : X \to X$ be a bounded linear transformation on X which has V as an invariant subspace (i.e. $Tx \in V$ for all $x \in V$). Let X/V be the quotient space $X/V := \{x + V : x \in X\}$ of translates of V, endowed with the norm

$$\|x + V\|_{X/V} := \inf\{\|y\|_X : y \in x + V\}.$$

You may assume without proof that X/V is a Banach space with this norm.

Let $T_1 : V \to V$ be the restriction of T to V, i.e. $T_1 x := Tx$ for all $x \in V$, and let $T_2 : X/V \to X/V$ be the map $T_2(x + V) := (Tx) + V$. You may assume without proof that T_1 and T_2 are well-defined bounded linear transformations.

Suppose that T_1 and T_2 are boundedly invertible on V and X/V respectively, i.e. there exists bounded linear operators $S_1 : V \to V$ and $S_2 : X/V \to X/V$ such that $S_1T_1 = T_1S_1$ is the identity on V, and $S_2T_2 = T_2S_2$ is the identity on X/V. Prove that T is also boundedly invertible.

Q6. Let (X, d) be a compact metric space, and let $\mathcal{F}$ be the set of non-empty compact subsets of X. If K_1, K_2 are two elements of $\mathcal{F}$ (i.e. two non-empty compact subsets of X), define the *Hausdorff distance* $d_H(K_1, K_2)$ to be the quantity

$$d_H(K_1, K_2) := \max\Big(\sup_{x\in K_1} \inf_{y\in K_2} d(x,y), \ \sup_{y\in K_2} \inf_{x\in K_1} d(x,y) \Big).$$

You may assume without proof that $(\mathcal{F}, d_H)$ is a metric space. Show that $(\mathcal{F}, d_H)$ is complete.
Hint: It may help to keep the following intuition in mind: if $d_H(K_1, K_2) \le \varepsilon$, this means that K_1 lies in the ε-neighborhood of K_2, and K_2 lies in the ε-neighborhood of K_1.

Q7. For each integer n, let $f_n : [0,1] \to \mathbf{R}$ be an everywhere differentiable (and hence continuous) function. Suppose that the sequence of functions f_n converges pointwise, and that the sequence of functions f_n' converges uniformly. Prove that the sequence of functions f_n converges uniformly, and that the limit is everywhere differentiable.
Hints: the fundamental theorem of calculus requires that f_n be *continuously* differentiable, and so a proof using that theorem may only earn partial credit. However, the mean value theorem is valid for functions which are merely differentiable, and may be used without proof.

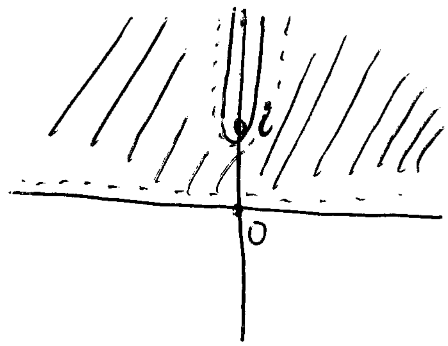
Q8. Evaluate the integral

$$I := \int_{-\infty}^{+\infty} \frac{\cos x}{(1+x^2)^2}\, dx.$$

You will need to briefly justify any limits you take.

Q9. Find the number of zeros of the polynomial $z^6 + 12z^4 + z^3 + 2z + 6$ in the first quadrant inside the unit circle (the domain $\{z = x + iy : x > 0, y > 0, |z| < 1\}$), and also in the first quadrant outside the unit circle (the domain $\{z = x + iy : x > 0, y > 0, |z| > 1\}$).

Q10. Let $D := \{z = x + iy : y > 0\} \setminus [i, i\infty]$ be the domain obtained from the open upper half-plane by excising the vertical ray from i to $i\infty$, that is,



Find a conformal map $w = f(z)$ of D onto the open unit disk $\{|w| < 1\}$. You may represent the map $f(z)$ as a composition of other maps. Include a sketch or diagram with your solution.

Q11. Let $f(z)$ be a bounded analytic function on the open right half-plane $\{z = x + iy : x > 0\}$ such that $f(x) \to 0$ as x tends to 0 along the positive real axis. Suppose $0 < \theta_0 < \pi/2$. Prove that $f(z) \to 0$ as $z \to 0$, uniformly in the sector $|\arg z| \leq |\theta_0|$.

Q12. Let D be a domain, i.e. a connected open set. Let $f : D \to \mathbf{R}$ be a continuous function with the property that whenever $\overline{B(z_0, r)} := \{z : |z - z_0| \leq r\}$ is a closed disk contained in D, and $h : \overline{B(z_0, r)} \to \mathbf{R}$ is a harmonic function such that $f(z) \leq h(z)$ for all $z \in \partial B(z_0, r) := \{z : |z - z_0| = r\}$, we have $f(z_0) \leq h(z_0)$. (Such functions are called *subharmonic*).

Show that f cannot attain its maximum at any point in D unless it is a constant function.

• UCLA Department of Mathematics •
Graduate Analysis Exam Winter 2004

No Smoking
No Drinking

There are altogether twelve porblems – six from "real analysis" and six from "complex analysis". Attempt all problems. Start each problem on a new page. Be sure to label your problems.

[A1] ———

Let $f: R^n \to [0,B]$ denote a bounded function about which you may make no additional assumptions. For any $\varepsilon > 0$, let us define $\phi_\varepsilon(x)$ by

$$\phi_\varepsilon(x) = \sup_{y:|x-y|<\varepsilon} f(y).$$

(a) Show that the $\varepsilon \to 0$ limit of $\phi_\varepsilon(x)$ exists.

(b) Denoting this limit by $\phi(x)$, show that ϕ is upper–semicontinuous.

[A2] ———

Let $f: [0,1] \to (0,1)$ be a function of class $\mathscr{C}^1$ – that is to say both f and f′ are continuous functions – with

$$\max_{0\le x\le 1} |f'(x)| \le 1-\varepsilon$$

for some $\varepsilon > 0$.

(a) Show that there is exactly one solution to the equation

$$f(x^*) = x^*.$$

(b) Let x_o denote any point in [0,1]. Define $x_1 = f(x_o)$ and, in general,

$$x_{n+1} = f(x_n).$$

Show that regardless of the choice of x_o, we have

$$\lim_{n\to\infty} x_n = x^*.$$

[A3] ———

Let $f: [0,1] \to [0,1]$ denote a Borel function and consider the subset of the square, A_f, defined by

$$A_f = \{(x,y) \in [0,1]\times[0,1] \mid y \le f(x)\}.$$

Show that A_f is a Borel set and that in fact,

$$m_2(A_f) = \int_{[0,1]} f(x)dm_1$$

where m_1 and m_2 denote one and two dimensional Lebesgue measure respectively.

[A4]

Let $\langle \mathbf{X},\mu,\mathcal{B}\rangle$ denote a finite measure space and let $f\in L^\infty(d\mu)$. Define

$$\alpha_n = \int_{\mathbf{X}} |f|^n d\mu .$$

Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty .$$

[A5]

Let μ denote a finite Borel measure supported on a countable set $\mathbf{Q}\subset \mathbb{R}$ and let

$$F(t) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(x)$$

denote its Fourier transform. Show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |F(t)|^2 dt = \sum_{q\in\mathbf{Q}} |\mu(q)|^2 .$$

[A6]

Consider the Hilbert space ℓ^2 of all complex valued square–summable sequences. Let $\mathbf{T}$ be the operator that shifts the sequence to the right and places a zero in the first slot:

$$\mathbf{T}(\zeta_1,\zeta_2,\zeta_3,\dots) = (0,\zeta_1,\zeta_2,\zeta_3,\dots).$$

(a) For any $\underline{a}\in\ell^2$, let $\underline{a}^{[n]} = \mathbf{T}^n\underline{a}$. Show that the $\underline{a}^{[n]}$ converge weakly to zero, i.e. that for any $\underline{b}\in\ell^2$, the numbers $\langle \underline{a}^{[n]}|\underline{b}\rangle$ converge to zero.

