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Time-periodic solutions of the Einstein's field equations I: general framework

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Abstract In this paper, we develop a new algorithm to find the exact solutions of the Einstein's field equations. Time-periodic solutions are constructed by using the new algorithm. The singularities of the time-periodic solutions are investigated and some new physical phenomena, such as degenerate event horizon and time-periodic event horizon, are found. The applications of these solutions in modern cosmology and general relativity are expected.

Keywords Einstein's field equations, time-periodic solution, Riemann curvature tensor, singularity, event horizon

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1 Introduction

The Einstein's field equations are the fundamental equations in general relativity and cosmology. The general version of the gravitational field equations or the Einstein's field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.1}$$

where $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, Λ is the cosmological constant, G stands for the Newton's gravitational constant, c is the velocity of the light and $T_{\mu\nu}$ is the energy-momentum tensor. In a vacuum, i.e., in regions of space-time in which $T_{\mu\nu} = 0$, the Einstein's field equations (1.1) reduce to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \tag{1.2}$$

or equivalently,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{1.3}$$

In particular, if the cosmological constant Λ vanishes, i.e., $\Lambda = 0$, then equation (1.2) becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \tag{1.4}$$

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or equivalently,

$$R_{\mu\nu} = 0. \tag{1.5}$$

Each of the equations (1.2)–(1.5) can be called the vacuum Einstein's field equations.

The mathematical study on the Einstein's field equations includes, roughly speaking, the following two aspects: (i) establishing the well-posedness theory of solutions; (ii) finding exact solutions with physical background. Up to now, very few results on the well-posedness for the Einstein's field equations have been established. In their classical monograph [4], Christodoulou and Klainerman proved the global nonlinear stability of the Minkowski space for the vacuum Einstein's field equations, i.e. they showed the nonlinear stability of the trivial solution of the vacuum Einstein's field equations. Recently, by using wave coordinates, Lindblad and Rodnianski gave a simpler proof (see [18]). In her Ph.D. thesis [25], Zipser generalizes the result of Christodoulou and Klainerman [4] to the Einstein-Maxwell equations. As for finding exact solutions, many works have been done and many interesting results have been obtained (see, e.g., [3, 22, 2]). In what follows, we will briefly recall some basic facts about the exact solutions of the Einstein's field equations.

The exact solutions are very helpful to understand the theory of general relativity and the universe. The typical examples are the Schwarzschild solution and the Kerr solution (see [3]). These solutions provide two important physical space-times: the Schwarzschild solution describes a stationary, spherically symmetric and asymptotically flat space-time, while the Kerr solution provides a stationary, axisymmetric and asymptotically flat space-time.

The study on exact solutions of the Einstein's field equations has a long history. In December 1915, Schwarzschild discovered the first non-trivial solution to the vacuum Einstein's field equations which is a static solution with zero angular momentum (see [21]). In 1951, Vaidya [23] generalized the Schwarzschild solution to non-vacuum Einstein's field equations, the Vaidya solution is neither static nor stationary, but it can be used to describe the gravitational field of a radiating star. The unique two-parameter family of solutions which describes the space-time around black holes is the Kerr family discovered by Roy Partrick Kerr in 1963 (see [12]). These solutions are very important in studying black holes in the Nature which is just the study of these solutions (see [3]). Various generalizations of the Kerr solution have been done (see, e.g., [22] and [2]). Gowdy [7, 8] constructed a new kind of solutions of the vacuum Einstein's equations, these solutions provide a new type of cosmological model. This model describes a closed inhomogeneous universe, space sections of these universes have either the three-sphere topology S^3 or the wormhole (hypertorus) topology $S^1 \otimes S^2$. Recently, Ori [19] presented a class of curved-spacetime vacuum solutions which develop closed timelike curves at some particular moment, and used these vacuum solutions to construct a time-machine model. The Ori model is regular, asymptotically flat, and topologically trivial.

From the above discussions, we see that the exact solutions play a crucial role in general relativity and cosmology, so it is always interesting to find new exact solutions for the Einstein's equations. Although many interesting and important solutions have been obtained, there are still many fundamental but open problems. One interesting open problem is *if there exists a "time-periodic" solution to the vacuum Einstein's field equations*. One of the main results in this paper is a solution of this problem.

In this paper, we focus on finding the exact solutions of the vacuum Einstein's field equations (1.3) and (1.5). We will present a general framework to find exact solutions. Under this framework, we can construct some interesting and important exact solutions, for example, the time-periodic solution of the vacuum Einstein's field equations. We analyze the singularities of time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times. We find that the new time-periodic solutions have degenerate event horizon and the time-periodic event horizon, which are two new phenomena in space-time geometry. The applications of these solutions and their new properties in modern cosmology and general relativity may be expected.

