Local Estimates for Some Elliptic Equations
Arising from Conformal Geometry

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Abstract

We present some local gradient and Hessian estimates from $C^0$ estimates for some elliptic equations arising from conformal deformations on the manifolds with totally geodesic boundary, which generalize Sophie Chen’s corresponding results.

1 Introduction

Let $(M^n, g)$ be a smooth, compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor of $g$ is defined by

$$A_g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{R_g}{2(n-1)} g \right),$$

where $\text{Ric}_g$ and $R_g$ are the Ricci and scalar curvatures of $g$, respectively. The $k$-curvature (or $\sigma_k$ curvature) is defined to be the $k$-th elementary symmetric function $\sigma_k$ of the eigenvalues $\lambda(g^{-1}A_g)$ of $g^{-1}A_g$. If $\tilde{g} = e^{-2u}g$ is a metric conformal to $g$, the Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g,$$

where $\nabla u$ and $\nabla^2 u$ denote the gradient and Hessian of $u$ with respect to $g$. Consequently, the problem of conformally deforming a given metric to one with prescribed $\sigma_k$-curvature reduces to solving the partial differential equation

$$\sigma_k \left( \lambda \left( g^{-1} \left[ \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g \right] \right) \right) = \psi(x)e^{-2ku}. \quad (1.1)$$

For compact manifolds without boundary, the existence of the solutions to the equation (1.1) has been studied by many authors (see [CGY1], [CGY2], [GW2], [GW3], [GW4], [GW5],

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[LL1], [LL2], [GV1], [GV2], [TW], [STW], [GeW], [V2] etc.) since these equations were first introduced by J. A. Viaclovsky [V1]. $C^1$ and $C^2$ estimates have also been studied extensively, see [Cn1], [GW1], [GW2], [LL1], [STW], [W2] for local interior estimates and [V2] for global estimates.

Another interesting problem is to study the fully nonlinear equation (1.1) on a compact Riemannian manifold $(M^n, g)$ with boundary $\partial M$. In [G], Bo Guan studied the existence problem under the Dirichlet boundary condition. There are many pioneer works on the Dirichlet problems for fully nonlinear elliptic equations, see [CNS], [Tr2] etc.. The Neumann problem for (1.1) has been studied by S. S. Chen [Cn2, Cn3], Jin-Li-Li [JLL] and Li-Li [LL3], etc.. Under various conditions, they derive local estimates for solutions and establish some existence results. Before introducing the problem, we need the following definitions.

Define

$$\Gamma_k = \{ \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, 1 \leq j \leq k \},$$

and $1 \leq k \leq n$, where $\sigma_k$ is the $k$-th elementary symmetric function defined by

$$\sigma_k(\Lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

for all $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. We also denote $\sigma_0 = 1$. Therefore we have the relation $\Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1$. For a 2-symmetric form $S$ defined on $(M^n, g)$, $S \in \Gamma_k$ means that the eigenvalues of $S, \lambda(g^{-1}S)$ lie in $\Gamma_k$. We also denote $\Gamma_{-k} = -\Gamma_k$.

Let $(M^n, g), n \geq 3,$ be a smooth compact Riemannian manifold with nonempty smooth boundary $\partial M$. We denote the mean curvature and the second fundamental form of $\partial M$ by $h_g$ and $L_{\alpha\beta}$, where $\{x^a\}_{1 \leq a \leq n-1}$ is the local coordinates on the boundary $\partial M$, and $\frac{\partial}{\partial x^n}$ the unit inner normal with respect to the metric. In this paper we use Fermi coordinates in a boundary neighborhood. In this local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and $(x^1, ..., x^{n-1})$ forms a local chart on the boundary. The metric can be expressed as $g = g_{\alpha\beta}dx^\alpha dx^\beta + (dx^n)^2$. The Greek letters $\alpha, \beta, \gamma, ...$ stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma, ... \leq n-1$, while the Latin letters $i, j, k, ...$ stand for the full indices, $1 \leq i, j, k, ... \leq n$. In Fermi coordinates, the half ball is defined by $\overline{B}_r = \{x_n \geq 0, \sum_i x_i^2 \leq r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \leq r^2\}$. Under the conformal change of the metric $\tilde{g} = e^{-2u}g$, the second fundamental form satisfies

$$\tilde{L}_{\alpha\beta}e^u = \frac{\partial u}{\partial \nu}g_{\alpha\beta} + L_{\alpha\beta}.$$
The boundary is called umbilic if the second fundamental form $L_{\alpha\beta} = \tau_g g_{\alpha\beta}$, where $\tau_g$ is the function defined on $\partial M$. Since the boundary $\partial M$ is connected, by Schur Theorem, $\tau_g = \text{const.}$. A totally geodesic boundary is umbilic with $\tau_g = 0$. Note that the umbilicity is conformally invariant. When the boundary is umbilic, the above formula becomes

$$\tau_g e^{-u} = \frac{\partial u}{\partial \nu} + \tau_g.$$ 

The $k$-Yamabe problem with umbilic boundary becomes to consider the following equation:

$$\left\{ \begin{array}{l}
\frac{1}{k} \left( \lambda \left( g^{-1} \left[ \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g \right] \right) \right) = e^{-2u} \quad \text{in } M, \\
\frac{\partial u}{\partial \nu} = \tau g e^{-u} - \tau_g \quad \text{on } \partial M.
\end{array} \right. \quad (1.2)$$

In [Cn2, Cn3] and [JLL], the authors established the a priori estimates and obtained some existence results for (1.2).

In this paper, we will generalize their results to more general equations, which in particular include the equation (1.2). In [GV3], Gursky and Viaclovsky introduced a modified Schouten tensor $A^t_g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{t R_g}{2(n-1)} g \right)$, where $t \in \mathbb{R}$ is a parameter. When $t = 1$, $A^1_g$ is just the Schouten tensor; $t = n - 1$, $A^{n-1}_g$ is the Einstein tensor; while $t = 0$, $A^0_g$ is the Ricci tensor. Under the conformal change of the metric $\tilde{g} = e^{-2u} g$, $A^t_g$ satisfies

$$A^t_g = A^t_g + \nabla^2 u + \frac{1 - t}{n - 2} (\Delta u) g + du \otimes du - \frac{2 - t}{2} |\nabla u|^2 g.$$ 

In [LS] and [SZ], we have studied

$$\sigma_k \left( \lambda \left( g^{-1} \left[ A^t_g + \nabla^2 u + \frac{1 - t}{n - 2} (\Delta u) g + du \otimes du - \frac{2 - t}{2} |\nabla u|^2 g \right] \right) \right) = f(x) e^{-2ku}$$

for $t \leq 1$ or $t \geq n - 1$. By use of the parabolic approach, we obtained some existence results. Let $(M, g)$ be a compact, connected Riemannian manifold of dimension $n \geq 3$ with umbilic boundary $\partial M$, $W$ be a $(0, 2)$ symmetric tensor on $(M^n, g)$. Motivated by [Cn1], in this paper we study the following equation

$$\left\{ \begin{array}{l}
F(g^{-1} W) = f(x, u) \quad \text{in } M \\
\frac{\partial u}{\partial \nu} = \tau g e^{-u} - \tau \quad \text{on } \partial M
\end{array} \right. \quad (1.3)$$

where $F$ satisfies some fundamental structure conditions listed later, and $\tau$ is the principal curvature of the boundary $\partial M$. We will establish local a priori estimates for the solutions to the equation (1.3). After that, we will give some applications.
We now describe the fundamental structure conditions for \( F \).

Let \( \Gamma \) be an open convex cone with vertex at the origin satisfying \( \Gamma_n \subset \Gamma \subset \Gamma_1 \). Suppose that \( F(\lambda) \) is a homogeneous symmetric function of degree one in \( \Gamma \) normalized with \( F(e) = F((1, \cdots , 1)) = 1 \). Moreover, \( F \) satisfies the following in \( \Gamma \):

(A1) \( F \) is positive.

