Wavelets and framelets from dual pseudo splines *

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Abstract

Dual pseudo splines constitute a new class of refinable functions with B-splines as special examples, which was introduced in [8]. In this paper, we shall construct Riesz wavelet associated with dual pseudo splines. Furthermore, we use dual pseudo splines to construct tight frame systems with desired approximation order by applying the unitary extension principle.
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1 Introduction

Pseudo splines were shown to be an important family of refinable functions and their properties were extensively studied by some researches ([2, 4, 5, 6, 19]). Dual pseudo splines were introduced in [8] in order to derive the maximum degree of polynomial reproduction by subdivision schemes. The lower bounds for the regularity exponents of the dual pseudo-splines were derived in [3]. The masks of dual pseudo splines were also used in [13] to construct symmetric orthonormal complex wavelets. In this paper, we construct Riesz wavelets and tight framelets from dual pseudo splines. We shall study several approximation properties of dual pseudo splines, which are important in server areas of approximation theory and wavelet theory. Before proceeding further, let us introduce some notations and definitions.

Throughout this paper, let $L_2(\mathbb{R})$ denote the Banach space of all measurable functions f on $\mathbb{R}$ such that

$$\|f\|_2 := \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2} < \infty.$$

The Fourier transform of a compactly supported measurable function f is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}.$$

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Similarly, if c is a (complex-valued) summable sequences on $\mathbb{Z}$, then its *Fourier series* is defined by

$$\hat{c}(\xi) := \sum_{k\in\mathbb{Z}} c(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}.$$

A compactly supported function $\phi \in L_2(\mathbb{R})$ is *refinable* if it satisfies the *refinement equation*

$$\phi = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2\cdot -k), \tag{1.1}$$

where $a : \mathbb{Z} \to \mathbb{R}$ is a sequence and is called the *mask* for the refinement equation. In this paper, we always assume $\hat{\phi}(0) \neq 0$.

For positive integers $N, l \in \mathbb{N}$ with $l < N$, the mask of a *dual pseudo spline* of order (N, l) is given by

$$\widehat{a_{N,l}}(\xi) := \frac{1+e^{-i\xi}}{2}\cos^{2N}(\xi/2)\sum_{i=0}^{l}\binom{N-1/2+i}{i}\sin^{2i}(\xi/2). \tag{1.2}$$

The corresponding dual pseudo splines $\phi_{N,l}$ can be defined in terms of their masks. It was showed in [3] that the dual pseudo splines are compactly supported refinable functions in $L_2(\mathbb{R})$. Unless it is necessary, we drop the subscript "(N, l)" in $\widehat{a_{N,l}}$ and $\phi_{N,l}$ for simplicity. This new family masks have some well-known special cases. For examples, when $l = 0, 1, N-1$, these masks were designed for stationary subdivision scheme that generates smooth curves (see e.g. [7, 16]) and when $l = 0$, the corresponding refinable function is a B-spline with order $2N + 1$. Recall that a B-spline with order N and its mask are defined by

$$\widehat{B_N}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^N \quad \text{and} \quad \hat{a}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N, \quad \xi \in \mathbb{R}.$$

Note that the mask $\hat{a}$ defined in (1.2) can be factorized as

$$\hat{a}(\xi) := \left(\frac{1+e^{-i\xi}}{2}\right)^{2N+1}\mathcal{L}(\xi), \tag{1.3}$$

where $\mathcal{L}(\xi)$ is a bounded trigonometric polynomial with $\mathcal{L}(0) = 1$ and $\mathcal{L}(\xi) > 0$ for all $\xi \in \mathbb{R}$. Therefore, the dual pseudo spline is the convolution of the B-spline B_{2N+1} with a distribution.

The rest of the paper is organized as follows. In Section 2, we construct Riesz wavelets from dual pseudo splines by a nature choice in wavelets analysis. In Section 3, we give a brief discussion of the approximation order of the truncated tight frame series, where the tight frame are obtained via the unitary extension principle from the dual pseudo splines.

