HARMONIC MAPS FROM $\mathbb{C}^n$ TO KÄHLER MANIFOLDS

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Abstract. In this paper, we prove that, under an assumption, a harmonic map of finite $\bar{\partial}$-energy from $\mathbb{C}^n$ $(n \geq 2)$ to any Kähler manifold must be a holomorphic map. It can be seen as a complex analogue of the Liouville type theorem for harmonic maps.

1. Introduction

The classical Liouville theorem says that a real-valued nonconstant harmonic function on R^n has to be unbounded. Sealey (see [2] or [6]) gave an analogue for harmonic maps states that any harmonic map of finite energy from R^n $(n \geq 2)$ to any Riemannian manifold must be a constant map. In this paper, we consider the complex analogue, i.e. the following question:

Quention: *Does a harmonic map with finite $\bar{\partial}$-energy from $\mathbb{C}^n$ $(n \geq 2)$ to any Kähler manifold have to be a holomorphic map?*

From Siu and Yau's proof of Frankel conjecture [3], we know that it is very important to study the holomorphism of harmonic maps. So the above question is obviously very interesting. We also hope that it is true. But we do not know how to prove it. Our slightly weak result can be stated as follows:

Theorem 1.1. *Let f be a harmonic map of finite $\bar{\partial}$-energy from $\mathbb{C}^n$ $(n \geq 2)$ to any Kähler manifold. If f satisfies that $\lim_{R\to\infty} R\int_{\partial B_R}(|f_*\frac{\partial}{\partial r}|^2 - < J^{'} f_*\frac{\partial}{\partial r}, f_* J\frac{\partial}{\partial r} >)dv \geq 0$, where J and $J^{'}$ denote the complex structures of $\mathbb{C}^n$ and target manifold, $\frac{\partial}{\partial r}$ is the radical vector field, then f is a holomorphic map.*

If we consider harmonic map $f = (f_1, ..., f_{2m})$ (clearly in this case every f_i is a harmonic function) from $\mathbb{C}^n$ to $\mathbb{C}^m$, then the answer of above question is true (see [4]).

Theorem 1.2. *Let f be any harmonic map of finite $\bar{\partial}$-energy from $\mathbb{C}^n$ to $\mathbb{C}^m$, then f is a holomorphic map.*

The proof of theorem 1.2 is not difficult but is different to theorem 1.1. We do not give it here.

The main idea of the proof is to define an one parameter family of maps and deduce the $\bar{\partial}$-energy variation formula. Then comparing with the directly computation via derivative.

2000 *Mathematics Subject Classification.* Primary 53C55.

Key words and phrases. harmonic maps, holomorphic maps.

The rest three sections is organized as follows: In section 2, we introduce briefly the harmonic maps. In section 3, we study the first variation of $\bar{\partial}$-energy. we would give the proof of theorem 1.1 in last section.

Acknowledgement: I would like to thank my teacher Liu Kefeng for many encouragements.

2. PRELIMINARIES

In this section we introduce some basic concepts of harmonic maps we need. For details, one may see the book of Xin [6].

Let (M, g), (N, h) be two Riemannian manifolds. f is a smooth map from M to N. We can define the energy density of f by

$$e(f) = \frac{1}{2} trace|df|^2 = \frac{1}{2} < f_* e_i, f_* e_i >,$$

where $\{e_i\}$ $(i = 1, ..., m = dimM)$ is the local orthonormal frame field of M. The energy integral is defined by

$$E(f) = \int_M e(f) dv.$$

If we choose the natural coordinates $\{x^i\}$ and $\{y^\alpha\}$, the energy density can be written as

$$e(f)(x) = \frac{1}{2} g^{ij}(x) \frac{\partial f^\alpha(x)}{\partial x^i} \frac{\partial f^\beta(x)}{\partial x^j} h_{\alpha\beta}(f(x)). \tag{2.1}$$

The tensor field of f is

$$\tau(f) = (\nabla_{e_i} df)(e_i),$$

where ∇ is the Levi-Civita connection of M.

Definition 2.1. If $\tau(f) = 0$, we say that f is a harmonic map.

From the variation view, harmonic maps can be regarded as the critical points of energy integral functional. Let f_t be an one parameter family of maps. We can regard it as a smooth map from $M \times (-\epsilon, \epsilon) \to N$. Let $f_0 = f$, $\frac{df_t}{dt}|_{t=0} = v$. Then we have the following first variation formula [6]

$$\frac{d}{dt} E(f_t)|_{t=0} = \int_M divW dv - \int_M < v, \tau(f) > dv, \tag{2.2}$$

where $W =< v, f_* e_j > e_j$. If M is compact, then $\int_M divW dv = 0$, we know that harmonic maps are the critical points of energy functional.