(b) Compute the adjoint, $\mathbf{T}^*$, of $\mathbf{T}$ and show that for any $\underline{a}\in\ell^2$, the vectors $[\mathbf{T}^*]^n\underline{a}$ converge *strongly* (i.e. in norm) to zero.

[CA 1]

Let $f(z) = u(x,y) + iv(x,y)$ denote a non–constant analytic function on some open domain $D \subset \mathbb{C}$. Show that at each point of D, the "level curves" $u(x,y) = \text{constant}$ and $v(x,y) = \text{constant}$ intersect at right angles.

[CA 2]

Let $K(z)$ denote a real–valued function of the complex variable z defined in some open domain $D \subset \mathbb{C}$. Then $K(z)$ is said to be *strictly* subharmonic if the inequality

$$K(z_o) < \frac{1}{2\pi}\int_0^{2\pi} K(z_o + \rho e^{i\phi})d\phi$$

holds for all ρ which are smaller than the distance from z_o to the boundary of D. Let $f(z)$ denote a *non–constant* analytic function on D. Show that $|f(z)|$ is strictly subharmonic.

[CA 3]

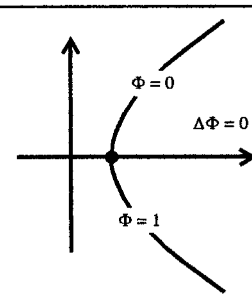
A function $f(z)$ is entire and has the property that for some fixed $\lambda > 0$, the inequality

$$|\mathrm{Re}[f(z)]| \geq \lambda |\mathrm{Im}[f(z)]|$$

holds for all $z \in \mathbb{C}$. Show that this function must be a constant.

[CA 4]

A bounded harmonic function $\Phi(x,y)$ is defined in the region of the plane $x^2 \geq y^2 + 1$. The function Φ satisfies the boundary condition that for $x^2 - y^2 = 1$ with $y > 0$, $\Phi = 0$ while for $x^2 - y^2 = 1$ with $y < 0$, $\Phi = 1$. Find this function. You may express your answer in terms real and/or imaginary parts of well known analytic functions.



[CA 5]

Using contour (or any other) methods, compute the integral

$$\int_0^\infty \frac{x\,dx}{1+x^3}.$$

Hint: Find a "path of return" where you compute an integral related to the one you want.

[CA 6]

Let $f(z)$ denote a function which is analytic at all points on a simple closed curve, γ, and everywhere inside γ except for the (isolated) points $b_1, b_2, \ldots b_r$ where it has poles of order $\beta_1, \beta_2, \ldots \beta_r$. Furthermore, f does not vanish on γ but does have zeros at the points $a_1, a_2, \ldots a_s$ which are inside γ and these zeros are of multiplicity $\alpha_1, \alpha_2, \ldots \alpha_s$. Then, according to a well known formula,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{s}\alpha_k - \sum_{k=1}^{r}\beta_k .$$

Now suppose that $g(z)$ is analytic inside and on γ. What is the generalization of the above formula when the integrand on the left hand side is replaced by $g(z)\frac{f'(z)}{f(z)}$? Provide justification for your answer.

Analysis Qualifying exam
Fall 2005

Answer all questions.

R1 Suppose that f is a bounded function on $[a, b]$ which is Riemann integrable.

a) Prove that f is a Lebesgue measurable function.

b) Must f be a Borel measurable function? Prove your assertion.

R2 Let X be a set with $\mathcal{S}$ a σ-algebra of sets in X, and F a signed measure on $\mathcal{S}$ with $F(S) > -\infty$ for all $S \in \mathcal{S}$. A subset $B \in \mathcal{S}$ is said to be *purely positive* if $F(C) \geq 0$ for all $C \in \mathcal{S}$ with $C \subseteq B$, and *purely negative* if $F(C) \leq 0$ for all $C \in \mathcal{S}$ with $C \subseteq B$. Prove that there exist $P, N \in \mathcal{S}$ with $X = P \cup N$ and $P \cap N = \emptyset$, where P is purely positive and N is purely negative. Hint: Begin by showing that $\beta = \inf\{F(B) : B \text{ is purely negative}\} > -\infty$.

R3 Consider real numbers $a_{n,m}$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ and assume that the inner and outer sums in the expressions

$$A := \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} a_{n,m}\right]$$

$$B := \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} a_{n,m}\right]$$

are absolutely convergent.

a) Give an example that shows that we may have $A \neq B$.

b) Under what reasonable additional assumption on $a_{n,m}$ can we conclude that $A = B$? Prove your assertion.

R4 Suppose that V is a complex normed space and that $f : V \to \mathbb{C}$ is a linear functional.

a) Prove that if V is finite-dimensional, then f must be bounded.

b) Give an example of an unbounded linear functional on a normed vector space. Prove your assertion.

R5 (see hint below) If $f \in L^1(\mathbb{R})$, consider the Fourier transform

$$\widehat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\alpha} dx$$

and the inverse Fourier transform

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{ix\alpha} d\alpha$$

(you may use any of the alternative standard definitions for the Fourier transform and its inverse).

(a) Let $A(\mathbb{R})$ be the image of the mapping $f \mapsto \widehat{f}$. Does one have that $A(\mathbb{R}) \subseteq L^1(\mathbb{R})$ or $A(\mathbb{R}) \supseteq L^1(\mathbb{R})$? Fully prove your assertions.

(b) Prove that

$$f, g \in L^2(\mathbb{R}) \Longrightarrow (\hat{f}\hat{g})\check{} = f * g.$$

Hint: Let $h(y) = \overline{g(x-y)}$ and observe that $\hat{h}(\alpha) = \overline{\hat{g}(\alpha)} e^{-2\pi i\alpha}$. Then use the fact that $f \mapsto \widehat{f}$ uniquely determines an isometry of $L^2(\mathbb{R})$ onto itself (the Plancherel theorem).

C1 Suppose that $f(z)$ is analytic and non-constant on a connected open set G in the complex plane. Prove that $f(G)$ is an open subset of the complex plane.

C2 Find an explicit conformal mapping from the upper half-plane slit along the vertical segment

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \backslash (0, i]$$

to the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

C3 Consider the meromorphic function

$$f(z) = \frac{(1-z^2)}{2i(z^2 - (a + \frac{1}{a})z + 1)}, \quad |a| < 1.$$

Find the Laurent series expansion for $f(z)$ valid in a neighborhood of the unit circle $|z| = 1$.

C4 Using the residue calculus, evaluate the integral

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}dx.$$

Hint: Use the positively oriented contour $\Gamma_{r,R}$ $0 < r < 1 < R$ consisting of the line segment $[r, R]$, followed by the arc $\Gamma_{r,R} = \{z = Re^{i\varphi} : 0 \le \varphi \le \pi\}$, the segment $-R, -r]$, and finally completed by the arc

$$\Gamma_r = \left\{z = re^{i\varphi} : 0 \le \varphi \le \pi\right\}.$$

Include a proof of the limiting arguments.

C5 Let $J \subseteq \mathbb{R}$ be a compact interval, and let μ be a measure on the real line whose support lies in J. For $z \in \mathbb{C}\backslash J$, define

$$C_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}.$$

a) Prove that $C_\mu(z)$ is analytic on $\mathbb{C}\backslash J$.

b) Find a power series expansion for $C_\mu(z)$ at ∞ in terms of the moments $m_k = \int_{\mathbb{R}} t^k d\mu(t)$.

c) Show that the mapping $\mu \mapsto C_\mu(z)$ is one-to-one (Hint: use the Stone-Weierstrass theorem to prove that the moments m_k determine μ).

ANALYSIS QUALIFYING EXAMINATION FALL 2006

THURSDAY SEPT. 21, 2006

INSTRUCTIONS. Each problem is worth 10 points. There are six real analysis problems, numbered 1–6, and six complex analysis problems, numbered 7–12. You are to solve five problems from each section.

! **You must indicate which five problems from each section are to be graded.** !

Please note that a complete solution to a single problem will be valued more highly than two half solutions to two problems.

