Algebra Qualifying Exam

Fall 2000

Everyone must do two problems in each of the four sections. To pass at the PhD level, you must attempt at least three 20-point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

A1. (10 points) Let $D_{2n}$ be the dihedral group of order $2n$ with $n > 1$. Determine the number of subgroups of $D_{2n}$ of index 2, and justify your answer.

A2. (15 points) A group of order a power of a prime $p$ is called a $p$–group. Let $G$ be a finite group. Prove that for any given prime $p$, there exists a unique normal subgroup $N$ of $G$ such that (i) $G/N$ is a $p$–group and (ii) any homomorphism $\pi$ of $G$ into a $p$–group is trivial on $N$ (that is, $\pi(N) = 1$).

A3. (20 points) Let $G$ be a finite group of order $n$. Suppose that $G$ has a unique subgroup of order $d$ for each positive divisor $d$ of $n$. Prove that $G$ is a cyclic group.
Rings

B1. (10 points) Let $M$ be a module over a commutative ring $A$. If every increasing (resp.
    decreasing) sequence of $A$–submodules of $M$ terminates after finite steps, the $A$–module $M$ is
called noetherian (resp. artinian).

(a) Prove that the $\mathbb{Z}$–module $\mathbb{Z}$ is noetherian and non-artinian.
(b) Prove that the $\mathbb{Z}$–module $\bigcup_{n=1}^{\infty} (p^n \mathbb{Z}/\mathbb{Z})$ is artinian and non-
     noetherian.

B2. (15 points) Let $A$ be a commutative ring with identity. Suppose that $a \in A$ is not nilpotent (that is,
    $a^n \neq 0$ for all $n > 0$).

(a) Prove that there exists a prime ideal $P \subset A$ such that $a \not\in P$;
(b) Give an example of a ring $A$ and a non-nilpotent $a \in A$ such that $a$ is contained in $M$ for all
     maximal ideals $M \subset A$. Justify your example.

B3. (20 points) Let $A$ be a commutative noetherian ring with identity $1 \neq 0$. Write $X(a)$ for the
    set of prime ideals of $A$ containing a given ideal $a$. Suppose that $X(0) = X(a) \cup X(b)$ and
    $X(a) \cap X(b) = \emptyset$ for two ideals $a$ and $b$. Prove the following facts:

(a) $A = a + b$;
(b) $a \cap b = ab$;
(c) The ideal $ab$ consists of nilpotent elements.
    Hint: You may use the assertion of B2 (a);
(d) There exists a positive integer $n$ such that $A$ is isomorphic to the
    product ring $(A/a^n) \times (A/b^n)$.
Fields

C1. (10 points) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over the field of rational numbers $\mathbb{Q}$.

C2. (15 points) Let $a$ be a primitive 16-th root of unity over the field $F$. Determine the dimension $[F(a) : F]$ when $F$ is:

a) The field of 9 elements,

b) The field of 7 elements,

c) The field of 17 elements,

d) What can you say in the case $p = 2$?

C3. (20 points) Let $F$ be an infinite field of characteristic $p > 0$. Recall that a finite dimensional extension $L/F$ is said to be simple if $L = F(u)$ for some element $u \in L$.

a) Suppose $L/F$ has a finite number of intermediate fields. Show that $L/F$ is simple.

b) Let $K$ be an intermediate field $F \subset K \subset L$, and suppose that $L = F(u)$ with $x^p + a_1 x^{p-1} + \cdots + a_r$ the monic irreducible polynomial of $u$ over $K$, $a_i \in K$. Show that $K = F(a_1, a_2, \ldots, a_r)$.

c) Conclude that $L/F$ is simple if and only if there are a finite number of intermediate fields.

d) Let $E = F(x, y)$ where $x$ and $y$ are indeterminates, and set

$$M = E(x^{1/p}, y^{1/p})$$

Show that $M/E$ has an infinite number of intermediate fields.
Linear Algebra

D1. (10 points) Let $V$ be a finite dimensional vector space over a field $F$ and $T : V \to V$ a linear map whose characteristic polynomial has distinct roots, all in $F$. Show that $T$ is diagonalizable.

D2. (15 points) Describe all non-diagonalizable $4 \times 4$ matrices over $\mathbb{Q}$ such that $A^5 + A^2 = 0$ up to similarity. Justify your answer.

D3. (20 points) Let $V$ be a complex vector space with positive definite inner product $(\ , \ )$, and $T : V \to V$ a linear map. Recall that the adjoint $T^*$ of $T$ is defined by:

$$(T(x), y) = (x, T^*(y))$$

for all $x, y \in V$, and that $T$ is called normal if $T$ and $T^*$ commute. Suppose that $T^3 = TT^*$. Let $U$ be the kernel of $T$ and $W$ the orthogonal complement of $U$ in $V$.

(a) Show that $W$ is $T^*$-invariant.
(b) Show that $U$ is $T^*$-invariant.
(c) Conclude that $W$ is $T$ invariant.
(d) Show that the restrictions of $T$ and $T^*$ to $W$ commute.
(e) Show that $T$ is normal.
Spring 2000

Algebra Qualifying Exam

Everyone must do two problems in each of the four sections. To pass at the Ph.D. level, you must attempt at least three 20 point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

A. GROUPS

A1. (10 points) State a theorem which classifies (i.e. lists) all finite abelian groups up to isomorphism. This means that each finite abelian group should be isomorphic to exactly one group of your list. Use your classification to list abelian groups of order 24.

A2. (15 points) Let $S_5$ be the symmetric group on 5 letters. For each positive integer $n$, list the number of elements of $S_5$ of order $n$. Justify your answer.

A3. (20 points) Let $\mathbb{F}_4$ be the field with 4 elements. Let $G = SL(2, \mathbb{F}_4)$ be the group of 2 by 2 invertible matrices with entries in $\mathbb{F}_4$. What is the order of $G$? Show, by analysing the action of $G$ on the lines containing the origin in $(\mathbb{F}_4)^2$, that $G$ is a simple group. (Hint: how many lines containing the origin are there?)

B. RINGS

B1. (10 points) List, up to isomorphism, all commutative rings with 4 elements. Prove your answer.

B2. (15 points) Let $p$ be a prime number. Show that a free $\mathbb{Z}$ module of rank 2 has $p+1$ submodules of index $p$.

B3. (20 points) Let $R$ be a commutative noetherian ring in which each ideal $I$ is principal and satisfies $I^2 = I$. Show that $R$ is isomorphic to a finite product of fields.

C. FIELDS

C1. Let $\alpha = 1 + \sqrt[3]{2} + \sqrt[3]{4}$.
   (a) Find the degree of $\alpha$ over $\mathbb{Q}$. Justify your answer.
(b) Find a normal closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. Justify your answer.

C2. Let $q$ be a power of a prime integer, $n \in \mathbb{N}$. Let $k$ be the least positive integer such that $q^k \equiv 1 \pmod{n}$. Prove that the finite field $\mathbb{F}_{q^k}$ is a splitting field of the polynomial $X^n - 1$ over $\mathbb{F}_q$.

C3. Find a subfield $F$ in the field of rational functions $\mathbb{C}(X)$ such that $\mathbb{C}(X)/F$ is a Galois extension with the Galois group isomorphic to the symmetric group $S_3$. (Hint: Consider automorphisms of $\mathbb{C}(X)$ given by $X \mapsto \frac{aX+b}{cX+d}, a, b, c, d \in \mathbb{Z}$.)

D. LINEAR ALGEBRA

D1. Let $A$ be a linear operator on a vector space of dimension $n$ such that $A^m$ is the zero operator for some $m$.
(a) Prove that all eigenvalues of $A$ are equal to zero.
(b) Prove that $A^n = 0$.

D2. Describe, up to similarity, all $4 \times 4$ matrices $A$ over $\mathbb{Q}$ such that $A^5 = -A^3$ but $A^3 \neq 0$. Justify your answer.

D3. Let $\text{End}(V)$ be the ring of all linear operators on a finite dimensional vector space $V$ (with respect to the addition and composition of operators). For an operator $A \in \text{End}(V)$ let $L_A$ be the subspace in $\text{End}(V)$ generated by the powers $A^i$, $i \geq 0$.
(a) Show that $L_A$ is a subring in $\text{End}(V)$.
(b) Prove that if $L_A$ is a field then the characteristic polynomial of $A$ is a power of an irreducible polynomial.
Algebra Qualifying Exam

Fall 2001

Everyone must do two problems in each of the four sections.
To pass at the Ph.D. level, you must attempt at least three 20-point problems. On multiple
part problems, do as many parts as you can; however, not all parts count equally.

Groups

G1. (10 points) Let $G$ be a finite group whose center has index $n$. Show
that every conjugacy class in $G$ has at most $n$ elements.

G2. (15 points) Let $G$ be a subgroup of $S_n$ that acts transitively on the set
$\{1, 2, \ldots, n\}$. Let $H$ be the stabilizer in $G$ of an element $x \in \{1, 2, \ldots, n\}$.
Prove that

$$ \bigcap_{g \in G} gHg^{-1} = \{e\} $$

G3. (20 points) Let $G$ be the group of matrices of the form

$$ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} $$

where $a \in (\mathbb{Z}/p)^*$ and $a \in \mathbb{Z}/p$. Describe all normal subgroups of $G$. Hint:
find a convenient normal subgroup of order $p$. 
Rings

R1. (10 points) Let $R$ be a commutative ring, $I \subset R$ a nonzero ideal. Prove that if $I$ is a free $R$-module then $I = aR$ for an element $a \in R$ which is not a zero divisor in $R$. Hint: consider the rank of $I$.

R2. (15 points) (a) Give an example of prime ideal in a commutative ring that is not maximal.

(b) Let $R$ be a commutative ring with identity. Suppose for every element $x \in R$ there exists an integer $n = n(x) > 1$ such that $x^n = x$. Show that every prime ideal in $R$ is maximal.

R3. (20 points) Let $R$ be a ring.

(a) Prove that if $a$ is a nilpotent element in a ring $R$ with identity, then the element $1 + a$ is invertible.

In the next two parts, let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be a polynomial in $R[X]$ of degree $n$, that is, $a_n \neq 0$.

(b) Show that if $R$ is an integral domain, then $f(X)$ is invertible in $R[X]$ if and only if $n = 0$.

(c) Show that if $R$ is a commutative ring, $f(X)$ is invertible in $R[X]$ if and only if all $a_0$ is invertible and $a_i$ are nilpotent in $R$ for every $i \geq 1$. 
Fields

**F1.** (10 points) Let $f(x) = x^3 - 2x - 2$.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
(b) Let $\theta$ be a complex root of $f(x)$. Express $\theta^{-1}$ as a polynomial in $\theta$ with coefficients in $\mathbb{Q}$.

**F2.** (15 points) Let $f(x) = x^3 + nx + 2$ where $n$ is an integer. Determine the (infinitely many) values of $n$ for which $f$ is irreducible over $\mathbb{Q}$.

**F3.** (20 points) Let $G$ be the Galois group of $x^p - 2$ over $\mathbb{Q}$ where $p$ is a prime. Show that $G$ is isomorphic to the group of matrices of the form

$$
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}
$$

where $a \in (\mathbb{Z}/p)^*$ and $a \in \mathbb{Z}/p$. 

