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ON A PROOF OF THE LABASTIDA-MARIÑO-OOGURI-VAFA CONJECTURE

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Abstract. We outline a proof of a remarkable conjecture of Labastida-Mariño-Ooguri-Vafa about certain new algebraic structures of quantum link invariants and the integrality of infinite family of new topological invariants. Our method is based on the cut-and-join analysis and a special rational ring characterizing the structure of the Chern-Simons partition function.

1. Introduction

For decades, we have witnessed the great development of string theory and its powerful impact on the development of mathematics. There have been a lot of marvelous results revealed by string theory, which deeply relate different aspects of mathematics. All these mysterious relations are connected by a core idea in string theory called "duality". It was found that string theory on Calabi-Yau manifolds provided new insight in geometry of these spaces. The existence of a topological sector of string theory leads to a simplified model in string theory, the topological string theory.

A major problem in topological string theory is how to compute Gromov-Witten invariants. There are two major methods widely used: mirror symmetry in physics and localization in mathematics. Both methods are effective when genus is low while having trouble in dealing with higher genera due to the rapidly growing complexity during computation. However, when the target manifold is Calabi-Yau threefold, large N Chern-Simons/topological string duality opens a new gate to a complete solution of computing Gromov-Witten invariants at all genera.

The study of large N Chern-Simons/topological string duality was originated in physics by an idea that gauge theory should have a string theory explanation. In 1992, Witten [28] related topological string theory of T^*M of a three dimensional manifold M to Chern-Simons gauge theory on M. In 1998, Gopakumar and Vafa [4] conjectured that, at large N, open topological A-model of N D-branes on T^*S^3 is dual to topological closed string theory on resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$. Later, Ooguri and Vafa [21] showed a picture on how to describe Chern-Simons invariants of a knot by open topological string theory on resolved conifold paired with lagrangian associated with the knot.

Though large N Chern-Simons/topological string duality still remains open, there have been a lot of progress in this direction demonstrating the power of this idea. Even for the simplest knot, the unknot, Mariño-Vafa formular [19, 15] gives a beautiful closed formula for Hodge integral up to three Chern classes of Hodge bundle. Furthermore, using topological vertex theory [1, 16, 13], one is able to compute Gromov-Witten invariants of any toric Calabi-Yau threefold by reducing the computation to a gluing algorithm of topological vertex. This thus leads to a closed formula of topological string partition function, a generating function of Gromov-Witten invariants, in all genera for any toric Calabi-Yau threefolds.

On the other hand, after Jones' famous work on polynomial knot invariants, there had been a series of polynomial invariants discovered (for example, [8, 3, 9]), the generalization of which was provided by quantum group theory [25] in mathematics and by Chern-Simons path integral with the gauge group $SU(N)$ [27] in physics.

Based on the large N Chern-Simons/topological string duality, Ooguri and Vafa [21] reformulated knot invariants in terms of new integral invariants capturing the spectrum of M2 branes ending on M5 branes embedded in the resolved conifold. Later, Labastida, Mariño and Vafa [12, 10] refined the analysis of [21] and conjectured the precise integrality structure for open Gromov-Witten invariants. This conjecture predicts a remarkable new algebraic structure for the generating series of general link invariants and the integrality of infinite family of new topological invariants. In string theory, this is a striking example that two important physical theories, topological string theory and Chern-Simons theory, exactly agree up to all orders. In mathematics this conjecture has interesting applications in understanding the basic structure of link invariants and three manifold invariants, as well as the integrality structure of open Gromov-Witten invariants. Recently, X.S. Lin and H. Zheng [14] verified LMOV conjecture in several lower degree cases for some torus links.

In this paper, we describe an outline of the proof of Labastida-Mariño-Ooguri-Vafa conjecture for any link. The details of the proofs are given in [17].

2. The Labastida-Mariño-Ooguri-Vafa conjecture

2.1. Quantum group invariants of links. Let $\mathcal{L}$ be a link with L components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$, represented by the closure of an element of braid group $\mathcal{B}_m$. We associate to each $\mathcal{K}_\alpha$ an irreducible representation R_α of quantized universal enveloping algebra $U_q(\mathfrak{sl}(N,\mathbb{C}))$[1], labeled by its highest weight Λ_α. Denote the corresponding module by V_{Λ_α}. The j-th strand in the braid will be associated with the irreducible module $V_j = V_{\Lambda_\alpha}$, if this strand belongs to the component $\mathcal{K}_\alpha$. The braiding is defined through the following *universal R-matrix* of $U_q(\mathfrak{sl}_N)$

$$\mathcal{R} = q^{\frac{1}{2}\sum_{i,j} C^{-1}_{ij} H_i \otimes H_j} \prod_{\text{positive root } \beta} \exp_q[(1-q^{-1})E_\beta \otimes F_\beta]\,.$$

Here $\{H_i, E_i, F_i\}$ are the generators of $U_q(\mathfrak{sl}_N)$, (C_{ij}) is the Cartan matrix and

$$\exp_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{4}k(k+1)} \frac{x^k}{\{k\}_q!}\,,$$

where

$$\{k\}_q = \frac{q^{-k/2} - q^{k/2}}{q^{-1/2} - q^{1/2}}\,, \qquad \{k\}_q! = \prod_{j=1}^{k} \{j\}_q\,.$$

Define *braiding* by $\check{\mathcal{R}} = P_{12}\mathcal{R}$, where $P_{12}(v \otimes w) = w \otimes v$.

[1]Later, we simply write $U_q(\mathfrak{sl}_N)$.

Now for a given link $\mathcal{L}$ of L components, one chooses a closed braid representative in braid group $\mathcal{B}_m$ whose closure is $\mathcal{L}$. In the case of no confusion, we also use $\mathcal{L}$ to refer its braid representative in $\mathcal{B}_m$. We will assign each crossing by the braiding as follows. Let U, V be two $U_q(\mathfrak{sl}_N)$-modules labeling two outgoing strands of the crossing, the braiding $\check{R}_{U,V}$ (resp. $\check{R}^{-1}_{V,U}$) is assigned as in Figure 1.