Problem 1. Let $\theta \in [0,1]$ be an irrational number. Let X be the unit circle $\{e^{2\pi it} : t \in [0,1]\}$ endowed with the arclength measure, and let

$$\alpha : X \to X$$

be the rotation by $2\pi\theta$, given by $\alpha(e^{2\pi it}) = e^{2\pi i(t+\theta)}$.

(1) Show that if $f \in L^2(X)$ and $f \circ \alpha = f$, then f is a.e. constant. (*Hint:* consider the Fourier transform of f, viewed as a 1-periodic function on $\mathbb{R}$).
(2) Use part 1 to show that if $Y \subset X$ is a Lebesgue-measurable subset so that $\alpha(Y) = Y$, then either Y has measure zero, or $X \setminus Y$ has measure zero.

Problem 2. Let E, F be two Lebesgue-measurable subsets of $\mathbb{R}$, and let χ_E, χ_F be their respective characteristic functions.

(1) Show that the convolution $\chi_E * \chi_F$ defined by
$$\chi_E * \chi_F(x) = \int_{\mathbb{R}} \chi_E(y)\chi_F(x-y)dy$$
is a continuous function.
(2) Show that
$$n\left(\chi_E * \chi_{[0,1/n]}\right) \to \chi_E$$
as $n \to \infty$ pointwise a.e.

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mu(X) = 1$. Let $f \in L^\infty(X, \mathcal{M}, \mu)$. Prove that

$$\lim_{p\to\infty} \left(\int |f|^p d\mu\right)^{1/p} = \|f\|_\infty.$$

Problem 4. Assume that f is a continuously differentiable 2π periodic function on $\mathbb{R}$. Show that the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \exp(int), \qquad t \in \mathbb{R}$$

is absolutely convergent (here $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)\exp(-int)dt$).

Problem 5. Let ℓ^2 be the space of all square-summable sequences of complex numbers, and let $T : \ell^2 \to \ell^2$ be a linear operator. Let e_n be the sequence

$$e_n = (0\,0\,\cdots\,0\,1\,0\,\cdots),$$

where 1 is in the n-th position. (The vectors e_n form a "standard" orthonormal basis for ℓ^2). Let $a_{nm} = \langle Te_m, e_n \rangle$ be the "matrix coefficients" of T.

(1) Assume that $\sum_{n,m=1}^{\infty} |a_{nm}|^2 < \infty$. Show that T is a bounded operator.

(2) Assume instead that $\sup\{|a_{nm}| : 1 \leq n, m < \infty\}$ is finite. Must T be bounded?

Problem 6. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane, endowed with the usual Lebesgue measure. Let $H = L^2(\mathbb{D})$ be the space of square-integrable complex-valued functions on $\mathbb{D}$.

(1) Show that the functions $\{z^n : n \geq 0\}$ are orthogonal in H.

(2) Do the functions $\{\|z^n\|^{-1}_{L^2(\mathbb{D})} z^n : n \geq 0\}$ form an orthonormal basis for H?

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and injective (i.e. univalent). Prove that f is linear: $f(z) = az + b$.

Problem 8. Suppose that f is holomorphic on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. Suppose also that for $z \in \mathbb{D}$, $\text{Re}(f(z)) > 0$ and that $f(0) = 1$. Prove that for all $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

Problem 9. Evaluate

$$\int_0^{\pi} \frac{d\theta}{2 + \cos\theta}$$

using the residue theorem.

Problem 10. Find an explicit conformal mapping from the slit disk

$$S = \{z : |z| < 1 \text{ and } z \notin [\tfrac{1}{2}, 1)\}$$

onto the disk $\mathbb{D} = \{z : |z| < 1\}$. (You may express your answer as an explicit composition of explicit maps).

Problem 11. Show that if $\alpha \in \mathbb{C}$ satisfies $0 < |\alpha| < 1$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ then the equation

$$e^z (z - 1)^n = \alpha$$

has exactly n simple roots in the right half-plane $\{z : \text{Re}(z) > 0\}$.

Problem 12. For $a_n = 1 - \dfrac{1}{n^2}$ let

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}.$$

(1) Show that f defines a holomorphic function on $\mathbb{D} = \{z : |z| < 1\}$.

(2) Prove that f does not have an analytic continuation to any larger disk $\{z : |z| < r\}$ where $r > 1$.

ANALYSIS QUALIFYING EXAMINATION

Sunday, April 2, 2006

Instructions: Work any 10 problems. To pass the exam, you must show a satisfactory knowledge of both Real Analysis (Problems 1–6) and Complex Analysis (Problems 7–12). All problems are worth ten points; parts of a problem do not carry equal weight. You need to tell us which 10 problems you want us to grade.
Great emphasis will be placed on your attention to detail.

Problem 1. Let μ and μ_n, $n \in \mathbb{N}$, be finite positive Borel measures on $\mathbb{R}$ such that

$$\int f(x)\mu_n(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int f(x)\mu(\mathrm{d}x)$$

for all continuous funtions f with compact support.

(1) Show that then for each compact set $K \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \mu_n(K) \leq \mu(K).$$

(2) Give an example that shows $\mu(\mathbb{R}) < \limsup_{n\to\infty} \mu_n(\mathbb{R})$ is possible.

Problem 2. Let f_n be a sequence of $L^1(\mathbb{R})$ functions such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\,dx = g(0),$$

for each $g \in C_0(\mathbb{R})$ that is, continuous functions vanishing at infinity. Show that $\|f_n\|_{L^1}$ is uniformly bounded but that f_n is not Cauchy in L^1.

Problem 3. Consider the space $L^\infty([0,1],\lambda)$ where λ denotes the Lebesgue measure. Let

$$d(f,g) = \inf_{\epsilon>0}\left[\epsilon + \lambda(\{x : |f(x) - g(x)| > \epsilon\})\right]$$

Prove that d is a metric and that $f_n \to f$ if and only if $f_n \to f$ in measure.
Recall that $f_n \to f$ in measure iff $\lambda(\{x : |f_n(x) - f(x)| > \delta\}) \to 0$ for all $\delta > 0$.

Problem 4. Prove (one direction of) the Ascoli-Arzelà Theorem: Suppose $f_n \colon [0,1] \to \mathbb{R}$ is a sequence of functions such that

(1) $\exists M : \forall x \in [0,1],\ \forall n \geq 1,\ |f_n(x)| \leq M$

(2) $\forall \epsilon > 0,\ \exists \delta > 0\ :\ \forall n \geq 1,\ |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$

Then (f_n) has a subsequence that converges uniformly.

Problem 5. Suppose $f \in L^p(\mathbb{R}^n, \mathrm{d}x)$ and $g \in L^q(\mathbb{R}^n, \mathrm{d}x)$ where

$$\tfrac{1}{p} + \tfrac{1}{q} = 1$$

and $1 < p < \infty$. Show that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

is a continuous function with

$$\lim_{|x|\to\infty} (f * g)(x) = 0.$$

Problem 6. Let $\mathcal{H}^2(\mathbb{D})$ denote the space of power series $f(z) = \sum_{n\geq 0} a_n z^n$ where $a_n \in \mathbb{C}$ form an ℓ^2-sequence. These can be regarded as analytic functions on the open unit disc $\mathbb{D}$. Note that $\mathcal{H}^2(\mathbb{D})$ is a complex Hilbert space with the norm $\|f\|^2 = \sum_{n\geq 0} |a_n|^2$.

(1) Show that $L(f) := \int_{-1}^{1} f(x)\,\mathrm{d}x$ defines a bounded linear functional on $\mathcal{H}^2(\mathbb{D})$.

(2) Find $g \in \mathcal{H}^2(\mathbb{D})$ so that $L(f) = \langle f, g\rangle$.

Problem 7. Show that every non-negative harmonic function on $\mathbb{R}^2$ is constant.

Problem 8. Compute the limit

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{\mathrm{i}x^2}\,\mathrm{d}x.$$

You may use the fact that

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{-x^2}\,\mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}.$$

without proof.

Problem 9. Show that any analytic function $f\colon \mathbb{C} \to \mathbb{C}$ that obeys

$$|f(z)| = 1 \text{ for all } z \in \mathbb{R}$$

can be written as $f(z) = e^{g(z)}$ for some analytic function $g(z)$.
Hint: Prove an analogue of the Schwarz reflection principle.

