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PRESCRIBING THE SYMMETRIC FUNCTION OF THE EIGENVALUES OF THE SCHOUTEN TENSOR

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ABSTRACT. In this paper we study the problem of conformally deforming a metric to a prescribed symmetric function of the eigenvalues of the Schouten tensor on compact Riemannian manifolds with boundary. We prove its solvability and the compactness of the solution set, provided the Ricci tensor is non-negative definite.

1. INTRODUCTION

Let (M^n, g) be a smooth, compact Riemannian manifold with totally geodesic boundary of dimension $n \geq 3$. The Schouten tensor of g is defined by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric and R are the Ricci and scalar curvatures of g, respectively.

Let $\sigma_k : R^n \to R$ be the k-th elementary symmetric function $(1 \leq k \leq n)$

$$\sigma_k(x) = \sum_{1\leq i1<\cdots<ik\leq n} x_{i1}\cdots x_{ik},$$

Γ_k the corresponding open, convex cone, i.e. $\Gamma_k = \{x \in R^n | \sigma_i(x) > 0, 1 \leq i \leq k\}$. Let

$$\Sigma_\theta = \{x = (x_1, \cdots, x_n) \in R^n | \quad \min x_i + \theta\Sigma x_i > 0\}.$$

Now let us consider the general symmetric function F defining on Γ ($\Gamma_n \subset \Gamma \subset \Sigma_{\frac{1}{n-2}}$) satisfying

(C_1) F is positive and $F = 0$ on $\partial\Gamma$;
(C_2) F is concave and monotone;
(C_3) F is invariant under exchange of variables;
(C_4) F is homogeneous of degree 1;
(C_5) $\lim_{s\to\infty} F(sx) = \infty, \ \forall\, x \in \Gamma$;
(C_6) $F(x) \leq \varrho\sigma_1(x)$ in Γ and $F(1, \cdots, 1) = n\varrho$, ϱ is a positive constant.

We need (C_1)-(C_4) to ensure that the elliptic equations are solvable. If F additionally satisfies (C_5) then Liouville Theorem in [12] is applicable. The condition (C_6) says that the Newton-Maclaurin inequality with respect to function F holds.

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We denote $[g] = \{\tilde{g} \mid \tilde{g} = e^{-2u}g\}$. We call the metric $\hat{g} = e^{-2u}g$ (as well as the function u) is Γ-admissible, or simply admissible, if $\hat{g} \in \left\{\tilde{g} \in [g] \quad | \quad \lambda(\tilde{g}^{-1}A_{\tilde{g}}) \in \Gamma\right\}$. Here, $\lambda(\tilde{g}^{-1}A_{\tilde{g}}) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of $\tilde{g}^{-1}A_{\tilde{g}}$.

In this paper we study the existence of some prescribing problems and the compactness of the solution set. The main result is as follows.

Theorem 1.1. *Let (M^n, g) be a compact n-dimensional Riemannian manifold with totally geodesic boundary. Let F be a symmetric function satisfying $(C_1) - (C_6)$ on Γ with $\Gamma_n \subset \Gamma \subset \Sigma_{\frac{1}{n-2}}$. If the manifold (M, g) is not conformal equivalent to a hemisphere, then for any positive function f, there exists an admissible conformal metric $\tilde{g} = e^{-2u}g$ with totally geodesic boundary satisfying*

$$F\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) = f.$$

Additionally, the set of all such solutions is compact in the C^m -topology for any $m \geq 0$.

We can get the following two corollaries from Theorem 1.1 immediately. One is to find a conformal metric $\tilde{g}$ with nonnegative $Ric_{\tilde{g}}$ such that

$$\det\left(\mu(\tilde{g}^{-1}Ric_{\tilde{g}})\right) = f^n, \tag{1.1}$$

where $\mu(\tilde{g}^{-1}Ric_{\tilde{g}}) = (\mu_1, \cdots, \mu_n)$ are the eigenvalues of $\tilde{g}^{-1}Ric_{\tilde{g}}$ and $f(x)$ is a positive function.