More precisely, the time-periodic solution to the vacuum Einstein's field equations in the spherical coordinates (t, r, θ, φ) can be written in the following form

$$ds^2 = (dt, dr, d\theta, d\varphi)(\eta_{\mu\nu})(dt, dr, d\theta, d\varphi)^{\mathrm{T}}, \tag{1.6}$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} G & -G+\dfrac{Mr}{r-m} & QK & 0 \\ -G+\dfrac{Mr}{r-m} & G-\dfrac{2Mr}{r-m} & -QK & 0 \\ QK & -QK & -K^2 & 0 \\ 0 & 0 & 0 & -K^2\sin^2\theta \end{pmatrix}, \tag{1.7}$$

in which

$$\begin{cases} G = 1 + 2\varepsilon\Omega^+ \sin\theta\cos(t-r), \\ K = r + m\ln|r-m| + \varepsilon\sin(t-r), \\ M = \Omega^+ \sin\theta, \\ Q = -\dfrac{1}{2}(1+2\sin\theta)\Omega^-. \end{cases} \tag{1.8}$$

In the above, $\varepsilon \in (-\frac{1}{8}, \frac{1}{8})$ and $m \in \mathbb{R}$ are two parameters, and $\Omega^\pm$ are defined by

$$\Omega^\pm = |\tan\theta/2|^{\frac{1}{2}} \pm |\tan\theta/2|^{-\frac{1}{2}}. \tag{1.9}$$

In Section 5 we analyze the singularity behaviors and obtain the following theorem which is one of the main results of this paper,

Theorem 1.1. *The vacuum Einstein's field equations have a time-periodic solution* (1.6)*, this solution describes a regular space-time, which has vanishing Riemannian curvature tensor but is not homogenous and not asymptotically flat. This space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to three event horizons: a degenerate event horizon, a steady event horizon and time-periodic event horizons.*

According to the authors' knowledge, (1.6) gives the first time-periodic solution to the vacuum Einstein's field equations.

We would like to point out that, by using our method, we can rederive almost all known exact solutions to the vacuum Einstein's field equations. Our method can also be used to find exact solutions of the Einstein's field equations in higher dimensions which will be of interests in string theory.

The paper is organized as follows. In Section 2, we present a general version of the Lorentzian metric to the Einstein's field equations. By using the version of the Lorentzian metric presented in Section 2, in Section 3 we describe a new method to construct exact solutions to the Einstein's field equations. Section 4 is devoted to constructing the time-periodic solution to the vacuum Einstein's field equations. In Section 5, we investigate the singularities behavior and the physical properties of the time-periodic solution. Section 6 is devoted to the discussion on other interesting exact solutions. Several general remarks on the method and results presented in this paper are given in Section 7.

2 The Lorentzian metric

The Einstein's field equations are a second-order global hyperbolic system of highly nonlinear partial differential equations with respect to the Lorentzian metric $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$. To solve the Einstein's field equations, one key point is to choose a suitable coordinate system. A good coordinate system can simplify the equations and make them easier to solve. In the study on the Einstein's field equations, there are two famous coordinate systems: harmonic coordinates and the Gaussian coordinates. The harmonic coordinates $\{x^\mu\}$ satisfy

$$g^{\mu\nu}\Gamma^\gamma_{\mu\nu} = 0, \quad \gamma = 0, 1, 2, 3,$$

where $\Gamma^\gamma_{\mu\nu}$ are the Christoffel symbols given by

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\delta} \left\{ \frac{\partial g_{\nu\delta}}{\partial x^\mu} + \frac{\partial g_{\mu\delta}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\delta} \right\},$$

in which $g^{\gamma\delta}$ is the inverse of $g_{\mu\nu}$. This kind of coordinate system plays an important role in the study of the theoretical aspects, such as the well-posedness, geometric and analysis properties of solutions, of the Einstein's field equations (see e.g., [5] and [18]). Another important coordinate system is the Gaussian coordinates. In the Gaussian coordinates, the Lorentzian metric of the (curved) space-time described by the Einstein's field equations can be written, at least locally, as

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{2.1}$$

where $(g_{ij})_{i,j=1}^3$ stands for a family of Riemannian metrics. Kossowski and Kriele [17] showed that any solution to the vacuum Einstein's field equations can be written as the form (2.1). Substituting (2.1) into the vacuum Einstein's field equations leads to

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t} = 0 \tag{2.2}$$

and some constraints (see [14]). In the class of solutions of the equations (2.2), we can further choose some solutions which satisfy other constraint equations, then these solutions solve the vacuum Einstein's field equations (1.5), thus we construct the solutions of the complete vacuum Einstein's field equations (see [10]). Motivated by this method as well as the Schwarzschild solution and the Kerr solution, in this paper we present an effective and very general framework to construct exact solutions of the vacuum Einstein's field equations. Our method and results can be generalized to general gravitational field equations (1.1).

Consider the metric of the following form

$$(g_{\mu\nu}) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & 0 & 0 \\ g_{20} & 0 & g_{22} & 0 \\ g_{30} & 0 & 0 & g_{33} \end{pmatrix}, \tag{2.3}$$

where $g_{\mu\nu}$ are smooth functions of the coordinates (x^0, x^1, x^2, x^3) and satisfy $g_{0i} = g_{i0}$. In the coordinate basis (x^0, x^1, x^2, x^3) the line element reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.4}$$