3
Linear Algebra

**LA1.** (10 points) Let $T$ be a linear operator on a finite-dimensional vector space $V$ such that $\text{Im}(T)$ and $\text{Im}(T^2)$ have the same dimension. Show that $\ker T \cap \text{Im}(T) = 0$.

**LA2.** (15 points) Find all similarity classes of $4 \times 4$ matrices $A$ over $\mathbb{Q}$ such that $A^2 \neq \pm A$ and $A^2 \neq I$ but $A^3 = I$ ($I$ is the identity $4 \times 4$ matrix).

**LA3.** (20 points) Let $V$ be a vector space over a field $k$. A bilinear form $f : V \times V \to k$ is called skew-symmetric if $f(u, v) = -f(v, u)$ for all $v, u \in V$ and is called alternating if $f(v, v) = 0$ for all $v \in V$.

(a) Prove that every alternating form is skew-symmetric.

(b) Give an example of a skew-symmetric form which is not alternating. Hint: choose $k$ of characteristic 2.

(c) Show that all alternating forms on $V$ form a vector space $\text{Alt}(V)$ and find $\dim \text{Alt}(V)$ if $\dim V = n$. 

4
Algebra Qualifying Exam

Spring 2001

Everyone must do two problems in each of the four sections.
To pass at the Ph.D. level, you must attempt at least three 20-point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

A. Groups.

A1. (10 points) Determine a complete set of groups of order eight up to isomorphism and show that every group of order eight is isomorphic to one of these.

A2. (15 points) A finite group $G$ acts on itself by conjugation. Determine all possible $G$ if this action yields precisely three orbits. Prove your result.

A3. (20 points) Let $G$ be a finitely generated group.
   a. Show for each integer $n$ there exist finitely many subgroups of index $n$.
   b. Suppose that there exists a subgroup of finite index in $G$. Prove that $G$ contains a characteristic subgroup of finite index.

B. Rings.

B1. (10 points) A commutative ring $R$ with unit is said to be a local ring if it has a unique maximal ideal. Show that a commutative ring $R$ with unit is a local ring if and only if for any two elements $u, v \in R$ satisfying $u + v = 1$ at least one of $u, v$ is a unit of $R$.

B2. (15 points) Let $R = \mathbb{R}[x, y]$. Find a finitely generated $R$-module $M$ that is not a direct sum of cyclic $R$-modules, and prove that it is not.

B3. (20 points) Let $f_1(z_1, \ldots, z_n), f_2(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n)$ be $n$ polynomials in $\mathbb{C}[z_1, \ldots, z_n]$. Assume that $f_i(0, 0, \ldots, 0) = 0$ for all $i = 1, \ldots, n$. Prove that the origin is the only point of $\mathbb{C}^n$ where all of the $f_i$ vanish if and only if the ideal $I$ generated by $f_1, \ldots, f_n$ contains all monomials of degree $N$ for some sufficiently large $N$.

C. Fields.

C1. (10 points) Let $F$ be a prime field, i.e., the rationals or a field with $p$ elements. Prove that an algebraic closure of $F$ has infinite degree over $F$.
   [Hint: You may want to do the two cases separately.]

C2. (15 points) Let $f \in \mathbb{Q}[x]$ be a polynomial of degree three. Let $K = \mathbb{Q}(\theta)$ be a splitting field of $f$. Determine all the possible galois groups of $K/F$, prove these are all such, and give explicit examples of $K$, i.e., determine a $\theta$ or $f$.

C3. (20 points) Prove that the polynomial $x^4 + 1$ is not irreducible over any field of positive characteristic.
D. Linear Algebra.

D1. (10 points) Let $V = M_{n \times n}(\mathbb{R})$ be the $n \times n$ real matrices. If $A, B \in V$, define $\langle A, B \rangle = \text{tr}(A^t B)$.

(a) Prove that $\langle , \rangle$ is a positive-definite symmetric inner product on $V$.

(b) If $E_{i,j}$ is the matrix with a 1 in the $i, j$ place and zeros elsewhere, prove that $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ is an orthonormal basis for $V$.

D2. (15 points) Let $k$ be a field. Prove that $\{x^i \otimes y^j \mid i, j \geq 0\}$ is a basis for the $k$-vector space $k[x] \otimes_k k[y]$. Use this to show that $k[x] \otimes_k k[y] \cong k[x, y]$ as vector spaces over $k$.

D3. (20 points) Let $SO(3) = \{A \in M_{3 \times 3}(\mathbb{R}) \mid A^t A = I \text{ and } \det(A) = 1\}$.

(a) Prove if $A \in SO(3)$ then $+1$ is an eigenvalue of $A$.

(b) Prove that if $W$ is the subspace of $\mathbb{R}^3$ orthogonal to a non-zero eigenvector of $A \in SO(3)$ with eigenvalue $+1$ then $A$ takes $W$ to $W$ and acts as rotation by some angle $\theta$ on $W$. 

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by dropping the lowest scoring problem in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

GROUP THEORY

PROBLEM 1.

a) Let $A$ be a free abelian group of rank $n$. If $H$ is a subgroup of $A$, show that $H$ is free abelian of rank $n$ if and only if $A/H$ is finite.

PROBLEM 2.

Let $G$ be a finite group of order 108. Show that $G$ has a normal subgroup of order 9 or 27.

PROBLEM 3.

Let $G$ be a finite group and $P$ a $p$-Sylow subgroup. Let $N_G(P)$ be the normalizer of $P$ in $G$. Show that:

a) $P$ is the unique $p$-Sylow subgroup of $N_G(P)$ (Do not quote a theorem that this is true!)

b) $N_G(P)$ is self normalizing in $G$
RING THEORY

PROBLEM 1.

Let $R$ be a commutative ring with 1, and let $S = R[x]$ be the polynomial ring in one variable. Suppose $M$ is a maximal ideal of $S$. Prove that $M$ cannot consist entirely of 0-divisors. Hint: you may want to distinguish the cases $x \in M$ or $x \notin M$.

PROBLEM 2.

Let $R$ be a commutative ring with 1, and suppose $I$ and $J$ are ideals of $R$ so that: $I + J = R$. Show that:

(i) $IJ = I \cap J$
(ii) $R/IJ \cong R/I \oplus R/J$.

PROBLEM 3.

Let $R$ be a commutative ring with 1, and let $S = R[x]$ be the polynomial ring in one variable. Let $f \in S$. If $f$ is a unit of $S$ (that is, $f$ is invertible in $S$), show that $f$ has the form $f = u + g$ where $u$ is a unit in $R$ and $g \in S$ is a nilpotent element without constant term.
FIELDS

PROBLEM 1.

a) Determine the minimal polynomial of \( u = \sqrt{3} + 2\sqrt{2} \) over \( Q \).
b) Determine the minimal polynomial of \( u^{-1} \) over \( Q \).

PROBLEM 2.

a) Let \( F \) be the field generated by the roots of the polynomial \( X^6 + 3 \) over \( Q \). Determine the Galois group of \( F/Q \).
b) Describe all subfields of \( F \).

PROBLEM 3.

Let \( p \) be a prime integer such that \( p \equiv 2 \) or \( 3 \) (mod 5). Prove that the polynomial

\[
1 + X + X^2 + X^3 + X^4
\]

is irreducible over \( \mathbb{Z}/p\mathbb{Z} \).
LINEAR ALGEBRA

PROBLEM 1.

Let $T$ be a linear operator on a finite-dimensional vector space $V$ such that $\text{im} T = \text{im} T^2$. Prove that $\ker T = \ker T^2$. (im and ker refer to the image and kernel of $T$)

PROBLEM 2.

Determine, up to similarity, all $3 \times 3$ matrices $A$ over $Q$ such that $A^2 + 2A^3 + A^4 = 0$ but $A + A^2 \neq 0$.

PROBLEM 3.

Let $T_1, T_2, \ldots, T_m$ be linear operators on a vector space of dimension $n$. Assume that
(i) $\dim \text{im}(T_i) = 1$ and
(ii) $T_i^2 \neq 0$ and $T_i T_j = 0$ for every $i \neq j$.

Prove that $m \leq n$. 
Algebra Qualifying Exam (Spring 2002)

Test Instructions: All problems are worth 20 points. Your total score will be computed by dropping the the lowest scoring problem. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

GROUP THEORY

PROBLEM 1.

Show that a group $G$ of order $2m$, where $m$ odd, has a normal subgroup of order $m$.

PROBLEM 2.

List, up to isomorphism, all finite abelian groups $A$ satisfying the following two conditions:

(i) $A$ is a quotient of $\mathbb{Z}^2$, and

(ii) $A$ is annihilated by 18, i.e. $18a = e$ for all $a$ in $A$.

Your list should contain a representative of each isomorphism class exactly once. How many groups are there?

PROBLEM 3.

Prove that a group $G$ of order 120 is not simple.
RING THEORY

PROBLEM 1.
Let $R$ be a ring and $A$ and $B$ be two non-isomorphic simple, left $R$-modules (a left-module is simple if it has no proper submodules, i.e., submodules other than $\{0\}$ and itself). Show that the only proper submodules of $M = A \oplus B$ are $\{(a,0) : a \in A\}$ and $\{(0,b) : b \in B\}$.

PROBLEM 2.
Let $R$ be a commutative local ring, that is, $R$ has a unique maximal ideal $M$.

(i) Show that if $x$ lies in $M$, then $1 - x$ is invertible.

(ii) Show that if $R$ is Noetherian and $I$ is an ideal satisfying $I^2 = I$, then $I = 0$. Hint: consider a minimal set of generators for $I$.

PROBLEM 3.
Let $\mathbb{F}_2$ be the field with 2 elements and let $R = \mathbb{F}_2[X]$. List, up to isomorphism, all $R$-modules of order 8.
LINEAR ALGEBRA

PROBLEM 1.
Let $\varphi : M_3(\mathbb{Q}) \rightarrow M_3(\mathbb{Q})$ be the map sending $m$ to $\varphi(m) = m^2 + 3m + 3$. Show that $\varphi(m) \neq 0$ for all $m \in M_3(\mathbb{Q})$.

PROBLEM 2.
Let $A$ be a real matrix with column vectors $A_1, A_2, ..., A_n$. If the $A_j$ are mutually orthogonal, then

$$|\det A| = \prod_{j=1}^{n} |A_j|$$

This follows because $|\det(^tA \cdot A)| = |\det A|^2$ and $^tA \cdot A$ is a diagonal matrix with diagonal entries $|A_1|^2, |A_2|^2, ..., |A_n|^2$. Prove that a general matrix satisfies the inequality

$$|\det A| \leq \prod_{j=1}^{n} |A_j|$$

Hint: apply the Gram-Schmidt orthogonalization process to the columns.