Problem 10. Prove that

$$\frac{\pi^2}{\tan^2(\pi z)} = a + \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

for some constant $a \in \mathbb{C}$.

Problem 11. Let $\epsilon = \frac{1}{100}$ and let $\mathcal{U}$ be an ϵ-neighborhood of the spiral $\{\theta e^{\mathrm{i}\theta}\colon 0 \leq \theta \leq 4\pi\}$ and let $\mathcal{O}$ be an ϵ-neighborhood of the spiral $\{2\theta e^{\mathrm{i}\theta}\colon 0 \leq \theta \leq 2\pi\}$. Let $f\colon \mathcal{U} \to \mathbb{C}$ and $g\colon \mathcal{O} \to \mathbb{C}$ be the corresponding analytic continuations of

$$z \mapsto \log\left(\frac{\cos(z)}{1-z^2}\right)$$

from the ϵ-neighborhood of the origin on the real axis such that $f(0) = 0 = g(0)$. Find the imaginary part of $f(4\pi) - g(4\pi)$.

Problem 12. Prove that the infinite product

$$\prod_{n=0}^{\infty} \left(1 + z^{2^n}\right)$$

converges and equals $(1-z)^{-1}$ for all z in the open unit disc.

ANALYSIS QUALIFYING EXAM FALL 2007

Instructions: Do 10 of the 13 problems, at least three from problems 9-13. All problems are worth the same amount.

1 Let $f: R \to R$ be a function of class C^1, periodic of period 2π, and satisfying $\int_{-\pi}^{+\pi} f(t)\, dt = 0$. Prove that $\int_{-\pi}^{+\pi} |f(t)|^2\, dt \le \int_{-\pi}^{+\pi} |f'(t)|^2\, dt$. Find all such functions for which equality holds.

2 Is there a closed uncountable subset of R which contains no rational numbers? Prove your answer.

3(a) Prove that a complete normed vector space(a Banach space) is either finite-dimensional or has uncountable dimension in the vector space sense, i.e., it is not generated as finite linear combinations of elements of some countable subset.
(b) Use part (a) to give an example of a vector space that cannot be given the structure of a Banach space, that is, it is not complete in any norm.

4 Prove that not every subset of [0,1] is Lebesgue measurable

5 (a) Prove that if a_n is a decreasing sequence of positive numbers converging to 0, then the series $\Sigma\, (-1)^n a_n$ is convergent.
(b) Prove that if $f:[0, \infty) \to (0, \infty)$ is decreasing and $\lim_{x\to+\infty} f(t) = 0$, then the improper integral $\int_0^{+\infty} f(x) \cos x\, dx$ is convergent.

6 Derive, proving the validity carefully, some series or product expansion(your choice) that yields a method of calculating π.

7 Suppose H is a separable Hilbert space.
(a) Prove: If $T:H\to H$ is a linear mapping such that $\|I-T\| < 1$,where I is the identity map of H to itself, then T is invertible.
(b) Suppose e_n, n=1,2,3,...,is a complete, orthonormal set in H(a Hilbert space basis). Suppose also that f_n ,n=1,2,3,...is an orthonormal set in H such that $\sum \|e_n - f_n\|^2 < 1$. Prove that f_n is a complete orthonormal set in H.

8 Let $f:R\to R$ be a continuous function that is periodic with period 2π. Show that if all the Fourier coefficients of f (for $[-\pi, +\pi]$) are 0, then f is identically 0.

9 Suppose that h: D-{(0,0)}→R is a harmonic function, where D is the unit disc in the complex plane.
(a) Show that there is exactly one real number A such that h(z)- A ln|z| is the real part of a holomorphic function on D −{(0,0)} . (Hint: Consider a candidate for a harmonic conjugate constructed by contour integration.)
(b) Use part (a) to show that if h is bounded then h extends to be a harmonic function on D.

10 Let U be a bounded connected open set in C. Prove that if K is a compact subset of U, then there is a constant C_K such that for every point z in K and every holomorphic L^2 function f on U, $|f(z)| \leq C_K \left(\int_U |f|^2 \right)^{\frac{1}{2}}$.

11 Let U be a bounded connected open set in C. Suppose that 0 belongs to U and that F:U→U is a holomorphic function with F(0)=0 and with F'(0)=1. Prove that F is the identity map. [Suggestion: Consider the power series of F composed with itself many times].

12 Find a conformal map to the unit disc of the half disc $\{z : |z| < 1, \operatorname{Re} z > 0\}$. You may write your answer as a composition of simpler conformal maps.

13 Suppose R is a positive number and U={ z : $|z| < R$}

(a) Show that for each holomorphic function f on U, there is a power series $\sum_{n=o}^{+\infty} a_n z^n$ which converges at each point z in U to f(z).

(b) Show that there is only one such power series.

ANALYSIS QUALIFYING EXAMINATION

March 31, 2007

Instructions: Solve any 10 problems and therefore at least 4 from Problems 1–6 and at least 4 from Problems 7–12. All problems are worth the same amount. Turn in only the 10 problems you want us to grade.

Notation: Throughout, $L^p(A)$ denotes the standard L^p space defined relative to the Lebesgue measure on A and $\|f\|_p$ denotes the corresponding norm.

Problem 1. Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be non-negative integrable functions with $\|f_n\|_1 = 1$. Suppose $f_n \to f$ pointwise a.e. with $\|f\|_1 = 1$. Show that

$$\int_A f_n(x)\mathrm{d}x \underset{n\to\infty}{\longrightarrow} \int_A f(x)\mathrm{d}x$$

uniformly in the choice of Borel set $A \subset \mathbb{R}$. *Hint*: First prove that $f_n \to f$ in L^1.

Problem 2. Consider the function $f \colon (0,\infty) \times (0,\infty) \to \mathbb{R}$ defined by

$$f(x,y) = \sum_{n=0}^{\infty} \frac{x}{x^2 + yn^2}$$

Show that the limit $g(y) := \lim_{x\to\infty} f(x,y)$ exists for all $y > 0$ and compute $g(y)$.

Problem 3. Let $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$ denote the convolution of f and g. Fix $g \in L^1(\mathbb{R})$. Do the following:

(1) Show that $A_g(f) := f * g$ is a bounded operator $L^1(\mathbb{R}) \to L^1(\mathbb{R})$.
(2) Suppose in addition $g \geq 0$. Find the corresponding norm $\|A_g\|$.
(3) Show that the only $f \in L^1(\mathbb{R})$ for which $f * f = f$ is $f = 0$.

Problem 4. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $T \colon L^2([0,1]) \to L^2([0,1])$ be defined by

$$(Tf)(x) = f(x + \alpha \bmod 1).$$

Denote $S_n f = f + Tf + T^2 f + \cdots + T^{n-1} f$. Do the following:

(1) For any $f \in L^2([0,1])$, prove that $\frac{1}{n} S_n f$ converges in L^2. Identify the limit.
(2) Suppose $f \colon [0,1] \to \mathbb{R}$ is continuous with $f(1) = f(0)$. Show that the convergence in (1) is uniform.

Problem 5. Let $A_n(f) = \frac{1}{n}\int_0^n f(x)\mathrm{d}x$. Show that there exists a continuous linear functional $A\colon L^\infty(\mathbb{R}_+) \to \mathbb{R}$ such that

$$A(f) = \lim_{n\to\infty} A_n(f)$$

whenever the limit exists. Here $\mathbb{R}_+ = (0, \infty)$.

Problem 6. Let X be a Banach space and let $A\colon X \to X$ be a linear map. Define

$$\varrho(A) = \{\lambda \in \mathbb{C}\colon (\lambda - A) \text{ maps } X \text{ onto } X\}$$

Show that $\varrho(A)$ is an open subset of $\mathbb{C}$.

Problem 7. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\tan\theta + \mathrm{i}a}$$

where $a \in (0, \infty)$.