Since $Ric_{\tilde{g}} = (n-2)A_{\tilde{g}} + \sigma_1(\lambda(\tilde{g}^{-1}A_{\tilde{g}}))\tilde{g}$, if we define $F(\lambda) = \sigma_n^{1/n}\left((n-2)\lambda + (\Sigma_{i=1}^n \lambda_i)\right)$ and $\Gamma = \{\lambda \quad | \quad F(\lambda) > 0\}$, then

$$\begin{aligned} & \det{}^{1/n}\left(\mu(\tilde{g}^{-1}Ric_{\tilde{g}})\right) \\ = \; & \sigma_n^{1/n}\Big(\mu\Big(g^{-1}\Big[(n-2)(du \otimes du - |\nabla u|^2 g) + (n-2)\nabla^2 u + \triangle u g + Ric_g\Big]\Big)\Big) \\ = \; & F\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big), \end{aligned}$$

where $\mu = (n-2)\lambda + \Sigma_{i=1}^n \lambda_i$. From the definition, it is easy to verify that F satisfying $(C_1) - (C_5)$, since

$$\frac{\partial F}{\partial \lambda_i} = \frac{\partial\left(\sigma_n^{1/n}\right)}{\partial \mu_s}\left(1 + (n-2)\delta_i^s\right),$$

and

$$\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} = \left(1 + (n-2)\delta_i^s\right)\frac{\partial^2\left(\sigma_n^{1/n}\right)}{\partial \mu_s \partial \mu_t}\left(1 + (n-2)\delta_j^t\right).$$

Moreover, from

$$\begin{aligned} F\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) &= \sigma_n^{1/n}\left(\mu(\tilde{g}^{-1}Ric_{\tilde{g}})\right) \\ &\leq \frac{1}{n}\sigma_1\left(\mu(\tilde{g}^{-1}Ric_{\tilde{g}})\right) \\ &= \frac{2n-2}{n}\sigma_1\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right), \end{aligned}$$

we know F satisfies (C_6) with $\varrho = \frac{2n-2}{n}$. Thus (1.1) turns out to be a proper equation with respect to Schouten tensor. Furthermore, as [6] and [15], by use of the volume comparison theorem, C^0 estimate of the solutions of such equation can be derived if $Ric \geq 0$. In other words, the condition $\Gamma \subset \Sigma_{\frac{1}{n-2}}$ ensures that the volume comparison theorem is applicable, where the eigenvalues of Schouten tensor λ satisfy $(n-2)\lambda + \Sigma_{i=1}^{n}\lambda_i \geq 0$ if and only if the eigenvalues of Ricci tensor $\mu \geq 0$. Similarly, on the manifold with totally geodesic boundary, based on the boundary C^1, C^2 estimates with Neumann boundary condition for general symmetric function ([2] or [9], etc.), we can get

Corollary 1.2. *Let (M, g) be a compact n dimension Riemannian manifold with totally geodesic boundary and the Ricci tensor is semi-positive definite. If it is not conformal equivalent to a hemisphere, then for any positive function f, there exists a conformal metric $\tilde{g} = e^{-2u}g$ with totally geodesic boundary and $Ric_{\tilde{g}} \geq 0$ and*

$$\det\left(\mu(\tilde{g}^{-1}Ric_{\tilde{g}})\right) = f^n.$$

Additionally, the set of all such solutions is compact in the C^m -topology for any $m \geq 0$.

Remark 1.3. The conformal problem with respect to the Ricci tensor has been studied extensively. In [13] and [8], the authors studied the negative Ricci curvature and proved that there exists a conformal metric $\tilde{g}$ with negative Ricci tensor $Ric_{\tilde{g}}$ such that

$$\det\left(\mu(\tilde{g}^{-1}Ric_{\tilde{g}})\right) = const..$$

When the Ricci tensor is positive definite, in [5], Guan and Wang derived a conformal metric with a constant smallest eigenvalue of Ricci tensor. In [15], Trudinger and Wang proved the prescribing problem of positive Ricci tensor on closed manifold.

Another corollary is to find a conformal metric $\tilde{g}$ in Γ_k $(n/2 \leq k \leq n)$ such that

$$\left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}} \left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) = f\left(x\right),$$

where $l < k$ and $\lambda(\tilde{g}^{-1}A_{\tilde{g}})$ the eigenvalues of $\tilde{g}^{-1}A_{\tilde{g}}$, and $f(x)$ is a positive function as well. Since it is easy to verify that $F = \left(\frac{\sigma k}{\sigma 1}\right)^{\frac{1}{k-l}}$ on $\Gamma = \Gamma_k$ satisfies $(C_1) - (C_6)$ with $\varrho = \frac{1}{n}\binom{n}{k}^{\frac{1}{k-l}}\binom{n}{l}^{-\frac{1}{k-l}}$ and $\Gamma_k \subset \Gamma_{n/2} \subset \Sigma_{\frac{1}{n-2}}$. If the manifold is closed, the existence of the solutions to the equation and the compactness of the solution set have been involved in [15]. Here we state the result for the manifolds with boundary.