FIGURE 1. Assign crossing by $\check{\mathcal{R}}$.

The above assignment will give a representation of $\mathcal{B}_m$ on $U_q(\mathfrak{g})$-module $V_1 \otimes \cdots \otimes V_m$. Namely, for any generator, $\sigma_i \in \mathcal{B}_m$, define[2]

$$h(\sigma_i) = \mathrm{id}_{V_1} \otimes \cdots \otimes \check{\mathcal{R}}_{V_i,V_{i+1}} \otimes \cdots \otimes id_{V_N}\,.$$

Therefore, any link $\mathcal{L}$ will provide an isomorphism

$$h(\mathcal{L}) \in \mathrm{End}_{U_q(\mathfrak{sl}_N)}(V_1 \otimes \cdots \otimes V_m)\,.$$

Let $K_{2\rho}$ be the enhancement of $\check{\mathcal{R}}$ in the sense of [24], where ρ is the half-sum of all positive roots of $\mathfrak{sl}_N$. The irreducible representation R_α is labeled by the corresponding partition A^α.

Definition 2.1. *Given L labeling partitions $A^1, \ldots, A^L$, the quantum group invariant of $\mathcal{L}$ is defined as follows:*

$$W_{(A^1,\ldots,A^L)}(\mathcal{L}) = q^{d(\mathcal{L})}\,\mathrm{tr}_{V_1\otimes\cdots\otimes V_m}(K_{2\rho} \circ h(\mathcal{L}))\,,$$

where

$$d(\mathcal{L}) = -\frac{1}{2}\sum_{\alpha=1}^{L} \omega(\mathcal{K}_\alpha)(\Lambda_\alpha, \Lambda_\alpha + 2\rho) + \frac{1}{N}\sum_{\alpha<\beta} \mathrm{lk}(\mathcal{K}_\alpha, \mathcal{K}_\beta)|A^\alpha| \cdot |A^\beta|\,,$$

and $\mathrm{lk}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ is the linking number of components $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$. A substitution of $t = q^N$ is used to give a two-variable framing independent link invariant.

2.2. Labastida-Mariño-Ooguri-Vafa conjecture. Let $\mathcal{L}$ be a link with L components and $\mathcal{P}$ be the set of all partitions. The Chern-Simons partition function of $\mathcal{L}$ is a generating function of quantum group invariants of links given by

$$Z_{\mathrm{CS}}(\mathcal{L}; q, t) = \sum_{\vec{A}\in\mathcal{P}^L} W_{\vec{A}}(\mathcal{L}; q, t) \prod_{\alpha=1}^{L} s_{A^\alpha}(x^\alpha) \tag{2.1}$$

for any arbitrarily chosen sequence of variables

$$x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots,)\,.$$

[2] In the case of σ_i^{-1}, use $\check{\mathcal{R}}^{-1}_{V_{i+1},V_i}$ instead.

In (2.1), $\vec{A}=(A^1,\dots,A^L)\in\mathcal{P}^L$ and $s_{A^\alpha}(x^\alpha)$ is the Schur function.

Free energy is defined to be

$$F=\log Z_{\mathrm{CS}}\,.$$

Use plethystic exponential, one can obtain

$$F=\sum_{d=1}^{\infty}\sum_{\vec{A}\neq 0}\frac{1}{d}f_{\vec{A}}(q^d,t^d)\prod_{\alpha=1}^{L}s_{A^\alpha}\big((x^\alpha)^d\big)\,, \tag{2.2}$$

where

$$(x^\alpha)^d=\big((x_1^\alpha)^d,(x_2^\alpha)^d,\dots\big)\,.$$

Based on the duality between Chern-Simons gauge theory and topological string theory, Labastida, Mariño, Ooguri, Vafa conjectured that $f_{\vec{A}}$ have the following highly nontrivial structures.

For any A, $B\in\mathcal{P}$, define the following function

$$M_{AB}(q)=\sum_{\mu}\frac{\chi_A(C_\mu)\chi_B(C_\mu)}{\mathfrak{z}_\mu}\prod_{j=1}^{\ell(\mu)}(q^{-\mu_j/2}-q^{\mu_j/2})\,. \tag{2.3}$$

Conjecture (LMOV). *For any $\vec{A}\in\mathcal{P}^L$,*

(i). *there exist $P_{\vec{B}}(q,t)$ for $\vec{B}\in\mathcal{P}^L$, such that*

$$f_{\vec{A}}(q,t)=\sum_{|B^\alpha|=|A^\alpha|}P_{\vec{B}}(q,t)\prod_{\alpha=1}^{L}M_{A^\alpha B^\alpha}(q). \tag{2.4}$$

Furthermore, $P_{\vec{B}}(q,t)$ has the following expansion

$$P_{\vec{B}}(q,t)=\sum_{g=0}^{\infty}\sum_{Q\in\mathbb{Z}/2}N_{\vec{B};\,g,Q}(q^{-1/2}-q^{1/2})^{2g-2}t^Q\,. \tag{2.5}$$

(ii). *$N_{\vec{B};\,g,Q}$ are integers.*

2.3. Related notations. For a given link $\mathcal{L}$ of L components, we will fix the following notations in this paper. Given $\lambda\in\mathcal{P}$, $\vec{A}=(A^1,\dots,A^L)$, $\vec{\mu}=(\mu^1,\dots,\mu^L)\in\mathcal{P}^L$. Let $x=(x^1,...,x^L)$ where x^α is a set of variables

$$x^\alpha=(x_1^\alpha,x_2^\alpha,\cdots)\,.$$

The following notations will be used throughout the paper.