PROBLEM 3.
Let $T \in M_3(\mathbb{C})$ and let $\mathcal{A}_T$ be the centralizer of $T$ in $M_3(\mathbb{C})$. Show that $\dim(\mathcal{A}_T) \geq 3$ and describe (up to similarity) the linear transformations $T$ such that $\dim(\mathcal{A}_T) = 3$. 
GALOIS THEORY

PROBLEM 1.
Let $\mathbb{F}_7$ be the field with 7 elements and let $L$ be the splitting field of the polynomial $X^{171} - 1 = 0$ over $\mathbb{F}_7$. Determine the degree of $L$ over $\mathbb{F}_7$, explaining carefully the principles underlying your computation.

PROBLEM 2.
Show that there exists a Galois extension of $\mathbb{Q}$ of degree $p$ for each prime $p$. State precisely all results which are needed to justify your answer.

PROBLEM 3.
Let $\alpha = \sqrt{i + 2}$ where $i = \sqrt{-1}$.

(a) Compute the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

(b) Let $F$ be the splitting field and compute the degree of $F$ over $\mathbb{Q}$;

(c) Show that $F$ contains 3 quadratic extensions of $\mathbb{Q}$;

(d) Use this information to determine the Galois group.
Algebra Qualifying Exam
Winter 2002

Everyone must do two problems in each of the four sections. If three problems of a section are tried, only two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups
A1. Let \( G \) be a free abelian group of rank \( n \) for a positive integer \( n \) (therefore \( G \cong \mathbb{Z}^n \) as groups).
   (a) Prove for a given integer \( m > 1 \), there are only finitely many subgroups \( H \) of index \( m \) in \( G \);
   (b) Find a formula of the number of subgroups of \( G \) of index 3. Justify your answer.

A2. Prove or disprove: there exists a finite abelian group \( G \) whose automorphism group has order 3.

A3. Let \( S \) and \( G \) be \( p \)-groups (with \( G \neq \{e\} \)), and assume that \( S \) acts on \( G \) by automorphisms. Show that the fixed subgroup \( G^S = \{g \in G|s(g) = g \text{ for all } s \in S\} \) is non-trivial (i.e., is not the trivial subgroup \( \{e\} \)).

Rings
B1. Let \( F \) be a field and \( A \) be a commutative \( F \)-algebra. Suppose \( A \) is of finite dimension as a vector space of \( F \).
   (a) Prove all prime ideals of \( A \) are maximal. Hint: consider maps \( R/P \to R/P \) (\( P \) prime) of the form \( x \to ax \) with \( a \in R \).
   (b) Prove that there are only finitely many maximal ideals of \( A \).

B2. Let \( A = M_n(F) \) be the ring of \( n \times n \) matrices with entries in an infinite field \( F \) for \( n > 1 \). Prove the following facts:
   (a) There are only 2 two-sided ideals of \( A \);
   (b) There are infinitely many maximal left ideals of \( A \). Hint: show that \( Ax = Ay \) (\( x, y \in A \)) if and only if \( \text{Ker}(x) = \text{Ker}(y) \).

B3. Let \( \mathbb{F}_2 \) be the field with 2 elements and \( A = \mathbb{F}_2[T, \frac{1}{T}] \) for an indeterminate \( T \). Prove the following facts:
   (a) The group of units in \( A \) is generated by \( T \).
   (b) There are infinitely many distinct ring endomorphisms of \( A \).
   (c) The ring automorphism group \( \text{Aut}(A) \) is of order 2.
Fields

C1. The discriminant of the special cubic polynomial \( f(x) = x^3 + ax + b \) is given by \(-4a^3 - 27b^2\). Determine the Galois group of the splitting field of \( x^3 - x + 1 \) over
(a) \( \mathbb{F}_3 \), the field with 3 elements.
(b) \( \mathbb{F}_5 \), the field with 5 elements.
(c) \( \mathbb{Q} \), the rational numbers.

C2. A field extension \( K/\mathbb{Q} \) is called \textit{biquadratic} if it has degree 4 and if \( K = \mathbb{Q}(\sqrt{a}, \sqrt{b}) \) for some \( a, b \in \mathbb{Q} \).
(a) Show that a biquadratic extension is normal with Galois group \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and list all sub-extensions.
(b) Prove that if \( K/\mathbb{Q} \) is a normal extension of degree 4 with \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) then \( K/\mathbb{Q} \) is biquadratic.

C3. Let \( K \) be a finite extension of the field \( F \) with no proper intermediate subfields.
(a) If \( K/F \) is normal, show that the degree \([K; F]\) is a prime.
(b) Give an example to show that \([K; F]\) need not be prime if \( K/F \) is not normal, and justify your answer.

Linear Algebra

D1. Let \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) where \( I \) is the \( n \times n \) identity matrix. Suppose that \( S \) is a \( 2n \times 2n \) symplectic matrix, meaning that \( S \) is real and satisfies \( ^tSJS = J \), where \( ^tS \) is the transpose of \( S \).
(a) Show that \( ^tS \) is symplectic.
(b) Show that \( S \) is similar to \( S^{-1} \).
(c) It is always true that \( \det S = 1 \). Prove this in case \( n = 1 \).

D2. Suppose that \( A \) is a linear operator on the vector space \( \mathbb{C}^n \) and that \( v \in \mathbb{C}^n \) satisfies \((A - aI)^2v = 0 \) for some \( a \in \mathbb{C} \), so that \( v \) is a \textit{generalized} eigenvector of \( A \) with eigenvalue \( a \). Suppose that \( |a| < 1 \). Show that
\[
\| A^m v \| \to 0
\]
as \( m \to \infty \), where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{C}^n \).

D3. Let the \( n \times n \) matrix \( A \) be defined over the field \( F \). Suppose that \( A \) has finite order:
\[
A^m = I
\]
for some positive integer \( m \).
(a) If the characteristic of \( F \) is 0, show that \( A \) may be diagonalized over \( F \).
(b) Show that the conclusion of (a) is not true for an arbitrary field \( F \).
Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by dropping the lowest scoring problem in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

GROUP THEORY

PROBLEM 1.

Let $G$ be a finitely generated group, and $n > 1$ an integer. Show that $G$ has at most a finite number of subgroups of index $n$.

PROBLEM 2.

Let $G$ be a finite group, $K$ a normal subgroup, and $P$ a $p$-Sylow subgroup of $K$ for some prime $p$. Prove that $G = KN_G(P)$.

PROBLEM 3.

Suppose $G$ is the free abelian group on generators $x, y, z, w$, considered as an additive group. Let $a = x - z + 2w, b = x - y + w, c = 3x - y - 2z + 5w, d = 2x - 2y + 4w$. If $H = \langle a, b, c, d \rangle$, determine the structure of $G/H$. 
RING THEORY

PROBLEM 1.

a) Let $R$ be a commutative ring with 1. Suppose $f \in R[x]$ is a non-zero 0-divisor in the polynomial ring $R[x]$. Assume that $R$ has no non-zero nilpotent elements. Show there is a non-zero element $a \in R$ so that $a \cdot f = 0$.

b) Give an example of an $R$ and $f$ so that all coefficients of $f$ are 0-divisors in $R$, but $f$ is not a 0-divisor in $R[x]$.

PROBLEM 2.

Let $R$ be a ring, not necessarily commutative, and $M$ a Noetherian left $R$-module. Suppose $f : M \to M$ is a surjective $R$-module map from $M$ to $M$. Prove that $f$ is an isomorphism.

PROBLEM 3.

a) Let $R$ be a commutative ring with 1, and $S$ a multiplicatively closed subset of $R$ not containing 0. Suppose $I$ is an ideal of $R$ maximal with respect to exclusion of $S$ (i.e. $I \cap S$ is empty and $I$ is largest such). Prove that $I$ is a prime ideal of $R$.

b) Show that every prime ideal of $R$ arises as in part a).
FIELDS

PROBLEM 1.

Determine the Galois group of the polynomial \( X^4 + 3X^2 + 1 \) over \( Q \).

PROBLEM 2.

Let \( f(X) \) be a polynomial of degree \( n > 0 \) over a field \( F \).

a) Prove that there is a field homomorphism \( \alpha : F(X) \to F(X) \) such that \( \alpha(X) = f(X) \).

b) Let \( L \) be the image of \( \alpha \). Prove that the field extension \( F(X)/L \) is finite and find its degree.

c) Find the minimal polynomial of \( X \) over \( L \).

PROBLEM 3.

Let \( p \) be a prime integer. Suppose that the degree of every finite extension of a field \( F \) is divisible by \( p \). Prove that the degree of every finite extension of \( F \) is a power of \( p \).
PROBLEM 1.

Let $A$ be a linear operator in a finite dimensional vector space. Prove that if $A^2 = A$ then $\text{Trace}(A) = \text{Rank}(A)$.

PROBLEM 2.

Let $L/F$ be a field extension and let $A$ and $B$ be $n \times n$ matrices over $F$. Prove that if $A$ and $B$ are conjugate in $M_n(L)$, then $A$ and $B$ are conjugate in $M_n(F)$.

PROBLEM 3.

Let $V$ be a finite dimensional vector space over a field $F$. Let $B$ be a non-degenerate bilinear form on $V$. (For every nonzero $v \in V$ there is $u \in V$ such that $B(u, v) \neq 0$.)

a) For every $v \in V$ define a linear form $l_v : V \to F$ by $l_v(u) = B(u, v)$. Prove that the map $V \to V^*$ given by $v \mapsto l_v$ is an isomorphism of vector spaces.

b) Prove that for every linear operator $a$ of $V$ there is a linear operator $a^*$ such that $B(a(u), v) = B(u, a^*(v))$ for all $u, v \in V$.

c) Prove that $(ab)^* = b^*a^*$ for every two linear operators $a$ and $b$.

d) Suppose that $B$ is either symmetric or skew-symmetric (that is $B(u, v) = B(v, u)$ or $B(u, v) = -B(v, u)$ respectively). Prove that $a^{**} = a$ for a linear operator $a$. 
Algebra Qualifying Exam
Winter 2003

Everyone must do two problems in each of the four sections. If three problems of a section are tried, only two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

A1. List, up to isomorphism, all abelian groups \( A \) which satisfy the following three conditions:
   (i) \( A \) has 108 elements;
   (ii) \( A \) has an element of order 9;
   (iii) \( A \) has no element of order 24.

A2. Let \( N \geq 1 \) be a positive integer. Show that a finitely generated group \( G \) has only finitely many subgroups of index at most \( N \).

A3. Let \( N \geq 2 \) be an integer. Show that a subgroup of index 2 in \( S_N \) is \( A_N \). Here \( S_N \) and \( A_N \) are the symmetric and alternating groups for \( N \), respectively.
Rings

B1. Give an example of two integral domains $A$ and $B$ which contain a field $F$ such that $A \otimes_F B$ is not an integral domain. Justify your answer. Hint: Take $A$ to be the field of rational functions $\mathbb{F}_p(X)$ for the field $\mathbb{F}_p$ with $p$ elements.

B2. Let $\mathbb{F}_q$ be the finite field of $q$ elements, and put $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$. Write $\sigma : K \to K$ for the field automorphism given by $x^\sigma = x^q$. Let

$$B = \left\{ \begin{pmatrix} a & b \\ db^\sigma & a^\sigma \end{pmatrix} \mid a, b \in K \right\}$$

for a given $d \in F^\times$. Prove the following three facts:

(a) $B$ is a subalgebra of dimension 4 over $F$ inside the $F$-algebra of $2 \times 2$ matrices over $K$.