For simplicity of notations, we denote the coordinates (x^0, x^1, x^2, x^3) by (t, x, y, z) and rewrite (2.3) as

$$(g_{\mu\nu}) = \begin{pmatrix} u & v & p & q \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix}, \tag{2.5}$$

where u, v, p, q, w, ρ and σ are smooth functions of the coordinates (t, x, y, z). It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \triangleq \det(g_{\mu\nu}) = uw\rho\sigma - v^2\rho\sigma - p^2 w\sigma - q^2 w\rho \tag{2.6}$$

and the inverse of $(g_{\mu\nu})$ reads

$$(g^{\mu\nu}) = \frac{1}{g} \begin{pmatrix} w\rho\sigma & -v\rho\sigma & -wp\sigma & -wq\rho \\ -v\rho\sigma & u\rho\sigma - p^2\sigma - q^2\rho & vp\sigma & vq\rho \\ -wp\sigma & vp\sigma & uw\sigma - v^2\sigma - q^2 w & pqw \\ -wq\rho & vq\rho & pqw & uw\rho - v^2\rho - p^2 w \end{pmatrix}. \tag{2.7}$$

Throughout this paper, we assume that

$$g < 0. \tag{H$_1$}$$

On the other hand, it is easy to see that at least two of these functions w, ρ, σ have the same sign. Without loss of generality, we may suppose that ρ and σ keep the same sign, for example,

$$\rho > 0 \quad \text{and} \quad \sigma > 0. \tag{H$_2$}$$

Remark 2.1. (H_1) is a very natural assumption for the Lorentzian metrics, while the assumption (H_2) is motivated by the Minkowski space-time and the Schwarzschild metric.

By a direct calculation, the characteristic polynomial of $g_{\mu\nu}$ reads

$$\begin{aligned} f(\lambda) =& \lambda^4 - (w + u + \sigma + \rho)\,\lambda^3 \\ &+ \left(u\rho + w\rho + w\sigma + \rho\,\sigma + uw + u\sigma - v^2 - q^2 - p^2\right)\lambda^2 \\ &+ \left(p^2\sigma + v^2\rho - uw\rho - u\sigma\,\rho - w\sigma\,\rho + q^2\rho + v^2\sigma + p^2 w - uw\sigma + q^2 w\right)\lambda \\ &+ uw\sigma\,\rho - wq^2\rho - p^2 w\sigma - v^2\sigma\,\rho. \end{aligned} \tag{2.8}$$

It follows from (2.8) that

$$\begin{cases} f(w) = -v^2(\rho - w)(\sigma - w), \\ f(\rho) = -p^2(w - \rho)(\sigma - \rho), \\ f(\sigma) = -q^2(w - \sigma)(\rho - \sigma). \end{cases} \tag{2.9}$$

Note that the assumption (H_1) is equivalent to

$$f(0) = uw\sigma\,\rho - wq^2\rho - p^2 w\sigma - v^2\sigma\,\rho < 0. \tag{2.10}$$

On the other hand, by (2.8) we have

$$f(\lambda) \to \infty \qquad \text{as} \quad |\lambda| \to \infty. \tag{2.11}$$

We divide the discussions into three cases as follows.

Case I. w, ρ, σ are distinct from each other

For simplicity, we assume that

$$w < \rho < \sigma \tag{2.12}$$

always holds in the whole region under consideration. For other cases, we have a similar discussion. In the present situation, it follows from the last two equations in (2.9) that

$$f(\rho) > 0 \quad \text{and} \quad f(\sigma) < 0. \tag{2.13}$$

Thus, combining (2.10) and (2.11), we know that the characteristic polynomial $f(\lambda)$ has four distinct real roots, denoted by λ_i $(i = 1, 2, 3, 4)$, these roots have the same regularity as $(g_{\mu\nu})$ and satisfy

$$\lambda_1 < 0 < \lambda_2 < \rho < \lambda_3 < \sigma < \lambda_4. \tag{2.14}$$

This implies that the metric $(g_{\mu\nu})$ is Lorentzian, provided that the assumptions (H_1) and (H_2) are satisfied.

The other cases (e.g., $\rho < w < \sigma$) can be treated in the the same way.

Remark 2.2. Typical examples of Case I are the Schwarzschild metric, the Kerr metric and the Gowdy metric.

Case II. $w = \rho = \sigma$

In this situation, the roots of the characteristic polynomial $f(\lambda)$ read

$$\begin{cases} \lambda_1 \triangleq \dfrac{w+u}{2} - \dfrac{\sqrt{w^2 - 2\,uw + u^2 + 4\,q^2 + 4\,p^2 + 4\,v^2}}{2}, \\ \lambda_2 \triangleq \dfrac{w+u}{2} + \dfrac{\sqrt{w^2 - 2\,uw + u^2 + 4\,q^2 + 4\,p^2 + 4\,v^2}}{2}, \\ \lambda_3 = \lambda_4 = w. \end{cases} \tag{2.15}$$

Noting the assumptions (H_1) (namely (2.10)), (H_2) and the fact $w = \rho = \sigma$, we have

$$\lambda_1 < 0 < \lambda_2, \quad \lambda_3 = \lambda_4 = w > 0. \tag{2.16}$$

(2.16) implies that the metric $(g_{\mu\nu})$ is Lorentzian, provided that the assumptions (H_1) and (H_2) are satisfied.

Remark 2.3. In this case, the eigenvalues λ_1 and λ_2 are continuous but may not be C^1 smooth.

Remark 2.4. For Case Ⅱ, a typical example is the Minkowski metric, i.e.,

$$(g_{\mu\nu}) = \mathrm{diag}\,\{-1, 1, 1, 1\}.$$

Case Ⅲ. $\rho = \sigma \neq w$ or $\rho = w \neq \sigma$ or $\sigma = w \neq \rho$

We only consider the case: $\rho = \sigma \neq w$. The other cases can be treated similarly.