$$[n]_q=q^{-\frac{n}{2}}-q^{\frac{n}{2}}\,,\qquad [\lambda]_q=\prod_{j=1}^{\ell(\lambda)}[\lambda_j]_q\,,\qquad \mathfrak{z}_{\vec{\mu}}=\prod_{\alpha=1}^{L}\mathfrak{z}_{\mu^\alpha}\,,$$

$$|\vec{A}|=(|A^1|,...,|A^L|)\,,\qquad \|\,\vec{A}\,\|=\sum_{\alpha=1}^{L}|A^\alpha|\,,\qquad \ell(\vec{\mu})=\sum_{\alpha=1}^{L}\ell(\mu^\alpha)\,,$$

$$\vec{A}^t=\big((A^1)^t,\dots,(A^L)^t\big)\,,\qquad \chi_{\vec{A}}(\vec{\mu})=\prod_{\alpha=1}^{L}\chi_{A^\alpha}(C_{\mu^\alpha})\,,\qquad s_{\vec{A}}(x)=\prod_{\alpha=1}^{L}s_{A^\alpha}(x^\alpha)\,.$$

One can define an order on $\mathcal{P}^L$ lexicographically. Therefore, one can generalize the concept of partition from the set of all non-negative integers to $\mathcal{P}^L$. We denote by $\mathcal{P}(\mathcal{P}^L)$ the set of all partitions on $\mathcal{P}^L$. Given $\Lambda \in \mathcal{P}(\mathcal{P}^L)$, the following quantity

$$\theta_\Lambda = \frac{(-1)^{\ell(\Lambda)-1}(\ell(\Lambda)-1)!}{|\operatorname{Aut}\Lambda|}$$

plays an important role in the relationship of topological string partition function and free energy.

Rewrite free energy as

$$F = \log Z = \sum_{\vec{\mu}\neq 0} F_{\vec{\mu}} p_{\vec{\mu}}(x)\,. \tag{2.6}$$

Here in the similar usage of notation,

$$p_{\vec{\mu}}(x) = \prod_{\alpha=1}^{L} p_{\mu^\alpha}(x^\alpha)\,.$$

We also rewrite Chern-Simons partition function as

$$Z_{\mathrm{CS}}(\mathcal{L}) = 1 + \sum_{\vec{\mu}\neq 0} Z_{\vec{\mu}} p_{\vec{\mu}}(x)\,.$$

3. Sketch of the proof

In this section we will give an outline of the proof of Labastida-Mariño-Ooguri-Vafa conjecture[3] which contains two parts in correspondence of (i) and (ii) in the description of LMOV conjecture: the **existence and integrality**.

3.1. Existence of the algebraic structure.

The existence of the algebraic structure (2.5) includes the following two steps:

- The symmetry of q and q^{-1} in $P_{\vec{B}}(q,t)$.
- The pole structure of $P_{\vec{B}}$.

The symmetry of q and q^{-1} in $P_{\vec{B}}(q,t)$ can be obtained from the lemma.

Lemma 3.1. $W_{\vec{A}^t}(q,t) = (-1)^{\|\vec{A}\|} W_{\vec{A}}(q^{-1},t)$.

To prove the existence of the pole structure, we will consider the following framed generating series. Substitute

$$W_{\vec{A}}(\mathcal{L};\, q,t,\tau) = W_{\vec{A}}(\mathcal{L};\, q,t) \cdot q^{\sum_{\alpha=1}^{L} \kappa_{A^\alpha}\tau/2}$$

in the Chern-Simons partition function, we have the following framed partition function

$$Z(\mathcal{L};\, q,t,\tau) = 1 + \sum_{\vec{A}\neq 0} W_{\vec{A}}(\mathcal{L};\, q,t,\tau)\cdot s_{\vec{A}}(x)\,.$$

Similarly, framed free energy

$$F(\mathcal{L};\, q,t,\tau) = \log Z(\mathcal{L};\, q,t,\tau)\,.$$

[3] Briefly, we call it LMOV conjecture.

It satisfies the following cut-and-join equation

$$\frac{\partial F(\mathcal{L};q,t,\tau)}{\partial\tau}=\frac{u}{2}\sum_{\alpha=1}^{L}\sum_{i,j\geq 1}\left(ijp^{\alpha}_{i+j}\frac{\partial^2 F}{\partial p^{\alpha}_i\partial p^{\alpha}_j}+(i+j)p^{\alpha}_i p^{\alpha}_j\frac{\partial F}{\partial p^{\alpha}_{i+j}}+ijp^{\alpha}_{i+j}\frac{\partial F}{\partial p^{\alpha}_i}\frac{\partial F}{\partial p^{\alpha}_j}\right).$$

Restrict the equation to $\vec{\mu}$, we have

$$\frac{\partial F_{\vec{\mu}}}{\partial\tau}=\frac{u}{2}\Bigg(\sum_{|\vec{\nu}|=|\vec{\mu}|,\,\ell(\vec{\nu})=\ell(\vec{\mu})\pm 1}\alpha_{\vec{\mu}\vec{\nu}}F_{\vec{\nu}}+\text{nonlinear terms}\Bigg),\tag{3.1}$$

where $\alpha_{\vec{\mu}\vec{\nu}}$ is some constant.

Denote by $\deg_u$ the lowest degree of u in a Laurent polynomial of u. The pole structure of $P_{\vec{B}}$ follows from the following degree lemma:

Lemma 3.2. $\deg_u F_{\vec{\mu}}\geq\ell(\vec{\mu})-2$.

Let $\bigcirc$ be the unknot. Given any $\vec{A}=(A^1,\dots,A^L)\in\mathcal{P}^L$, we will obtain the following limit behavior of quantum group invariants of links at $q\to 1$.

$$\lim_{q\to 1}\frac{W_{\vec{A}}(\mathcal{L};q,t)}{W_{\vec{A}}(\bigcirc^{\otimes L};q,t)}=\prod_{\alpha=1}^{L}\xi_{\mathcal{K}_\alpha}(t)^{d_\alpha},\tag{3.2}$$

where $|A^\alpha|=d_\alpha$, $\mathcal{K}_\alpha$ is the α-th component of $\mathcal{L}$, and $\xi_{\mathcal{K}_\alpha}(t)$, $\alpha=1,\dots,L$, are independent of $\vec{A}$.