(A2) \( F \) is concave (i.e., \( \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \) is negative semi definite).

(A3) \( F \) is monotone (i.e., \( \frac{\partial F}{\partial \lambda_i} \) is positive).

(A4) \( \frac{\partial F}{\partial \lambda_i} \geq \epsilon \frac{F}{\sigma_1}, \) for some constant \( \epsilon > 0 \), for all \( i \).

We also need an additional condition on the boundary \( \partial M \) in the Fermi coordinates:

(B) either \( \rho_1 F_{nn} \geq \sum F^{\alpha \alpha} \) or \( \rho_2 F_{nn} \leq \sum F^{\alpha \alpha} \), for some \( \rho_1, \rho_2 > 0 \).

The conditions (A1), (A2), (A3) and (A4) are similar with those in Szu-yu Sophie Chen [Cn2] and [Cn3]. While condition (B) is in some extent different, however, we will show it is reasonable later for the case that \( F = \sigma_1 / k \).

Before stating the theorems, we introduce the following notations. Let \( f(x,z) : M^n \times \mathbb{R} \rightarrow \mathbb{R} \) be given positive function. Let \( u(x) : M^n \rightarrow \mathbb{R} \) be a solution to

\[(1.3)\]

We define

\[c^{\sup}(r) = \sup_{B_r^+} (f + |\nabla x f(x,u)| + |f_x(x,u)| + |\nabla^2 x f(x,u)| + |\nabla z f(x,u)| + |f_z(x,u)|)\]

or

\[c^{\sup}(r) = \sup_{B_r} (f + |\nabla x f(x,u)| + |f_x(x,u)| + |\nabla^2 x f(x,u)| + |\nabla z f(x,u)| + |f_z(x,u)|),\]

which varies with boundary or interior estimates.

Now we turn to the first equation we are going to discuss: let

\[ W = \nabla^2 u + \frac{1-t}{n-2} (\triangle u) g + a(x) du \otimes du + b(x) |\nabla u|^2 g + S, \quad (1.4) \]

where \( t \) is a constant satisfying \( t < 1 \), \( S \) a 2-symmetric form defined on \( M \), and \( a(x) \), \( b(x) \) are two smooth functions on \( M \). The derivatives are covariant derivatives with respect to the metric \( g \). We have

**Theorem 1.** Let \( F \) satisfy the structure conditions (A1)-(A4), additional condition (B) in a corresponding cone \( \Gamma \). Suppose that the boundary \( \partial M \) is totally geodesic on the boundary. Let \( u(x) \) be a \( C^4 \) solution to the equation

\[
\begin{align*}
\{ F(\nabla^2 u + \frac{1-t}{n-2} \triangle u g + a(x) du \otimes du + b(x) |\nabla u|^2 g + S) = f(x,u) & \quad \text{in } \overline{B_r^+}, \\
\frac{\partial u}{\partial x_n} = 0 & \quad \text{on } \Sigma_r.
\end{align*}
\]

(1.5)
Suppose that $|\nabla f| < \Lambda f$, $|f_0| < \Lambda f$ for some constant $\Lambda > 0$.

Case (a)

I. If $\Gamma \subset \Gamma_2^+$, $\frac{1}{n-2} a(x) - b(x) > \delta_1 > 0, \min \{(a + b)^2, b^2\} \geq \delta_2 > 0$ and $t < 1$ then
\[
\sup_{B_2^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,
\]
where $C$ depends only on $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_2, ||a||_{C^2(\overline{B}^+)}$, $||b||_{C^2(\overline{B}^+)}$, $||S||_{C^2(\overline{B}^+)}$, $||g||_{C^3(\overline{B}^+)}$ and $\overline{r}_{sup}(r)$.

II. If $t = 1, \Gamma \subset \Gamma_2^+$, $-b > \delta_1 > 0$, $\min \{(a + b)^2, b^2\} \geq \delta_2 > 0$, and $a = \text{const.}, b = \text{const.}, S = A$ (Schouten tensor), then
\[
\sup_{B_2^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,
\]
where $C$ depends only on $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_2, ||a||_{C^2(\overline{B}^+)}$, $||b||_{C^2(\overline{B}^+)}$, $||S||_{C^2(\overline{B}^+)}$, $||g||_{C^3(\overline{B}^+)}$ and $\overline{r}_{sup}(r)$.

Case (b)

I. If we have $\frac{1}{n-2} a(x) - b(x) > \delta_1 > 0$, $a(x) + nb(x) < -\delta_3 < 0, a(x) \geq 0$ and $t < 1$, then
\[
\sup_{B_2^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,
\]
where $C$ depends on $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(\overline{B}^+)}$, $||b||_{C^2(\overline{B}^+)}$, $||S||_{C^2(\overline{B}^+)}$, $||g||_{C^3(\overline{B}^+)}$ and $\overline{r}_{sup}(r)$.

II. If we have $t = 1, -b > \delta_1 > 0$, $a + nb < -\delta_3 < 0, a \geq 0$ and $a = \text{const.}, b = \text{const.}, S = A$ (Schouten tensor), then
\[
\sup_{B_2^+} (|\nabla^2 u| + |\nabla u|^2) \leq C,
\]
where $C$ depends on $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(\overline{B}^+)}$, $||b||_{C^2(\overline{B}^+)}$, $||S||_{C^2(\overline{B}^+)}$, $||g||_{C^3(\overline{B}^+)}$ and $\overline{r}_{sup}(r)$.

When $t = 1$, the local interior estimates have been discussed by S. Chen in [Cn1].

When $t = 1$, and $a(x) = 1, b(x) = -\frac{1}{2}$, the boundary estimates have been obtained by S. Chen [Cn2, Cn3] and Jin-Li-Li [JLL]. If we just focus on the interior estimates for the same equation, we may get

**Corollary 1.** Let $F$ satisfy the structure conditions (A1)-(A4) in a corresponding cone $\Gamma$. And $u(x)$ be a $C^4$ solution to the equation
\[
F(g^{-1}(-\nabla^2 u + \frac{1}{n-2} \Delta u - g + a(x)du \otimes du + b(x)|\nabla u|^2g + S)) = f(x, u) \quad \text{in} \quad M \quad (1.6)
\]
in a local geodesic ball $B_r$. Suppose that $|\nabla f| < \Lambda f$, $|f_z| < \Lambda f$ for some constant $\Lambda > 0$.

Case (a). If $\Gamma \subset \Gamma^+_2$, $1 - \frac{1}{n-2} a(x) - b(x) > \delta_1 > 0$ and $\min \{(a + b)^2, b^2\} \geq \delta_2 > 0$, then

$$\sup_{B^+_2} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where $C$ depends only on $r, n, \Lambda, \delta_1, \delta_2, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}, ||g||_{C^3(B_r)}$ and $c_{\sup}(r)$.

Case (b). If we have $\frac{1}{n-2} a(x) - b(x) > \delta_1 > 0$ and $a(x) + nb(x) < -\delta_3 < 0$. Then we have

$$\sup_{B^+_2} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where $C$ depends on $r, n, \epsilon, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}, ||g||_{C^3(B_r)}$ and $c_{\sup}(r)$.

The a priori estimate in Theorem 1 and Corollary 1 relies on the signs of $a(x)$ and $b(x)$. In fact, in [STW] the authors give a counterexample to show that there is no regularity if $a(x) = 0$ and $b(x)$ is positive. It is well known that the equation (1.6) has another elliptic branch, namely when the eigenvalues $\lambda$ lie in the negative cone $\Gamma^-_k$. Now we consider the second equation. Let

$$V = \frac{t - 1}{n - 2} (\triangle u) g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S,$$

where $t$ is a constant satisfying $t > n - 1$. The derivatives are covariant derivatives with respect to the metric $g$. We have

**Theorem 2.** Let $F$ satisfy the structure conditions (A1)-(A4), additional condition (B) in a corresponding cone $\Gamma$. Suppose that the boundary $\partial M$ is totally geodesic on the boundary. Let $u(x)$ be a $C^4$ solution to the equation

$$\begin{cases}
F(g^{-1}(\frac{t - 1}{n - 2}) (\triangle u) g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S) = f(x, u) & \text{in } B^+_2,
\frac{\partial u}{\partial x^r} = 0 & \text{on } \Sigma_r.
\end{cases}$$

(1.7)

Suppose that $|\nabla f| < \Lambda f$, $|f_z| < \Lambda f$ for some constant $\Lambda > 0$.