2 Riesz wavelet basis

Let $\psi \in L_2(\mathbb{R})$. We say that the wavelet ψ generates a *Riesz wavelet basis* in $L_2(\mathbb{R})$ if there exist two positive constants C_1 and C_2 such that

$$C_1\left(\sum_{j,k\in\mathbb{Z}}|c_{j,k}|^2\right)^{1/2} \leq \left\|\sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}\right\|_2 \leq C_2\left(\sum_{j,k\in\mathbb{Z}}|c_{j,k}|^2\right)^{1/2}$$

for all finitely supported sequences $\{c_{j,k}\}_{j,k\in\mathbb{Z}}$ and the linear span of

$$X(\psi) := \{\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k) : \ j,k \in \mathbb{Z}\}$$

is dense in $L_2(\mathbb{R})$. The generator ψ is called *Riesz wavelet*. It is well known that the order of vanishing moments is one of the most important factors for success of wavelet in various applications. In particular, vanishing moments guarantee the approximation order. We say that ψ has *vanishing moments* of order K if

$$\widehat{\psi}^{(k)}(0) = 0 \quad \forall \quad k = 0, \ldots, K-1,$$

where $\widehat{\psi}^{(k)}$ denotes the kth derivative of $\widehat{\psi}$.

Due to some desirable properties, Riesz wavelets have been found to be of interest in many applications such as image processing, computer graphical and numerical algorithms. Riesz wavelets are generally obtained from a refinable function via the multiresolution analysis. Let ϕ be a refinable function whose shifts are stable, a natural choice of ψ is

$$\hat{\psi}(2\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}\hat{\phi}(\xi), \quad \xi \in \mathbb{R}. \tag{2.1}$$

See [1, 4, 11, 12, 13, 14]) for some discussions for Riesz wavelets.

When ϕ is a compactly supported *orthonormal* refinable function, it is well known that ψ defined in (2.1) generates an orthonormal wavelet basis for $L_2(\mathbb{R})$ [1]. When ϕ is chosen to be the B-splines or interpolatory refinable function, Han and Shen showed in [14] that the wavelet defined in (2.1) is a Riesz wavelet. In [13], Han generalized this results to fractional B-splines, which are refinable functions introduced by Unser and Blu in [22]. It was shown in [4] that when the refinable function ϕ is chosen to be a pseudo spline, the wavelet ψ defined in (3.1) is also a Riesz wavelet.

In this section, we will show that for dual pseudo spline ϕ, the wavelet defined by (2.1) is a Riesz wavelet. In order to show this, we need the following Lemma 3.1 that was obtained by Han and Shen.

Lemma 2.1. *[14, Theorem 2.1] Let a be a finitely supported mask of a refinable function $\phi \in L_2(\mathbb{R})$ with $\hat{a}(0) = 1$ and $\hat{a}(\pi) = 0$, such that $\hat{a}$ can be factorized into the form*

$$|\hat{a}(\xi)| = \cos^n(\xi/2)|\mathcal{L}(\xi)|, \quad \xi \in [-\pi, \pi],$$

where $\mathcal{L}$ is the Fourier series of a finitely supported sequence with $\mathcal{L}(\xi) \neq 0$. Suppose that

$$|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 \neq 0, \quad \xi \in [-\pi, \pi].$$

Define

$$\hat{\psi}(2\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}\hat{\phi}(\xi)$$

and

$$\tilde{\mathcal{L}}(\xi) := \frac{\mathcal{L}(\xi)}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2}.$$

Assume that

$$\rho_{\mathcal{L}} := \|\mathcal{L}\|_\infty < 2^{n-1/2} \quad \text{and} \quad \rho_{\tilde{\mathcal{L}}} := \|\tilde{\mathcal{L}}\|_\infty < 2^{n-1/2}. \tag{2.2}$$

Then ψ generates a Riesz wavelet basis for $L_2(\mathbb{R})$.

We need following lemmas, which are important to the proof of main result.

Lemma 2.2. *For the given integers l, N with $0 \le l < N$, we have*

(1) $\binom{N+1}{i} = \binom{N}{i} + \binom{N}{i-1}$.

(2) $\sum_{i=0}^{l} \binom{N-\alpha+i}{i} x^i = \sum_{i=0}^{l} \binom{N+1-\alpha+l}{i} x^i (1-x)^{l-i}$, *where* $0 \le \alpha < 1$.