3. $\bar{\partial}$-ENERGY VARIATION

In this section we consider complex case.

Let f be a harmonic map from $\mathbb{C}^n$ to any Kähler manifold N. J is the standard complex structure of $\mathbb{C}^n$ and J' is the complex structure of N. ω and ω^N are the corresponding Kähler forms of $\mathbb{C}^n$ and N (i.e. $\omega(.,.) =< J.,. >$ and $\omega^N(.,.) =<$

$J^{'}.,.>)$. The ∂-energy density is defined by [6]

$$\begin{aligned} e'(f) = |\partial f|^2 &= |f_*J + J'f_*|^2 \\ &= \frac{1}{4}(|f_*e_i|^2 + |f_*Je_i|^2 + 2 < J^{'}f_*e_i, f_*Je_i >) \\ &= \frac{1}{2}(e(f) + < f^*\omega^N, \omega^M >) \end{aligned}$$

and the $\bar\partial$-energy density is defined by

$$\begin{aligned} e''(f) = |\bar\partial f|^2 &= |f_*J - J'f_*|^2 \\ &= \frac{1}{4}(|f_*e_i|^2 + |f_*Je_i|^2 - 2 < J^{'}f_*e_i, f_*Je_i >) \\ &= \frac{1}{2}(e(f) - < f^*\omega^N, \omega^M >), \end{aligned}$$

where $\{e_i, Je_i\}$ $(i = 1, ..., n)$ is the Hermitian frame of $\mathbb{C}^n$ and $< f^*\omega^N, \omega >$ denotes the induced norm. We call that f is holomorphic (resp. conjugate holomorphic), if $f_*J = J^{'}f_*$ (resp. $f_*J = -J^{'}f_*$). Obviously f is holomorphic if and only if $|\bar\partial f|^2 \equiv 0$. We denote $\bar\partial$-energy by

$$E_{\bar\partial}(f) = \int_{\mathbb{C}^n} |\bar\partial f|^2 dv.$$

Now we define an one parameter family of maps $f_t(x) = f(tx) : \mathbb{C}^n \longrightarrow N, t \in (1-\epsilon, 1+\epsilon)$ and $f_1 = f$. The relation between $E_{\bar\partial}(f)$ and $E_{\bar\partial}(f_t)$ is given by the following lemma:

Lemma 3.1. *If $E_{\bar\partial}(f) < \infty$, then $\int_{\mathbb{C}^n} |\bar\partial f_t|^2 dv = t^{2-2n} \int_{\mathbb{C}^n} |\bar\partial f|^2 dv$.*

Proof. Under the standard Hermitian metric of $\mathbb{C}^n$, $g^{ij} = \delta_{ij}$. From (2.1), clearly we have

$$e(f_t)(x) = t^2 e(f)(tx).$$

Under the natural coordinates, it is also easy to show that

$$< f_t^*\omega^N, \omega > (x) = t^2 < f^*\omega^N, \omega > (tx).$$

Hence we have

$$|\bar\partial f_t|^2(x) = t^2 |\bar\partial f|^2(tx).$$

Then

$$\int_{B_R} |\bar\partial f_t|^2 dv = t^{2-2n} \int_{B_{Rt}} |\bar\partial f|^2 dv$$

where B_R is the Euclid ball of radius R around 0. Let $R \longrightarrow \infty$, we obtain the lemma. □

Let us consider the first variation of $\bar\partial$ energy :

Lemma 3.2. $\frac{d}{dt} E_{\bar\partial}(f_t)|_{t=1} = \lim_{R\to\infty} \frac{R}{2} \int_{\partial B_R} (|f_*\frac{\partial}{\partial r}|^2 - < J^{'}f_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r} >) dv$, *where ∂B_R is the Euclid sphere of radius R around* 0.

The proof will be separated in two steps.

Proof. Let $\{e_1, ..., e_{2n} = \frac{\partial}{\partial r}\}$ be a local orthonormal frame field, where $\frac{\partial}{\partial r}$ denotes unit radial vector field. By the definition of $f_t(x)$, it is easy to see that the variation vector field of f_t at $t = 1$ is $v = \frac{df_t}{dt}|_{t=1} = rf_*\frac{\partial}{\partial r}$.

Step 1: From (2.2), we have

$$\begin{aligned}\frac{d}{dt}\int_{B_R} e(f_t)dv|_{t=1} &= \int_{B_R} div < v, f_*e_j > e_j dv - \int_{B_R} < v, \tau(f) > dv\\ &= \int_{\partial B_R} < v, f_*\frac{\partial}{\partial r} > dv\\ &= R\int_{\partial B_R} |f_*\frac{\partial}{\partial r}|^2 dv.\end{aligned}$$

For f is harmonic, we know that the tensor field $\tau(f) = 0$ and the second "=" follows from divergence theorem [5].