By (2.10) and (2.11), there exist $\lambda_\pm$ such that

$$\lambda_- = \max\{\lambda \mid f(\lambda) = 0,\ \lambda \in \mathbb{R}^-\} < 0 \tag{2.17}$$

and

$$\lambda_+ = \min\{\lambda \mid f(\lambda) = 0,\ \lambda \in \mathbb{R}^+\} > 0. \tag{2.18}$$

Case Ⅲ-1. $w < \rho = \sigma$

In the present situation, we consider

$$(g^\varepsilon_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & q \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \rho+\varepsilon \end{pmatrix}, \tag{2.19}$$

where $\varepsilon > 0$ is a small parameter. Obviously,

$$w < \rho < \rho + \varepsilon.$$

Similarly to Case I, the roots of the characteristic polynomial $f^\varepsilon(\lambda)$ associated with $g^\varepsilon_{\mu\nu}$ has four distinct real roots, denoted by λ^ε_i $(i = 1, 2, 3, 4)$, these roots have the same regularity as $(g_{\mu\nu})$ and satisfy

$$\lambda^\varepsilon_1 < 0 < \lambda^\varepsilon_2 < \rho < \lambda^\varepsilon_3 < \rho + \varepsilon < \lambda^\varepsilon_4. \tag{2.20}$$

Letting $\varepsilon \to 0$, we get

$$\lambda^\varepsilon_1 \to \lambda_1 \triangleq \lambda_-, \quad \lambda^\varepsilon_2 \to \lambda_2 \triangleq \lambda_+, \quad \lambda^\varepsilon_3 \to \lambda_3, \quad \lambda^\varepsilon_4 \to \lambda_4, \tag{2.21}$$

where $\lambda_{3,4}$ are functions of u, v, p, q, w, ρ and satisfy

$$\lambda_3,\ \lambda_4 \geqslant \lambda_+. \tag{2.22}$$

Thus, we have

$$\lambda_1 < 0 < \lambda_2,\ \lambda_3,\ \lambda_4. \tag{2.23}$$

Moreover, by a direct calculation we observe that there exists at least one of $\lambda_2,\ \lambda_3,\ \lambda_4$ which is ρ. (2.23) implies that the metric $(g_{\mu\nu})$ is Lorentzian, provided that the assumptions (H_1) and (H_2) are satisfied.

Case Ⅲ-2. $\rho = \sigma < w$

In the present situation, we consider

$$(\hat{g}^{\varepsilon}_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & q \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \rho-\varepsilon \end{pmatrix}, \tag{2.24}$$

where $\varepsilon > 0$ is a small parameter. Obviously,

$$\rho - \varepsilon < \rho < w.$$

Similarly to Case I, the characteristic polynomial $\hat{f}^{\varepsilon}(\lambda)$ associated with $\hat{g}^{\varepsilon}_{\mu\nu}$ has four real roots, denoted by $\hat{\lambda}^{\varepsilon}_i$ $(i = 1, 2, 3, 4)$, these roots satisfy

$$\hat{\lambda}^{\varepsilon}_1 < 0 < \hat{\lambda}^{\varepsilon}_2 < \rho < \hat{\lambda}^{\varepsilon}_3 < w < \lambda^{\varepsilon}_4. \tag{2.25}$$

Similarly to Case Ⅲ-1, letting $\varepsilon \to 0$, we get

$$\hat{\lambda}^{\varepsilon}_1 \to \lambda_1 \triangleq \lambda_-, \quad \hat{\lambda}^{\varepsilon}_2 \to \lambda_2 \triangleq \lambda_+, \quad \hat{\lambda}^{\varepsilon}_3 \to \lambda_3, \quad \hat{\lambda}^{\varepsilon}_4 \to \lambda_4, \tag{2.26}$$

where $\lambda_{3,4}$ are functions of u, v, p, q, w, ρ and satisfy (2.22). Therefore,

$$\lambda_1 < 0 < \lambda_2,\ \lambda_3 < w < \lambda_4. \tag{2.27}$$

Moreover, there exists (at least) one of $\lambda_2,\ \lambda_3$ is ρ. (2.27) implies that the metric $(g_{\mu\nu})$ is Lorentzian, provided that the assumptions (H_1) and (H_2) are satisfied.

Remark 2.5. For Case Ⅲ, the typical examples are the Gödel metric (see [6]) and the Ori metric (see [19]).

Summarizing the above discussions, we have the following theorem.

Theorem 2.1. *Under the assumptions* (H_1) *and* (H_2)*, the metric* $(g_{\mu\nu})$ *is Lorentzian.*

Throughout this paper, we always suppose that the assumptions (H_1) and (H_2) are satisfied.

It is rather easy to see that the forms of the metric $(g_{\mu\nu})$ given in (2.3) or (2.5) include the Minkowski metric, the Schwarzschild, the Kerr metric, the Gowdy metric, the Gödel metric as well as the metric associated with the time machine introduced by Ori recently, etc.

On the other hand, for any $(1 + 3)$-dimensional Lorentzian metric $(g_{\mu\nu})$, we have

Theorem 2.2. *By a suitable transformation on* $g_{\mu\nu}$*, any* $(1+3)$*-dimensional Lorentzian metric* $(g_{\mu\nu})$ *can be rewritten in the form* (2.3) *with properties* (H_1) *and* (H_2).