(3) $\binom{N+l}{i} < \binom{N+\alpha+l}{i} < \binom{N+1+l}{i}$, *where* $0 \le i \le l$ *and* $0 \le \alpha < 1$.

Before we prove the main result of this section, we also need the following lemmata later.

Lemma 2.3. *For the given integers l, N with $0 \le l < N$, we have*

(1) $(2N+4) \sum_{i=0}^{l-1} \binom{N+l}{i} - l \sum_{i=0}^{l} \binom{N+l}{i} \ge 0$.

(2) $2^{2l} \binom{N+\alpha+l}{l} \le \left(\sum_{i=0}^{l} \binom{N+\alpha+l}{i} \right)^2$, *where* $0 \le \alpha < 1$.

Proof. For (1), it is suffices to show that

$$(N+1) \sum_{i=0}^{l-1} \binom{N+l}{i} - l \binom{N+l}{l} \ge 0.$$

The above equality follows from the identity $(N+1)\binom{N+l}{l-1} = l\binom{N+l}{l}$.

For (2), we first prove that

$$2^{2l} \binom{N+1+l}{l} \le \left(\sum_{i=0}^{l} \binom{N+l}{i} \right)^2. \tag{2.3}$$

(2.3) is obviously true for $N = 1$. Suppose (2.3) holds for N_0 with $0 \le l < N$. Consider the case $N = N_0 + 1$. For $0 \le l < N_0$, we deduce that

$$\begin{aligned}
2^{2l} \binom{N_0+2+l}{l} &= \frac{N_0+2+l}{N_0+2} 2^{2l} \binom{N_0+1+l}{l} \\
&\le \frac{N_0+2+l}{N_0+2} \left(\sum_{i=0}^{l} \binom{N_0+l}{i} \right)^2 \\
&= \left(\sum_{i=0}^{l} \binom{N_0+l}{i} + \left(\sqrt{\frac{N_0+2+l}{N_0+2}} - 1 \right) \sum_{i=0}^{l} \binom{N_0+l}{i} \right)^2 \\
&= \left(\sum_{i=0}^{l} \binom{N_0+l}{i} + \left(\frac{l}{N_0+2+\sqrt{N_0+2}\sqrt{N_0+2+l}} \right) \sum_{i=0}^{l} \binom{N_0+l}{i} \right)^2 \\
&\le \left(\sum_{i=0}^{l} \binom{N_0+l}{i} + \left(\frac{l}{2N_0+4} \right) \sum_{i=0}^{l} \binom{N_0+l}{i} \right)^2.
\end{aligned}$$

It follows from (2) in Lemma 2.2 and (1) in Lemma 2.3 that

$$\begin{aligned} 2^{2l}\binom{N_0+2+l}{l} &\leq \left(\sum_{i=0}^{l}\binom{N_0+l}{i}+\sum_{i=0}^{l-1}\binom{N_0+l}{i}\right)^2 \\ &\leq \left(\sum_{i=0}^{l}\binom{N_0+1+l}{i}\right)^2. \end{aligned}$$

It remains to show that (2.3) holds for $l = N_0$, i.e.

$$2^{2N_0}\binom{2N_0+2}{N_0} \leq \left(\sum_{i=0}^{N_0}\binom{2N_0+1}{i}\right)^2.$$

By computation, we have

$$\sum_{i=0}^{N_0}\binom{2N_0+1}{i} = \frac{1}{2}\sum_{i=0}^{2N_0+1}\binom{2N_0+1}{i} = \frac{1}{2}(1+1)^{2N_0+1} = 2^{2N_0}.$$

By (2) in Lemma 2.2, we have

$$\binom{2N_0+2}{N_0} = \binom{2N_0+1}{N_0} + \binom{2N_0+1}{N_0-1} \leq \sum_{i=0}^{N_0}\binom{2N_0+1}{i}.$$

Now we conclude that (2.3) holds. This, together with (3) in Lemma 2.2, implies that

$$2^{2l}\binom{N+\alpha+l}{l} \leq 2^{2l}\binom{N+1+l}{l} \leq \left(\sum_{i=0}^{l}\binom{N+l}{i}\right)^2 \leq \left(\sum_{i=0}^{l}\binom{N+\alpha+l}{i}\right)^2.$$