(b) $B$ is a division algebra if and only if there exists no $c \in K$ such that $d = cc^\sigma$.

(c) $B$ cannot be a division algebra.

B3. Let $A$ be a discrete valuation ring with maximal ideal $M$, and define

$$B = \left\{ (a, b) \in A \times A \mid a \equiv b \mod M \right\}.$$ 

Prove the following facts:

(a) $B$ has only one maximal ideal;

(b) $B$ has exactly two non-maximal prime ideals.
Fields

C1. Let $\mathbb{F}_q$ be the finite field of $q$ elements. Answer the following questions:
   (a) List all subfields of $\mathbb{F}_{p^d}$ for a prime $p$. Justify your answer.
   (b) Find a formula for the number of monic irreducible polynomials of degree 6 in $\mathbb{F}_p[X]$. Justify your answer.

C2. Let $K/F$ be a quadratic extension of fields and $M/F$ be a Galois extension over $F$ containing $K$ such that $\text{Gal}(M/K)$ is a cyclic group of odd prime order $p$. Answer the following two questions:
   (a) Determine the possible groups $\text{Gal}(M/F)$ up to isomorphisms, and justify your answer.
   (b) Find the number of intermediate fields $L$ between $F$ and $M$ with $[L : F] = p$. Justify your answer.

C3. Find the degree of the splitting field $E$ of $X^6 - 3$ over the following fields:
   (a) $\mathbb{Q}[\sqrt{-3}]$ ($\mathbb{Q}$: the field of rational numbers);
   (b) $\mathbb{F}_7$, the field with 7 elements;
   (c) $\mathbb{F}_5$, the field with 5 elements,
   and justify your answer.
Linear Algebra

D1. Let $L$ be the subgroup of $\mathbb{Z}^3$ with generators $(3, 2, 1)$ and $(2, 2, 6)$. Represent the quotient group $A = \mathbb{Z}^3/L$ as a product of cyclic groups.

D2. List suitably the Jordan canonical forms of all matrices $A$ that satisfy
   (i) $tr(A) = 1$;
   (ii) $tr(A)^2 = 2\det(A) + tr(A^2)$.

D3. Let $M$ be a matrix with complex entries. Deduce, using the structure theory of modules, that:
   (i) $M = S + N$, where $S$ is semisimple (i.e. diagonalizable), $N$ is nilpotent, and $N$ and $S$ commute.
   (ii) $N = P(M)$ where $P(X)$ is a polynomial with complex coefficients.
1. Groups

(a) Is there a finite group $G$ such that $G/Z(G)$ has 143 elements? ($Z(G)$ is the center of $G$.)

(b) Prove that every group of order 30 has a subgroup of order 15.

(c) Find all finite groups that have exactly three conjugacy classes.

2. Rings

(a) Let $X$ be a finite set and let $A$ be the ring of all functions from $X$ to the field $R$ of real numbers. Prove that an ideal $M$ of $A$ is maximal if and only if there is an element $x \in X$ such that

$$M = \{ f \in A | f(x) = 0 \}.$$ 

(b) Describe all $n \in \mathbb{Z}$ such that the ring $\mathbb{Z}/n\mathbb{Z}$ has no idempotents other than 0 and 1.

(c) A (non-commutative) ring $R$ is called local if for every $a \in R$ either $a$ or $1 - a$ is invertible. Prove that non-invertible elements of a local ring form a (two-sided) ideal.

3. Linear Algebra

(a) Determine whether it is true in general that in $GL_n(\mathbb{C})$ every matrix is conjugate to its transpose.

(b) Determine the number of conjugacy classes in $GL_3(\mathbb{C})$ whose elements $A$ satisfy the polynomial $X^2 - 2X + 1 = 0$.

(c) Let $A$ be an $n$-by-$n$ symmetric matrix with real entries and let, for $j \in [1,,n]$, $A_j$ be the submatrix consisting of the entries of $A$ in the first $j$ rows and columns of $A$. Show $A$ is positive definite iff $det(A_j) > 0$ for all $j \in [1,,n]$.

4. Fields

(a) Let $a$ be an integer and let $p$ be a prime. Show that if $a$ is not a $p$-th power, then $X^p - a$ is irreducible over $\mathbb{Q}$. 

(b) Show that if $K$ and $L$ are finite separable extensions of $F$ with $K$ Galois over $F$, such that $K \cap L = F$, then $[KL:F]=[L:F][K:F]$. Show that if neither $K$ nor $L$ are Galois over $F$, then this fact need not be true.

(c) By using several quadratic extensions of the rational function field in two variables $F = F_2(X, Y)$ where $F_2$ is the field with 2 elements, give an example of a field extension of finite degree of $F$ that possesses infinitely many intermediate fields.
Algebra Qualifying Exam (Spring 2004)

Test Instructions: Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

GROUP THEORY

PROBLEM 1.

A group $G$ is said to act transitively on a set $S$ if for any element $s \in S$, then

$$S = Gs.$$ 

Suppose $G$ is finite and that $G$ acts transitively on $S$. Let $f(g)$ be the number of elements of $S$ fixed by the action of $g \in G$ on $S$. Prove

$$|G| = \sum_{g \in G} f(g).$$

PROBLEM 2.

Classify all groups of order $2 \cdot 7 \cdot 11$.

PROBLEM 3.

Let $G$ be a finite group and $H$ a subgroup of $G$. Let $n = (G : H)$ be the index of $H$ in $G$.

(a) Show that

$$(G : \cap_{x \in G} xHx^{-1})$$

is a factor of $n!$.

(b) Suppose that the index $(G : H)$ is the minimal prime factor of the order of $G$. Show $H$ is a normal subgroup.
RING THEORY

PROBLEM 1.

Let $R$ be a commutative noetherian ring with unity $1$ and $f : R \rightarrow R$ a surjective ring homomorphism, i.e. $f(R) = R$. Show $f$ is an isomorphism.

PROBLEM 2.

Let $R$ be the ring $\mathbb{Q}[x]$ and let $M$ be the submodule of $R^2$ generated by the elements $(1 - 2x, -x^2)$ and $(1 - x, x - x^2)$. According to the theory of modules over principal ideal domains, $R^2/M$ is a direct sum of cyclic $R$ modules of the form $R/P(x)$ for monic polynomials $P(x)$. Find such a direct sum decomposition explicitly in this case.

PROBLEM 3.

Suppose we are given a collection of polynomials in $r$ variables with rational coefficients:

$$f_1, \ldots, f_N \in \mathbb{Q}[T_1, \ldots, T_r].$$

We define the complex algebraic set $V_\mathbb{C} \subset \mathbb{C}^r$ by

$$V_\mathbb{C} = \{(a_1, \ldots, a_r) \mid f_i(a_1, \ldots, a_r) = 0 \text{ for all } i \text{ from } 1 \text{ to } N\}.$$ 

Suppose $V_\mathbb{C}$ is not empty. Show that there is a finite extension $K$ of $\mathbb{Q}$ and a point

$$(a_1, \ldots, a_r) \in V_\mathbb{C}$$

with all $a_k \in K$. 

2
LINEAR ALGEBRA

PROBLEM 1.

(a) For which \( z \in \mathbb{C} \) is
\[
\begin{pmatrix}
1 & 2z \\
z - 1 & 1
\end{pmatrix}
\]
not similar over \( \mathbb{C} \) to a diagonal matrix? Justify your answer.

(b) Let \( J_n \) be the \( n \times n \) matrix each of whose entries is 1. Determine those \( n \in \mathbb{Z}^+ \) for which \( J_n \) is diagonalizable over \( \mathbb{C} \) and give a diagonal matrix that is similar to \( J_n \) for such \( n \).

PROBLEM 2.

Find an explicit formula for the determinant of a \( 3 \times 3 \) complex matrix \( A \) as a polynomial in the traces \( t_n = \text{Tr}(A^n) \) for \( n = 1, 2, \ldots \).

PROBLEM 3.

Let \( V \) be a vector space over \( \mathbb{C} \) of dimension \( d > 0 \). Suppose that \( A, B, C \) are linear operators on \( V \) such that
\[
AB - BA = C.
\]
Suppose also that \( C \) commutes with both \( A \) and \( B \). If \( V \) has no proper non-zero subspace that is left stable under all three operators, show that \( d = 1 \).
FIELD THEORY

PROBLEM 1.
Let $K$ be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(x) = x^6 + 3$.

(a) Show that $K$ contains a primitive 6-th root of unity.

(b) Show that $K$ is a Galois extension of $\mathbb{Q}$.

(c) Determine the number of fields $F$ of degree 3 over $\mathbb{Q}$ with $F \subseteq K$.

PROBLEM 2.
Suppose that $f(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree $d > 1$ with $d$ roots $x_1, \ldots, x_d$ in $\mathbb{C}$. If $x_2 = ax_1$ for $a \in \mathbb{Q}$ different from $-1$, prove that $f(x)$ is reducible.

PROBLEM 3.
Let $K$ be a field and $L$ a finite extension of $K$. Consider the set $A$ of all elements $x \in L$ with the property that $K[x]$ is a Galois extension of $K$ with an abelian Galois group $\text{Gal}(K[x]/K)$. Show that $A$ is a subfield of $L$ containing $K$.
Algebra Qualifying Exam  
Fall 2005

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by taking the two best scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

(1) Groups
(a) Let $G$ be an abelian group generated by $n$ elements. Prove that every subgroup of $G$ can also be generated by $n$ elements.
(b) Let $N$ be a normal subgroup of $G$. Prove that for a Sylow $p$-subgroup $P$ of $G$, the intersection $P \cap N$ is a Sylow $p$-subgroup of $N$.
(c) Is there a nontrivial action of the alternating group $A_4$ on a set of two elements?

(2) Rings
(a) Let $I$ and $J$ be ideals of a commutative ring $R$ with unit such that $I + J = R$. Prove that $I \cdot J = I \cap J$.
(b) Prove that the factor ring $R[x,y]/(y^2-x^3)R[x,y]$ is not a P.I.D.
(c) Let $x, y, z, t$ be elements of a (non-commutative) ring $R$ such that $xz = yt = 1$, $xt = yz = 0$ and $zx + ty = 1$. Prove that the left $R$-modules $R$ and $R \oplus R$ are isomorphic.

(3) Linear Algebra
(a) Prove that if three distinct real numbers $\lambda_i$ and three arbitrary numbers $\mu_j$ are given, then there exists a unique polynomial $f(x) \in \mathbb{R}[x]$ of degree at most 2 such that $f(\lambda_i) = \mu_i$.
(b) Let $K/F$ be a field extension of finite degree $n$ and assume $K = F(\alpha)$ where $\alpha$ satisfies a polynomial $f(x)$ of degree $n$ in $F[x]$. Let $\varphi : K \rightarrow K$ be the $F$-linear map $\varphi(x) = \alpha x$. Show that the eigenvalues of $\varphi$ coincide with the roots of $f(x)$.
(c) A bilinear form $A(x,y)$ on a vector space $V$ over $\mathbb{C}$ is called alternating if $A(v,w) = -A(w,v)$ for all $v, w \in V$ and it is called non-degenerate if, for each nonzero $v \in V$ there exists $w \in V$ such that $A(v,w) \neq 0$.
   (i) Prove that any two non-degenerate alternating forms on $V = \mathbb{C}^2$ differ by a scalar multiple.
   (ii) Does this remain true for $V = \mathbb{C}^n$ for $n > 2$? Either prove true or give a counterexample.
4) Fields

(a) Let $F_q$ be the finite field with $q = p^n$ elements and let $N : F_q \rightarrow F_p$ be the norm map, defined by

$$Nx = \prod \sigma(x)$$

where $\sigma$ runs over the Galois group $G = \text{Gal}(F_q/F_p)$. Prove that $N$ is surjective.