Problem 8. Determine the number of zeros of the polynomial

$$p(z) = z^4 + z^3 + 4z^2 + 2z + 3$$

in the right half-plane $\{z\colon \mathrm{Re} z > 0\}$.

Problem 9. Let $f(z)$ be analytic for $0 < |z| < 1$. Suppose there are $C > 0$ and $m \geq 1$ such that

$$\left|f^{(m)}(z)\right| \leq \frac{C}{|z|^m}, \qquad 0 < |z| < 1.$$

Show that $f(z)$ has a removable singularity at $z = 0$.

Problem 10. Let $J = \{\mathrm{i}y\colon 1 \leq y < \infty\}$ and let $\mathbb{H} = \{z\colon \mathrm{Im} z > 0\}$ be the open upper half plane. Consider the domain $D = \mathbb{H} \setminus J$. Find a bounded harmonic function $u\colon D \to \mathbb{R}$ such that $u(x + \mathrm{i}y) \to 0$ as $y \downarrow 0$ and $u(z) \to 1$ as $z \to J$. It is fine to represent to solution in terms of a composition of conformal maps.

Problem 11. Prove that a meromorphic function $f(z)$ in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is the sum of the principal parts at its poles.

Problem 12. Let D be a domain (connected open set) in $\mathbb{C}$ and let (u_n) be a sequence of harmonic functions $u_n\colon D \to (0, \infty)$. Show that if $u_n(z_0) \to 0$ for some $z_0 \in D$, then $u_n \to 0$ uniformly on compact subsets of D.

Analysis Qualifying Examination

Wednesday, September 17, 2008
9am-1pm

Instructions: Work any 10 problems. To pass the exam, you must show a satisfactory knowledge of both Real Analysis (Problems 1-6) and Complex Analysis (Problems7-12). All problems are worth ten points; parts of a problem may not carry equal weight. You need to tell us which 10 problems you want us to grade. Great emphasis will be placed on your attention to detail.

1. Fix $1 \leq p < \infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{C}$. Suppose there exists f in $L^p([0,1])$ so that $f_n \to f$ in the L^p sense, that is ,

$$\int_0^1 |f_n(x) - f(x)|^p \, dx \to 0 .$$

   (a) Show that $f_n \to f$ in measure, that is, $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$.

   (Here μ=Lebesgue measure.)

   (b) Show that there is a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ almost everywhere.

2. Is every vector space isomorphic as a vector space to some Banach space? Prove your answer. (Banach space=complete normed vector space, as usual).

3. Prove: If $f : [0,1] \to \mathbb{R}$ is an arbitrary function, not necessarily measurable, then the set of points at which f is continuous is a Lebesgue-measurable set.
(Suggestion: for $x \in \mathbb{R}$, $\delta > 0$, set $S_x(\delta) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x| < \delta, |x_2 - x| < \delta\}$.
Consider the function of $x \in \mathbb{R}$, $\omega(x) = \lim_{\delta \to 0^+} S_x(\delta)$. Caution: it might be $+\infty$ for some x values.)

4. Let X be a subset of $\ell^2(\mathbb{Z})$. Show that X is precompact (i.e., has compact closure) in the $\ell^2(\mathbb{Z})$ topology if and only if X is bounded and

$$\forall \varepsilon > 0, \exists\, N \geq 1 \text{ such that } \forall x \in X, \sum_{|n| \geq N} |x_n|^2 < \varepsilon$$

5. Let $d\mu$ be a finite positive Borel measure on $[0,2\pi]$ and suppose

$$\limsup_{n\to\pm\infty}\left|\int e^{in\theta}d\mu(\theta)\right| = 0.$$

Show that for any $f \in L^1(d\mu)$,

$$\limsup_{n\to\pm\infty}\left|\int e^{in\theta}f(\theta)d\mu(\theta)\right| = 0.$$

6. Define for each n= 1,2,3.., the Cantor-like set C_n as $[0,1]$ with its central open interval of length $\frac{1}{2^n}\cdot\frac{1}{3}$ removed, then with the two central open intervals of length $\frac{1}{2^n}\cdot\frac{1}{3^2}$ removed from the remaining two closed intervals and so on(at the j^{th} stage, 2^{j-1} intervals of length $\frac{1}{2^n}\cdot\frac{1}{3^j}$ are removed), continuing with j= 1,2,3...

(a) With μ= Lebesgue measure, show that $\mu([0,1]-\bigcup_{n=1}^{+\infty}C_n) = 0$

(b) Show that if E is a subset of [0,1] which is not Lebesgue measurable (you may assume such an E exists without proof), then for some $n\geq 1$, $E \cap C_n$ fails to be Lebesgue measurable.

(c) Use part (b) to show that there is a continuous, strictly increasing function $f: \mathbb{R}\to\mathbb{R}$ with $f(\mathbb{R})=\mathbb{R}$ and a Lebesgue measurable set $A\subset\mathbb{R}$ such that $f(A)$ is not Lebesgue measurable.

7. If $h:\{z\in\mathbb{C}:1<|z|<2\}\to\mathbb{R}$ is a continuous function, set for $1<r<2$:

$$M_h(r) = \frac{1}{2\pi}\int_0^{2\pi} h(re^{i\theta})d\theta$$

(a) Show that if h= Re F, $F:\{z\in\mathbb{C}:1<|z|<2\}\to\mathbb{C}$ holomorphic, then $M_h(r)$ is constant on $\{r: 1<r<2\}$.

(b) Show that if h is a real-valued harmonic function on $\{z\in\mathbb{C}:1<|z|<2\}$, then there are constants $c_1, c_2\in\mathbb{R}$ such that $M_h(r) = c_1\ln r + c_2$ for all $r \in (1,2)$.

8. Suppose f: $\{z \in \mathbb{C} : 0 < |z| < 1\} \to \mathbb{C}$ is a holomorphic function with $\int_U |f|^2 < +\infty$ where $U = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and the integral is the usual $\mathbb{R}^2$ – area integral. Prove that f has a removable singularity at z=0.

9. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane and let $H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the upper half plane .

(a) Explicitly describe all conformal mappings g from H onto D that obey $g(i)=0$.

(b) Suppose f: D→H has $f(0) = i$, f holomorphic. Show that $\operatorname{Im} f(x) \geq \frac{1-x}{1+x}$ for all $x \in (0,1)$.

10. Suppose U is a bounded connected open set in $\mathbb{C}$ and $z_0 \in U$.

Let $F = \{f : U \to D, f \text{ holomorphic}, f(z_0) = 0\}$ where $D=\{z \in \mathbb{C}: |z|<1\}$.

(a) Show that if K is a compact subset of U, then there is a constant $M_K>0$ such that $|f'(z)| \leq M_K$ for all $z \in K$, $f \in F$

(b) Use part (a) to show that if $\{f_n : f_n \in F\}$ is a sequence in F, then there is a subsequence $\{f_{n_j}\}$ which converges uniformly on every compact subset of U to a function $f_0 \in F$.

(Note: Part of this is to show $f_0(U) \subset D$.)

11. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane and let $\overline{D}$ denote its closure. Suppose f: D→$\mathbb{C}$ is continuous on $\overline{D}$ and analytic (holomorphic) in its interior. Show that if f takes only real values on $\partial\overline{D} := \{z: |z| = 1\}$, then f must be constant.

12. Evaluate $\int_0^{\pi} \frac{d\theta}{a^2 + \sin^2\theta}$ for all real numbers a>0.

Analysis S08

Instructions: Attempt ten of the thirteen questions, including at least three from Q9-13. Each question is worth 10 points.

Q1. Let $S^1 := \{z \in \mathbf{C} : |z| = 1\}$ denote the unit circle. Show that there exists a measurable function $f : S^1 \to S^1$ whose Fourier coefficients $\hat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{2\pi i\theta})e^{-2\pi i\theta n}\, d\theta$ are non-zero for every integer $n \in \mathbf{Z}$. (Hint: use the Baire category theorem.)