$\square$

Lemma 2.4. *For the given nonnegative integers l, N with $l < N$, let $R(y) := \sum_i^l \binom{N-\alpha+i}{i} y^i$ where $0 \leq \alpha < 1$. Define*

$$S(y) := (1-y)^{2N+1}R^2(y) + (y)^{2N+1}R^2(1-y).$$

Then,

- *(i)* $\max_{y\in[0,1]} R(y) = \binom{N+1-\alpha+l}{l}$.
- *(ii)* $\min_{y\in[0,1]} S(y) = (\frac{1}{2})^{2N+2l}\left(\sum_{i=0}^{l}\binom{N+1-\alpha+l}{i}\right)^2$.

Proof. Since all the coefficients of the polynomial $R(y)$ are positive, it is easy see that

$$\max_{y\in[0,1]} R(y) = R(1).$$

By item (2) in Lemma 2.3,

$$R(1)=\sum_{i=0}^{l}\binom{N+1-\alpha+l}{i}1^i(1-1)^{l-i}=\binom{N+1-\alpha+l}{l}.$$

Denote $\tilde{R}(y):=R^2(y)$, then $\tilde{R}(y)=\sum_{i=0}^{2l}c(i)y^i$. The positive coefficients $c(i)$ are uniquely determined by the coefficients of $R(y)$. It follows from $l<N$ that

$$\sum_{i=0}^{2l}c(i)\left(\frac{1}{y}\right)^{2N-i}\geq\sum_{i=0}^{2l}c(i)\left(\frac{1}{1-y}\right)^{2N-i},\quad\forall\, y\in[0,1/2].$$

This show that,

$$-(2N+1)(1-y)^{2N}R^2(y)+(2N+1)y^{2N}R^2(1-y)\leq 0\quad\forall\, y\in[0,1/2].$$

With the similar arguments, we have that

$$-2y^{2N+1}R(1-y)R'(1-y)+2(1-y)^{2N+1}R(y)R'(y)\leq 0\quad\forall\, y\in[0,1/2].$$

Taking the derivative of $S(y)$, we obtain

$$\begin{aligned}S'(y)&=-(2N+1)(1-y)^{2N}R^2(y)+2(1-y)^{2N+1}R(y)R'(y)\\&\quad+(2N+1)y^{2N}R^2(1-y)-2y^{2N+1}R(1-y)R'(1-y)\\&<0\end{aligned}$$

for all $y\in[0,1/2]$.

By directly calculate, we have that $S(y)=S(1-y)$. Hence, $S(y)$ is symmetric about 1/2. Now we conclude that

$$S'(y)\begin{cases}\leq 0, & y\in[0,1/2],\\ \geq 0, & y\in[1/2,1].\end{cases}$$

Thus, $\min_{y\in[0,1]}S(y)=S(1/2)$. By item (1) in Lemma 2.3, we deduce that

$$\begin{aligned}S\left(\frac{1}{2}\right)&=\left(\frac{1}{2}\right)^{2N+1}R^2\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^{2N+1}R^2\left(\frac{1}{2}\right)\\&=\left(\frac{1}{2}\right)^{2N}R^2\left(\frac{1}{2}\right)\\&=\left(\frac{1}{2}\right)^{2N}\left(\sum_{i=0}^{l}\binom{N+1-\alpha+l}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{l-i}\right)^2\\&=(\frac{1}{2})^{2N+2l}\left(\sum_{i=0}^{l}\binom{N+1-\alpha+l}{i}\right)^2.\end{aligned}$$

□

Remark 2.5. *For the case that $\alpha = 0$, Lemma 2.3 and Lemma 2.4 were proved by Dong and Shen in [4].*

Now we state the main theorem of this section which says that the system $X(\psi)$ derived from dual pseudo splines forms a Riesz wavelet basis for $L_2(\mathbb{R})$.

Theorem 2.6. *Let ϕ be the dual pseudo spline of order (N, l) with mask $\hat{a}$ defined in* (1.2). *Define*

$$\widehat{\psi}(2\xi) := e^{-i\xi}\overline{\hat{a}(\xi + \pi)}\widehat{\phi}(\xi).$$

Then ψ generates a Riesz wavelet basis in $L_2(\mathbb{R})$. The wavelet ψ has $2N + 1$ vanishing moments.