(b) Let $\varphi(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an irreducible polynomial of degree 4 in $\mathbb{Q}[x]$ and let $K$ be the field generated by the complex roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $\varphi$. Let $F$ be the field generated by:

$$\beta_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4),$$
$$\beta_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4),$$
$$\beta_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3).$$

Prove that $K/F$ is an abelian extension, that is, the Galois group $H = \text{Gal}(K/F)$ is abelian. Hint: prove that the $\beta_i$ are distinct and determine the possible elements of $\text{Gal}(K/\mathbb{Q})$ that fix them.

(c) Let $K$ be the splitting field of $\varphi(x) = x^{11} - 7$ over $\mathbb{Q}$. Describe the Galois group $G = \text{Gal}(K/\mathbb{Q})$ by giving generators and relations. Determine the number of quadratic subfields of $K$ (a quadratic subfield is a subfield $E \subset K$ such that $[E : \mathbb{Q}] = 2$).
Algebra Qualifying Exam
Winter 2005

Test Instructions: Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest is ignored). For multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups
A1. (a) If $G$ is a simple group that has subgroup of index $n$, prove that the order of $G$ is a factor of $n!$.
   (b) Prove that there is no simple nonabelian group of order $p^e m$ with $e > 0$ for a prime $p > m$.
A2. An additive abelian group is called divisible if multiplication by $n$ for every positive integer $n$ is a surjective endomorphism.
   (a) Show that if $G$ is divisible, $G/H$ is divisible for any subgroup $H$ of $G$.
   (b) Give an example (with a proof) of a divisible group for which the multiplication by $n$ is not an automorphism for every positive integer $n$.
   (c) Prove or disprove that there is only one isomorphism class of finitely generated divisible groups.
A3. Let $G$ be a free abelian group of finite rank $r$.
   (a) Show that there are only finitely many homomorphisms of $G$ into $\mathbb{Z}/n\mathbb{Z}$ for each positive integer $n$.
   (b) Find a formula of the number of surjective homomorphisms of $G$ onto $\mathbb{Z}/p\mathbb{Z}$ for a prime $p$ if $r = 2$.

Linear Algebra
B1. An $n \times n$ real symmetric matrix $P$ is positive definite if the inner product $P(x, y) = ^t xPy$ is positive definite (that is, $P(x, x) > 0$ for all $0 \neq x \in \mathbb{R}^n$). Let $S$ be an $n \times n$ invertible real symmetric matrix. Let $W \subset \mathbb{R}^n$ be a subspace such that the inner product $S(x, y) = ^t xSy$ is positive definite on $W$ but $S$ is not positive definite on $W + \mathbb{R}x$ for any $x \notin W$.
   (a) Show that $\mathbb{R}^n = W \oplus W^\perp$ for $W^\perp = \{ x \in \mathbb{R}^n | S(x, W) = 0 \}$.
   (b) For each $x \in \mathbb{R}^n$, writing $x = x_W \oplus x_W^\perp$ for $x_W \in W$ and $x_W^\perp \in W^\perp$, define $P(x, y) = S(x_W, y_W) - S(x_W^\perp, y_W^\perp)$. Show that $PS^{-1} = SP^{-1}$ and $P$ is positive definite.
(c) If $P$ is symmetric positive definite and satisfies $PS^{-1} = SP^{-1}$, there exists a subspace $W$ such that $P = S$ on $W$ and $P = -S$ on $W^\perp$.

B2. Let $V$ be a two dimensional vector space over a field $F$. Let $T : V \to V$ be a linear transformation of finite order $m$. Prove the following facts:
(a) If $F = \mathbb{Q}$, then $m \leq 6$.
(b) For any given positive integer $N$, there exists a finite field $F$ and a nondiagonalizable $T$ of order greater than $N$.

B3. Let $V$ be a finite dimensional vector space over a field $F$ and $T : V \to V$ be a linear transformation. Let $v \in V$ be a non-zero vector in $V$. Prove the following facts:
(a) There exists a monic polynomial $P(X)$ in $F[X]$ such that $P(T)v = 0$.
(b) Among monic polynomials $P(X) \in F[X]$ with $P(T)v = 0$, there exists a unique polynomial $P_0(X)$ of minimal degree.
(c) If $P(T)v = 0$, then $P_0(X)$ is a factor of $P(X)$ in $F[X]$.

Rings

C1. Let $R$ be an integral domain. If $m$ is a maximal ideal in $R$, view the localization $R_m := S^{-1}R$, with $S = R \setminus m$, in the quotient field of $R$. Show that
\[
R = \bigcap_{m \in \text{Max}(R)} R_m.
\]

C2. Let $R$ be a commutative Artinian ring. Show that there are only finitely many prime ideals in $R$ and every one of them is maximal.

C3. Let $R \subseteq A \subseteq B$ be commutative rings. Suppose that $R$ is noetherian and $B$ is a finitely generated $R$-algebra. Suppose that as an $A$-module $B$ is finitely generated. Show that $A$ is a finitely generated $R$-algebra.

Fields

D1. Show that the identity map is the only field automorphism of the real numbers. Show this is not true of the complex numbers.

D2. Let $F$ be a field of positive characteristic $p$ and $f$ the polynomial $x^p - x - a \in F[x]$. Let $K/F$ be a splitting field of $f$. Show that $K/F$ is galois and determine explicitly (with proof) the Galois group of $K/F$.

D3. Let $K/F$ be a finite extension of finite fields. Prove that the norm map $N_{K/F} : K \to F$ is surjective.
Algebra Qualifying Exam
Fall 2006

Test Instructions: Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest is ignored). For multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

G1. List all finite groups $G$ whose automorphism group has prime order. Justify your answer.

G2. Let $G$ be a finite group and $H$ be a non-normal subgroup of $G$ of index $n > 1$.
   (a) Show that if $|H|$ is divisible by a prime $p \geq n$, then $H$ cannot be a simple group;
   (b) Show that there is no simple group of order $504 = 2^3 \cdot 3^2 \cdot 7$.
      (Hint: Choose a good prime $\ell$, and let $G$ act on the set of Sylow $\ell$-subgroups getting an embedding of $G$ into a permutation group, and apply (a). You may use the fact that the alternating group $A_n$ is simple if $n \geq 5$.)

G3. Let $SL_2(\mathbb{Z}/p\mathbb{Z}) = \{(a \ b \ c \ d) \mid a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, \ ad - bc = 1\}$, where $p$ is an odd prime.
   (a) Prove that any subgroup of a cyclic group is cyclic.
   (b) Compute the order of $G$.
   (c) Prove that for any odd prime $\ell$, the Sylow $\ell$-subgroup of $G$ is cyclic. (Hint: You may use the fact that the multiplicative group $F^\times$ of a finite field $F$ is cyclic).

Rings

R1. Determine all prime ideals in the polynomial ring $\mathbb{Z}[x]$. Justify your work.

R2. Let $R$ be a noetherian domain. A non-zero element $x$ in $R$ is called a prime element if $(x)$ is a prime ideal. Prove all of the following:
   (a) Every nonzero non-unit in $R$ is a product of irreducible elements.
   (b) Every nonzero ideal $I \neq R$ in $R$ contains a (finite) product of non-zero prime ideals.
(c) If every nonzero prime ideal in $R$ contains a prime element
then every irreducible element in $R$ is a prime element.

[You may not use theorems about UFD's]

R3. Let $R$ be a commutative ring and $M$ a finitely generated $R$-
module. Suppose there exists a positive integer $n$ and a surjec-
tive $R$-module homomorphism $\varphi : M \to R^n$. Show that $\ker \varphi$
is also a finitely generated $R$-module.

Fields

F1. Let $F$ be a finite field of positive characteristic $p$. Show that
the unit group $F \setminus \{0\}$ of $F$ is a cyclic group and that $F$ is a
Galois extension of $\mathbb{Z}/p\mathbb{Z}$.

F2. Let $f(x)$ be the polynomial $x^6 + 3$ over $\mathbb{Q}$ (the field of rational
numbers). Determine the Galois group of $f(x)$, i.e., the Galois
group of $K/\mathbb{Q}$ where $K$ is a splitting field of $f(x)$.

F3. Let $f(x)$ be an irreducible polynomial over $F$ and $K/F$ a normal
extension. Show that $f(x)$ factors into irreducible polynomials
over $K$ all of the same degree.

Linear Algebra

L1. Let $V$ be an $n$-dimensional vector space over a field $F$ (for
finite $n$) and $\sigma : F \to F$ be a field homomorphism (sending
1 to 1). We regard $V \otimes_{F,\sigma} F$ as an $F$-vector space via the $F$-
multiplication given by $a(v \otimes \alpha) = (av) \otimes \alpha$ for $a, \alpha \in F$ and
$v \in V$.

(a) Compute the formula of $\dim_F(V \otimes_{F,\sigma} F)$ if $F$ is a finite
extension of a field $k$ fixed by $\sigma$, where $V \otimes_{F,\sigma} F$ is the
tensor product over $F$ regarding $F$ as $F$-module via $\sigma$.

(b) Let $p$ be a prime. If $F = F_p(x_1, \ldots, x_m)$ ($F_p = \mathbb{Z}/p\mathbb{Z}$)
and $\sigma(\phi) = \phi^p$ for all $\phi \in F$, compute the formula of
$\dim_F(V \otimes_{F,\sigma} F)$. Here $F_p(x_1, \ldots, x_m)$ is the field of fractions
of the polynomial ring $F_p[x_1, \ldots, x_m]$ of $m$ variables.

(c) Give an example of a field $F$ of characteristic 3 and a homo-
morphism $\sigma : F \to F$ such that $\dim_F V \otimes_{F,\sigma} F = 2 \dim_F V$,
and justify your example.

L2. Let $V$ be a finite dimensional vector space over the rational
number field $\mathbb{Q}$ and $I : V \times V \to \mathbb{Q}$ be an alternating bilinear
map (that is, $I(x,y) = -I(y,x)$ for all $x,y \in V$). We call $I$
degenerate if there exists nonzero $x \in V$ such that $I(x,V) = 0$,
and $I$ is non-degenerate if $I$ is not degenerate.

(a) Show that if $V$ is two dimensional and $I$ is non-degenerate
then there exists a basis $x, y$ such that $I(x,y) = 1.$
(b) Show that $I$ is degenerate if $\dim_{\mathbb{Q}} V$ is odd.