Q2. Let $\mathbf{R}/\mathbf{Z}$ be the unit circle with the usual Lebesgue measure. For each $n = 1, 2, 3, \ldots$, let $K_n : \mathbf{R}/\mathbf{Z} \to \mathbf{R}^+$ be a non-negative integrable function such that $\int_{\mathbf{R}/\mathbf{Z}} K_n(t)\, dt = 1$ and $\lim_{n\to\infty} \int_{\varepsilon \le |t| \le 1/2} K_n(t)\, dt = 0$ for every $0 < \varepsilon < 1/2$, where we identify $\mathbf{R}/\mathbf{Z}$ with $(-1/2, 1/2]$ in the usual manner. (Such a sequence of K_n are known as *approximations to the identity.*) Let $f : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ be continuous, and define the convolutions $f * K_n : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ by

$$f * K_n(x) := \int_{\mathbf{R}/\mathbf{Z}} f(x-t)K_n(t)\, dt.$$

Show that $f * K_n$ converges uniformly to f.

Q3. Let X be a compact metric space.

- (a) Show that X is separable (i.e. it has a countable dense subset).
- (b) Show that X is second countable (i.e. there exists a countable base for the topology).
- (c) Show that $C(X)$ (the space of continuous functions $f : X \to \mathbf{R}$ with the uniform topology) is separable. (Hint: use part (b), Urysohn's lemma and the Stone-Weierstrass theorem.)

Q4. Let $f, g \in L^2(\mathbf{R})$ be two square-integrable functions on $\mathbf{R}$ (with the usual Lebesgue measure). Show that the convolution

$$f * g(x) := \int_{\mathbf{R}} f(y)g(x-y)\, dy$$

of f and g is a bounded continuous function on $\mathbf{R}$.

Q5. Let H be a Hilbert space, and let $T : H \to H$ be a bounded linear operator on H.

- Show that if the operator norm $\|T\|$ of T is strictly less than 1, then the operator $1 - T$ is invertible.
- Let $\sigma(T)$ denote the set of all complex numbers z such that $T - zI$ is not invertible. (This set is known as the *spectrum* of T.) Show that $\sigma(T)$ is a compact subset of $\mathbf{C}$.

Q6. Let μ_n be a sequence of Borel probability measures on $[0,1]$, thus each μ_n is a non-negative finite measure on the Borel σ-algebra of $[0,1]$ (the σ-algebra generated by the open sets in $[0,1]$) with $\mu_n([0,1]) = 1$. Show that there exists a subsequence μ_{n_j}, as well as another Borel probability measure μ, such that $\lim_{j\to\infty} \int_{[0,1]} f(x)\, d\mu_{n_j}(x) = \int_{[0,1]} f(x)\, d\mu(x)$ for all continuous functions $f : [0,1] \to \mathbf{R}$. (Hint: use the Riesz representation theorem and Q3.)

Q7. Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a bounded smooth function, and suppose that the Laplacian $\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$ of u is rotationally symmetric, which means that $\Delta u(R_\theta(x,y)) = \Delta u(x,y)$ for any rotation $R_\theta : (x,y) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. Show that u is also rotationally symmetric. (Hint: You may use without proof the fact that the Laplacian Δ commutes with all rotations R_θ.)

Q8. Let H be a real Hilbert space, let K be a closed non-empty subset of H, and let v be a point in H. Show that there exists a unique $w \in K$ which minimizes the distance to v in the sense that $\|v - w\| < \|v - w'\|$ for all $w' \in K \backslash \{w\}$. (Hint: you may find the parallelogram law $\frac{\|a\|^2 + \|b\|^2}{2} = \|\frac{a+b}{2}\|^2 + \|\frac{a-b}{2}\|^2$ to be useful.)

Q9. Show using the residue theorem that

$$\int_0^\infty \frac{\log^2 x}{1 + x^2} = \frac{\pi^3}{8}.$$

Q10. Let the power series $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence $r > 0$. For each ρ with $0 < \rho < r$ let $M_f(\rho) := \sup\{|f(z)|; |z| = \rho\}$. Show that the following holds for each such ρ:

$$\sum_{n=0}^\infty |a_n|^2 \rho^{2n} \le M_f(\rho)^2.$$

Q11. Let $\mathbb{D} := \{z; |z| < 1\}$ be the unit disc. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map having 2 unequal fixed points $a, b \in \mathbb{D}$. Show that $f(z) = z$ for all $z \in D$. (Hint: use Schwartz's lemma.)

Q12. Consider the annulus $A := \{z \in \mathbf{C} : r < |z| < R\}$, where $0 < r < R$. Show that the function $f(z) := 1/z$ cannot be uniformly approximated in A by complex polynomials.

Q13. Let $\Omega \subset \mathbf{C}$ be an open set containing the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbf{C} : |z| \leq 1\}$, and let $f_n : \Omega \to \mathbf{C}$ be a sequence of holomorphic functions on Ω which converge uniformly on compact subsets of Ω to a limit $f : \Omega \to \mathbf{C}$. Suppose that $|f(z)| \neq 0$ whenever $|z| = 1$. Show that there is a positive integer N such that for $n \geq N$, the functions f_n and f have the same number of zeros in the unit disk $\mathbb{D} := \{z \in \mathbf{C} : |z| < 1\}$.

ANALYSIS QUALIFYING EXAM, FALL 2009

Instructions: Work any 10 problems and therefore at least 4 from Problems 1-6 and at least 4 from Problems 7-12. All problems are worth ten points. Full credit on one problem will be better than part credit on two problems. If you attempt more than 10 problems, indicate which 10 are to be graded.

1: Find a non-empty closed set in the Hilbert space $L^2([0,1])$ that does not contain an element of smallest norm. Prove your assertion.

2: Let v be a trigonometric polynomial in two variables, that is,

$$v(x,y) = \sum_{n,m\in\mathbb{Z}} a_{n,m} e^{2\pi i(nx+my)} ,$$

with only finitely many non-zero coefficients $a_{n,m}$. If

$$u = v - \Delta v$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian, prove that

$$\|v\|_{L^\infty([0,1]\times[0,1])} \leq C\|u\|_{L^2([0,1]\times[0,1])}$$

for some constant C independent of v.

3: Let $f : [0,1] \to \mathbb{R}$ be continuous with

$$\min_{0\leq x\leq 1} f(x) = 0 .$$

Assume that for all $0 \leq a < b \leq 1$ we have

$$\int_a^b \left[f(x) - \min_{a\leq x\leq b} f(y)\right] dx \leq \tfrac{1}{2}|b-a|$$

a) Prove that for all $\lambda \geq 0$:

$$|\{x : f(x) > \lambda + 1\}| \leq \tfrac{1}{2}|\{x : f(x) > \lambda\}|$$

Here $|S|$ denotes the Lebesgue measure the set S.
b) Prove that for all $1 \leq c < 2$,

$$\int_0^1 c^{f(x)}\, dx \leq \frac{100}{2-c}$$

[The constant 100 is not optimal, but is certainly large enough.]

4: Prove the following variant of the Lebesgue Differentiation theorem: Let μ be a finite Borel measure on $\mathbb{R}$, singular with respect to Lebesgue measure. Then for Lebesgue-almost every $x \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \frac{\mu([x-\epsilon, x+\epsilon])}{2\epsilon} = 0$$

5: Construct a Borel subset E of the real line $\mathbb{R}$ such that for all intervals $[a,b]$ we have

$$0 < m(E \cap [a,b]) < |b-a|$$

where m denotes Lebesgue measure.

6: The Poisson kernel for $0 \le \rho < 1$ is the 2π periodic function on $\mathbb{R}$ defined by

$$P_\rho(\theta) = \operatorname{Re}\left(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}}\right)$$

For functions h continuous on and harmonic inside the closed disc of radius R about the origin one has (you need not prove this)

$$h(re^{i\eta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\eta-\theta) h(Re^{i\theta})\, d\theta$$

Assume that h is harmonic and positive on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Prove that there exists a positive Borel measure μ on $[0, 2\pi]$ such that for all $re^{i\theta} \in D$ one has

$$h(re^{i\eta}) = \int_0^{2\pi} P_r(\eta-\theta)\, d\mu(\theta)$$

7: a) Define *unitary operator* on a complex Hilbert space.
b) Let S be a unitary operator on a complex Hilbert space. Using your definition, prove that for every complex number $|\lambda| < 1$ the operator $S - \lambda I$ is invertible. Here I denotes the identity operator.
c) For a fixed vector v in the Hilbert space and all $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, we define

$$h(\lambda) = \langle (S+\lambda I)(S-\lambda I)^{-1} v, v \rangle .$$

Show that $\operatorname{Re} h$ is a positive harmonic function. [You may *not* invoke the spectral theorem — this is part of a proof of that theorem.]