Case (a). If $\Gamma \subset \Gamma^+_2$, $\frac{t - 1}{n - 2} a(x) + b(x) > \delta_1 > 0$, $\min \{(a + b)^2, b^2\} \geq \delta_2 > 0$, and $\frac{1}{\rho^2} - \frac{t - 1}{n - 2} < 0$, then

$$\sup_{B^+_2} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where $C$ depends on $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_2, ||a||_{C^2(B^+_2)}, ||b||_{C^2(B^+_2)}, ||S||_{C^2(B^+_2)}, ||g||_{C^3(B^+_2)}$ and $c_{\sup}(r)$. 

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Corollary 2. Let $F$ satisfy the structure conditions (A1)-(A4) in a corresponding cone $\Gamma$. And $u(x)$ be a $C^4$ solution to the equation

$$F(\Lambda^{-1}(\frac{n-1}{n-2}\triangle u) g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S) = f(x,u) \quad \text{in } M$$

(1.8)

in a local geodesic ball $B_r$. Suppose that $|\nabla f| < \Lambda f$, $|f_z| < \Lambda f$ for some constant $\Lambda > 0$.

Case (a). If $\Gamma \subset \Gamma^+_{\delta_1}$, $\frac{t-1}{n-2}a(x) + b(x) > \delta_1 > 0$, $\min \{ (a + b)^2, b^2 \} \geq \delta_2 > 0$, then

$$\sup_{\overline{B}_r} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where $C$ depends only on $r, n, \Lambda, \delta_1, \delta_2, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}$, and $c_{sup}(r)$.

Case (b). If we have $\frac{t-1}{n-2}a(x) + b(x) > \delta_1 > 0$, $a(x) + nb(x) > \delta_3 > 0$, then

$$\sup_{\overline{B}_r} (|\nabla^2 u| + |\nabla u|^2) \leq C,$$

where $C$ depends on $r, n, \Lambda, \delta_1, \delta_3, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}$, and $c_{sup}(r)$.

This paper is organized as follows. We would begin with some background in Section 2. In Section 3, we will discuss the applications which are based on the a priori estimates in Theorem 1 and corollary 1 to get the existent result of $k$-Yamabe problem. The proof of Theorem 1 and Theorem 2 are in Section 4 and Section 5 respectively.

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2 Preliminaries

In this section, we give some basic facts about homogeneous symmetric functions and show some outcomes by direct calculation under Fermi coordinates.

From Lemma 1 and Lemma 2 below, we can conclude that $F$ satisfies (A1)-(A4).

Lemma 1. (see [Cn1]) Let $\Gamma$ be an open convex cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma$, and let $e = (1, \cdots, 1)$ be the identity. Suppose that $F$ is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that $F$ is concave in $\Gamma$. Then

(a) $\sum_i \lambda_i \frac{\partial F}{\partial \lambda_i} = F(\lambda)$, for $\lambda \in \Gamma$;
(b) $\sum_i \frac{\partial F}{\partial \lambda_i} \geq F(e) = 1$, for $\lambda \in \Gamma$.

Lemma 2. (see [Tr2, LT]) Let $G = (\sigma_k^\gamma)_{1 \leq l < k \leq n}$, $0 \leq l < k \leq n$.

(a) $G$ is positive and concave in $\Gamma_k$;
(b) $G$ is monotone in $\Gamma_k$, i.e., the matrix $G_{ij} = \frac{\partial G}{\partial W_{ij}}$ is positive definite;
(c) Suppose $\lambda \in \Gamma_k$. For $0 \leq l < k \leq n$, the following is the Newton-Maclaurin inequality

$$k(n-l+1)\sigma_{l-1} \sigma_k \leq l(n-k+1)\sigma_l \sigma_{k-1}.$$  

The next lemma is a modification of Lemma 4 in [Cn2], which implies that $(\frac{n}{k})^{\frac{1}{2}} \sigma_k$ satisfies the additional condition (B) with $\rho_1 = \rho_2 = (n-k)$.

Lemma 3. (see [Cn2]) For $1 \leq k \leq n-1$, if $\lambda \in \Gamma_k$.

(a) if $\lambda_i \leq 0$, then $\sum_{j \neq i} \frac{\partial \sigma_k}{\partial \lambda_j} \leq (n-k) \frac{\partial \sigma_k}{\partial \lambda_i}$;
(b) if $\lambda_i \geq 0$, then $\sum_{j \neq i} \frac{\partial \sigma_k}{\partial \lambda_j} \geq (n-k) \frac{\partial \sigma_k}{\partial \lambda_i}$.

The following two lemmas will be used in proving Theorem 1 and 2. Let us review some formulae on the boundary under Fermi coordinates (see [Cn3]). The metric is expressed as $g = g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2$. The Christoffel symbols satisfy

$$\Gamma^\gamma_{\alpha\beta} = L_{\alpha\beta}, \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma} g^{\gamma\beta}, \Gamma^n_{\alpha n} = 0, \Gamma^n_{\alpha\beta} = \Gamma^n_{\beta\alpha},$$
on the boundary, where we denote the tensors and covariant derivatives with respect to the induced metric on the boundary by a tilde (e.g. $\Gamma^\gamma_{\alpha\beta}$, $\tau_{\alpha\beta}$). When the boundary is umbilic, we have

$$\Gamma^n_{\alpha\beta} = \tau g_{\alpha\beta}, \Gamma^\beta_{\alpha n} = -\tau \delta_{\alpha\beta}, \Gamma^n_{\alpha n} = 0.$$

Lemma 4. (see [Cn3]) Suppose boundary $\partial M$ is umbilic. Let $u$ satisfy $u_n := \frac{\partial u}{\partial x^n} = 8$.
$-\tau + \tilde{\tau} e^{-u}$, where $\tilde{\tau}$ is constant. Then on the boundary we have

$$u_{n\alpha} = -\tau_\alpha + \tau u_\alpha - \tilde{\tau} u_\alpha e^{-u};$$  \hspace{1cm} (2.1)

and

$$u_{\alpha\beta n} = (2\tau - \tilde{\tau} e^{-u})u_{\alpha\beta} - \tau u_{mn}g_{\alpha\beta} + \tilde{\tau} u_\alpha u_\beta e^{-u} - \tau_{\alpha\beta} + \tau u_\beta + \tilde{\tau} u_\alpha - \tau u_\gamma g_{\alpha\gamma} + R_{n\beta\alpha n}(-\tau + \tilde{\tau} e^{-u}) - \tau(-\tau + \tilde{\tau} e^{-u})^2 g_{\alpha\beta}. \hspace{1cm} (2.2)$$

**Lemma 5.** Suppose the boundary $\partial M$ is totally geodesic and $u_n = 0$ on the boundary. Then we have on the boundary

$$W_{\alpha\beta n} = \frac{1-t}{n-2} u_{n\alpha n} g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n \left( \Sigma_\gamma u_\gamma^2 \right) g_{\alpha\beta} + S_{\alpha\beta n}, \hspace{1cm} (2.3)$$

$$V_{\alpha\beta n} = \frac{t-1}{n-2} u_{n\alpha} u_\beta - a_n u_\alpha u_\beta - b_n \left( \Sigma_\gamma u_\gamma^2 \right) g_{\alpha\beta} + S_{\alpha\beta n}. \hspace{1cm} (2.4)$$