(c) If $\dim_{\mathbb{Q}} V = 2m$ is even and $I$ and $I'$ are two nondegenerate alternating forms on $V$, show that there exists an invertible linear transformation $T : V \to V$ such that $I'(x, y) = I(Tx, Ty)$ for all $x, y \in V$.

L3. Let $T : V \to V$ be a linear transformation on an $n$-dimensional vector space $V$ over $\mathbb{C}$ with $n \geq 1$. Suppose that $T^n = 0$ but $T^{n-1} \neq 0$.

(a) Compute $\dim \ker(T^{101})$.

(b) If $S : V \to V$ is a linear transformation with $ST = TS$ and $\dim S(V) = n - 2$, compute $\dim S^{101}(V)$. 
Algebra Qualifying Exam
Spring 2006

Test instructions: All problems are worth 20 points. You are expected to do two problems from each of the four sections. Your total score will be computed by taking the two best scoring problems from each section. In problems where an argument must be given, you will lose points if you fail to state clearly the basic results you use.

Groups

G1. Let $G$ be a finite group. Let $K$ be a normal subgroup of $G$ and $P$ a $p$-Sylow subgroup of $K$. Show that

$$G = KN_G(P).$$

G2.

(a) What is the order of $SL_2(F_4)$?

(b) Show there is an isomorphism from $SL_2(F_4)$ to $A_5$.

Hint: Consider the action of $SL_2(F_4)$ on the set of one dimensional subspaces of the vector space $(F_4)^2$ of dimension two over $F_4$.

G3. Let $G$ be a group of order 2000 and suppose that $P$ and $P'$ are two distinct Sylow 5 subgroups of $G$. Let

$$I = P \cap P'.$$

(a) Prove that $|I| = 25$.

(b) Show that the index of $N_G(I)$ is at most 2.
Rings

R1. Suppose $D$ is an integral domain and suppose that $D[x]$ is a principal ideal domain. Show $D$ is a field.

R2. Let $R$ be a commutative Noetherian ring with unit, and suppose $M$ is a finitely generated $R$ module. Suppose $f : M \to M$ is an $R$ module homomorphism which is onto. Show that $f$ is an isomorphism.

R3. Let $R$ be a commutative ring with unit and $m$ a maximal ideal of $R$.

(a) Suppose $I_1 \cdots I_n$ are ideals of $R$ and that

$$m \supseteq I_1 \cdots I_n,$$

where $I_1 \cdots I_n$ is the product of the ideals. Show

$$m \supseteq I_k$$

for some $k$.

(b) Suppose that $R$ satisfies the descending chain condition (dcc) on ideals, i.e. every strictly decreasing sequence of ideals is finite. Show $R$ has only a finite number of maximal ideals. You may use part (a), but not theorems on the structure of rings satisfying the dcc.
Fields

F1.

(a) Show that the Galois group of the splitting field of

\[ X^4 - 2 \]

over \( \mathbb{Q} \) has order 8.

(b) Is this Galois group isomorphic to the dihedral group, the quaternion group or one of the three abelian groups of order 8?

F2. Let \( F \) be a finite field.

(a) Show that more than half the elements of \( F \) are squares.

(b) Show that every element of \( F \) is the sum of two squares.

F3. Let \( K \) be a finite extension of the field \( F \) with no proper intermediate fields.

(a) If \( K/F \) is normal, show \([K : F]\) must be prime.

(b) Give an example to show that \([K : F]\) need not be prime if \( K/F \) is not normal, explaining why your example works.
Linear Algebra

L1. Let $A$ be a $2 \times 2$ complex matrix and let $W_A$ be the space of all $2 \times 2$ matrices that commute with $A$.

(a) What is the minimal possible dimension of $W_A$ as $A$ varies over all $2 \times 2$ complex matrices?
(b) Classify those $A$ such that $W_A$ has minimal dimension.

L2. Let $A$ be a $3 \times 3$ matrix over a field $F$ that satisfies

$$A^4 = I \quad \text{and} \quad A^2 \neq I,$$

where $I$ is the identity matrix. Find all similarity classes of such $A$ when

(a) $F = \mathbb{Q}$
(b) $F$ is the field of two elements.

L3. Let $T_1, T_2, \ldots, T_n$ be linear operators on a vector space of dimension $m$ over a field $F$. Assume that

(a) $\dim \text{im}(T_i) = 1$ for each $i$ and
(b) $T_i^2 \neq 0$ and $T_i T_j = 0$ for $i \neq j$.

Show

$$n \leq m$$
Algebra Qualifying Exam
Fall 2007

Test Instructions: Each problem is worth 20 points. You must attempt to do at least two problems in each of the four sections. Your total score will be computed using only the two best scoring problems in each section. Whether you pass or fail depends on your performance in each section, not only on the total score. In each problem you may lose points if you do not explain clearly your reasoning or any theorems which you quote.

Groups

G1. Let $F$ be a finite field of characteristic $p$ and let $G$ be a subgroup of order $p^n$ of the group $GL(N, F)$ of invertible $N$ by $N$ matrices with entries in $F$. Show that there is a non-zero vector $v$ in $F^N$ such that $gv = v$ for every $g \in G$.

G2. Let $M$ be the submodule of $Z^3$ generated by elements $(0, 3, 2)$, $(6, 48, 24)$ and $(6, 24, 12)$. Describe the quotient group $Z^3/M$ by giving a product of cyclic groups to which $Z^3/M$ is isomorphic.

G3. A group $G$ acts doubly transitively on a set $X$ if for each pair $(x_1, x_2)$ and $(y_1, y_2)$, with $x_1 \neq x_2$ and $y_1 \neq y_2$ of pairs of points of $X$, there exists a $g \in G$ such that $gx_1 = y_1$ and $gx_2 = y_2$. Show that if a finite $G$ acts non-trivially and doubly transitively on $X$, then the stabilizer $S_x$ of any point $x$ in $X$ is a maximal proper subgroup of $G$. (Here a subgroup $M$ of $G$ is maximal proper if $M$ is not equal to $G$ and if, for any subgroup $H$ of $G$ which contains $M$, either $H = M$ or $H = G$ holds.)

Rings

R1. Let $F$ be a field and $A$ be a commutative $F$-algebra. Suppose $A$ is of finite dimension as a vector space of $F$.
   (a) Prove that if $A$ is a domain, $A$ is a field.
   (b) Prove that even if $A$ is not a domain, there are only finitely many prime ideals of $A$.

R2. Let $A$ be a commutative ring with identity, and write $V$ for the set of all prime ideals of $A$. Put $D(x) = \{ P \in V | x \notin P \}$ for $x \in A$. Prove
   (a) $D(a) = D(a^n)$ for integers $n > 0$;
   (b) $V = D(a) \cup D(b) \cup D(c)$ if $a^3 + b^5 + c^7$ is invertible in $A$.

R3. Determine all isomorphism classes of modules over the polynomial ring $F_2[X]$ which are of dimension 2 over $F_2$, and justify your answer. Here $F_2$ is a field of two elements.
Fields

F1. Let $F$ be a field. Show that the unit group $F \setminus \{0\}$ of $F$ is finitely generated if and only if $F$ is finite.

F2. Let $f(x)$ be the polynomial $x^4 - 2x^2 - 2$ over $\mathbb{Q}$ and $K$ be a splitting field of $f(x)$. Determine the Galois group $\text{Gal}(K/\mathbb{Q})$, and find the number of Galois extensions of $\mathbb{Q}$ inside $K$. Prove your answer.

F3. Let $f(x)$ be an irreducible polynomial over the field $F$ and let $K/F$ be a finite extension.
   (a) Define what it means for the extension $K/F$ to be normal.
   (b) Show that if $K$ is normal over $F$, then, in $K[X]$, $f(x)$ factors into a product of irreducible polynomials of the same degree.
   (c) Show by example that this result does not hold for $K$ not normal.

Linear Algebra

LA1. Let $A$ be an $N$ by $N$ matrix with entries in $\mathbb{C}$.
   (i) Let $g$ be an invertible $N$ by $N$ matrix. Show that
       $$\lim_{n \to \infty} A^n = 0$$
       if and only if $\lim_{n \to \infty} (gAg^{-1})^n = 0$.
   (ii) Give necessary and sufficient conditions in terms of the conjugacy class of $A$ only for
       $$\lim_{n \to \infty} A^n = 0$$
       to hold.

   Here, if $A_n = (a_{n(i,j)})$ is a sequence of $N$ by $N$ matrices with
   entries $a_{n(i,j)}$, $\lim_{n \to \infty} A_n = 0$ if and only if $\lim_{n \to \infty} a_{n(i,j)} = 0$ for all $i$ and $j$.

LA2. Let $A$ be a $3$ by $3$ matrix with complex entries. Suppose that $A$ satisfies the relation $A^2 + A + I_3 = 0$, where $I_3$ denotes the $3$ by $3$ identity matrix.
   (i) List the possible Jordan normal forms of $A$.
   (ii) Suppose $A$ has entries in $\mathbb{R}$. List the possible Jordan normal forms of $A$.

LA3. Let $F$ be a field of characteristic $0$ and let $Q$ be an invertible $n$ by $n$ symmetric matrix with entries in $F$.
   (i) Show that there exists an invertible matrix $A$ such that $AQ A'$ is diagonal.
   (ii) Does the same remain true if $Q$ is not invertible? Explain.
Algebra Qualifying Exam
Spring 2007

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by taking the two best scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

Groups

1. Let $G$ be a simple group containing an element of order 21. Prove that every proper subgroup of $G$ has index at least 10.

2. Find the number of subgroups of $\mathbb{Z}^n$ of index 5.

3. Let $G$ be a group with cyclic automorphism group $\text{Aut}(G)$. Prove that $G$ is abelian.

Rings

1. Let $D$ be a division ring (a ring with identity in which every non-zero element is invertible). Let $R = \text{Mat}_n(D)$ be the ring of $n \times n$ matrices with entries from $D$. Prove that $R$ has no two-sided ideals other than $R$ itself and $\{0\}$.

2. Let $R = \text{End}(V)$ be the ring of all linear endomorphisms of an infinite dimension complex vector space $V$ with countable basis $\{e_1, e_2, \ldots \}$. Prove that $R$ and $R \oplus R$ are isomorphic as left $R$-modules.

3. (a) Give a description of all maximal ideals of the ring $\mathbb{C}[x, y]$. Justify your description. You may use the Nullstellensatz.

(b) Let $M = (x^2 - y, y^2 - 5)$ be an ideal in $R = \mathbb{Q}[x, y]$. Prove that $M$ is a maximal ideal.
Fields

1. Let $F = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i / 5}$ and let $E/F$ be a Cyclic Galois extension of degree 5. Prove that there exists $\alpha \in F$ such that $E = F(\sqrt[5]{\alpha})$. Hint: find $\alpha \in E$ such that $\sigma(\alpha) = \zeta \alpha$, where $\sigma$ is a generator of the Galois group $\text{Gal}(E/F)$.

2. Let $K = \mathbb{Q}(\sqrt[3]{3}, \sqrt[5]{5})$.
   (a) Prove that $K$ has only one subfield $F \subset K$ such that $[F : \mathbb{Q}] = 2$.
   (b) Find all subfields of $K$.
   (c) Find an element $u \in K$ such that $K = \mathbb{Q}(u)$.
   (d) Describe all elements $u \in K$ such that $K = \mathbb{Q}(u)$.