8: Let Ω be an open convex region in the complex plane. Assume f is a holomorphic function on Ω and the real part of its derivative is positive: $\operatorname{Re}(f'(z)) > 0$ for all $z \in \Omega$.
a) Prove that f is one-to-one.
b) Show by example that the word "convex" cannot be replaced by "connected and simply connected".

9: Let f be a non-constant meromorphic function on the complex plane $\mathbb{C}$ that obeys

$$f(z) = f(z + \sqrt{2}) = f(z + i\sqrt{2}) \ .$$

(In particular, the poles of these three functions coincide.) Assume f has at most one pole in the closed unit disc

$$\overline{D} = \{z : |z| \leq 1\} \ .$$

a) Prove that f has exactly one pole in $\overline{D}$.
b) Prove that this is not a simple pole.

10: Consider the unit sphere in $\mathbb{R}^3$ with poles removed:

$$S := \big\{ (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) : 0 < \phi < \pi \text{ and } \theta \in \mathbb{R} \big\} \ .$$

Give an explicit formula for a conformal map from the complex plane onto S so that horizontal lines are mapped to circles of constant ϕ and vertical lines are mapped to arcs of constant θ.

11: Let f be an analytic function in the open unit disc $D = \{z : |z| < 1\}$ that obeys $|f(z)| \leq 1$ for all $z \in D$. Let g be the restriction of f to the real interval $(0, 1)$ and assume

$$\lim_{r \to 1} g(r) = 1$$

and

$$\lim_{r \to 1} g'(r) = 0 \ .$$

Prove that f is constant.

12: Let f be a non-constant meromorphic function in the complex plane. Assume that if f has a pole at the point $z \in \mathbb{C}$, then z is of the form $n\pi$ with an integer $n \in \mathbb{Z}$. Assume that for all non-real z we have the estimate

$$|f(z)| \leq (1 + |\mathrm{Im}(z)|^{-1})e^{-|\mathrm{Im}(z)|}$$

Prove that for every integer $n \in \mathbb{Z}$, f has a pole at the point πn.

Analysis Qualifying Examination - March 26, 2009

Instructions:

Work any 10 problems and therefore at least 4 from Problems 1 - 6 and at least 4 from Problems 7 - 12. All problems are worth ten points. Full credit on one problem will be better than part credit on two problems.

1. Let f and g be real-valued integrable functions on a measure space $(X, \mathcal{B}, \mu)$, and define

$$F_t = \{x \in X : f(x) > t\},\ G_t = \{x \in X : g(x) > t\}.$$

Prove

$$\int |f - g| d\mu = \int_{-\infty}^{\infty} \mu\big((F_t \setminus G_t) \cup (G_t \setminus F_t)\big) dt.$$

2. Let H be an infinite dimensional real Hilbert space.

a) Prove the unit sphere $S = \{x \in H : ||x|| = 1\}$ of H is weakly dense in the unit ball $B = \{x \in H : ||x|| \leq 1\}$ of H. (i.e. if $x \in B$, there is a sequence $\{x_n\} \in S$ such that for all $y \in H$, $< x, y >= \lim < x_n, y >$.)

b) Prove there is a sequence T_n of bounded linear operators from H to H such that $||T_n|| = 1$ for all n but $\lim T_n(x) = 0$ for all $x \in H$.

3. Let X be a Banach space and let X^* be its dual Banach space. Prove that if X^* is separable then X is separable.

4. Let $f(x)$ be a non-decreasing function on $[0, 1]$. You may assume the theorem that f is differentiable almost everywhere.

a) Prove that $\int_0^1 f'(x)dx \leq f(1) - f(0)$.

Hint: Fatou.

b) Let $\{f_n\}$ be a sequence of non-decreasing functions on the unit interval $[0, 1]$, such that the series $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [a, b]$. Prove that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost everywhere on $[0, 1]$.

Hint: Let $r_n(x) = \sum_{k \geq n} f_k(x)$. It is enough to show $r_n'(x) \to 0$ a.e. Take a subsequence r_{n_j} such that $r_{n_j}(1) - r_{n_j}(0) \to 0$ and use part (a).

5. Let $I = I_{0,0} = [0,1]$ be the unit interval, and for $n = 0, 1, 2, \ldots,$ and $0 \le j \le 2^n - 1$ let

$$I_{n,j} = [j2^{-n}, (j+1)2^{-n}].$$

For $f \in L^1(I, dx)$ define $E_n f(x) = \sum_{j=0}^{2^n-1}\Big(2^n \int_{I_{n,j}} f\,dt\Big)\chi_{I_{n,j}}$.
Prove that if $f \in L^1(I, dx)$ then $\lim_{n\to\infty} E_n f(x) = f(x)$ almost everywhere on I.

6. For $I_{n,j}$ as in Problem 5, define the Haar function $h_{n,j} = 2^{n/2}\Big(\chi_{I_{n+1,2j}} - \chi_{I_{n+1,2j+1}}\Big)$.

a) Carefully draw $I_{2,1}$ and graph $h_{2,1}$.

b) Prove that if $f \in L^2(I)$ with respect to Lebesgue measure and $\int_I f\,dt = 0$, then

$$\int_I |f(x)|^2 dx = \sum_{n,j} \Big| \int f(t) h_{n,j}(t) dt \Big|^2.$$

c) Prove that if $f \in L^1(I)$ and $\int_I f(t)dt = 0$, then almost everywhere on I,

$$f(x) = \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \Big(\int f(t) h_{n,j}(t) dt\Big) h_{n,j}(x).$$

Hint: Compare the n-th partial sum to $E_n f$ from Problem 5.

7. Let μ be a finite positive Borel measure on the complex plane $\mathbb{C}$.

a) Prove that $F(z) = \int_{\mathbb{C}} \frac{1}{z-w} d\mu(w)$ exists for almost all $z \in \mathbb{C}$ and that $\int_K |F(z)| dxdy < \infty$ for every compact $K \subset \mathbb{C}$.

b) Using (a), prove that for almost every horizontal line L (almost everywhere measured by y intercept), and all compact $K \subset L$, $\int_K |F(x+iy)| dx < \infty$.

c) By "almost all squares in $\mathbb{C}$" we mean all squares in $\mathbb{C}$ with sides parallel to the axes except for those squares whose lower left and upper right vertices (z_1, z_2) belong to a Lebesgue measure zero subset of $\mathbb{C}^2$. Prove that for almost all open squares S,

$$\mu(S) = \frac{1}{2\pi i} \int_{\partial S} F(z) dz.$$

Hint: Use b) and the analagous result for vertical lines.

8. Let f be an entire non-constant function that satisfies the functional equation

$$f(1-z) = 1 - f(z)$$

for all $z \in \mathbb{C}$. Show that $f(\mathbb{C}) = \mathbb{C}$.

9. Let $f(z)$ be an analytic function on the entire complex plane $\mathbb{C}$ and assume $f(0) \neq 0$. Let $\{a_n\}$ be the zeros of f, counted with their multiplicities.

a) Let $R > 0$ be such that $|f(z)| > 0$ on $|z| = R$. Prove

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta = \log |f(0)| + \sum_{|a_n| < R} \log(\frac{R}{|a_n|}).$$

b) Assume $|f(z)| \leq Ce^{|z|^\lambda}$ for positive constants C and λ. Prove that

$$\sum (\frac{1}{|a_n|})^{\lambda+\epsilon} < \infty$$

for all $\epsilon > 0$.

Hint: Estimate $\#\{n : |a_n| < R\}$ by using (a) for the circle of radius $2R$.