Corollary 1.4. *Let (M, g) be a compact n-dimensional and Γ_k admissible Riemannian manifold with totally geodesic boundary. If it is not conformal equivalent to a hemisphere, then for any positive function f, there exists a conformal Γ_k admissible metric $\tilde{g} = e^{-2u}g$ with totally geodesic boundary and*

$$\left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}} \left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) = f,$$

where $l < k$ and $\lambda(\tilde{g}^{-1}A_{\tilde{g}})$ the eigenvalues of $\tilde{g}^{-1}A_{\tilde{g}}$. Additionally, the set of all such solutions is compact in the C^m -topology for any $m \geq 0$.

This paper is organized as follows. We begin with some preliminaries in Section 2. In Section 3, we will discuss the deformation and a priori estimates. The proof of Theorem 1.1. is in Section 4.

2. PRELIMINARIES

We introduce Fermi coordinates in a boundary neighborhood at first. In this local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and $\left(x^1, ..., x^{n-1}\right)$ forms a local chart on the boundary where $x^n = 0$. The metric can be expressed as

$$g = g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2 .$$

The Greek letters $\alpha, \beta, \gamma, ...$stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma, ... \leq n-1$, while the Latin letters $i, j, k, ...$ stand for the full indices, $1 \leq i, j, k, ... \leq n$ (See [4] and [1]).

We denote the functions, tensors and covariant differentiations with respect to the induced metric on the boundary by a *bar*(e.g. $\bar{\Gamma}^\alpha_{\beta\gamma}$, $\bar{R}_{\alpha\beta}$). Then the Christoffel symbols on the boundary satisfy

$$\bar{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta} \right) = \Gamma^\gamma_{\alpha\beta},$$

and $\Gamma^n_{nn} = 0$, $\Gamma^\alpha_{nn} = 0$, $\Gamma^n_{n\alpha} = 0$.

Let us denote $\frac{\partial}{\partial x^i}$ by ∂_i. The boundary is called umbilic if the second fundamental form $L_{\alpha\beta} = \tau g_{\alpha\beta}$, where τ is a function defined on ∂M. Since the boundary ∂M is connected, by Schur Theorem, $\tau = const.$. A totally geodesic boundary is umbilic with $\tau = 0$.

Thus $\Gamma^n_{\alpha\beta}|_{\partial M} = L_{\alpha\beta} = \tau g_{\alpha\beta}$ and $\Gamma^\alpha_{n\beta}|_{\partial M} = -L_{\alpha\gamma} g^{\gamma\beta} = -\tau \delta^\beta_\alpha$.

Under the conformal metric $\tilde{g} = e^{-2u} g$, the functions, tensors and the covariant differentiations with respect to $\tilde{g}$ denoted by a *tilde* (e.g. $A_{\tilde{g}}$, $\tilde{L}_{\alpha\beta}$).

Let $[g]$ be the set of metrics conformal to g. For $\tilde{g} = e^{-2u} g \in [g]$, we consider the equation

$$F\left(\lambda(\tilde{g}^{-1} A_{\tilde{g}})\right) = f. \tag{2.1}$$

The Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g,$$

where ∇u and $\nabla^2 u$ denote the gradient and Hessian of u with respect to g. Consequently, (2.1) is equivalent to

$$F\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = f(x) e^{-2u}.$$

Then the second fundamental form satisfies

$$\tilde{L}_{\alpha\beta} e^u = \frac{\partial u}{\partial \nu} g_{\alpha\beta} + L_{\alpha\beta}.$$

Note that the umbilicity is conformally invariant. When the boundary is umbilic, the above formula becomes

$$\tilde{\tau} e^{-u} = \frac{\partial u}{\partial \nu} + \tau,$$

where $\tilde{L}_{\alpha\beta} = \tilde{\tau} \tilde{g}_{\alpha\beta}$.

Therefore, whence the initial metric g on manifold M is with totally geodesic boundary ∂M, the boundary of the manifold M with conformal metric $\tilde{g} = e^{-2u} g$ is still totally geodesic if and only if $\frac{\partial u}{\partial \nu} = 0$.

Therefore, in order to prove Theorem 1.1, we need to find admissible solutions of the following equation

$$(2.2)\qquad \begin{cases} F\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = f(x)e^{-2u} & \text{in } M,\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.\end{cases}$$

3. DEFORMATION, C^1 AND C^2 ESTIMATES

To prove the existence of solution to the equation (2.2), we employ the following deformation which defined in [7]
(3.1)

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[\varsigma(1-\psi(t))g + \psi(t)A_g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) & \\ = \psi(t)f(x)e^{-2u} + (1-t)(\int e^{-(n+1)u})^{2/n+1} & \text{in } M,\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M,\end{cases}$$

where $\psi \in C^1[0,1]$ satisfies $0 \le \psi(t) \le 1, \psi(0) = 0,\ \psi(t) = 1$ for $t \ge \frac{1}{2}$; and $\varsigma = (n\varrho)^{-1} vol(M_g)^{\frac{2}{n+1}}$, where $F(1,\cdots,1) = n\varrho$.