3. Let $F = \mathbb{Z}/3$. First explain why $F[x]/(x^2 - 2)$ is isomorphic to $F[x]/(x^2 - 2x - 1)$. Then find an explicit isomorphism:
   $$\phi : F[x]/(x^2 - 2) \rightarrow F[x]/(x^2 - 2x - 1).$$

Linear Algebra

1. Let $A$ be a linear operator in a $\mathbb{Q}$-vector space $V$ of dimension $n$ such that the minimal polynomial of $A$ has degree $n$. Prove that every linear operator on $V$ that commutes with $A$ is a polynomial in $A$ over $\mathbb{Q}$.

2. Let $G = \text{GL}_n(\mathbb{C})$ be the multiplicative group of invertible $n \times n$ matrices over $\mathbb{C}$. Prove that every element of finite order in $G$ is conjugate to a diagonal matrix.

3. Let $A(x, y)$ be a bilinear form on a vector space $V$ of finite dimension and
   $$V_i = \{ x \in V \text{ such that } A(x, y) = 0 \text{ for all } y \in V \},$$
   $$V_r = \{ y \in V \text{ such that } A(x, y) = 0 \text{ for all } x \in V \}.$$
   Prove that $\dim V_i = \dim V_r$. 
Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by taking the two best scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

Groups

G. Let $G$ be a finite group of order $g$ and $\mathbb{Z}[G] \subset \mathbb{Q}[G]$ be the group algebras of $G$ with integer and rational coefficients, respectively. Let

$$e\mathbb{Z}[G] = \{ea \in \mathbb{Q}[G] \mid a \in \mathbb{Z}[G]\}$$

for $e = g^{-1} \sum_{h \in G} h \in \mathbb{Q}[G]$, and define a group

$$G' = e\mathbb{Z}[G] / (\mathbb{Z}[G] \cap e\mathbb{Z}[G])$$

Prove that $G'$ is a group of order $g$. Find a necessary and sufficient condition to have $G' \cong G$ as groups, and justify your answer.

G2. Prove or disprove:
(a) the group $\text{GL}_2(\mathbb{Q})$ of $2 \times 2$ matrices with rational coefficients has finite cyclic subgroups of order bigger than any given positive integer $N$.
(b) the group $\text{GL}_2(\mathbb{R})$ of $2 \times 2$ matrices with real coefficients has finite cyclic subgroups of order bigger than any given positive integer $N$.

G3. Let $G$ be an additive abelian group such that multiplication by $n$: $x \mapsto nx$ is surjective for all positive integers $n$. Let

$$G[n] = \{x \in G \mid nx = 0\}$$

and $p$ be a prime.
(a) Prove that for a given integer $m \geq 1$, there are only finitely many subgroups $H$ of order $p^m$ in $G$ if $G[p]$ is finite;
(b) Find a formula of the number of subgroups of $G$ of order $p$ if the order of $G[p]$ is $p^3$, and justify your answer.

Rings

R1. Let $A = M_n(F)$ be the ring of $n \times n$ matrices with entries in a field $F$.
(a) Prove that any left ideal of $A$ is principal of the form $Ax$;
(b) How many left ideals of $A$ if $n = 2$ and $F$ is a finite field? (Give a simple formula of the number of maximal left ideals of $A$, and justify your answer.)
R2. Let $A$ be a domain and $B = A[T, \frac{1}{T}]$ for an indeterminate $T$. Prove that the ring automorphism group $\text{Aut}(B/A)$ of $B$ inducing the identity on $A$ is finite if and only if the group of invertible elements of $A$ is finite.

R3. Consider the covariant functor $F : A \mapsto A^\times$ from the category ALG of commutative rings with identity to the category of sets. Here $A^\times$ is the group of invertible elements of $A$. Give an explicit form of a commutative ring $R$ such that the functor $F$ is isomorphic to the functor $A \mapsto \text{Hom}_{\text{ALG}}(R, A)$.

**Fields**

F1. Let $L/F$ be a cubic (of degree 3) field extension of characteristic zero. Prove that there is an element $a \in F$ and a cubic field extension $L_0$ of the field $F_0 = \mathbb{Q}(a)$ such that $L$ is the composite $F L_0$ of $F$ and $L_0$ over $F_0$.

F2. A field extension $L/F$ is said to be balanced if every field homomorphism $L \to L$ over $F$ is an isomorphism.

(a) Prove that every algebraic (possibly infinite) field extension is balanced;

(b) Give an example of a balanced non-algebraic field extension.

F3. Let $p$ be a prime integer and $F$ a field such that the degree of every nontrivial finite field extension of $F$ is divisible by $p$. Prove that for any finite field extension $L/F$, there exists a tower of field extensions $F = F_0 \subset F_1 \subset \cdots \subset F_n = L$ such that $[F_{i+1} : F_i] = p$ for any $i = 0, \ldots, n - 1$.

**Linear Algebra**

L1. Let $V$ be a finite dimensional vector space of dimension $n$ over a field of characteristic 2. A bilinear form $b$ on $V$ is called symmetric (respectively, alternating) if $b(v, v') = b(v', v)$ for all $v, v' \in V$ (respectively, $b(v, v) = 0$ for all $v \in V$). Prove that the space $\text{Alt}(V)$ of all alternating bilinear forms on $V$ is a subspace of the space $\text{Sym}(V)$ of all symmetric bilinear forms on $V$ and find dimension of the factor space $\text{Sym}(V)/\text{Alt}(V)$.

L2. Find the number of conjugacy classes of elements of order 4 in the general linear group $\text{GL}_4(\mathbb{Q})$.

L3. An $n \times n$ matrix $A$ over a field $F$ is called regular over $F$ if the minimal and characteristic polynomials of $A$ coincide. Prove that for a field extension $L/F$, an $n \times n$ matrix $A$ over $F$ is regular over $F$ if and only if $A$ is regular over $L$. 

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Test Instructions: Each problem is worth 20 points. You should do two problems in each section. In particular, do not submit more than two problems in each section. In each problem, you must explain clearly your reasoning and any theorems that you quote. Whether you pass or fail depends on your performance in each section, not only on the total score.

GROUPS

PROBLEM G1
Let $p$ be a prime number. Show that a subgroup $G$ of $S_p$ which contains an element of order $p$ and which contains a transposition must be the whole of $S_p$.

PROBLEM G2
Let $G = D_{2n}$ be the dihedral group of order $2n$ where $n \geq 3$. Prove that $\text{Aut}(G)$ is isomorphic to the group of $2 \times 2$ matrices of the form

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha \in (\mathbb{Z}/n)^*, \beta \in \mathbb{Z}/n \right\}.$$

PROBLEM G3
Let $K$ be a normal subgroup of a finite group $G$. Let $p$ be a prime. Let $N_p$ be the number of $p$-Sylow subgroups ($p$-SSG) of $G$ and $N_p'$ the number in $K$.

(a) Show that $N_p = [G : N_G(P)]$ where $P$ is any $p$-SSG of $G$ and $N_G(P)$ is the normalizer of $P$ in $G$.

(b) Prove that $N_p'$ divides $N_p$. 

1
RINGS

PROBLEM R1
Let $D$ be an associative ring with unit having no zero divisors. Assume that the center of $D$ contains a field $k$ such that $\dim_k(D) < \infty$. Prove that $D$ is a division algebra (i.e., every non-zero element is invertible).

PROBLEM R2
Let $G$ be a finite group of order $|G| > 1$. The rational group ring $\mathbb{Q}[G]$ of $G$ is the $\mathbb{Q}$-algebra consisting of all finite linear combinations

$$\sum_{g \in G} a_g g$$

where $a_g \in \mathbb{Q}$. Multiplication in $\mathbb{Q}[G]$ is defined by extending the group multiplication linearly.

(a) Show that $\mathbb{Q}[G]$ has a non-trivial idempotent: $\exists a \in \mathbb{Q}[G], a \neq 0, 1$ with $a^2 = a$.
Hint: reduce to the case of a cyclic group $G = \langle x : x^n = 1 \rangle$.

(b) Show that $\mathbb{Q}[G]$ contains an invertible element $u$ that is non-trivial, that is, not of the form $u = a g$ where $a \in \mathbb{Q}$ and $g \in G$.

(FYI: The Kadison-Kaplansky Conjecture claims that for $G$ without torsion, the group algebra $\mathbb{Q}[G]$ contains no non-trivial idempotent. This conjecture is open in general.)

PROBLEM R3
Let $R$ be a Noetherian ring and $I$ any ideal of $R$. Prove that there exist prime ideals $P_1, ..., P_m$ of $R$ such that

$$P_1 P_2 \cdots P_m \subseteq I$$

Hint: Show that if $J$ is any non-prime ideal, then there exist $a, b \notin J$ such that $(J + a)(J + b) \subseteq J$. Then use the Noetherian property.
FIELDS

PROBLEM F1
Consider the polynomial $P(X) = X^5 - 4X + 2$ in $\mathbb{Q}[X]$.

(a) Show that $P$ is irreducible and has 3 real roots and 2 complex ones.

(b) Show that the Galois group of $P$ is $S_5$.

PROBLEM F2
Let $\zeta_n = \exp(2\pi i / n)$ be a primitive $n$th root of unity. Let $F_n = \mathbb{Q}(\zeta_n)$. Set

$$d_n = [F_n : \mathbb{Q}]$$

(a) Let $n = 6$. Find an irreducible polynomial of degree $d_6$ in $\mathbb{Q}[x]$ whose roots generate $F_6$.

(b) Let $n = 12$. Find an irreducible polynomial of degree $d_{12}$ in $\mathbb{Q}[x]$ whose roots generate $F_{12}$.

PROBLEM F3
Let $E/F$ be a finite, separable extension of fields. Prove that there exists $\alpha \in E$ such that $E = F(\alpha)$. State clearly any theorems you use in the proof.
LINEAR ALGEBRA

PROBLEM LA1
Let $V$ and $W$ be vector spaces over a field $F$ and let $V^*$ be the dual space of $V$. Let $\operatorname{Hom}(V, W)$ be the space of linear maps from $V$ to $W$. There exists a natural linear map

$$T : V^* \otimes_F W \to \operatorname{Hom}(V, W)$$

defined by $T(f \otimes w)(v) = f(v) \cdot w$. Show that $V$ is finite dimensional if and only if $T$ is an isomorphism for all $W$.

PROBLEM LA2
Let $M_4(\mathbb{Q})$ be the ring of all $4 \times 4$ matrices with coefficients in $\mathbb{Q}$. Find a set of representatives for the conjugacy classes of elements $X \in M_4(\mathbb{Q})$ satisfying the equation $X^4 = 2X^2$.

PROBLEM LA3
Let $V$ be a finite dimensional $F$-vector space and $T : V \to V$ a linear endomorphism. Show that there exists a decomposition

$$V = V_1 \oplus V_2$$

with the properties:

1. $T(V_i) \subset V_i$ for $i = 1, 2$
2. $T$ is an isomorphism on $V_1$
3. $T$ is nilpotent on $V_2$.