Proof. Since $(g_{\mu\nu})$ is a $(1+3)$-dimensional Lorentzian metric, without loss of generality, we may assume that $\mathcal{G} \triangleq (g_{ij})^3_{i,j=1}$ is Riemannian. Therefore, there exists an invertible matrix $\mathcal{Q} = (q_{i,j})$ such that

$$\mathcal{Q}\mathcal{G}\mathcal{Q}^{\mathrm{T}} = \mathrm{diag}\{\lambda_1,\ \lambda_2,\ \lambda_3\}. \tag{2.28}$$

Thus, there exists an invertible matrix

$$\tilde{\mathcal{Q}} \triangleq \begin{pmatrix} q_{00} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathcal{Q} \end{pmatrix}$$

such that

$$\tilde{\mathcal{Q}} \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tilde{\mathcal{Q}}^T = \begin{pmatrix} \tilde{g}_{00} & \tilde{g}_{01} & \tilde{g}_{02} & \tilde{g}_{03} \\ \tilde{g}_{10} & \tilde{g}_{11} & 0 & 0 \\ \tilde{g}_{20} & 0 & \tilde{g}_{22} & 0 \\ \tilde{g}_{30} & 0 & 0 & \tilde{g}_{33} \end{pmatrix}, \tag{2.29}$$

where $\mathbf{0} = (0, 0, 0)$ and q_{00} is a non-vanishing function of $g_{\mu\nu}$. (2.29) implies that there exists an invertible transformation on $g_{\mu\nu}$ such that the $(1 + 3)$-dimensional Lorentzian metric $(g_{\mu\nu})$ can be rewritten in the form (2.3) with properties (H_1) and (H_2). This proves Theorem 2.2. □

We now consider the solutions of the Einstein's field equations. By Bianchi identities and Theorems 2.1 and 2.2, we believe that the general form of the solutions of the Einstein's field equations takes one of the following forms

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & q \\ v & 0 & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix}, \tag{Type I}$$

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} 0 & v & p & q \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix} \tag{Type II}$$

or

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & 0 \\ v & w & 0 & 0 \\ p & 0 & \rho & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix}. \tag{Type III}$$

For Type I, the assumption (H_1) is equivalent to

$$v \neq 0. \tag{2.30}$$

Therefore, by Theorem 2.1 we have

Theorem 2.3. *If the assumptions* (H_1) *and* (H_2) *are satisfied, namely,*

$$\rho > 0, \quad \sigma > 0 \quad and \quad v \neq 0, \tag{2.31}$$

then the metric $(\eta_{\mu\nu})$ *is Lorentzian.*

For Type II, we have

Theorem 2.4. *If* w, ρ, σ *keep the same sign, then the hypotheses* (H_1) *and* (H_2) *are satisfied, and the metric* $(\eta_{\mu\nu})$ *is Lorentzian.*

Similarly, for Type III we have

Theorem 2.5. *If* u *is negative, namely,* $u < 0$, *and* w, ρ, σ *are positive, then the hypotheses* (H_1) *and* (H_2) *are satisfied, and the metric* $(\eta_{\mu\nu})$ *is Lorentzian.*

We are interested in finding exact solutions of the Einstein's field equations of the above Types I–III.

3 General framework to find exact solutions

The aim of this paper is to construct exact solutions of the vacuum Einstein's field equations. In order to illustrate our method, as an example, we use the Lorentzian metric Type I to construct some interesting exact solutions for the vacuum Einstein's field equations (1.2). More precisely, we use the metric

$$(\eta_{\mu\nu}) \triangleq \begin{pmatrix} u & v & p & q \\ v & 0 & 0 & 0 \\ p & 0 & \rho & 0 \\ q & 0 & 0 & \sigma \end{pmatrix} \tag{3.1}$$

to solve the vacuum Einstein's field equations (1.2), namely,

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R + \Lambda\eta_{\mu\nu} = 0. \tag{3.2}$$

where u, v, p, q, ρ and σ are smooth functions of the coordinates (t, x, y, z). The equations (3.2) can be rewritten as

$$G_{\mu\nu} = -\Lambda\eta_{\mu\nu}, \tag{3.3}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$ stands for the Einstein tensor. Noting (3.1), we have

$$G_{11} = -\frac{1}{2}\left\{\frac{v_x}{v}\left(\frac{\rho_x}{\rho} + \frac{\sigma_x}{\sigma}\right) + \frac{1}{2}\left[\left(\frac{\rho_x}{\rho}\right)^2 + \left(\frac{\sigma_x}{\sigma}\right)^2\right] - \left(\frac{\rho_{xx}}{\rho} + \frac{\sigma_{xx}}{\sigma}\right)\right\}, \tag{3.4}$$

One of the equations (3.2) reads

$$G_{11} = -\Lambda\eta_{11}, \tag{3.5}$$

namely,

$$\frac{v_x}{v}\left(\frac{\rho_x}{\rho} + \frac{\sigma_x}{\sigma}\right) + \frac{1}{2}\left[\left(\frac{\rho_x}{\rho}\right)^2 + \left(\frac{\sigma_x}{\sigma}\right)^2\right] - \left(\frac{\rho_{xx}}{\rho} + \frac{\sigma_{xx}}{\sigma}\right) = 0. \tag{3.6}$$