Similar as [7], at $t = 1$, (3.1) becomes (2.2). While at $t = 0$, it becomes

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) = (\int e^{-(n+1)u})^{\frac{2}{n+1}} & \text{in } M,\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.\end{cases}$$

We can show that the above equation has a unique solution $u(x) \equiv 0$.

In fact, it is obvious that $u \equiv 0$ is a solution. Now we are going to prove its uniqueness.

At the maximum point x_0 of u, no matter x_0 is interior or boundary point, we always have that $\nabla u|_{x_0} = 0$, and $\nabla^2 u|_{x_0}$ is non-positive definite. In fact if x_0 is interior point, it is clear; if x_0 is boundary point, we have $\frac{\partial u}{\partial \nu}|_{\partial M} = 0$ by equation (3.1), and $\frac{\partial u}{\partial x^\alpha}|_{x_0} = 0$, where $\{x^\alpha\}_{1\le\alpha\le n-1}$ is a local coordinates on the boundary ∂M around x_0. Therefore $\nabla^2 u|_{x_0}$ is non-positive definite. Now at x_0 we have

$$\begin{aligned} vol(M_g)^{\frac{2}{n+1}} &= \varsigma\cdot n\varrho = \varsigma F(\lambda(g^{-1}\cdot g))\\ &\ge F\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big)\\ &= (\int e^{-(n+1)u})^{\frac{2}{n+1}}.\end{aligned}$$

Similarly, at the minimum point of u, we can get $\varsigma\cdot n\varrho \le (\int e^{-(n+1)u})^{\frac{2}{n+1}}$. As a result, we have $vol(M_g)^{\frac{2}{n+1}} = \varsigma\cdot n\varrho = (\int e^{-(n+1)u})^{\frac{2}{n+1}}$.

By (C_6), we know $F \le \varrho\sigma_1$. Hence,

$$\begin{aligned} \varsigma\cdot n\varrho &= F\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big)\\ &\le \varrho\ \sigma_1\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big)\\ &= \varrho\ \Big(n\varsigma + \triangle u + (1-\frac{n}{2})|\nabla u|^2\Big).\end{aligned}$$

Then

$$(\frac{n}{2}-1)\int_M |\nabla u|^2 \le \int_M \triangle u = \int_{\partial M} \frac{\partial u}{\partial \nu} = 0,$$

and $u \equiv const. = 0$.

Thus the operator

$$\begin{aligned}\Psi_t[u] &= F\Big(\lambda\Big(g^{-1}\Big[\varsigma(1-\psi(t))g + \psi(t)A_g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big)\\ &\quad -\psi(t)f(x)e^{-2u} - (1-t)(\int e^{-(n+1)u})^{\frac{2}{n+1}}\end{aligned}$$

satisfies Leray-Schauder degree $deg(\Psi_0, \mathcal{O}_0, 0) \neq 0$ at $t = 0$, where the Leray-Schauder degree is defined by [11](see [2] for the boundary case) and $\mathcal{O}_0$ is a neighborhood of the zero solution in $\{u \in C^{4,\alpha}(M) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M\}$. Thus whence we obtain the homotopy-invariance of degree, we can derive that the Leray-Schauder degree is nonzero at $t = 1$. This shows that equation (2.2) is solvable.

The C^1 and C^2 estimates of the solutions to (3.1) have been proved in [9], we may obtain

Lemma 3.1. *For any fixed $0 < \delta < 1$, there is a constant $C = C(\delta, n, g, f)$ such that any solution of (3.1) with $t \in [0, 1-\delta]$ satisfies $\|u\|_{C^{4,\alpha}} \le C$.*

So without loss of generality, we may assume that u_{t_i} tends to $-\infty$ at $t_i \to 1$, where u_{t_i} is the solution of (3.1) at $t = t_i$ which will be denoted by u_i in what follows. Thus equation (3.1) turns to be

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[A_g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) & \\ = (1-t)o + f(x)e^{-2u} & \text{in } M,\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{3.2}$$

where u is assumed to be admissible, and $o \ge 0$ is a constant.