Proof. Since $\hat{a}$ does not has symmetric zeros, which implies that

$$|\hat{a}(\xi)|^2 + |\hat{a}(\xi + \pi)|^2 \neq 0.$$

Define $R_{N,l}(y) := \sum_i^l \binom{N-1/2+i}{i} y^i$, where $y = \sin^2(\xi/2)$ and N, l are nonnegative integers with $l < N$. Then

$$\begin{aligned} |\hat{a}(\xi)|^2 &= \cos^{2N+2}(\xi/2)\left[\sum_{i=0}^{l}\binom{N-1/2+i}{i}\sin^{2i}(\xi/2)\right]^2 \\ &= (1-y)^{2N+1}R^2(y). \end{aligned}$$

It follows from Lemma 3.4

$$\begin{aligned} \|\tilde{\mathcal{L}}(\xi)\|_\infty &= \sup_{y\in[0,1]} \frac{R(y)}{(1-y)^{2N+1}R^2(y) + (y)^{2N+1}R^2(1-y)} \\ &\leq \frac{\max_{y\in[0,1]} R(y)}{\min_{y\in[0,1]} S(y)} \\ &= \frac{\binom{N+1/2+l}{l}}{(\frac{1}{2})^{2N+2l}\left(\sum_{i=0}^{l}\binom{N+1/2+l}{i}\right)^2} \\ &\leq 2^{2N} \qquad \text{(from Lemma 3.2)} \\ &< 2^{2N+1-1/2}. \end{aligned}$$

By [13], we have $|\hat{a}|^2 + |\hat{a}(\cdot + \pi)|^2 \leq 1$, which implies that $|\mathcal{L}(\xi)| \leq |\tilde{\mathcal{L}}(\xi)|$ for all $\xi \in \mathbb{R}$. By Lemma 2.1, we conclude that ψ generates a Riesz wavelet basis in $L_2(\mathbb{R})$. The second claim follows from the fact that mask $\hat{a}$ has the factor

$$\left(\frac{1-e^{-i\xi}}{2}\right)^{2N+1}.$$

□

Example 2.1. *Let $N = 2$ and $l = 1$. Define ψ by (2.1) with dual pseudo spline $\phi_{2,1}$. Then by Theorem 2.6, $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$. The wavelet ψ has 5 vanishing moments. See Figure 1 for the graphs of function $\phi_{2,1}$ and ψ.*

3 Tight Frame from dual pseudo splines

In this section, we discuss the approximation order of tight wavelets frame from dual pseudo splines via the unitary extension principle. For given $\Psi := \{\psi^1, ...\psi^L\}$, we say that the system

$$X(\Psi) := \{\psi_{j,k},\ \psi \in \Psi,\ j, k \in \mathbb{Z}\}$$

is a *tight frame* for $L_2(\mathbb{R})$ if

$$\|f\|_2^2 = \sum_{g \in X(\Psi)} |\langle f, g\rangle|^2, \quad \forall\ f \in L_2(\mathbb{R}).$$

The generator ψ is called *framelet*. Let ϕ be a compactly supported refinable function in $L_2(\mathbb{R})$ with a mask a such that $\hat{\phi}(0) = 1$, suppose that there exist some 2π-periodic trigonometric polynomials $\widehat{b^l}, l = 1, \ldots, L$, such that

$$|\hat{a}(\xi)|^2 + \sum_{l=1}^{L} |\widehat{b^l}(\xi)|^2 = 1 \quad \text{and} \quad \hat{a}(\xi)\overline{\hat{a}(\xi + \pi)} + \sum_{l=1}^{L} \widehat{b^l}(\xi)\overline{\widehat{b^l}(\xi + \pi)} = 0. \tag{3.1}$$

Define $\psi^l, l = 1, \ldots, L$, by

$$\widehat{\psi^l}(2\xi) = \widehat{b^l}(\xi)\widehat{\phi}(\xi), \quad l = 1, \ldots, L. \tag{3.2}$$