The following *cut-and-join analysis* will give the desired results.

If $\|\vec{\mu}\|=1$, it can be verified through the degree of u in HOMFLY polynomial.

If $\|\vec{\mu}\|>1$, by (3.2) We can prove that

$$\deg_u\left(\frac{\partial F_{\vec{\mu}}}{\partial\tau}\right)=\deg_u F_{\vec{\mu}}\,.$$

Suppose Lemma 3.2 holds for any $\|\vec{\nu}\|<d$. Now, consider

$$S=\{\vec{\nu}:\,|\vec{\nu}|=\vec{d}\}\,,$$

where $\vec{d}=(d^1,\dots,d^L)$ and $\sum_{\alpha=1}^L d^\alpha=d$. If $\vec{\mu}\in S$, one important fact is that the degree of u in the nonlinear terms in (3.1) is no less than $\ell(\vec{\mu})-3$. We will prove Lemma 3.2 holds for $\forall\vec{\mu}\in S$ by contradiction.

Denote by

$$S_r=\{\vec{\nu}:\vec{\nu}\in S\text{ and }\ell(\vec{\nu})=r\}\subset S$$

and

$$D_r=\min\{\deg_u F_{\vec{\mu}}:\,\vec{\mu}\in S_r\}\,.$$

Assume k is the smallest integer such that there exists a $\vec{\nu}\in S_k$ satisfying

$$\deg_u F_{\vec{\nu}}<\ell(\vec{\nu})-2=k-2.$$

Choose $\vec{\mu}\in S_k$ such that $\deg_u F_{\vec{\mu}}=D_k$. Rewrite (3.1) into three parts

$$\frac{\partial F_{\vec{\mu}}}{\partial\tau}=\frac{u}{2}\sum_{|\vec{\nu}|=|\vec{\mu}|,\,\ell(\vec{\nu})=\ell(\vec{\mu})-1}\alpha_{\vec{\mu}\vec{\nu}}F_{\vec{\nu}}+\frac{u}{2}\sum_{|\vec{\nu}|=|\vec{\mu}|,\,\ell(\vec{\nu})=\ell(\vec{\mu})+1}\alpha_{\vec{\mu}\vec{\nu}}F_{\vec{\nu}}+*\tag{3.3}$$

where $*$ represents the nonlinear part and $\deg_u *=k-2$. Here in the equation, $\vec{\nu}$ runs in S.

However,

$$\deg_u \frac{\partial F_{\vec{\mu}}}{\partial \tau} = \deg_u F_{\vec{\mu}} = D_k < k-2\,.$$

$D_{k-1} \geq (k-1)-2$, so the first part of the sums in the r.h.s of (3.3) is of the lowest order of u at least $k-2$ and the lowest order of u in $\frac{\partial F_{\vec{\mu}}}{\partial \tau}$ must come from the summation over $\ell(\vec{\nu}) = \ell(\vec{\mu}) + 1$. Then we have

$$1 + D_{k+1} \leq D_k, \tag{3.4}$$

and

$$D_{k+1} \leq D_k - 1 < k - 2 - 1 < (k+1) - 2\,.$$

Choose $\vec{\xi} \in S_{k+1}$ such that $\deg_u F_{\vec{\xi}} = D_{K+1}$, run the similar analysis as above. Since (3.4), one will get

$$1 + D_{k+2} \leq D_{k+1}.$$

Thus we obtain

$$D_k > D_{k+1} > D_{k+2} > ...D_{d-1} > D_d\,.$$

Remember that there is still one last equation we have not used yet. Consider the cut-and-join equation for the following partition

$$\vec{\eta} = \big((1^{d_1}), \ldots, (1^{d_L})\big) \in S_d\,.$$

Note that this partition has the longest length in S, there will be only the first summand in (3.3) left. In particular, no non-linear terms will appear in the equation. Therefore,

$$\frac{\partial F_{\vec{\eta}}}{\partial \tau} = \frac{u}{2} \sum_{\ell(\vec{\nu})=d-1} \alpha_{\vec{\eta}\vec{\nu}} F_{\vec{\nu}}\,.$$

This implies

$$D_d \geq D_{d-1} + 1\,,$$

which contradicts with $D_{d-1} > D_d$.This gives a proof of the existence of (2.5).

Define

$$\widetilde{F}_{\vec{\mu}} = \frac{F_{\vec{\mu}}}{\phi_{\vec{\mu}}(q)}, \qquad \widetilde{Z}_{\vec{\mu}} = \frac{Z_{\vec{\mu}}}{\phi_{\vec{\mu}}(q)}\,.$$

where

$$\phi_{\vec{\mu}}(q) = \prod_{\alpha=1}^{L} \prod_{j=1}^{\ell(\mu^\alpha)} [\mu_j^\alpha]\,.$$

Lemma 3.2 directly implies the following:

Corollary 3.3. *For any $\vec{\mu}$, $\widetilde{F}_{\vec{\mu}}$ is of the following form:*

$$\widetilde{F}_{\vec{\mu}}(q,t) = \sum_{\textit{finitely many } n_\alpha} \frac{a_\alpha(t)}{[n_\alpha]^2} + \textit{ polynomial.}$$

Let $\mathcal{L}$ be the closure of a braid β with writhe number 0. Cable the i-th component of β, β_i, by substituting $\ell(\mu^i)$ parallel strands for each strand of β_i and the $\ell(\mu^i)$ parallel components are colored by partitions (μ^i_1),...,$(\mu^i_{\ell(\mu^i)})$. We take the closure of this new braid and obtain a new link, denoted by $\mathcal{L}_{\vec{\mu}}$.