**Proof.** By the boundary condition we know that $\tilde{\tau} = \tau = 0$. From formulas (2.1) and (2.2) we have $u_{n\alpha} = 0$ and $u_{\alpha\beta n} = 0$. Then

$$W_{\alpha\beta n} = u_{\alpha\beta n} + \frac{1-t}{n-2} \Sigma_k u_{k\alpha n} g_{\alpha\beta} + a_n u_\alpha u_\beta + a u_{\alpha\beta n} + a_n u_\alpha u_\beta + 2b \Sigma_k u_{k\alpha n} g_{\alpha\beta} + b_n \left( \Sigma_\gamma u_\gamma^2 \right) g_{\alpha\beta} + S_{\alpha\beta n}$$

$$= 1 - \frac{t}{n-2} u_{n\alpha n} g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n \left( \Sigma_\gamma u_\gamma^2 \right) g_{\alpha\beta} + S_{\alpha\beta n}. \hspace{1cm} (2.3)$$

For $V_{ij}$ we can get the equalities in the same way.

\[ \square \]

**3 Applications**

We denote $[g] = \{ \hat{g} \mid \hat{g} = e^{-2u} g \}$ and $[g]_k = \{ \hat{g} \mid \hat{g} \in [g] \cap \Gamma \}$. We call $g$ is $k$-admissible if and only if $[g]_k \neq \emptyset$. Now the first Yamabe constant on Riemannian manifold $(M^n, g)$ with nonempty boundary $\partial M$ can be defined as

$$\mathcal{Y}_1[g] = \inf_{\hat{g} \in [g], \text{vol}(\hat{g})=1} \left( \int_M R_{\hat{g}} + \frac{1}{\text{vol}(\hat{g})} \int_{\partial M} h_{\hat{g}} \right).$$
We may define the boundary curvature $B^k$ for the manifold with umbilical boundary and higher order Yamabe constants $\mathcal{Y}_k[g]$ for $2 \leq k < n/2$ as follows (see [Cn3, GLW, S], compare with the definition in [STW]):

$$B^k = \sum_{i=0}^{k-1} C(n, k, i) \sigma_i \left( g^{-1} A^T \right) \tau^{2k-2i-1},$$

and

$$\mathcal{Y}_k[g] = \left\{ \begin{array}{ll} \inf_{g \in [g]_{k-1}; \text{vol}(\hat{g})=1} \mathcal{F}_k & \text{if } [g]_k \neq \emptyset \\
-\infty & \text{if } [g]_k = \emptyset \end{array} \right.$$

where $C(n, k, i) = \frac{(n-i-1)!}{(n-k)!(2k-2i-1)!}$, $A^T = [A_{\alpha\beta}]$ is the tangential part of the Schouten tensor, $\tau$ is a constant satisfying $L_{\alpha\beta} = \tau g_{\alpha\beta}$, and

$$\mathcal{F}_k(\hat{g}) = \int_M \sigma_k(\hat{g}^{-1} A_{\hat{g}}) + \int_{\partial M} B^k_{\hat{g}}.$$

By Theorem 1 we can get the following Theorem 3 and 4.

**Theorem 3.** ([Cn3]) Let $(M, \partial M, g)$ is a compact manifold of dimension $n \geq 3$ with boundary and $\partial M$ is totally geodesic. Suppose that $2 \leq k < n/2$ and $Y_1, Y_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $A_{\hat{g}} \in \Gamma_k$ and $B^k_{\hat{g}} = 0$.

**Proof.** Following proof is mainly from [GV1]. Compare with [S], we may prove this theorem by continuity method. Consider a family of equations involving a parameter $t$,

$$\left\{ \begin{array}{ll} \sigma^{1/k}(g^{-1} A_{\hat{g}}^t) = f(x) e^{2ut} & \text{in } M \\
\frac{\partial u}{\partial t} = 0 & \text{on } \partial M \end{array} \right. \quad (3.1)$$

where $\hat{g} = e^{-2u} g$, $f(x) > 0$ and $t \leq 1$. Since $\mathcal{Y}_k > 0$, then $[g]_{k-1} \neq \emptyset$. We may assume $g \in [g]_{k-1}$. Therefore the scalar curvature $R_{\hat{g}} > 0$. Then there exists $a > -\infty$ so that $A_{\hat{g}}^a$ is positive definite. For $t \in [a, 1]$, we consider the deformation

$$\left\{ \begin{array}{ll} \sigma^{1/k}(g^{-1} A_{u_t}^t) = f(x) e^{2ut} & \text{in } M \\
\frac{\partial u}{\partial t} = 0 & \text{on } \partial M \end{array} \right. \quad (3.2)$$

where $A_{u_t}^t = A_{\hat{g}}^t$ with $\hat{g} = e^{-2ut} g$, $f(x) = \sigma^{1/k} (g^{-1} A_{u_a}^a) > 0$ and $u_a \equiv 0$ is a solution of (3.2) for $t = a$. Let

$$I = \left\{ t \in [a, t_0] | \exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (3.2) with } A_{u_t} \in \Gamma_k \right\}.$$

It is easy to prove that the linearized operator $L_t : C^{2,\alpha}(M) \cap \{ \frac{\partial u}{\partial t} = 0 \} \rightarrow C^{\alpha}(M)$ is invertible. This together with the implicit theorem imply that the set $I$ is open.

Since Theorem 1 case (b) implies the $C^1$ and $C^2$ estimates of the solution to (3.2) which depend only on the upper bound of $u$. Since $A^t = A^1 + \frac{1-t}{n-2} \sigma_1(A^1) g$, at the
maximal point $x_0$ of $u$, we have $|\nabla u| = 0$ and $\nabla^2 u_t(x_0)$ is negative semi-definite, no matter $x_0$ being interior or boundary point. Hence,

$$f(x_0)^k e^{2ku(x_0)} = \sigma_k \left( g^{-1} A_g^2 u \right) \leq \sigma_k \left( g^{-1} \left( A + \frac{1-t}{n-2} \sigma_1(A) g \right) \right) \leq C,$$

where we use $\sigma_1(A) > 0$ and $a \leq t \leq 1$. We then get the upper bound. By the gradient estimate, we may easily get the lower bound of $u$. Therefore we conclude that $I = [a, 1]$. We thus finish the proof.

\[\square\]

If $(M, g)$ is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Then by [E], we may assume that the background metric $g$ is a Yamabe metric with its constant scalar curvature $R > 0$ and the boundary is totally geodasic. Then using the same proof of Theorem 3, we may prove that there exists a metric $\hat{g} \in [g]$ such that $A_{\hat{g}} \in \Gamma_k$ and $B_{\hat{g}}^k = 0$. By [Cn2], we can get the following existence result.

**Theorem 4.** ([Cn3]) Let $(M, \partial M, g)$ is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Suppose that $2 \leq k < n/2$ and $Y_1, Y_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $\sigma_k(A_{\hat{g}}) = 1$ and $B_{\hat{g}}^k = 0$.

4 Proof of Theorem 1

(1) Proof in case (a).

Let $K = \triangle u + a|\nabla u|^2$. Note that $\Gamma \subset \Gamma_1^+$, we can immediately get

$$0 \leq tr(W) = (1 + n \frac{1-t}{n-2}) \triangle u + (a + nb)|\nabla u|^2 + tr S \leq (1 + n \frac{1-t}{n-2}) K - n\delta_1 |\nabla u|^2 + C.$$ 

Then

$$(1 + n \frac{1-t}{n-2}) K \geq n\delta_1 |\nabla u|^2 - C > -C.$$ 

Hence, $K$ has lower bound. We also have

$$|\nabla u|^2 \leq \frac{(1 + n \frac{1-t}{n-2}) K + C}{n\delta_1}. \quad (4.1)$$

Without loss of generality, we may assume $K > 0$. Otherwise, by the above inequality (4.1), we know that $|\nabla u|^2 \leq C$. Then we have the $C^1$ estimates. Furthermore, we have $\triangle u \leq C$. And from the condition $\Gamma \subset \Gamma_2^+$, we know that $(trW)^2 - |W|^2 =
$2\sigma_2(W) > 0$. Therefore $|W| \leq Ctr(W) \leq C$ which implies $|\nabla^2 u| \leq C$. We then get $C^2$ estimates.