Then the *unitary extension principle* (UEP) asserts that $\Psi = \{\psi^1, \ldots, \psi^L\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$ (see [2, 23, 24]). Following [2], we define the *truncated operator* as

$$\mathcal{Q}_n :\ f \to \sum_{\ell=1}^{L} \sum_{j<n} \sum_{k \in \mathbb{Z}} \langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}.$$

We say that the *operator* $\mathcal{Q}_n$ *provides approximation order* m_1, if for all f in the Sobolev space $W_2^{m_1}(\mathbb{R})$, such that

$$\|f - \mathcal{Q}_n f\|_2 = O(2^{-nm_1}).$$

The approximation order of the truncated operator $\mathcal{Q}_n$ was studied in [2]. Recall that a function ϕ satisfies the Strang-Fix condition of order m if

$$\hat{\phi}(0) \neq 0, \quad \hat{\phi}^{(j)}(2\pi k) = 0, \quad j = 0, 1, 2, \ldots, m - 1,\ k \in \mathbb{Z} \setminus \{0\}.$$

Suppose that the refinable function ϕ satisfies the Strang-Fix condition of order m_0 and the corresponding mask $\hat{a}$ satisfies

$$1 - |\hat{a}|^2 = O(|\cdot|^{m_2})$$

at the origin, then it was pointed out in [2] that $m_1 = \min\{m_0, m_2\}$.

Theorem 3.1. *Let ϕ be a dual pseudo spline of order (N, l) with mask a as in* (1.2). *Let $\Psi := \{\psi^1, ..., \psi^L\}$, where*

$$\hat{\psi}^l(2\xi) = \hat{b}^l(\xi)\hat{\phi}(\xi), l = 1, 2, ..., L$$

and $\hat{b}^l$ satisfy the conditions in (4.1) *for* $l = 1, 2, ..., L$. *Then the system*

$$X(\Psi) := \{\psi_{j,k},\ \psi \in \Psi,\ j, k \in \mathbb{Z}\}$$

is a tight frame *for* $L_2(\mathbb{R})$. *Furthermore, the corresponding truncated operator* $\mathcal{Q}_n$ *provides approximation order* $2l + 2odd_l$, *where* $odd_l := \frac{1-(-1)^l}{2}$.

Proof. It was showed in [8, Theorem 8] that

$$\hat{a}(\xi) = e^{-i\xi/2} + O(|\xi|^{2l+2odd_l}).$$

This implies that

$$1 - |\hat{a}| = O(|\cdot|^{2l+2odd_l}).$$

Note that

$$1 - |\hat{a}|^2 = (1 - |\hat{a}|)(1 + |\hat{a}|),$$

therefore, there exists a positive constant C such that

$$C|\cdot|^{2l+2odd_l} \leq 1 - |\hat{a}|^2 \leq 2C|\cdot|^{2l+2odd_l}.$$

It means that $1 - |\hat{a}|^2 = O(|\cdot|^{2l+2odd_l})$.

Since mask $\hat{a}$ has the factor $\left(\frac{1-e^{-i\xi}}{2}\right)^{2N+1}$, ϕ satisfies the Strang-Fix condition of order $2N+1$. Hence, the truncated operator $\mathcal{Q}_n$ provides approximation order

$$\min\{2N + 1, 2l + 2odd_l\} = 2l + 2odd_l.$$

□

In the end of this section, we provide some examples to illustrate Theorem 3.1. The construction follows from what was given in [4].

Example 3.1. *We choose $\hat{a}$ to be the mask of the dual pseudo spline with order* (2, 1), *i.e.*

$$\hat{a}(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^5 \left(1 + \frac{5}{2}\sin^2(\xi/2)\right).$$

We define

$$\begin{aligned} \widehat{b^1}(\xi) &:= e^{-i\xi}\overline{\hat{a}(\xi + \pi)}, \\ \widehat{b^2}(\xi) &:= \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi), \\ \widehat{b^3}(\xi) &= e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi), \end{aligned}$$

where

$$\mathcal{A}(\xi) = \frac{1}{128}\sqrt{\frac{35}{9 + 4\sqrt{5}}}(-1 + e^{-2i\xi})^2(-9 - 4\sqrt{5} + e^{-2i\xi}).$$

The above masks satisfy the UEP conditions (3.1), hence the system $X(\Psi) := \{\psi^1_{j,k}, \psi^2_{j,k}, \psi^3_{j,k}, j, k \in \mathbb{Z}\}$ *is a tight frame. Furthermore, the approximation order of the truncated operator* $\mathcal{Q}_n$ *is* 4.