We assume that

$$(\rho\sigma)_x \neq 0. \tag{H$_3$}$$

Solving the ODE (3.6) gives

$$v = v_0 \exp\left\{\int \left[\frac{\rho_{xx}}{\rho} + \frac{\sigma_{xx}}{\sigma} - \frac{1}{2}\left(\frac{\rho_x}{\rho}\right)^2 - \frac{1}{2}\left(\frac{\sigma_x}{\sigma}\right)^2\right]\frac{\rho\sigma}{(\rho\sigma)_x}dx\right\}, \tag{3.7}$$

where $v_0 = v_0(t, y, z)$ is an integral function depending on t, y and z. In particular, by taking the ansatz

$$\rho = \tilde{\rho}(t, y, z)\exp\{2f(t, x)\}, \quad \sigma = \tilde{\sigma}(t, y, z)\exp\{2f(t, x)\}, \tag{3.8}$$

(3.7) becomes

$$v = v_0 f_x e^f. \tag{3.9}$$

We now calculate G_{12}.

Noting the ansatz (3.8) and (3.9), we have

$$G_{12} = -\frac{1}{2v_0}\left\{\frac{p_{xx}}{f_x e^f} - \frac{p_x(f_{xx} + f_x^2)}{f_x^2 e^f} - \frac{2pf_x}{e^f} + 2\frac{\partial v_0}{\partial y}f_x\right\}. \tag{3.10}$$

By (3.2), we have $G_{12} = 0$, namely,

$$\frac{p_{xx}}{f_x e^f} - \frac{p_x(f_{xx} + f_x^2)}{f_x^2 e^f} - \frac{2pf_x}{e^f} + 2\frac{\partial v_0}{\partial y}f_x = 0. \tag{3.11}$$

(3.11) is an ODE for the unknown p. Solving (3.11), we can obtain

$$p = p(f(t, x), v_0(t, y, z), p_0(t, y, z), p_1(t, y, z)), \tag{3.12}$$

where p depends on f, v_0 and two arbitrary integral functions p_0 and p_1, which depends on (t, y, z).

Similarly, we solve $G_{13} = 0$ and get

$$q = q(f(t, x), v_1(t, y, z), q_0(t, y, z), q_1(t, y, z)). \tag{3.13}$$

Solving $G_{23} = 0$, we can get the relations on the functions $p_0, q_0, \tilde{\rho}$ and $\tilde{\sigma}$

$$R(p_0, q_0, \tilde{\rho}, \tilde{\sigma}) = 0 \quad \text{and} \quad q_1 = p_1 = 0. \tag{3.14}$$

On the other hand, solving $G_{22} = -\Lambda\rho$, and $G_{33} = -\Lambda\sigma$, namely

$$G_{22} = -\Lambda\tilde{\rho}\exp\{2f\}, \qquad G_{33} = -\Lambda\tilde{\sigma}\exp\{2f\}, \tag{3.15}$$

we can get one relation (because of the symmetry on y and z) about $f, p_0, q_0, \tilde{\rho}, \tilde{\sigma}$, denoted by

$$R_2(p_0, q_0, f, \tilde{\rho}, \tilde{\sigma}) = 0. \tag{3.16}$$

We next solve $G_{01} = -\Lambda v$, i.e., $G_{01} = -\Lambda v_0 f_x e^f$ and obtain $u = u(p_0, q_0, \tilde{\rho}, \tilde{\sigma}, f, u_0, u_1)$, where u_0, u_1 are two arbitrary integral functions depending on (t, y, z).

We now solve $G_{02} = -\Lambda p$ and $G_{03} = -\Lambda q$, namely,

$$G_{02} = -\Lambda p(v_0, f, p_0, q_0) \quad \text{and} \quad G_{03} = -\Lambda q(v_0, f, p_0, q_0). \tag{3.17}$$

This gives two relations

$$R_i(p_0, q_0, f, v_0, \tilde{\rho}, \tilde{\sigma}, u_0, u_1) = 0, \quad i = 3, 4. \tag{3.18}$$

Finally, we solve $G_{00} = -\Lambda u$ and get the following relation

$$R_5(p_0, q_0, f, v_0, \tilde{\rho}, \tilde{\sigma}, u_0, u_1) = 0. \tag{3.19}$$

Thus, we can successfully solve the full Einstein's field equations (3.2).

At the end of this section, for Type I metrics, our method can be described by the following algorithm:

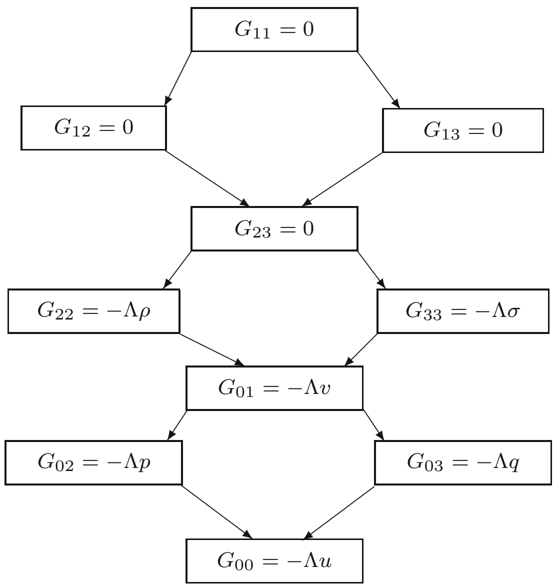

Figure 1 The algorithm to construct the exact solutions

For Type II and Type III metrics, we can develop a similar algorithm. Here we omit the details.