Now by the assumption and (4.1), we have

$$|\nabla u|^2 \leq C(K + 1),$$

(4.2)

where $C$ depends on $||a||_\infty$ and $||b||_\infty$.

By (4.2), we can consequently obtain

$$\Delta u = K - a|\nabla u|^2 \leq K + ||a||_\infty|\nabla u|^2 \leq C(K + 1).$$

From the condition $\Gamma \subset \Gamma^+_2$ again, we know that $|W_{ij}| \leq Ctr(W)$ which implies

$$\pm u_{ij} \leq C(\Delta u + C(K + 1)) + C(K + 1),$$

thus

$$|\nabla^2 u| \leq C(K + 1),$$

(4.3)

where $C$ depends only on $||a||_\infty$ and $||b||_\infty$ as well.

As a matter of fact, (4.2) and (4.3) are the fundamental inequalities that we would use once and once again. As a result, the terms $u_{ij}$ and $u_iu_j$ can be controled by $C(K + 1)$ later on.

Let $L = \zeta Ke^pxn$, where $0 \leq \zeta \leq 1$ is a cutoff function such that $\zeta = 1$ in $B_{2r}$ and $\zeta = 0$ outside $B_r$, and also $|\nabla \zeta| \leq C\zeta^{1/2}$, $|\nabla^2 \zeta| \leq \frac{C}{r^2}$. If we can prove $K$ at the maximum point of $L$, $x_0$, is bounded, then $K$ is also bounded in $B_{2r}$. Furthermore, by (4.2), (4.3), we can get $C^1$ and $C^2$ estimates.

To prove $K(x_0)$ is bounded, let us consider an auxiliary function $H = \eta Ke^pxn$ in a neighborhood $B_r(x_0) \subset B_s$, where $0 \leq \eta \leq 1$ is a cutoff function depending only on $r$ such that $\eta = 1$ in $B_{2r}$ and $\eta = 0$ outside $B_r$. Besides, we may assume $F^{ij}$ at the point $x_0$ is diagonal. Now let us go on with the proof of case (a) to get $C^1$ and $C^2$ estimates.

**Step 1.** We will prove the maximum point of $H$, says $x_0$, must be in the interior of $M$.

To prove this conclusion we argue by contradiction. That is, we assume $H$ arrives at its maximum point $x_0$ on the boundary, undertake a direct calculation of $H_n$, we can show that $H_n|_{x_0} > 0$, which violates the assumption.
Note that \( \eta \) is a function depending only on \( r \), thus at the boundary point we have \( \eta_n = 0 \). Besides, by use of (2.1), (2.2), (4.2) and (4.3), we can get

\[
\begin{align*}
    u_{\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) \\
    = a_n u_\gamma u_\gamma \geq -C(K+1).
\end{align*}
\]

and

\[
\begin{align*}
    H_n |_{x_0} &= \eta e^{px_n}(K_n + pK) \\
    &= \eta e^{px_n}(u_{nnn} + u_{\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) + pK) \\
    \geq \eta e^{px_n}(u_{nnn} - C(K+1) + pK)
\end{align*}
\]

Now we need the following claim:

**Claim.** There exist the constants \( C \) depending only on \( ||a||_\infty, ||b||_\infty, \Lambda, t \) and \( \varepsilon \), such that at \( x_0 \) we have \( u_{nnn} \geq -C(K+1) \),

By the claim we know that if \( p \) is large enough, then

\[
\begin{align*}
    H_n |_{x_0} &= \eta e^{px_n}(u_{nnn} - C(K+1) + pK) \\
    \geq \eta e^{px_n}(-CK - C + pK) \\
    \geq \eta e^{px_n}((p-C)K - C) > 0,
\end{align*}
\]

which completes the proof of step 1.

**Proof of Claim.**

We wish to get the estimates of \( u_{nnn} \) from the equation (1.5) itself. Since

\[
f_n = F^{\alpha \beta} W_{\alpha \beta n} + F^{\alpha n} W_{\alpha nn} + F^{nn} W_{nnn}.
\]

Noting that \( u_n = 0 \) and \( \mu = 0 \) on the boundary, the formula (2.3) in Lemma 5 combining with (4.2), (4.3) tells us that the first term \( W_{\alpha \beta n} \) can be controled by \( C(K+1)g_{\alpha \beta} + \frac{1}{n-2} u_{nnn}g_{\alpha \beta} \) and \( C \) depends only on \( ||a||_\infty \) and \( ||b||_\infty \). We restrict the argument at the point \( x_0 \). Then the second term vanishes since \( F^{ij} \) is diagonal at that point. For the third term, we know that \( W_{nnn} = u_{nnn} + \frac{1}{n-2} u_{nnn} + 2au_{nn}u_n + 2bu_{kn}u_k + S_{nnn} = u_{nnn} + \frac{1}{n-2} u_{nnn} + S_{nnn} \). Since \( F^{ij} \) is positive definite, an algebraic fact says that \(- (n - 2) \sigma_1 \leq \lambda_i \leq \sigma_1 \). As a result, \( |F^{ij}| \leq C \sum F^{ii} \), where \( C \) depends only on \( n \).

Without loss of generality, we may assume \( u_{nnn} \leq 0 \).
Case I : $t < 1$.

\[
    f_n = F^{\alpha\beta} W_{\alpha\beta n} + F^{nn} W_{nnn} \\
    \leq C \sum F^{\alpha\alpha} (K + 1) + \frac{1 - t}{n - 2} F^{\alpha\beta} u_{\alpha\beta n} g_{\alpha\beta} + F^{nn} W_{nnn} \\
    \leq F^{nn} u_{nnn} + \sum F^{\alpha\alpha} \left( \frac{1 - t}{n - 2} u_{\alpha\alpha n} + C(K + 1) \right).
\]

Then by the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, and the condition (B) that either 
\[
    \sum F^{\alpha\alpha} F^{nn} \leq \rho_1 \quad \text{or} \quad F^{nn} \sum F^{\alpha\alpha} \leq \frac{1}{\rho_2},
\]
we know either
\[
    -C f_{nnn} \leq \left( \frac{1 - t}{n - 2} \frac{\sum F^{\alpha\alpha}}{F^{nn}} + 1 \right) u_{nnn} + C \frac{\sum F^{\alpha\alpha}}{F^{nn}} (K + 1)
\]
or
\[
    -C \frac{f}{\sum F^{\alpha\alpha}} \leq \left( \frac{1 - t}{n - 2} + \frac{F^{nn}}{\sum F^{\alpha\alpha}} \right) u_{nnn} + C (K + 1).
\]

Besides, by the assumption $u_{nnn} \leq 0$ and the condition (A4) $\frac{\partial F}{\partial \lambda_i} > \varepsilon F^\sigma_{\sigma_1}$, then we have either
\[
    -C (K + 1) \leq -C \frac{f}{F^{nn}} \leq u_{nnn} + C (K + 1)
\]
or
\[
    -C (K + 1) \leq -C \frac{f}{\sum F^{\alpha\alpha}} \leq \frac{1 - t}{n - 2} u_{nnn} + C (K + 1).
\]

Thus $u_{nnn} \geq -C \frac{n - 2}{1 - t} (K + 1)$ where $C$ depends only on $||a||_\infty, ||b||_\infty, \Lambda$, and $\varepsilon$.

Case II : $t = 1$.