Example 3.2. *We choose $\hat{a}$ to be the mask of the dual pseudo spline with order* $(3,1)$*, i.e.*

$$\hat{a}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^7 \left(1 + \frac{7}{2}\sin^2(\xi/2)\right).$$

We define

$$\begin{aligned}
\widehat{b^1}(\xi) &:= e^{-i\xi}\overline{\hat{a}(\xi+\pi)},\\
\widehat{b^2}(\xi) &:= \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi),\\
\widehat{b^3}(\xi) &= e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}(\xi) = \frac{1}{2}e^{4i\xi}\big(&- 0.002950560718227624 + 0.012538064132321055e^{-2i\xi}\\
&+ 0.22812835732470788e^{-4i\xi} - 0.4820686641734686e^{-6i\xi}\\
&+ 0.2443528034346672e^{-8i\xi}\big).
\end{aligned} \tag{3.3}$$

The above masks satisfy the UEP conditions (3.1), hence the system $X(\Psi) := \{\psi^1_{j,k}, \psi^2_{j,k}, \psi^3_{j,k}, j, k \in \mathbb{Z}\}$ *is a tight frame. Furthermore, the approximation order of the truncated operator* $\mathcal{Q}_n$ *is* 4.

The dual pseudo-spline of order (N,l) is smoother than the pseudo-spline of order (N,l) for small N and l yet its support is larger by one [3]. As we can see from table, the dual pseudo splines fill the gaps between the pseudo splines with different orders in some sense. This give us a wide range of choices of the refinable functions that meets various demands for balancing the approximation power, the length of the support, and the regularity in applications.

Table 1:

	support of $\phi, \psi^1, \psi^2, \psi^3$	approximation order	regularity exponents
Example 3.1	$[0,7], [0,7], [0,9], [0,9]$	4	2.8301
Example in [4]	$[0,8], [0,8], [0,9], [0,9]$	4	3.6781
Example 3.2	$[0,9], [0,9], [0,9], [0,9]$	4	4.5406

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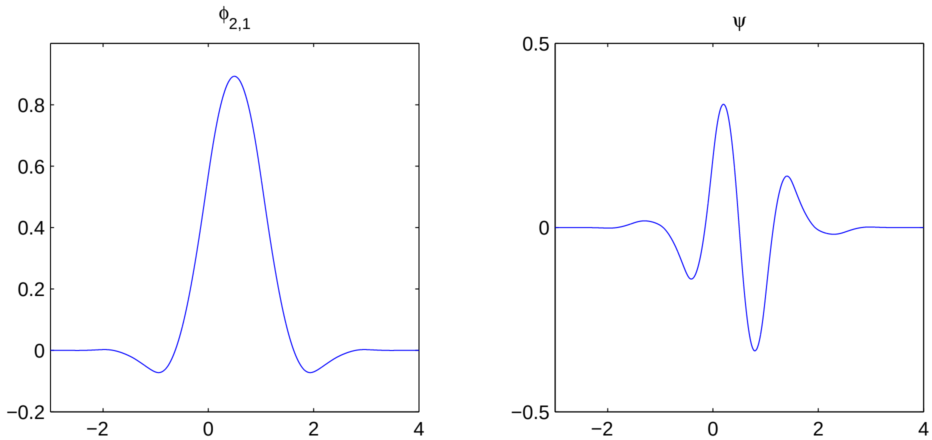

Figure 1: The graph of the dual pseudo spline $\phi_{2,1}$ (left) and the graph of the wavelet function (right) in Example 2.1. The wavelet system $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