10. Let $\mathbb{D}$ be the open unit disc and μ be Lebesgue measure on $\mathbb{D}$. Let H be the subspace of $L^2(\mathbb{D}, \mu)$ consisting of holomorphic functions. Show that H is complete.

11. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and injective in some annulus $\{z : r < |z| < 1\}$, where $\mathbb{D}$ is the open unit disc. Show that f is injective in $\mathbb{D}$.

12. Let Q be the closed unit square in the complex plane $\mathbb{C}$ and let R be the closed rectangle in $\mathbb{C}$ with vertices $\{0, 2, i, 2+i\}$. Prove there does *not* exist a surjective homeomorphism $f : Q \to R$ that is conformal on the interior Q^o and that maps corners to corners.

ANALYSIS QUAL: MARCH 24, 2010

Answer at most 10 questions, including at least 4 even numbered questions. On the front of your paper indicate which 10 problem you wish to have graded.

Problem 1. (a) Let $1 \leq p < \infty$. Show that if a sequence of real-valued functions $\{f_n\}_{n\geq 1}$ converges in $L^p(\mathbb{R})$, then it contains a subsequence that converges almost everywhere.

(b) Give an example of a sequence of functions converging to zero in $L^2(\mathbb{R})$ that does not converge almost everywhere.

Problem 2. Let $p_1, p_2, \ldots, p_n$ be distinct points in the complex plane $\mathbb{C}$ and let U be the domain

$$U = \mathbb{C} \setminus \{p_1, \ldots, p_n\}.$$

Let A be the vector space of real harmonic functions on U and let $B \subset A$ be the subspace of real parts of complex analytic functions on U. Find the dimension of the quotient vector space A/B, give a basis for this quotient space, and prove that it is a basis.

Problem 3. For an $f : \mathbb{R} \to \mathbb{R}$ belonging to $L^1(\mathbb{R})$, we define the Hardy-Littlewood maximal function as follows:

$$(Mf)(x) := \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)|\, dy.$$

Prove that it has the following property: There is a constant A such that for any $\lambda > 0$,

$$\left|\{x \in \mathbb{R} : (Mf)(x) > \lambda\}\right| \leq \frac{A}{\lambda} \|f\|_{L^1}$$

where $|E|$ denotes the Lebesgue measure of E. If you use a covering lemma, you should prove it.

Problem 4. Let $f(z)$ be a continuous function on the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ such that $f(z)$ is analytic on the open disk $\{|z| < 1\}$ and $f(0) \neq 0$.

(a) Prove that if $0 < r < 1$ and if $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \geq \log |f(0)|.$$

(b) Use (a) to prove that $\left|\{\theta \in [0, 2\pi] : f(e^{i\theta}) = 0\}\right| = 0$ where again $|E|$ is the Lebesgue measure of E.

Problem 5. (a) For $f \in L^2(\mathbb{R})$ and a sequence $\{x_n\}_{n\geq 1} \subset \mathbb{R}$ which converges to zero, define

$$f_n(x) := f(x + x_n).$$

Show that $\{f_n\}_{n\geq 1}$ converges to f in L^2 sense.

(b)Let $W \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure. Show that the set of differences

$$W - W = \{x - y : x, y \in W\}$$

contains an open neighborhood of the origin.

Problem 6. Let μ be a finite, positive, regular Borel measure supported on a compact subset of the complex plane $\mathbb{C}$ and define the Newtonian potential of μ to be

$$U_\mu(z) = \int_{\mathbb{C}} \left|\frac{1}{z-w}\right| d\mu(w).$$

(a) Prove that U_μ exists at Lebesgue almost all $z \in \mathbb{C}$ and that

$$\int\int_K U_\mu(z)dxdy < \infty$$

for every compact $K \subset \mathbb{C}$. Hint: Fubini.

(b) Prove that for almost every horizontal or vertical line $L \subset \mathbb{C}$, $\mu(L) = 0$ and $\int_K U_\mu(z)ds < \infty$ for every compact subset $K \subset L$, where ds denotes Lebesgue linear measure on L. Hint: Fubini and (a). (Here a. e. vertical line means the vertical lines through $(x, 0)$ for a.e. $x \in \mathbb{R}$. Likewise for horizontal lines.)

(c) Define the Cauchy potential of μ to be

$$S_\mu(z) = \int_{\mathbb{C}} \frac{1}{z-w} d\mu(w),$$

which trivially exists whenever $U_\mu(z) < \infty$. Let R be a rectangle in $\mathbb{C}$ whose four sides are contained in lines L having the conclusions of (b). Prove that

$$\frac{1}{2\pi i}\int_{\partial R} S_\mu(z)dz = \mu(R).$$

Hint: Fubini and Cauchy.

Problem 7. Let H be a Hilbert space and let E be a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$\|x\| = \inf_{y\in E} \|y\|.$$

Problem 8. Let $F(z)$ be a non-constant meromorphic function on the complex plane $\mathbb{C}$ such that for all $z \in \mathbb{C}$,

$$F(z + 1) = F(z) \text{ and } F(z + i) = F(z).$$

Let Q be a square with vertices $z, z + 1, z + i$ and $z + (1 + i)$ such that F has no zeros and no poles on ∂Q. Prove that inside Q the function F has the same number of zeros as poles (counting multiplicities).

Problem 9. Let
$$A = \{x \in \ell^2 : \sum_{n\geq 1} n|x_n|^2 \leq 1\}.$$

(a) Show that A is compact in the ℓ^2 topology.
(b) Show that the mapping from A to $\mathbb{R}$ defined by
$$x \mapsto \int_0^{2\pi} \Big|\sum_{n\geq 1} x_n e^{in\theta}\Big| \frac{d\theta}{2\pi}$$
achieves its maximum on A.

Problem 10. Let $\Omega \subset \mathbb{C}$ be a connected open set, let $z_0 \in \Omega$, and let $\mathcal{U}$ be the set of postive harmonic functions U on Ω such that $U(z_0) = 1$. Prove for every compact set $K \subset \Omega$ there is a finite constant M (depending on Ω, z_0 and K) such that
$$\sup_{U\in\mathcal{U}} \sup_{z\in K} U(z) \leq M.$$
You may use Harnack's inequality for the disk without proving it, provided you state it correctly.

Problem 11. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support.
(a) Prove there is a constant A such that
$$\|f * \phi\|_{L^q} \leq A\|f\|_{L^p} \quad \text{for all} \quad 1 \leq p \leq q \leq \infty \quad \text{and all} \quad f \in L^p.$$
If you use Young's (convolution) inequality, you should prove it.
(b) Show by example that such a general inequality cannot hold for $p > q$.

Problem 12. Let F be a function from the open unit disk $\mathbb{D} = \{|z| < 1\}$ to $\mathbb{D}$ such that whenever z_1, z_2 and z_3 are distinct points of $\mathbb{D}$ there exists an analytic function f_{z_1,z_2,z_3} from $\mathbb{D}$ into $\mathbb{D}$ such that
$$F(z_j) = f_{z_1,z_2,z_3}(z_j),\ j = 1, 2, 3.$$
Prove that F is analytic at every point of $\mathbb{D}$.

Hint: Fix $z \in \mathbb{D}$ and let $\mathbb{D} \ni z_n \to z, z_n \neq z$. Show that the sequence
$$\frac{F(z_n) - F(z)}{z_n - z}$$
is bounded and then prove that every two of its convergent subsequences have the same limit.

Problem 13. Let X and Y be two Banach spaces. We say that a bounded linear transformation $A : X \to Y$ is *compact* if for every bounded sequence $\{x_n\}_{n\geq 1} \subset X$, the sequence $\{Ax_n\}_{n\geq 1}$ has a convergent subsequence in Y.

Suppose X is reflexive (that is, $(X^*)^* = X$) and X^* is separable. Show that a linear transformation $A : X \to Y$ is compact if and only if for every bounded sequence $\{x_n\}_{n\geq 1} \subset X$, there exists a subsequence $\{x_{n_j}\}$ and a vector $\phi \in X$ such that $x_{n_j} = \phi + r_{n_j}$ and $Ar_{n_j} \to 0$ in Y.