Furthermore, we can get a more exact estimate on the geodesic ball $B(x, r) = \{y \in M \quad | \quad dist(x, y) < r\}$:

Lemma 3.2. *([9]). Let $u \in C^4(M)$ be a k -admissible solution of (3.1) in $B(x, r)$ and $0 \le r < 1$. Then there is a constant $C = C(n, g, f)$ such that*

$$\left(|\nabla^2 u| + |\nabla u|^2\right)(x') \le C\Big(r^{-2} + \exp\left(-2 \inf_{B(x, 2\sqrt{10}r)} u\right)\Big). \tag{3.3}$$

for all $x' \in B(x, r)$.

4. PROOF OF THEOREM 1.1.

We call $\{u_k\}$ the blow up sequence and $\bar{x} \in M$ the blow up point, if $u_k(x_{0,k}) \to -\infty$ as $x_{0,k} \to \bar{x}$, where $\{x_{0,k}\} \subset M$. Now let $\{u_k\}$ be a blow up solutions of (3.2) with the blow up point $\bar{x}$.

First of all, we would like to prove that $\bar{x}$ can be approximated by local minimum points of u_k. Let $v_k = e^{-(n-2)/2u_k}$, denote $v_k(x_{0,k})^{\frac{1}{n-2}}$ by $R_{0,k}$ and $\frac{1}{1-e^{-1/2}}$ by A_0.

Lemma 4.1. *In each geodesic ball $B(x_{0,k}, A_0 R_{0,k}^{-1}) \subset M$ we may find a local maximum point of v_k, named by x_k. Furthermore,*

$$v_k(x_k) = \sup_{B(x_k, v_k(x_k)^{-\frac{1}{n-2}})} v_k.$$

Proof. Let $e^{u_k(x_{0,k})} = \varepsilon_{0,k}$. We define a mapping:

$$\begin{array}{rcl} \mathcal{U}_{0,k} : \mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2}) \subset T_{x_{0,k}}(M) & \rightarrow & B(x_{0,k}, {\varepsilon_{0,k}}^{1/2}) \\ y & \longmapsto & \exp_{x_{0,k}}(\varepsilon_{0,k} y), \end{array}$$

where the metric on tangent space is $\breve{g}_k = \varepsilon_{0,k}^{-2} \mathcal{U}_{0,k}^* g$ and $\mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2})$ is a geodesic ball. Moreover, consider a sequence of functions $\mu_{0,k}(y) = u_k(\mathcal{U}_{0,k}(y)) - \log \varepsilon_{0,k}$. We may derive a equation that $\mu_{0,k}(y)$ satisfies. In fact, we have

$$\begin{cases} F\Big(\lambda\Big(\breve{g}_k^{-1}\Big[A_{\breve{g}_k} + \nabla^2 \mu_{0,k} + d\mu_{0,k} \otimes d\mu_{0,k} - \frac{1}{2}|\nabla \mu_{0,k}|^2 \breve{g}_k\Big]\Big)\Big) & \\ = \varepsilon_{0,k}^2 (1-t) o + f(\mathcal{U}_{0,k}(y)) e^{-2\mu_{0,k}} & \text{in } \mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2}), \\ \frac{\partial \mu_{0,k}}{\partial x^n} = 0 & \text{on } \mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2}) \cap \{x^n = 0\}. \end{cases}$$

where $\mu_{0,k}$ is admissible, and o is a nonnegative constant.

Let us begin with the easy case $u_k(x) \geq u_k(x_{0,k}) - 1$ in $B(x_{0,k}, {\varepsilon_{0,k}}^{1/2})$. In this case, $0 \leq e^{-\frac{n-2}{2}\mu_{0,k}} \leq e^{\frac{n-2}{2}}$ in $\mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2})$. Hence, $\mu_{0,k}$ converges in C^3 to μ_∞ with $0 \leq e^{-\frac{n-2}{2}\mu_\infty} \leq e^{\frac{n-2}{2}}$ on $\mathbb{R}^n$. And the limit function μ_∞ satisfies

$$F\Big(\lambda\Big(\delta^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 \delta)\Big]\Big)\Big) = f(\bar{x}) e^{-2u}.$$

Then by the Liouville Theorem [12], we know that 0 is the locally minimum point of $\mu_{0,k}$. Rescaling back, we see that $x_{0,k}$ is the locally minimum point of u_k in $B(x_{0,k}, {\varepsilon_{0,k}}^{1/2})$.