Let $\mu^i = (\mu^i_1, \cdots, \mu^i_{\ell_i})$. We use symbol

$$\hat{Z}_{\vec{\mu}} = Z_{\vec{\mu}} \cdot \mathfrak{z}_{\vec{\mu}}; \qquad \hat{F}_{\vec{\mu}} = F_{\vec{\mu}} \cdot \mathfrak{z}_{\vec{\mu}}\,.$$

Notice the following fact from the definition of quantum group invariants:

$$\hat{Z}_{(\mu^1,\cdots,\mu^L)}(\mathcal{L}) = \hat{Z}_{(\mu^1_1),\cdots,(\mu^1_{\ell_1}),\cdots,(\mu^L_1),\cdots,(\mu^L_{\ell_L})}(\mathcal{L}_{\vec{\mu}})\,, \tag{3.5}$$

$$\hat{F}_{(\mu^1,\cdots,\mu^L)}(\mathcal{L}) = \hat{F}_{(\mu^1_1),\cdots,(\mu^1_{\ell_1}),\cdots,(\mu^L_1),\cdots,(\mu^L_{\ell_L})}(\mathcal{L}_{\vec{\mu}})\,. \tag{3.6}$$

Consider $\delta_n = \sigma_1 \cdots \sigma_{n-1}$. Let $\mathfrak{S}_A$ be the minimal projection of the Hecke algebra $\mathcal{H}_n \to \mathcal{H}_A$, and

$$\mathfrak{P}_\mu = \sum_A \chi_A(C_\mu)\mathfrak{S}_A\,.$$

We will apply a lemma of Aiston-Morton [2] in the following computation:

$$\delta_n^n \mathfrak{S}_A = q^{\frac{1}{2}\kappa_A}\mathfrak{S}_A\,.$$

Let

$$\vec{d} = \big((d_1),\ldots,(d_L)\big), \qquad \frac{1}{\vec{d}} = \Big(\frac{1}{d_1},\ldots,\frac{1}{d_L}\Big)\,.$$

Due to the cabling formula to the length of partition (3.5) and (3.6), we can simply deal with all the color of one row without loss of generality. Take framing $\tau_\alpha = n_\alpha + \frac{1}{d_\alpha}$ and choose a braid group representative of $\mathcal{L}$ such that the writhe number of $\mathcal{L}_\alpha$ is n_α. Denote by $\vec{\tau} = (\tau_1,\ldots,\tau_L)$,

$$\begin{aligned}
\hat{Z}_{\vec{d}}\Big(\mathcal{L};q,t;\vec{\tau}\Big) &= \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{d}})W_{\vec{A}}(\mathcal{L};q,t)q^{\frac{1}{2}\sum_{\alpha=1}^L \kappa_{A^\alpha}\left(n_\alpha+\frac{1}{d_\alpha}\right)} \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha}\operatorname{Tr}\Big(\mathcal{L}_{\vec{d}}\sum_{\vec{A}}\chi_{\vec{A}}(C_{\vec{d}})q^{\frac{1}{2}\sum_\alpha \kappa_\alpha \frac{1}{d_\alpha}}\bigotimes_\alpha \mathfrak{S}_{A^\alpha}\Big) \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha}\operatorname{Tr}\Big(\mathcal{L}_{\vec{d}}\sum_{\vec{A}}\chi_{\vec{A}}(C_{\vec{d}})(\delta_{d_1}\otimes\cdots\otimes\delta_{d_L})\bigotimes_\alpha \mathfrak{S}_{A^\alpha}\Big) \\
&= t^{\frac{1}{2}\sum_\alpha d_\alpha n_\alpha}\operatorname{Tr}\Big(\mathcal{L}_{\vec{d}}\cdot\otimes_{\alpha=1}^L\delta_{d_\alpha}\cdot\mathfrak{P}_{(1)}^{(d_1)}\otimes\cdots\otimes\mathfrak{P}_{(1)}^{(d_L)}\Big)\,.
\end{aligned}$$

Here, $\mathfrak{P}_{(1)}^{(d_\alpha)}$ means that in the projection, we use $q^{d_\alpha}, t^{d_\alpha}$ instead of using q,t. If we denote by

$$\mathcal{L} * Q_{\vec{d}} = \mathcal{L}_{\vec{d}}\cdot\delta_{\vec{d}}\cdot\mathfrak{P}_{(1)}^{(d_1)}\otimes\cdots\otimes\mathfrak{P}_{(1)}^{(d_L)}\,,$$

we have

$$\hat{Z}_{\vec{d}}\Big(\mathcal{L};q,t;\frac{1}{\vec{d}}\Big) = \mathcal{H}(\mathcal{L}*Q_{\vec{d}})\,. \tag{3.7}$$

Here $\mathcal{H}$ is the homfly polynomial which is normalized as

$$\mathcal{H}(\text{unknot}) = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,.$$

With the above normalization, for any given link $\mathcal{L}$, we have

$$[1]^L \cdot \mathcal{H}(\mathcal{L}) \in \mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big]\,.$$

Substituting q by q^{d_α} in the corresponding component, it leads to the following:

$$\prod_{\alpha=1}^{L} [d_\alpha] \cdot \hat{Z}_{\vec{d}}(\mathcal{L}; q, t; \vec{\tau}) \in \mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big]\,. \tag{3.8}$$

On the other hand, given any frame $\vec{\omega} = (\omega_1, \ldots, \omega_L)$,

$$\begin{aligned}\hat{Z}_{\vec{\mu}}(\mathcal{L};\, q, t; \vec{\omega}) &= \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{\mu}}) W_{\vec{A}}(\mathcal{L};\, q, t; \vec{\omega}) \\ &= \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{\mu}}) \sum_{\vec{\nu}} \frac{\chi_{\vec{A}}(C_{\vec{\nu}})}{\mathfrak{z}_{\vec{\nu}}} \hat{Z}_{\vec{\nu}}(\mathcal{L};\, q, t) q^{\frac{1}{2}\sum_\alpha \kappa_{A^\alpha}\omega_\alpha}\,.\end{aligned}$$