At the end of this section, we would like to point out the following remarks:

Remark 3.1. Our method can be used to solve the Einstein's field equations with physically relevant energy-momentum tensors, e.g., the tensor for perfect fluid: $T_{\gamma\delta} = (\mu + p)u_\gamma u_\delta + pg_{\gamma\delta}$, where $\mu > 0$ is the density, p is the pressure, u stands for the space-time velocity of the fluid with $u_\gamma u^\gamma = -1$.

Remark 3.2. The method presented in this paper can also be used to solve the Einstein's field equations in higher space-time dimensions.

4 Time-periodic solutions

By the method presented in last section, we can construct many new exact solutions to the vacuum Einstein's field equations

$$G_{\mu\nu} = 0. \tag{4.1}$$

For example, in the coordinates $(\tau, \widetilde{r}, \widetilde{\theta}, \widetilde{\varphi})$, taking the ansatz in (3.8) as follows

$$\widetilde{\rho} = -1, \quad \widetilde{\sigma} = -\sin^2\widetilde{\theta}, \quad f = \ln[\widetilde{r} + \tau + \varepsilon\sin\tau],$$

one can obtain an interesting solution with the form

$$\widetilde{\eta}_{\mu\nu} = \begin{pmatrix} \widetilde{\eta}_{00} & \widetilde{\eta}_{01} & \widetilde{\eta}_{02} & 0 \\ \widetilde{\eta}_{01} & 0 & 0 & 0 \\ \widetilde{\eta}_{02} & 0 & \widetilde{\eta}_{22} & 0 \\ 0 & 0 & 0 & \widetilde{\eta}_{33} \end{pmatrix}, \tag{4.2}$$

where

$$\begin{cases} \widetilde{\eta}_{00} = 1 + 2\varepsilon\widetilde{\Omega}^+\sin\widetilde{\theta}\cos\tau + \dfrac{2\widetilde{\Omega}^+\sin\widetilde{\theta}}{m}\left\{\exp\left(\dfrac{\widetilde{r}+\tau}{m}\right) + m\right\}, \\ \widetilde{\eta}_{01} = \dfrac{\widetilde{\Omega}^+\sin\widetilde{\theta}}{m}\left\{\exp\left(\dfrac{\widetilde{r}+\tau}{m}\right) + m\right\}, \\ \widetilde{\eta}_{02} = \dfrac{1}{2}(1 + 2\sin\widetilde{\theta})\widetilde{\Omega}^-\left[\widetilde{r} + \tau + \varepsilon\sin\tau\right], \\ \widetilde{\eta}_{22} = -\left[\widetilde{r} + \tau + \varepsilon\sin\tau\right]^2, \\ \widetilde{\eta}_{33} = -\left[\widetilde{r} + \tau + \varepsilon\sin\tau\right]^2\sin^2\widetilde{\theta}, \end{cases} \tag{4.3}$$

in which ε is a parameter, $m \neq 0$ is a constant, and

$$\widetilde{\Omega}^+ = |\tan\widetilde{\theta}/2|^{\frac{1}{2}} + |\tan\widetilde{\theta}/2|^{-\frac{1}{2}}, \quad \widetilde{\Omega}^- = |\tan\widetilde{\theta}/2|^{\frac{1}{2}} - |\tan\widetilde{\theta}/2|^{-\frac{1}{2}}. \tag{4.4}$$

The solution (4.2) belongs to the class of type I, also belongs to the class of type Ⅲ.

Remark 4.1. When $\sin\widetilde{\theta} = \pm 1$, (2.12) is not satisfied. This is due to the choice of the polar coordinates. In this case, the resulting singularities are not essential.

Making the transformation

$$\begin{cases} t = \tau + \widetilde{r}, \\ r = m + \exp\left(\dfrac{\tau + \widetilde{r}}{m}\right), \\ \theta = \widetilde{\theta}, \\ \varphi = \widetilde{\varphi}. \end{cases} \tag{4.5}$$

Then the solution to (4.1) becomes, in the coordinates (t, r, θ, φ),

$$ds^2 = (dt, dr, d\theta, d\varphi)(\eta_{\mu\nu})(dt, dr, d\theta, d\varphi)^T, \tag{4.6}$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} G & -G + \dfrac{Mr}{r-m} & QK & 0 \\ -G + \dfrac{Mr}{r-m} & G - \dfrac{2Mr}{r-m} & -QK & 0 \\ QK & -QK & -K^2 & 0 \\ 0 & 0 & 0 & -K^2\sin^2\theta \end{pmatrix}, \tag{4.7}$$

in which

$$\begin{cases} G = 1 + 2\varepsilon\Omega^+ \sin\theta\cos(t-r), \\ K = r + m\ln|r-m| + \varepsilon\sin(t-r), \\ M = \Omega^+\sin\theta, \\ Q = -\dfrac{1}{2}(1+2\sin\theta)\Omega^-. \end{cases} \tag{4.8}$$

In (4.8), $\Omega^\pm$ are defined by

$$\Omega^\pm = |\tan\theta/2|^{\frac{1}{2}} \pm |\tan\theta/2|^{-\frac{1}{2}}. \tag{4.9}$$

An important property of the space-time described by (4.6) is given by the following theorem.