The alternative case is that there exists $x_{1,k} \in B(x_{0,k}, {\varepsilon_{0,k}}^{1/2})$ such that $u_k(x_{1,k}) < u_k(x_{0,k}) - 1$. Then we may consider the lower bound of u_k in $B(x_{1,k}, {\varepsilon_{1,k}}^{1/2})$, where $\varepsilon_{1,k} = e^{u_k(x_{1,k})} < e^{-1}\varepsilon_{0,k}$. If $u_k \geq u_k(x_{1,k}) - 1$ in $B(x_{1,k}, {\varepsilon_{1,k}}^{1/2})$, then $\mu_{1,k}(y) = u_k(\mathcal{U}_{1,k}(y)) - \log \varepsilon_{1,k} > -1$, where

$$\mathcal{U}_{1,k} : y \rightarrow \exp_{x_{1,k}}(\varepsilon_{1,k} y),$$

and $x_{1,k}$ is a locally minimum point of u_k.

Otherwise, we may repeat the previous proceedings with $u_k(x_{j,k}) < u_k(x_{j-1,k}) - 1$ ($x_{j,k} \in B(x_{j-1,k}, {\varepsilon_{j-1,k}}^{1/2})$), $\varepsilon_{j,k} = e^{u_k(x_{j,k})} < e^{-1}\varepsilon_{j-1,k}$ and $\mu_{j,k}(y) = u_k(\mathcal{U}_{j,k}(y)) - \log \varepsilon_{j,k}$, where

$$\mathcal{U}_{j,k} : y \rightarrow \exp_{x_{j,k}}(\varepsilon_{j,k} y).$$

For any given k, as $u_k \in C^\infty(M)$, there exists $j(k) \in \mathbb{N}$, $j(k) < \infty$ such that $u_k(x_{j(k),k}) < u_k(x_{j(k)-1,k}) - 1$ and $u_k \geq u_k(x_{j(k),k}) - 1$ in $B(x_{j(k),k}, {\varepsilon_{j(k),k}}^{1/2})$. Hence, we can find a locally minimum point of the u_k in $B(x_{j(k),k}, {\varepsilon_{j(k),k}}^{1/2}) \subset B(x_{0,k}, A_0 {\varepsilon_{0,k}}^{1/2})$. This completes the proof (See Lemma 3.2 in [15] for more details). □

Now we consider the rescaled sequence $w_k = u_k - \sup_M u_k$. Suppose x_k^0 is the maximum point of u_k. Since $e^{-2\sup u_k} f(x_k^0) = e^{-2u_k(x_k)} f(x_k^0) \leq C(\triangle u_k + A_g)(x_k^0) \leq C$ thus $\bar{x} = \lim x_k$ is the blow up point with respect to w_k as well. It is obviously that w_k satisfies the equation

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[A_g + \nabla^2 w_k + dw_k \otimes dw_k - \frac{1}{2}|\nabla w_k|^2 g\Big]\Big)\Big) & \\ = (1-t) o + f(x) e^{-2\sup_M u_k} e^{-2w_k} & \text{in } M, \\ \frac{\partial w_k}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

where w_k is admissible, and $o \geq 0$ is a constant.

By virtue of Lemme 4.1, we may assume $\bar{x} = \lim x_k$, where $\{x_k\}$ are locally minimum points of u_k. Hence $\{x_k\}$ are also locally minimum points of w_k and

$$w_k(x_k) = \inf_{B(x_k, e^{\frac{1}{2}w_k(x_k)})} w_k.$$

Note that F satisfies $(C_1)-(C_6)$ and w_k are Γ admissible, where $\Gamma \subset \Sigma_{\frac{1}{n-2}}$. Hence w_k are subharmonic and satisfy

$$W + \frac{1}{n-2}\sigma_1(W)g \geq 0, \tag{4.1}$$

where $W = \nabla^2 w_k + dw_k \otimes dw_k - \frac{1}{2}|\nabla w_k|^2 g + A_g$. We need the idea of the minimal radial functions of w in $B_R(x_0)$ ([15]):

$$\widehat{w}(x) = \sup\{w(y) : y \in \partial B_r(x_0), r = d(x, x_0) \leq R\},$$

and denote $\nabla^2 \widehat{w} + d\widehat{w} \otimes d\widehat{w} - \frac{1}{2}|\nabla \widehat{w}|^2 g + A_g$ by $\widehat{W}$. Now we are ready to prove the following

Propostion 4.2. *Let u_j be a blow up sequence of solutions to (3.2). Then $w_j = u_j - \sup_M u_j$ converges in $w^{1,p}$ (for any $1 < p < \frac{n}{n-1}$) to an admissible function w. Moreover, if $\bar{x}$ is a blow up point of w, then near $\bar{x}$,*

$$w(x) = 2\log d(x, \bar{x}) + o(1), \tag{4.2}$$

where $d(x, \bar{x})$ denotes the geodesic distance from x to $\bar{x}$ with respect to the metric g. Furthermore, each blow up point is isolated.