Note that

\[
    W_{\alpha\beta n} = a_n u_{\alpha} u_{\beta} + b_n \left( \sum \gamma u_{\gamma}^2 \right) g_{\alpha\beta} + S_{\alpha\beta n}.
\]

Thus under the assumption

\[
    a = \text{const.}, b = \text{const.}, S_{\alpha\beta n} = A_{\alpha\beta n},
\]
we have

\[
    f_n|_{x_0} \leq F^{\alpha\alpha} W_{\alpha\alpha n} + F^{nn} W_{nnn} \\
    = F^{\alpha\alpha} A_{\alpha\alpha n} + \sum F^{nn} W_{nnn} \\
    \leq F^{nn} u_{nnn} + C(K + 1)).
\]

The last inequality comes form the calculation in Lemma 8 of [Cn3]: $g^{\beta\gamma} A_{\beta\gamma n} = 0$. Then the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, and (A4) $\frac{\partial F}{\partial \lambda_i} > \varepsilon F^\sigma_{\sigma_1}$ implies $u_{nnn} \geq -C(K + 1)$ where $C$ depends only on $||a||_\infty, ||b||_\infty, \Lambda$, and $\varepsilon$. 

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Step 2. Now the $C^1$ and $C^2$ estimates will be completed by the rest proof in this part.

At the maximum point of $H$, $x_0$, after choosing normal coordinates, we have

$$0 = H_i = e^{px_n} (\eta_i K + \eta K_i + p\delta_{in}K\eta)$$

That is

$$K_i = -\left(\frac{\eta_i}{\eta} + p\delta_{in}\right)K.$$

We also have

$$0 \geq H_{ij} = e^{px_n}((\eta_i K + \eta K_i + p\delta_{in}K\eta) p\delta_{jn} + \eta_i \eta_j K_i + \eta_i \eta_j K_i + p\delta_{in}K_\eta + p\delta_{in}K\eta_j).$$

Note that $|\nabla \eta| \leq \frac{Cn^{3/2}}{r}$, $|\nabla^2 \eta| \leq \frac{C}{r^2}$, we have

$$0 \geq H_{ij} = e^{px_n}(\eta K_{ij} + \Lambda_{ij} K),$$

where

$$\Lambda_{ij} = \eta_{ij} - p\eta_i \delta_{jn} - p\eta_j \delta_{in} - p^2 \eta \delta_{in} \delta_{jn} - \frac{2\eta_i \eta_j}{\eta}$$

$$\geq -C (p^2 + 1) \delta_{ij}$$

and $C$ depends only on $r$.

Denote

$$P^{ij} = F^{ij} + \frac{1}{n - 2} \left(\sum_i F^{ii}\right) \delta^{ij}.$$

By the above inequalities, we have

$$0 \geq \eta P^{ij} H_{ij} e^{-px_n}$$

$$= \eta^2 P^{ij} K_{ij} + \eta \Lambda_{ij} P^{ij} K$$

$$\geq \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ik} u_{kj} + u_{ijk} u_k) + a_{ij} u_k^2 + 4a_i u_{jk} u_k]$$

$$- C \left(\sum_i F^{ii}\right) (1 + K)$$

$$\geq \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ik} u_{kj} + u_{ijk} u_k)] - C(\sum_i F^{ii})(1 + K^{3/2}). \quad (4.4)$$
Now we estimate the terms $\sum_k P^{ij}u_{i,jk}$ and $\sum_k P^{ij}(u_{k,i}u_{k,j} + u_{i,jk}u_k)$ respectively.

$$\sum_k P^{ij}u_{i,jk} = F^{ij} \sum_k [W_{i,jk} - 2a(u_{i,k}u_{j,k} + u_{i,k}u_{j}) - 2b(u_{i,k}u_l + u_{k,l}u_{i,j})g_{ij} - a_{kk}u_{i,j} - 4a_k u_{i,j} u_{j} - b_{kk}u_{i,j}^2g_{ij} - 4b_k u_{i}u_{j}g_{ij} - S_{i,j,k}]$$

$$\geq \sum_k f_{kk} + F^{ij} \sum_k [-2a(u_{i,k}u_{j,k} + u_{i,k}u_{j}) - 2b(u_{k,l}u_l + u_{k,l}u_{i,j})g_{ij}]$$

$$-C \sum F^{ii}(1 + K^{3/2})$$

$$\geq \sum_k f_{kk} + F^{ij} \sum_k [-2au_{i,k}u_{j,k} - 2au_{j}(-2au_{i,k}u_{k} - a_tu_k^2 - (\frac{\eta_i}{\eta} + p\delta_{in})K) - 2bu_{i,j}g_{ij} - 2bu_{i}( -2au_{i,k}u_{k} - a_tu_k^2 - (\frac{\eta_i}{\eta} + p\delta_{in})K)g_{ij}$$

$$-C \sum F^{ii}(1 + K^{3/2}).$$

We then get

$$\sum_k P^{ij}u_{i,jk} \geq \sum_k f_{kk} + F^{ij} \sum_k [-2b\Sigma_l (u_{l,k})^2g_{ij} - 2au_{i,k}u_{j,k} + 4a^2u_{j}u_{i,k}u_{k}$$

$$+ 4ab\Sigma_l (u_{l,k}u_{i,k}u_{j,k})] - C\eta^{-1/2} \sum F^{ii}(1 + K^{3/2}). \quad (4.5)$$

We also have

$$2aP^{ij}(u_{i,jk}u_k + u_{k,i}u_{j,k}) = 2aF^{ij}(W_{i,jk}u_k - 2au_{i,k}u_{j,k} - 2bu_{i,k}u_{j}g_{ij}u_k + u_{i,k}u_{j,k}$$

$$+ \frac{1}{n-2} \sum_{l,k} u_{l,k}^2\delta_{ij}a_ku_{i,j}u_k - b_ku_{i,j}^2g_{ij}u_k - S_{i,j,k}u_k)$$

$$\geq 2au_{i,k}f_{k} + F^{ij}(-4a^2u_{i,k}u_{j,k} - 4aBu_{i,k}u_{i,k}u_{j,k} + 2au_{i,k}u_{j,k})$$

$$+ \frac{1}{n-2} \sum_{l,k} F^{ii}\sum_{l,k} u_{l,k}^2 - C \sum_i F^{ii}(1 + K^{3/2}). \quad (4.6)$$

From (4.4), (4.5) and (4.6) we therefore have

$$0 \geq (\sum F^{ii}) (2(\frac{1-t}{n-2}a - b))\eta^2[\nabla^2 u]^2 - C\eta^{3/2}K^{3/2} - C\eta K - C. \quad (4.7)$$

If there exists a constant $A > 0$, such that $|\nabla u|^2(x_0) < A|\Delta u|(x_0)$.

By $|u_{ij}| \leq C(K + 1)$, we know that at the point $x_0$, $|u_{ij}| \leq C(|\Delta u| + 1)$. Thus (4.7) becomes

$$0 \geq \eta F^{ij}H_{ij}e^{-px_0} \geq \sum F^{ii}(\frac{2}{n-2}a - b)\eta^2|\nabla u|^2 - C\eta^{3/2}|\Delta u|^{3/2} - C\eta |\Delta u| - C.$$

Hence

$$|\Delta u|(x_0) \leq C$$

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and
\[ K \leq C. \]

Otherwise for any constant \( A > 0 \) large enough, \( |\nabla u|^2(x_0) \geq A|\Delta u|(x_0) \). From \( \min |u_{ij}| \leq C(K + 1) \), we know that at the point \( x_0 \), \( |u_{ij}| \leq C(|\nabla u|^2 + 1) \). Thus (4.7) becomes
\[
0 \geq \sum F^{ii}(2(\frac{1-t}{n-2}u - b)\eta^2 u_{ii}^2 - C\eta^{3/2}|\nabla u|^3 - C\eta|\nabla u|^2 - C). \tag{4.8}
\]

We may assume that \( W_{ij} \) is diagonal at the point \( x_0 \),
\[
W_{ii} = u_{ii} + \frac{1-t}{n-2}\Delta u + au_i^2 + b\Sigma_k u_k^2 + S_{ii},
\]
and
\[
0 = W_{ij} = u_{ij} + au_iu_j + S_{ij}, \quad (i \neq j).
\]