Exchange the order of summation, we have the following *convolution formula*:

$$\hat{Z}_{\vec{\mu}}(\mathcal{L};\, q, t; \vec{\omega}) = \sum_{\vec{\nu}} \frac{\hat{Z}_{\vec{\nu}}(\mathcal{L};\, q, t)}{\mathfrak{z}_{\vec{\nu}}} \sum_{\vec{A}} \chi_{\vec{A}}(C_{\vec{\mu}}) \chi_{\vec{A}}(C_{\vec{\nu}}) q^{\frac{1}{2}\sum_\alpha \kappa_{A^\alpha}\omega_\alpha}\,. \tag{3.9}$$

This property holds for arbitrary choice of n_α, $\alpha = 1, \ldots, L$. The coefficients of possible other poles vanish for arbitrary integer n_α. q^{n_α} is involved through certain polynomial relation, which implies the coefficients for other possible poles are simply zero. Therefore, (3.8) holds for any frame.

Now instead of canceling all the poles of $\hat{Z}(\mathcal{L})$ according to (3.8), we consider $[c]\hat{Z}_{(c),\vec{d}}$, which is equivalent to consider $[c]^2\widetilde{Z}_{(c),\vec{d}}$. Let $\mathcal{K}$ be the knot labeled by $[c]$. Multiplying $\widetilde{Z}$ by $[c]^2$ cancels all the poles related to $\mathcal{K}$ according to (3.7). Therefore, $[c]^2\widetilde{Z}_{(c),\vec{d}}$ has the same principle part as $\widetilde{Z}_{\vec{d}}$ except for multiplying by an element in $\mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big]$. Note that

$$\widetilde{Z}_{\vec{\mu}} = \sum_{\Lambda \vdash \vec{\mu}} \frac{\widetilde{F}_\Lambda}{\operatorname{Aut}|\Lambda|}\,.$$

$[c]^2(\widetilde{Z}_{(c),\vec{d}} - \widetilde{F}_{(c),\vec{d}})$ contains all the principle terms from $\mathcal{L}$ omitting $\mathcal{K}$. Therefore,

$$[c]^2\widetilde{F}_{(c),\vec{d}} \in \mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big]\,.$$

In the above discussion, $\mathcal{K}$ can be chosen to be any component of $\mathcal{L}$. We thus proved the following proposition:

Proposition 3.4. *Notation as above, we have:*

$$\prod_{\alpha=1}^{L} [d_\alpha] \cdot \hat{Z}_{\vec{d}}(\mathcal{L}; q, t) \in \mathbb{Q}[[1]^2, t^{\pm\frac{1}{2}}]; \tag{3.10}$$

$$[d_\alpha]^2 \widetilde{F}_{\vec{d}}(\mathcal{L}; q, t) \in \mathbb{Q}\big[[1]^2, t^{\pm\frac{1}{2}}\big], \forall\alpha\,. \tag{3.11}$$

Similarly, We can obtain

$$\widetilde{F}_{\vec{d}}(\mathcal{L};q,t)=\frac{H_{\vec{d}/D_{\vec{d}}}(t^{D_{\vec{d}}})}{D_{\vec{d}}\cdot[D_{\vec{d}}]^2}+\text{polynomial in }[D_{\vec{d}}]^2\text{ and }t^{\pm\frac{1}{2}D_{\vec{d}}}\,. \tag{3.12}$$

Once again, due to arbitrary choice of n_α, we know the above pole structure of $\widetilde{F}_{\vec{d}}$ holds for any frame.

Proposition 3.5. *Notations are as above. Assume $\mathcal{L}$ is labeled by the color $\vec{\mu}=(\mu^1,\ldots,\mu^L)$. Denote by $D_{\vec{\mu}}$ is the greatest common divisor of $\{\mu^1_1,\ \ldots\ ,\mu^1_{\ell(\mu^1)},\ldots,\mu^i_j,\ldots,\mu^L_{\ell(\mu^L)}\}$. $\widetilde{F}_{\vec{\mu}}$ has the following structure:*

$$\widetilde{F}_{\vec{\mu}}(q,t)=\frac{H_{\vec{\mu}/D_{\vec{\mu}}}(t^{D_{\vec{\mu}}})}{D_{\vec{\mu}}\cdot[D_{\vec{\mu}}]^2}+f(q,t)\,,$$

where $H_{\vec{\mu}/D_{\vec{\mu}}}(t)$ only depends on $\vec{\mu}/D_{\vec{\mu}}$ and $\mathcal{L}$, $f(q,t)\in\mathbb{Q}\big[[1]^2,t^{\pm\frac{1}{2}}\big]$.

Remark 3.6. *In Proposition 3.5, it is very interesting to interpret in topological string side that $H_{\vec{\mu}/D_{\vec{\mu}}}(t)$ only depends on $\vec{\mu}/D_{\vec{\mu}}$ and $\mathcal{L}$. The principle term is generated due to summation of counting rational curves and independent choice of k in the labeling color $k\cdot\vec{\mu}/D_{\vec{\mu}}$. This phenomenon simply tells us that contributions of counting rational curves in the labeling color $k\cdot\vec{\mu}/D_{\vec{\mu}}$ are through multiple cover contributions of $\vec{\mu}/D_{\vec{\mu}}$.*