Step 2: Another hand, from [6], we know $\frac{d}{dt}f_t^*\omega^N = d\theta_t$, here $\theta_t = f_t^* i(f_{t*}\frac{\partial}{\partial t})\omega^N$. Since $\frac{df_t}{dt}|_{t=1} = rf_*\frac{\partial}{\partial r}$, we have $\theta_1 = \theta = rf^* i(f_*\frac{\partial}{\partial r})\omega^N$. Then

$$\begin{aligned}&\frac{d}{dt}\int_{B_R} < f_t^*\omega^N, \omega > dv|_{t=1}\\ =& \int_{B_R} < d\theta, \omega > dv\\ =& \int_{B_R} d(\theta\wedge *\omega) + \int_{B_R} < \theta, \delta\omega > dv\\ =& \int_{\partial B_R} \theta\wedge *\omega - \int_{B_R} < \theta, *d\omega^{n-1} > dv\\ =& \int_{\partial B_R} \theta\wedge *\omega\\ =& -\int_{\partial B_R} \theta(e_i)\omega(e_i, \frac{\partial}{\partial r})dv\\ =& -R\int_{\partial B_R} \omega^N(f_*\frac{\partial}{\partial r}, f_*e_i)\omega(e_i, \frac{\partial}{\partial r})dv\\ =& -R\int_{\partial B_R} < J'f_*\frac{\partial}{\partial r}, f_*e_i >< Je_i, \frac{\partial}{\partial r} > dv\\ =& R\int_{\partial B_R} < J'f_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r} > dv.\end{aligned}$$

Note that $< d\theta, \omega > dv = d\theta\wedge *\omega$, the second "=" follows from the differential rules, where δ and $*$ are the co-differential and star operators. By the stokes theorem and the definition of δ, the third "=" holds. The fifth "=" follows from direct computation. Since we may choose $e_1 = J\frac{\partial}{\partial r}$, the last "=" holds. Hence we have

$$\frac{d}{dt}\int_{B_R} |\bar\partial f_t|^2 dv|_{t=1} = \frac{R}{2}\int_{\partial B_R} (|f_*\frac{\partial}{\partial r}|^2 - < J'f_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r} >)dv.$$

According to lemma 3.1, $E_{\bar\partial}(f_t) < \infty$. Let $R \longrightarrow \infty$, we get the lemma. □

Remark 3.3. If M is a compact manifold , $\int_M < f^*\omega^N, \omega^M > dv$ is a homotopy invariant. This was observed firstly by Lichnerowicz [1].

4. Proof of theorem 1.1

Now we can use lemma 3.1 and 3.2 to prove theorem 1.1: If f satisfies that $\lim_{R\to\infty} R\int_{\partial B_R}(|f_*\frac{\partial}{\partial r}|^2 - < J' f_*\frac{\partial}{\partial r}, f_* J\frac{\partial}{\partial r} >)dv \geq 0$. Then by lemma 3.2 we have $\frac{d}{dt}E_{\bar\partial}(f_t)|_{t=1} \geq 0$. But from lemma 3.1,

$$\begin{aligned}\frac{d}{dt}E_{\bar\partial}(f_t)|_{t=1} &= \lim_{t\to 1}\frac{\int_{\mathbb{C}^n}|\bar\partial f_t|^2 dv - \int_{\mathbb{C}^n}|\bar\partial f|^2 dv}{t-1}\\ &= \lim_{t\to 1}\frac{t^{2-2n}-1}{t-1}\int_{\mathbb{C}^n}|\bar\partial f|^2 dv\\ &= (2-2n)\int_{\mathbb{C}^n}|\bar\partial f|^2 dv.\end{aligned}$$

Since $n \geq 2$, this yields $\int_{\mathbb{C}^n}|\bar\partial f|^2 dv = 0$. Hence $|\bar\partial f|^2 \equiv 0$ and f is a holomorphic map.

Considering ∂-energy, with same arguments, we have

Theorem 4.1. *Let f be a harmonic map of finite ∂-energy (i.e.$\int_{\mathbb{C}^n}|\partial f|^2 < \infty$) from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold. If f satisfies that $\lim_{R\to\infty} R\int_{\partial B_R}(|f_*\frac{\partial}{\partial r}|^2 + < J' f_*\frac{\partial}{\partial r}, f_* J\frac{\partial}{\partial r} >)dv \geq 0$, where J and J' denote the complex structures of $\mathbb{C}^n$ and target manifold, $\frac{\partial}{\partial r}$ is the radical vector field, then f is a conjugate holomorphic map.*

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