Hint: Consider the sequences of subspaces $\operatorname{Im}(T) \supset \operatorname{Im}(T^2) \supset \cdots$ and that $\operatorname{Ker}(T) \subset \operatorname{Ker}(T^2) \subset \cdots$. 
Algebra Qualifying Exam
Fall 2009

Test Instructions: Each problem is worth 20 points. Attempt at least 8 problems in any section. All tried problems will be graded.

Part 1: Categories and Functors.
(Cat 1). Let $\textbf{Top}$ be the category of topological spaces. Recall that a morphism $f$ in some category is called a monomorphism if, for any two morphisms $g_1$ and $g_2$ that can be precomposed with $f$, $fg_1 = fg_2$ implies $g_1 = g_2$. Dually, $f$ is called an epimorphism if, for any $g_1$ and $g_2$ that can be post-composed with $f$, $g_1f = g_2f$ implies $g_1 = g_2$.

(a) Show that a continuous map $f : X \to Y$ is a monomorphism in $\textbf{Top}$ if and only if $f$ is one-to-one.
(b) Now show by example that an epimorphism in $\textbf{Top}$ need not be onto.

(Cat 2). Let $F : \textbf{Ab} \to \textbf{Sets}$ be the forgetful functor from abelian groups to sets. Show that $F$ does not have a right adjoint.

Part 2: Groups.
(Gr 1). Suppose $A$ is an abelian group that is generated by $n$ elements (or fewer). Show that any subgroup of $A$ also can be generated by $n$ elements (or fewer).
(Gr 2). Let $p < q$ be primes, $n \geq 0$ an integer and $G$ a group of order $pq^n$. Show that $G$ is solvable.

Part 3: Representations.
(Rep 1). Let $G$ be a finite group and $\rho : G \to \text{GL}(V)$ a complex representation. Prove that $(V, \rho)$ splits as a direct sum of irreducible representations of $G$. [Note: It does not suffice to just quote a theorem. You have to actually prove the statement.]
(Rep 2). Let $G$ be a finite $p$-group and $\rho : G \to \text{GL}(V)$ a representation in a $\mathbb{F}_p$-vector space.
(a) Show that $V$ has a one-dimensional $G$-invariant subspace $W$.
(b) Show by example that $(V, \rho)$ need not split into a direct sum of irreducible representations.

Part 4: Commutative Rings.
(C1). Find a homomorphism $A \to B$ of commutative rings (sending the identity of $A$ to the identity of $B$) and non-zero $A$-modules $M, N$ such that the canonical map

$$B \otimes_A \text{Hom}_A(M, N) \to \text{Hom}_B(B \otimes_A M, B \otimes_A N)$$

is the zero map, and justify your answer. Prove that the map is an isomorphism if $M$ is a finitely generated projective $A$-module.
(C2). Prove the following facts:
(a) Any subring of $\mathbb{Q}$ sharing the identity with $\mathbb{Q}$ is a PID.
(b) For a subring $A \subset \mathbb{Z}[\sqrt{-1}]$ sharing the identity with $\mathbb{Z}[\sqrt{-1}]$, if $A \neq \mathbb{Z}$ and $A \neq \mathbb{Z}[\sqrt{-1}]$, $A$ is not a PID.
Part 5: Non-commutative Rings.

(R1). Prove that every two-sided ideal of the ring $M_2(\mathbb{Z})$ is principal, i.e., generated by one element.

(R2). Let $B$ be a central simple algebra over $k$ of dimension 4 (so, the center of $B$ is $k$ and has no nontrivial two-sided ideals except for $(0)$ and $B$ itself). Prove the following facts.
   (a) All left ideals of $B$ have even dimensional.
   (b) $B \cong M_2(k)$ if and only if $B$ is not a division algebra, where $M_2(k)$ is the matrix algebra of $2 \times 2$ matrices with coefficients in $k$.

Part 6: Fields.

(F1). Prove that the multiplicative group $F \setminus \{0\}$ of a field $F$ is a cyclic group if and only if $F$ is a finite field.

(F2). Let $k = \mathbb{F}_2(t, s)$ be the field of fractions of two variable polynomial ring $\mathbb{F}_2[t, s]$, where $\mathbb{F}_2$ is the field with 2 elements. Write $\theta_a$ for a root of $T^2 + T + a = 0$ for $a \in k$ in an algebraic closure of $k$. An intermediate field $M$ between $K$ and $k$ for a field extension $K/k$ is a subfield in $K$ containing $k$.
   (a) How many intermediate fields between $k$ and $k(\theta_t, \theta_s)$?
   (b) How many intermediate fields between $k$ and $k(\sqrt{t}, \sqrt{s})$?

Justify all your answers.
ALGEBRA QUALIFYING EXAM: Spring 2009

TEST INSTRUCTIONS All problems are worth 20 points. You are expected to do 2 problems from each of the four sections. Your total score will be computed by taking the two best-scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results that you use.

GROUPS

G1. Let $N$ be a normal subgroup of a finite group $G$ and $P$ a Sylow $p$-subgroup of $G$. Prove that $P \cap N$ is a Sylow $p$-subgroup of $N$.

G2. Let $A$ be an abelian group generated by $n$ elements. Prove that any subgroup of $A$ can be generated by $n$ elements.

G3. Let $G$ be a finite group and $H \subseteq G$ a subgroup of index $n$. Suppose that $xH \cap Hy \neq \emptyset$ for any elements $x, y \in G \setminus H$. Prove that $|G| \geq n^2 - n$. (Hint: consider an action of $G$ on $(G/H) \times (G/H)$.)

RINGS

R1. Show the the ring $\mathbb{Z}[2i]$ consisting of all complex numbers $a + 2bi$ with $a, b \in \mathbb{Z}$ is not a PID.

R2. Let $M_n(F)$ be a the matrix ring of $n \times n$ matrices over a field $F$. Suppose that there is a subring of $M_n(F)$ isomorphic to $M_m(F)$ for some $m$. Prove that $m$ divides $n$.

R3. Two polynomials $f, g \in R[t]$ over a commutative ring $R$ are called coprime over $R$ if $f$ and $g$ generate the unit ideal in $R[t]$. Let $f, g \in \mathbb{Z}[t]$ be two polynomials such that $f$ and $g$ are coprime over $\mathbb{Q}$ and the residues of $f$ and $g$ in $(\mathbb{Z}/p\mathbb{Z})[t]$ are coprime for every prime integer $p$. Prove that $f$ and $g$ are coprime over $\mathbb{Z}$.

LINEAR ALGEBRA

LA1. A matrix $N$ is said to be nilpotent of order $k$ if $N^k = 0$, but $N^{k-1} \neq 0$. If $N$ is nilpotent of order $k$, prove that

$$k = \min\{m \mid \ker(N^m) = \ker(N^{m+1})\}.$$ 

LA2. Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

(A) Prove that $(I - A)(I + A)^{-1}$ is an orthogonal matrix.
(B) Compute $e^{At}$ as a function of $t$. 

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LA3. Let $\pi \in S_n$ be a permutation of $n$ elements. Let $P_\pi$ be the $n \times n$ matrix taking the standard basis vector $e_i \mapsto e_{\pi(i)}$ for all $i$. Describe the eigenvalues over $\mathbb{C}$ of $P_\pi$ in terms of the cyclic decomposition of $\pi$.

FIELDS

F1. Let $p(x) \in K[x]$ be a monic irreducible polynomial of degree $n$ whose discriminant $D \neq 0$, where $K$ has characteristic $\neq 2$. Prove that the Galois group of $p$ is contained in the alternating group $A_n$ if and only in $\sqrt{D} \in K$.

F2. Let $\alpha$ be transcendental over $\mathbb{Q}$. What is the minimal polynomial of $\alpha$ over $\mathbb{Q}(\frac{a^2+1}{a^2-1})$?

F3. (A) Which roots of unity are contained in quadratic extensions of $\mathbb{Q}$, and which extensions are these?

(B) If $K/\mathbb{Q}$ is any field extension that contains a primitive $n$'th root of unity, and $n$ is odd, then prove $K$ contains a primitive $2n$'th root of unity.
Algebra Qualifying Exam
Spring 2010

Test Instructions: Each problem is worth 20 points. Attempt at least 8 problems. All tried problems will be graded.

Part 1: Categories and Functors

Problem 1 Show that the functor from (unitary) rings to groups sending a ring $A$ to its group of units $A^\times$ is co-representable by a ring $R$. In other words, show there exists a ring $R$ and a natural isomorphism of functors $\text{Hom}_{\text{rings}}(R, A) \to A^\times$.

Problem 2 (i) Define what it means for two categories to be equivalent. (ii) A groupoid $\mathcal{G}$ is a category such that all morphisms are isomorphisms. $\mathcal{G}$ is called connected if for any two objects $x$ and $y$, $\text{Hom}_{\mathcal{G}}(x, y)$ is non-empty. Show that any non-empty connected groupoid is equivalent to a group, that is, a groupoid with one object.

Part 2: Groups

Problem 3 Determine, using the structure theory of abelian groups or otherwise, all finitely generated abelian groups $A$ such that the group $\text{Aut}(A)$ of automorphisms of $A$ is finite. State clearly any basic theorems that you use. Determine the order of the automorphism group of $A = \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Problem 4 Show that if $S_4$ denotes the symmetric group of degree 4, and $\sigma$ is an automorphism of $S_4$, and $\tau \in S_4$ is a transposition, then $\sigma(\tau)$ is also a transposition. By studying the action of $\sigma$ on transpositions, show that every automorphism of $S_4$ is inner. (Remark: this result holds for all $S_n$ except $n = 6$.)

Part 3: Representations

Problem 5 Give the total number and the dimensions of the irreducible complex representations of $S_4$. Prove your answer.

Problem 6 State and prove Schur’s Lemma which describes $\text{Hom}_{\mathcal{G}}(V, W)$ for $V$ and $W$ finite dimensional irreducible complex representations of a finite group.
Part 4: Commutative Rings and Modules

Problem 7 Show that the group of units of the ring $\mathbb{Z}/N\mathbb{Z}$ is cyclic iff $N$ is either a power of an odd prime number, twice a power of an odd prime number, or 4.

Problem 8 Let $F$ be the field with 2 elements and let $R = F[X]$. List up to isomorphism all $R$-modules with 8 elements that are cyclic.

Part 5: Non-Commutative Rings

Problem 9 Let $A$ be a left Noetherian ring. Show that every left invertible element $a \in A$ is two-sided invertible.

Problem 10 Let $F$ be a field and $V$ a finite-dimensional $F$-vector space. Show that $R = \text{End}_F(V)$ has no non-trivial two-sided ideals.

Part 6: Fields

Problem 11 Let $F$ be a field of characteristic zero containing the $p$-th roots of unity for $p$ a prime. Show that the cyclic extensions of degree $p$ of $F$ in any algebraic closure $\overline{F}$ of $F$ are in one to one correspondence with the subgroups of order $p$ of $F^*/(F^*)^p$.

Problem 12 Determine all fields $F$ such that the multiplicative group of $F$ is finitely generated.