Since
\[
F^{ii}(u_{ii} + \frac{1-t}{n-2}\Delta u g_{ii} + S_{ii})^2 \leq 2F^{ii}[u_{ii}^2 + (\frac{1-t}{n-2}\Delta u g_{ii} + S_{ii})^2],
\]
we obtain
\[
2\Sigma_i F^{ii} u_{ii} u_{ii} \geq \Sigma_i F^{ii}[u_{ii} + \frac{1-t}{n-2}\Delta u g_{ii} + S_{ii}]^2 - 2(\frac{1-t}{n-2}\Delta u g_{ii} + S_{ii})^2]
\[
\geq \Sigma_i F^{ii}[\sum_{j \neq i}(-au_iu_j)^2 + (W_{ii} - au_i^2 - b|\nabla u|^2)^2]
\[
- 2(\frac{1-t}{n-2})^2 \frac{1}{A^2} \Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii}
\]
\[
\geq \Sigma_i F^{ii}[(a + b)^2 u_i^2 + \sum_{j \neq i} b^2 u_j^2]|\nabla u|^2 - 2(a + b)\|f\|_{\infty}|\nabla u|^2
\]
\[
- 2(\frac{1-t}{n-2})^2 \frac{1}{A^2} \Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii}.
\]

By the assumption of the theorem caes (a), \( \min \{(a + b)^2, b^2\} \geq \delta_2 > 0 \), we then have
\[
2\Sigma_i F^{ii} u_{ii} u_{ii} \geq \left(\delta_2 - 2(\frac{1-t}{n-2})^2 \frac{1}{A^2}\right) \Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii}
\]

Therefore, by (4.8) we have
\[
0 \geq (\Sigma_i F^{ii}) \left[\delta_1 \left(\delta_2 - 2 \left(\frac{1-t}{n-2}\right)^2 \frac{1}{A^2}\right) \eta^2|\nabla u|^4 - C\eta^{3/2}|\nabla u|^{3/2} - C\eta|\nabla u|^2 - C\right].
\]

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Since $A > 0$ large enough, we have

$$|\nabla u|^2(x_0) \leq C,$$

therefore

$$K \leq C.$$

(2) Proof in case (b).

Since $a(x) + nb(x) < -\delta_3$, by the condition $\Gamma \subset \Gamma_1^+$, we have

$$0 \leq tr(W) = (1 + n \frac{1 - t}{n - 2}) \Delta u + a|\nabla u|^2 + nb|\nabla u|^2 + trB$$

$$\leq (1 + n \frac{1 - t}{n - 2}) \Delta u - \delta_3|\nabla u|^2 + C.$$

Then

$$|\nabla u|^2 \leq C(\Delta u + 1). \quad (4.9)$$

The proof is similar as the argument in case (a). We take the same auxiliary function $H = \eta(\Delta u + a|\nabla u|^2)e^{px} \triangleq \eta Ke^{px}$, where $\eta(r)$ is a cutoff function as in case (a), $p$ is a large positive constant which is determined in the proof. Without loss of generality, we may assume

$$K = \Delta u + a|\nabla u|^2 >> 1.$$

Since $a(x) \geq 0$, by (4.9), we have

$$\Delta u \leq C(K + 1)$$

and

$$|\nabla u|^2 \leq C(K + 1).$$

Therefore by using the same Claim, we may prove that the maximum point of $H$ does not achieve on the boundary and it must be in the interior. Suppose that the maximum point of $H$ achieves at $x_0$, an interior point. Then at this point, we need to note that $|\nabla u|^2, \Delta u$ and $K$ all can be controlled by $C(|\nabla^2 u| + 1)$. By the same computation as in case (a), (4.4), (4.5), (4.6) and (4.7) become

$$0 \geq \eta^2 P^{ij} \sum_k [u_{ijk} + 2a(u_{ki}u_{kj} + u_{ijk}u_{kk})] - C(\sum_i F^{ii})(1 + |\nabla^2 u|^{3/2}), \quad (4.10)$$

$$\sum_k P^{ij}u_{ijk} \geq \sum_k f_{kk} + F^{ij} \sum_k [-2b \Sigma_l (u_{lk})^2 g_{ij} - 2au_{uk}u_{jk} + 4a^2 u_{j}u_{ki}u_{k}$$

$$+ 4ab \Sigma_l (u_{lki}u_{k}) g_{ij}] - C\eta^{-1/2} \sum F^{ii}(1 + |\nabla^2 u|^{3/2}), \quad (4.11)$$
\[ 2aP^{ij}(u_{ijk}u_k + u_{ki}u_{kj}) \geq 2au_k f_k + F^{ij}(-4a^2 u_{ik}u_j u_k - 4abu_{ik}u_l u_k g_{lj} + 2au_{ik}u_{jk}) \]
\[ + \frac{1-t}{n-2} 2a \sum_i F^{ii} (1 + |\nabla^2 u|^{3/2}), \]

and

\[ 0 \geq (\sum F^{ii}) (2(\frac{1-t}{n-2}a - b)(\eta|\nabla^2 u|)^2 - C(\eta|\nabla^2 u|)^{3/2} - C\eta|\nabla^2 u| - C) \]

respectively. (4.13) gives \( \eta|\nabla^2 u|(x_0) \leq C \) and hence the bounds of \( K, |\nabla^2 u| \) and \( |\nabla u| \).

\[ \square \]

Proof of Corollary 1.
The proof of the case (a) is the corresponding part in Theorem 1 just ignoring the boundary argument. For the case (b), the difference with the corresponding part of Theorem 1 is that we needn’t ask \( a(x) \geq 0 \), which is necessary in the boundary argument in Theorem 1. By the same argument as case (b) of Theorem 1, we may get (4.9) and (4.13). Therefore the estimate can be easily obtained.

\[ \square \]

5 Proof of Theorem 2

(1) Proof in case (a).

Let \( H = \eta(\triangle u + a|\nabla u|^2) e^{px} \) and \( K = \triangle u + a|\nabla u|^2 \), where \( 0 \leq \eta \leq 1 \) is a cutoff function as before.

Note that \( \Gamma \subset \Gamma_1^+ \) and \( V = \frac{t-1}{n-2}(\triangle u) g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S \), we can immediately get

\[ 0 \leq tr(V) = (n \frac{t-1}{n-2} - 1)\triangle u - (a + nb)|\nabla u|^2 + tr S \leq (n \frac{t-1}{n-2} - 1)K - n\delta_1|\nabla u|^2 + C, \]

Hence, \( |\nabla u|^2 \leq \frac{(n \frac{t-1}{n-2} - 1)K + C}{n\delta_1} \). Thus we have

\[ |\nabla u|^2 \leq C(K + 1), \]

where \( C \) depends only on \( ||a||_\infty, ||b||_\infty \) and \( \delta_1 \).
By (5.1), we can obtain $\triangle u < C(K + 1)$ and

$$|\nabla^2 u| \leq C(K + 1). \quad (5.2)$$

Then we may prove that the maximum point of $H$ must be in the interior of $M$. As case (a) of Theorem 1, we prove this by contradiction. If the maximum point of $H$, $x_0$, is on the boundary, then by (2.1), (2.2) and (2.4), firstly we can prove

$$u_{\alpha \alpha n} + 2au_{\alpha n} u_{\alpha} + 2au_{\alpha n} u_{\alpha} + a_n (u_{\gamma} u_{\gamma} + u_n u_n) \geq - C(K + 1).$$

Furthermore, we can get the following Claim as well:

**Claim.** We can find some positive constant $C$, such that $u_{nnn}(x_0) \geq - C(K + 1)$.

Then from the Claim, we can compute the derivative of $H$ along the inner normal direction $x_n$ and show that $H|_{x_0} > 0$ as long as $p$ is chosen large enough, which is contradicts with the assumption that $x_0$ is a maximum point. Hence, $H$ achieves its maximum at an interior point.

**Proof of Claim.**

We may assume $u_{nnn} \leq 0$. Choose a normal coordinates at $x_0$.