3.2. Integrality. By the definition of $P_{\vec{B}}(q,t)$, comparing with Proposition 3.5, we have the following computation:

$$\begin{aligned}
P_{\vec{B}}(q,t)&=\sum_{\vec{\mu}}\frac{\chi_{\vec{B}}(\vec{\mu})}{\phi_{\vec{\mu}}(q)}\sum_{d|\vec{\mu}}\frac{\pi(d)}{d}F_{\vec{\mu}/d}(q^d,t^d)\\
&=\sum_{\vec{\mu}}\chi_{\vec{B}}(\vec{\mu})\sum_{d|\vec{\mu}}\frac{\mu(d)}{d}\widetilde{F}_{\vec{\mu}/d}(q^d,t^d)\\
&=\sum_{\vec{\mu}}\chi_{\vec{B}}(\vec{\mu})\sum_{d|D_{\vec{\mu}}}\frac{\mu(d)}{d}\frac{H_{\vec{\mu}/D_{\vec{\mu}}}(t^{D_{\vec{\mu}}})}{D_{\vec{\mu}/d}\cdot[D_{\vec{\mu}}]^2}+\text{polynomial}\\
&=\sum_{\vec{\mu}}\chi_{\vec{B}}(\vec{\mu})\delta_{1,D_{\vec{\mu}}}\frac{H_{\vec{\mu}/D_{\vec{\mu}}}(t^{D_{\vec{\mu}}})}{D_{\vec{\mu}}\cdot[D_{\vec{\mu}}]^2}+\text{polynomial}\,,
\end{aligned}$$

where $\delta_{1,n}$ equals 1 if $n=1$ and 0 otherwise. It implies that $P_{\vec{B}}$ is a rational function which only has pole at $q=1$. In the above computation, we used a fact of Möbius inversion,

$$\sum_{d|n}\mu(d)=\delta_{1,n}\,.$$

Therefore, for each $\vec{B}$,

$$\sum_{g=0}^{\infty}\sum_{Q\in\mathbb{Z}/2}N_{\vec{B};\,g,Q}(q^{-1/2}-q^{1/2})^{2g}t^Q\in\mathbb{Q}[(q^{-1/2}-q^{1/2})^2,t^{\pm\frac{1}{2}}]\,.$$

The integrality of $N_{\vec{B};\,g,Q}$ can be derived from the following theorem:

Theorem 1. *We have*

$$\sum_{g=0}^{\infty}\sum_{Q\in\mathbb{Z}} N_{\vec{B};\,g,Q}(q^{-1/2}-q^{1/2})^{2g}t^{Q} \in \mathbb{Z}[(q^{-1/2}-q^{1/2})^2, t^{\pm 1/2}]\,. \tag{3.13}$$

Remark 3.7. *The above theorem also implies the refined integral invariants $N_{\vec{B};\,g,Q}$ vanish at large genera (also for large Q).*

Define $y = (y^1, y^2, \dots, y^L)$ where

$$y^\alpha = (y_1^\alpha, y_2^\alpha, \dots)$$

is a set of arbitrarily chosen variables. Denote by $\Omega(y)$ the set of all symmetric function in $(y^1, y^2, \dots, y^L)$ with integral coefficients.

We start by defining the following special ring which characterizing the algebraic structure of Chern-Simons partition function.

$$\mathfrak{R}(y;q,t) = \left\{ \frac{a(y;q,t)}{b(q)} \,:\, a(y;q,t) \in \Omega(y)[[1]_q^2, t^{\pm 1/2}],\, b(q) = \prod_{n_k}[n_k]_q^2 \in \mathbb{Z}[[1]_q^2] \right\}.$$

Given $\frac{f(y;\,q,t)}{b(q)} \in \mathfrak{R}(y;\,q,t)$, if $f(y;\,q,t)$ is a primitive polynomial in terms of $q^{\pm 1/2}$, $t^{\pm 1/2}$ and Schur functions of y, we call $\frac{f(y;\,q,t)}{b(q)}$ is primitive.

Given any $\frac{r}{s}h(y;\,q,t)$ where $h(y;\,q,t) \in \mathfrak{R}(y;\,q,t)$ is primitive, define

$$\mathrm{Ord}_p\left(\frac{r}{s}h(y;\,q,t)\right) = \mathrm{Ord}_p\left(\frac{r}{s}\right)$$

for any prime number p.

We will consider the following generating series

$$T_{\vec{d}} = \sum_{|\vec{B}|=\vec{d}} s_{\vec{B}}(y)P_{\vec{B}}(q,t)$$

After some calculations, we have

$$T_{\vec{d}} = q^{|\vec{d}|}\sum_{k|\vec{d}}\frac{\mu(k)}{k}\sum_{\mathfrak{A}\in\mathcal{P}(\mathcal{P}^n),\|\mathfrak{A}\|=\vec{d}/k}\theta_{\mathfrak{A}}\prod_{j=1}^{\ell(\mathfrak{A})}W_{\mathfrak{A}_j}(q^k,t^k)s_{\mathfrak{A}_j}((z)^k) \tag{3.14}$$

Proposition 3.8. *$T_{\vec{d}}(y;\,q,t) \in \mathfrak{R}(y;v,t)$.*

Proposition 3.8 implies $q=0$ is a pole of $T_{\vec{d}}$. Since $T_{\vec{d}}$ can be written as

$$T_{\vec{d}} = \sum_{g=0}^{\infty}\sum_{Q\in\mathbb{Z}/2}\left(\sum_{|\vec{B}|=\vec{d}} N_{\vec{B};\,g,Q}s_{\vec{B}}(y)\right)(q^{-1/2}-q^{1/2})^{2g-2}t^{Q}$$

by the existence of (2.5). If there are infinitely many $N_{\vec{B};\,g,Q}$ nonzero, $q=0$ is then an essential singularity point of $T_{\vec{d}}$, which is a contradiction. Therefore, for each $\vec{B}$,

$$\sum_{g=0}^{\infty}\sum_{Q\in\mathbb{Z}/2} N_{\vec{B};\,g,Q}(q^{-1/2}-q^{1/2})^{2g}t^{Q} \in \mathbb{Q}[(q^{-1/2}-q^{1/2})^2, t^{\pm\frac{1}{2}}]\,.$$

On the other hand, by Proposition 3.8, $T_{\vec{d}} \in \mathfrak{R}(y;\, v, t)$ and $\mathrm{Ord}_p T_{\vec{d}} \geq 0$ for any prime number p. We have

$$\sum_{|\vec{B}|=\vec{d}} N_{\vec{B};\, g,Q} s_{\vec{B}}(y) \in \Omega(y)\,,$$

which implies $N_{\vec{B};\, g,Q} \in \mathbb{Z}$.