Proof. Since a similar proposition on manifold without boundary has appeared in [15], we only focus on the differences.

Step 1. We may get admissible solutions on the doubled manifold. Glue two copies of (M, g) along the totally geodesic boundary together and denote the doubling manifold by $\check{M}$. With the given smooth Riemannian metric g on M, there is a standard metric $\check{g}$ on $\check{M}$ induced from g. When ∂M is totally geodesic in (M, g), $\check{g}$ is $C^{2,1}$ on $\check{M}$ (see [3]).

We can extend w_k to a $C^2(\check{M})$ function $\check{w}_k$ as follows: Near the boundary we take Fermi coordinates, $\check{w}_k$ is then defined as

$$\check{w}_k(x_1, \cdots, x_n) = \begin{cases} w_k(x_1, \cdots, x_n), & x_n \geq 0, \\ w_k(x_1, \cdots, -x_n), & x_n \leq 0. \end{cases}$$

Since $\frac{\partial w_k}{\partial \nu} = 0$, it is easy to verify by definition that $\check{w}_k \in C^2(\check{M})$. As matter of fact,

$$\begin{aligned} &\lim_{x_n \to 0^+} \frac{\partial \check{w}_k}{\partial x^n}(x_1, \cdots, x_n) = \frac{\partial w_k}{\partial x^n}(x_1, \cdots, x_{n-1}, 0) \\ = \quad & 0 = -\frac{\partial w_k}{\partial x^n}(x_1, \cdots, x_{n-1}, 0) = \lim_{x_n \to 0^-} \frac{\partial \check{w}_k}{\partial x^n}(x_1, \cdots, x_n), \end{aligned}$$

and

$$\lim_{x_n \to 0^+} \frac{\partial^2 \check{w}_k}{\partial (x^n)^2}(x_1, \cdots, x_n) = \lim_{x_n \to 0^-} \frac{\partial^2 \check{w}_k}{\partial (x^n)^2}(x_1, \cdots, x_n).$$

Thus from the admissible property of w_k we know that $\check{w}_k$ is also admissible and satisfies (4.1).

Step 2. We can find convergent "minimal radial functions" on doubled manifold. Inequality (4.1) says $\check{w}_k$ is subharmonic. From Corollary 2.1 in [15], $\{\check{w}_k\}$ converges to a subharmonic function $\check{w}$ in $W^{1,p}$ (for any $1 < p < \frac{n}{n-1}$)). By Corollary 2.2 in [15], the corresponding minimal radial functions $\widehat{\check{w}}_k$ also converge to $\widehat{\check{w}}$. Note that the minimal radial functions depend only on distance to the center, by Corollary 2.1 and Corollary 2.2 in [15], we may obtain

$$\widehat{\check{w}}\,(r) = \lim_{k\to\infty} \widehat{\check{w}}_k\,(r)\,, \tag{4.3}$$

where

$$\widehat{\check{w}}_k(r) = \sup\{\check{w}_k(y) : y \in \partial B_r(x_k)\},$$

and

$$\widehat{\check{w}}(r) = \sup\{\check{w}(y) : y \in \partial B_r(\bar{x})\}.$$

On the one hand, based on (4.3) and (4.1), we can get the following estimates

$$\widehat{\check{w}}(x) \le 2\log d(x,\bar{x}) + C. \tag{4.4}$$

In fact, we may assume $\widehat{\check{w}}_k(r) = \check{w}_k(x_r)$, $x_r = (0,\cdots,0,r)$, $|A_g| \le Cr/2$. $\widehat{\check{w}}_k$ are still admissible and satisfy inequality (4.1). Thus

$$\begin{aligned}
0 &\le \left((n-2)\widehat{W}_{nn} + \Sigma_i\widehat{W}_{ii}\right)(x_r)\\
&\le (n-1)\left(\widehat{\check{w}}''_k + (\widehat{\check{w}}'_k)^2 - \frac{g_{nn}}{2}(\widehat{\check{w}}'_k)^2 + Cr/2\right)\\
&\quad + \Sigma_{i=1}^{n-1}\left((\frac{1}{r}+C)\widehat{\check{w}}'_k - \frac{g_{ii}}{2}(\widehat{\check{w}}'_k)^2 + Cr/2\right)\\
&\le (n-1)\left(\widehat{\check{w}}''_k + \frac{1}{r}\widehat{\check{w}}'_k + C(\widehat{\check{w}}'_k + r)\right),
\end{aligned}$$

where the last inequality comes from $\Sigma_i g_{ii} \ge n$. Hence,

$$\left(\log(r\widehat{\check{w}}'_k + r^2)\right)' + C \ge 0.$$

By taking a limit we get (4.4).