(1) If $\sum_{F_{\alpha \alpha}} \leq \rho_1$ :

$$f_n = F_{\alpha \beta} V_{\alpha \beta n} + F_{nnn} V_{nnn} \leq C \sum F_{\alpha \alpha} (K + 1) - \frac{1 - t}{n - 2} F_{\alpha \beta} u_{n n n} g_{\alpha \beta} + F_{nnn} V_{nnn} \leq F_{nnn} (-1 - \frac{1 - t}{n - 2}) u_{n n n} + \sum_{\alpha} F_{\alpha \alpha} (-\frac{1 - t}{n - 2} u_{n n n} + C(K + 1)).$$

Then by the conditions $|\nabla f| < \Lambda f$ and $|f| < \Lambda f$, we know

$$-C \frac{f}{\sum F_{\alpha \alpha}} \leq \left(-1 + \frac{1 - t}{n - 2} \frac{\sum F_{\alpha \alpha}}{F_{nnn}} - 1 - \frac{1 - t}{n - 2}\right) u_{n n n} + C \frac{\sum F_{\alpha \alpha}}{F_{nnn}} (K + 1) \leq \left(-1 - \frac{1 - t}{n - 2}\right) u_{n n n} + C(K + 1)$$

Since $\frac{1 - t}{n - 2} > 1$ thus $u_{n n n} \geq - C(K + 1)$.

(2) If $\frac{F_{nn}}{\sum F_{\alpha \alpha}} \leq \frac{1}{\rho_2}$ :

$$f_n = F_{\alpha \beta} V_{\alpha \beta n} + F_{nnn} V_{nnn} \leq C \sum F_{\alpha \alpha} (K + 1) - \frac{1 - t}{n - 2} F_{\alpha \beta} u_{n n n} g_{\alpha \beta} + F_{nnn} V_{nnn} \leq F_{nnn} (-1 - \frac{1 - t}{n - 2}) u_{n n n} + \sum_{\alpha} F_{\alpha \alpha} (-\frac{1 - t}{n - 2} u_{n n n} + C(K + 1)) \leq - F_{nnn} u_{n n n} + \sum_{\alpha} F_{\alpha \alpha} (-\frac{1 - t}{n - 2} u_{n n n} + C(K + 1)).$$
Then by the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, we know

$$-C \frac{f}{\sum F^{\alpha \alpha}} \leq - \left(1 - \frac{t}{n-2} + \frac{F_{nn}}{\sum F^{\alpha \alpha}}\right) u_{nnn} + C(K + 1)$$

$$\leq - \left(1 - \frac{t}{n-2} + \frac{1}{\rho Z}\right) u_{nnn} + C(K + 1).$$

Thus if $\frac{1-t}{n-2} + \frac{1}{\rho Z} < 0$ then $u_{nnn} \geq -C(K + 1)$.

□

Now let $x_0$ be an interior point where $H$ achieves its maximum. At $x_0$, we have

$$0 = H_i = e^{px}(\eta_i K + \eta K_i + p\delta_{in}K\eta),$$

that is

$$K_i = -(\frac{\eta_i}{\eta} + p\delta_{in})K.$$ 

We also have

$$0 \geq H_{ij} = e^{px}((\eta_i K + \eta K_i + p\delta_{in}K\eta)p\delta_{jn} + \eta_{ij}K$$

$$+ \eta K_{ij} + \eta_i K_j + \eta_j K_i + p\delta_{in}K\eta_j + p\delta_{in}K\eta_j).$$

Note that $|\nabla \eta| \leq \frac{C\eta^{1/2}}{r}, |\nabla^2 \eta| \leq \frac{C}{r^2}$, we have

$$0 \geq H_{ij} \triangleq e^{px}(\eta K_{ij} + \Lambda_{ij}K),$$

where $\Lambda_{ij}$ is bounded. If we take

$$Q^{ij} = \frac{t - 1}{n - 2}(\sum F^{\alpha \alpha})^2 \delta_{ij} - F^{ij},$$

which is also positive, we can obtain

$$0 \geq \eta Q^{ij} H_{ij} e^{-px} = -\eta F^{ij} H_{ij} e^{-px} + \eta \frac{1}{n-2}(\sum F^{ij})H_{kk} e^{-px}.$$ 

By the same computation as in the case (a) of Theorem 1, we may get

$$0 \geq \sum F^{ij}(2(\frac{t - 1}{n - 2}a + b)\eta^2|\nabla^2 u|^2 - C\eta^{3/2}K^{3/2} - C\eta K - C).$$

(5.3)

As in the case (a) of Theorem 1, we may discuss (5.3) in two cases. If there exists a constant $A > 0$, such that $|\nabla u|^2(x_0) < A|\Delta u|(x_0)$, we may prove

$$|\Delta u|(x_0) \leq C$$

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and 

\[ K \leq C. \]

Otherwise for any constant \( A > 0 \) large enough, \( |\nabla u|^2(x_0) \geq A|\Delta u|(x_0) \). By use of the assumption that \( \min \{(a + b)^2, b^2\} \geq \delta_2 > 0 \), we may prove

\[ |\nabla u|^2(x_0) \leq C; \]

therefore we have

\[ K \leq C. \]

By (5.2), we get the Hessian estimates.

(2) Proof in case (b).

We take the same auxiliary function \( H = \eta(\Delta u + a|\nabla u|^2)e^{p_x} \triangleq \eta Ke^{p_x} \) as in the case (a), where \( 0 \leq \eta \leq 1 \) is a cutoff function such that \( \eta = 1 \) in \( B_{\frac{r}{2}} \) and \( \eta = 0 \) outside \( B_r \), and also \( |\nabla \eta| \leq \frac{C}{r}, |\nabla^2 \eta| \leq \frac{C}{r^2}. \)

Since \( a(x) + nb(x) \geq \delta_3 \), by the condition \( \Gamma \subset \Gamma^+_1 \) again, we have

\[
0 \leq tr(V) = (n\frac{t - 1}{n - 2} - 1)\Delta u - a|\nabla u|^2 - nb|\nabla u|^2 + trS \leq (n\frac{t - 1}{n - 2} - 1)\Delta u - \delta_3|\nabla u|^2 + C,
\]

and then

\[ |\nabla u|^2 \leq C(\Delta u + 1). \] (5.4)

Without loss of generality, we may assume

\[ K = \Delta u + a|\nabla u|^2 >> 1. \]

Since \( a(x) \geq 0 \), by (5.4), we have

\[ \Delta u \leq C(K + 1) \] (5.5)

and

\[ |\nabla u|^2 \leq C(K + 1). \] (5.6)

Similar as the proof of the case (b) in Theorem 1, we may prove that the maximum point of \( H \) does not appear on the boundary and it must be in the interior. The condition \( \frac{1}{\rho^2} - \frac{t - 1}{n - 2} < 0 \) will be used to prove this. Now suppose that the maximum
point of $H$ achieves at $x_0$, an interior point, we may get an inequality just replacing $K$ in (5.3) by $|\nabla^2 u|$

$$0 \geq \sum F^{ui}(2\left(\frac{t-1}{n-2}a + b\right)\eta^2|\nabla^2 u|^2 - C(\eta|\nabla^2 u|)^{3/2} - C(\eta|\nabla^2 u|)^{3/2} - C(\eta|\nabla^2 u|) - C). \quad (5.7)$$

The coefficient of the highest order term $\frac{t-1}{n-2}a(x) + b(x) > \delta_2 > 0$ since $a(x) \geq 0$ and $a(x) + nb(x) > \delta_3 > 0$. Therefore we can get the bounds of $K, |\nabla^2 u|$ and $|\nabla u|^2$.

Proof of Corollary 2.

Here the proof is completely similar as the proof of Corollary 1. In the case (b), we needn’t ask $a(x) \geq 0$. We need an additional condition $\frac{t-1}{n-2}a(x) + b(x) > \delta_1 > 0$ to guarantee the coefficient of the highest order term in (5.7) is positive. We remain the detail to the readers.

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