Combining the above discussions, we have

$$\sum_{g=0}^{\infty} \sum_{Q\in\mathbb{Z}/2} N_{\vec{B};\, g,Q} (q^{-1/2} - q^{1/2})^{2g} t^Q \in \mathbb{Z}[(q^{-1/2} - q^{1/2})^2, t^{\pm 1/2}]\,.$$

To prove Proposition 3.8, we combine the study of multi-cover contribution and p-adic argument. For any give prime number p, the following observation is important for the p-adic argument

$$\left\{ \mathfrak{B} : \sum_{j=1}^{\ell(\mathfrak{B})} |\mathfrak{B}_j| = p\, \vec{d} \text{ and } \mathrm{Ord}_p(\theta_{\mathfrak{B}}) < 0 \right\} = \left\{ \mathfrak{A}^{(p)} : \sum_{j=1}^{\ell(\mathfrak{A})} |\mathfrak{A}_j| = \vec{d} \right\}.$$

Matching the following terms, finally we have

$$\mathrm{Ord}_p \left(\theta_{\mathfrak{A}^{(p)}} W_{\mathfrak{A}^{(p)}}(q,t) s_{\mathfrak{A}^{(p)}}(z) - \frac{1}{p} \theta_{\mathfrak{A}} W_{\mathfrak{A}}(q^p, t^p) s_{\mathfrak{A}}(z^p) \right) \geq 0\,. \tag{3.15}$$

Let

$$\Phi_{\vec{d}}(y;\, q, t) = \sum_{\mathfrak{A}\in\mathcal{P}(\mathcal{P}^n),\, \|\mathfrak{A}\|=\vec{d}} \theta_{\mathfrak{A}} W_{\mathfrak{A}}(q,t) s_{\mathfrak{A}}(z)\,.$$

The following inequality can be obtained from (3.15):

$$\mathrm{Ord}_p \left(\Phi_{p\vec{d}}(y;\, q, t) - \frac{1}{p} \Phi_{\vec{d}}(y^p; q^p, t^p) \right) \geq 0\,.$$

Therefore,

$$\begin{aligned} T_{\vec{d}} &= q^{|\vec{d}|} \sum_{k|\vec{d}} \frac{\mu(k)}{k} \sum_{\mathfrak{A}\in\mathcal{P}(\mathcal{P}^L),\, \|\mathfrak{A}\|=\vec{d}/k} \theta_{\mathfrak{A}} W_{\mathfrak{A}}(q^k, t^k) s_{\mathfrak{A}}(z^k) \\ &= q^{|\vec{d}|} \sum_{k|\vec{d},\, p\nmid k} \frac{\mu(k)}{k} \left(\Phi_{\vec{d}/k}(y^k;\, q^k, t^k) - \frac{1}{p} \Phi_{\vec{d}/(pk)}(y^{pk};\, q^{pk}, t^{pk}) \right). \end{aligned}$$

This implies

$$\mathrm{Ord}_p T_{\vec{d}} \geq 0\,. \tag{3.16}$$

Combining with (2.5), we thus prove Proposition 3.8.

4. Concluding remarks

In this section, we briefly discuss some interesting problems related to string duality which may be approached through the techniques developed in this paper.

Let $\mathbf{p} = (\mathbf{p}^1, \ldots, \mathbf{p}^L)$, where $\mathbf{p}^\alpha = (p_1^\alpha, p_2^\alpha, \ldots,)$. Defined the following generating series of open Gromov-Witten invariants

$$F_{g,\vec{\mu}}(t, \tau) = \sum_{\beta} K_{g,\vec{\mu}}^{\beta}(\tau) e^{\int_\beta \omega}$$

where ω is the Kähler class of the resolved conifold, τ is the framing parameter and

$$t = e^{\int_{\mathbb{P}^1}\omega}, \quad \text{and} \quad e^{\int_\beta \omega} = t^Q\,.$$

Consider the following generating function

$$F(\mathbf{p};\,u,t;\,\tau) = \sum_{g=0}^{\infty}\sum_{\vec{\mu}} u^{2g-2+\ell(\vec{\mu})} F_{g,\vec{\mu}}(t;\,\tau)\prod_{\alpha=1}^{L} p^{\alpha}_{\mu^\alpha}\,.$$

It satisfies the log cut-and-join equation

$$\frac{\partial F(\mathbf{p};\,u,t;\,\tau)}{\partial \tau} = \frac{u}{2}\sum_{\alpha=1}^{L}\mathfrak{L}_\alpha F(\mathbf{p};\,u,t;\,\tau)\,.$$

Therefore, duality between Chern-Simons theory and open Gromov-Witten theory reduces to verifying the uniqueness of the solution of cut-and-join equation.

Cut-and-join equation for Gromov-Witten side comes from the degeneracy and gluing procedure while uniqueness of cut-and-join system should in principle be obtained from the verification at some initial value. However, it seems very difficult to find a suitable initial value. A new hope might be found in our development of cut-and-join analysis. In the log cut-and-join equation the non-linear terms reveals the important recursion structure. For the uniqueness of cut-and-join equation, it will appear as the vanishing of all non-linear terms. We will put this in our future research.

Our proof of the LMOV conjecture has shed new light on the famous volume conjecture. See for example the discussions in [6]. The cut-and-join analysis we developed in this paper combined with rank-level duality in Chern-Simons theory seems to provide a new way to prove the existence of the limits of quantum invariants.

There are also other open problems related to LMOV conjecture. For example, quantum group invariants satisfy skein relation which must have some implications on topological string side as mentioned in [10]. One could also rephrase a lot of unanswered questions in knot theory in terms of open Gromov-Witten theory. We hope that the relation between knot theory and open Gromov-Witten theory will be explored much more in detail in the future. This will definitely open many new avenues for future research.

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