On the other hand, let $\widehat{\check{v}}_k = e^{-(n-2)/2\widehat{\check{w}}_k}$. From $\triangle\widehat{\check{v}}_k \le C\widehat{\check{v}}_k r$, we get

$$[r^{n-1}\widehat{\check{v}}'_k]' \le Cr^n\widehat{\check{v}}_k$$

Thus, by a direct calculation we know

$$\widehat{\check{w}}(x) \ge 2\log d(x,\bar{x}) + o(1).$$

Therefore

$$\widehat{\check{w}}(x) = 2\log d(x,\bar{x}) + o(1). \tag{4.5}$$

Then the comparison principle helps us to deduce (4.2) from (4.5). Roughly speaking, since $\check{w}$ equals $\widehat{\check{w}}$ at some points, the comparison principle implies they are equal everywhere. That is

$$\check{w}(x) = 2\log d(x,\bar{x}) + o(1).$$

(For more details, one may consult section 3 of [15].) □

Proof of Theorem 1.1.

As the proof of Proposition 4.2, we glue two copies of (M, g) together. Denote the doubled manifold and functions by a "check" (e.g. $\check{M}, \check{w}$). Since the Ricci curvature $Ric_{e^{-2\check{w}}g}$ is still non-negative, by (4.5) and Volume Comparison Theorem, there is at most one end away from the blow up points; the metric $e^{-2\check{w}}g$ is in fact a Euclidean one (see section 7 of [6] for details), namely (M, g) is conformally equivalent to the unit half sphere, which contradicts with the assumption in Theorem 1.1. Therefore there is a unform L^∞ bound for solutions. So the set of solutions is compact. This completes the proof of Theorem 1.1. □

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References

1. S. Chen, *Conformal Deformation on Manifolds with Boundary*, Geom. Funct. Anal., **19** (2009), no. 4, 1029–1064.
2. S. Chen, *Boundary value problems for some fully nonlinear elliptic equations*, Calc. Var. Partial Differential Equations, 2007, **30(1)**:1–15.
3. José F. Escobar, *The Yamabe problem on manifolds with boundary*, J. Differential Geom. **35** (1992), no. 1, 21–84.
4. A. Gray and L. Vanhecke, *The volumes of tubes about curves in a Riemannian manifold*, proc. London Math. Soc., **44** (1982), 215-243.
5. P.F. Guan and G.F. Wang, *Conformal deformations of the smallest eigenvalue of the Ricci tensor*, American Journal of Mathematics, **129** (2007), no.2, 499-526.
6. M. Gursky and J.Viaclovsky, *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, Ann. of Math., **166** (2007), 475–531.
7. M. Gursky and J. Viaclovsky, *Volume comparison and the σ_k-Yamabe problem*, Adv. Math. **187** (2004), no. 2, 447-487.
8. M. Gursky and J. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J., **52** (2003),399 -419.
9. Y. He and W.M. Sheng, *On existence of the prescribing k -curvature problem on manifolds with boundary*, preprint.
10. Y. He and W.M. Sheng, *Local estimates for some elliptic equations arising from conformal geometry*, Preprint.
11. Y. Y. Li, *Degree theory for second order nonlinear elliptic operators and its applications*, Comm. Partial Differential Equations, **14** (1989), no. 11, 1541-1578.
12. A. Li and Y.Y. Li, *On some conformally invariant fully nonlinear equations, part II: Liouville, Harnack, and Yamabe*, Acta Math., **195** (2005), 117–154, MR2233687, Zbl pre05039005.
13. J.Y. Li and W.M. Sheng, *Deforming metrics with negative curvature by a fully nonlinear flow*, Calc. Var. PDE., 2005, **23**: 33–50.
14. W.M. Sheng, N.S. *Trudinger and X-J. Wang, The Yamabe problem for higher order curvatures*, J. Diff. Geom. **77** (2007), 515-553.
15. N.S. Trudinger and X-J. Wang, *The intermediate case of the Yamabe problem for higher order curvatures*, International Mathematics Research Notices, **2009**.
16. X.-J. Wang, *A class of fully nonlinear elliptic equations and related functionals*, Indiana Univ. Math. J., **43** (1994), 25–54, MR1275451, Zbl 0805.35036.

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