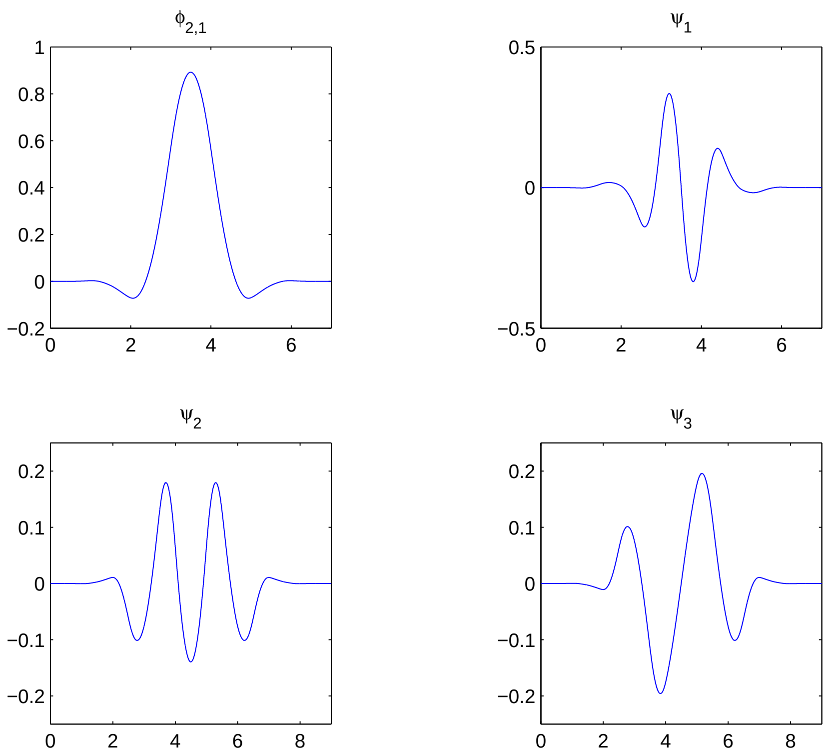

Figure 2: $\phi_{2,1}$ is the dual pseudo spline with order $(2,1)$. ψ^1, ψ^2 and ψ^3 are the graphs of the framelets in Example 3.1. The wavelet system $X(\psi)$ is a tight wavelet frame $L_2(\mathbb{R})$.

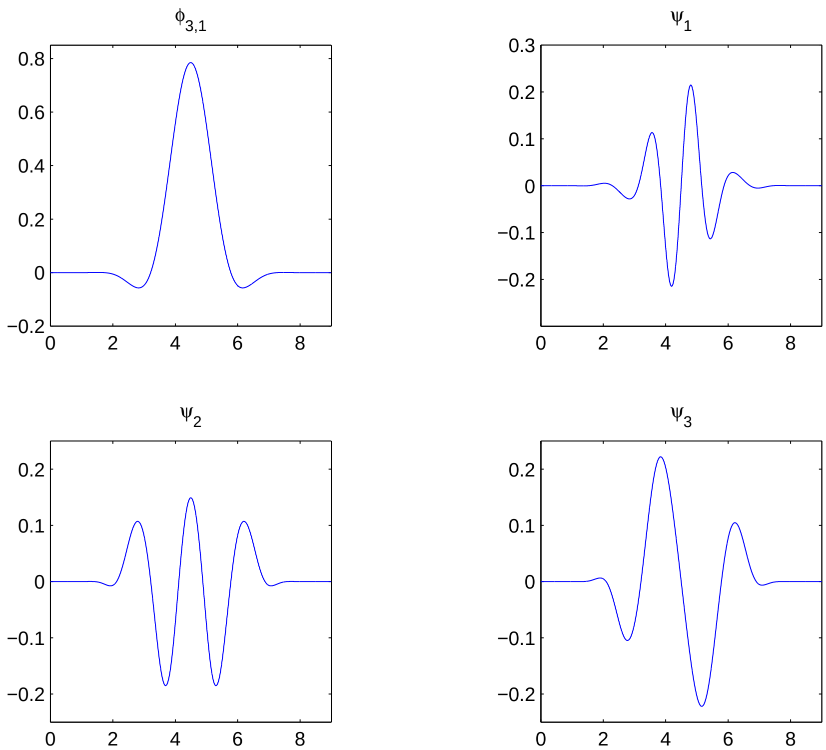

Figure 3: $\phi_{3,1}$ is the dual pseudo spline with order $(3,1)$. ψ^1, ψ^2 and ψ^3 are the graphs of the framelets in Example 3.2. The wavelet system $X(\psi)$ is a tight wavelet frame $L_2(\mathbb{R})$.

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