Theorem 4.1. *When ε takes its value in the interval $(-\frac{1}{8},\frac{1}{8})$, i.e., $\varepsilon\in(-\frac{1}{8},\frac{1}{8})$, the solution (4.6) to the vacuum Einstein's field equations is time-periodic.*

Proof. Noting $\varepsilon\in(-\frac{1}{8},\frac{1}{8})$, we have

$$\begin{aligned} \eta_{00} &= G = 1 + 2\varepsilon\Omega^+\sin\theta\cos(t-r) \\ &= 1 + 4\varepsilon\left(\sqrt{\left|\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right|} + \sqrt{\left|\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right|}\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos(t-r) \\ &\geqslant 1 - 4|\varepsilon|\left(\sqrt{\left|\sin^3\frac{\theta}{2}\right|\left|\cos\frac{\theta}{2}\right|} + \sqrt{\left|\cos^3\frac{\theta}{2}\right|\left|\sin\frac{\theta}{2}\right|}\right)|\cos(t-r)| \\ &\geqslant 1 - 8|\varepsilon| > 1 - 8\times\frac{1}{8} = 0. \end{aligned} \tag{4.10}$$

On the other hand, by calculations we have

$$\begin{vmatrix} G & -G + \dfrac{Mr}{r-m} \\ -G + \dfrac{Mr}{r-m} & G - \dfrac{2Mr}{r-m} \end{vmatrix} = -\frac{(Mr)^2}{(r-m)^2} < 0 \tag{4.11}$$

for $r\neq 0, m$ and $\theta\neq 0,\pi$,

$$\begin{vmatrix} G & -G + \dfrac{Mr}{r-m} & QK \\ -G + \dfrac{Mr}{r-m} & G - \dfrac{2Mr}{r-m} & -QK \\ QK & -QK & -K^2 \end{vmatrix} = K^2\frac{(Mr)^2}{(r-m)^2} > 0 \tag{4.12}$$

and

$$\begin{vmatrix} G & -G + \dfrac{Mr}{r-m} & QK & 0 \\ -G + \dfrac{Mr}{r-m} & G - \dfrac{2Mr}{r-m} & -QK & 0 \\ QK & -QK & -K^2 & 0 \\ 0 & 0 & 0 & -K^2\sin^2\theta \end{vmatrix} = -K^4\sin^2\theta\frac{(Mr)^2}{(r-m)^2} < 0 \tag{4.13}$$

for $r\neq 0, m$, $\theta\neq 0,\pi$ and $K\neq 0$. In next section, we will show that $\theta = 0,\pi$, $r = 0, m$ and $\{(t,r)|K = K(t,r) = 0\}$ are not essential singularities, the singularities at $\theta = 0,\pi$ are due to the use of the polar coordinates, the singularities $r = 0, m$ and $\{(t,r)|K = K(t,r) = 0\}$ correspond to the event horizons which can be removed by making suitable transformation of variables.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (4.7) and (4.8) that the Lorentzian metric (4.6) is indeed a time-periodic solution of the vacuum Einstein's field equations. This proves Theorem 4.1. □

Remark 4.2. The time-periodic solution (4.6) was first obtained by us in early 2007. The authors have presented this solution in several conferences for the past year. As mentioned above, according to the authors' knowledge, this is the first time-periodic solution to the vacuum Einstein's field equations.

Remark 4.3. In July 2006, the first author discussed with Dr Gu about some ideas in the present paper. We noted that for the case of Type I metric, Gu [9] constructed some interesting exact solutions, however none of which is time-periodic, because the variable t in these solutions can not be taken as the time coordinate.

Direct computations give us the following property of the time-periodic solutions.

Lemma 4.1. *In the geometry of the space-time* (4.6)*, it holds that*

$$\frac{\partial K}{\partial t}=\frac{G-1}{2M},\quad \frac{\partial^2 K}{\partial t^2}=\frac{1}{2M}\frac{\partial G}{\partial t},\tag{4.14}$$

$$\frac{\partial \Omega^+}{\partial \theta}=\frac{\Omega^-}{2\sin\theta},\quad \frac{\partial \Omega^-}{\partial \theta}=\frac{\Omega^+}{2\sin\theta},\quad \frac{\partial M}{\partial \theta}=Q,\tag{4.15}$$

$$\frac{\partial G}{\partial r}=-\frac{\partial G}{\partial t},\quad \frac{\partial^2 G}{\partial r\partial t}=-\frac{\partial^2 G}{\partial t^2},\quad \frac{\partial^2 G}{\partial \theta\partial r}=-\frac{\partial^2 G}{\partial \theta\partial t},\tag{4.16}$$

$$\frac{\partial K}{\partial t}+\frac{\partial K}{\partial r}=\frac{r}{r-m},\quad \frac{\partial^2 K}{\partial t\partial r}=-\frac{\partial^2 K}{\partial t^2},\quad \frac{\partial^2 K}{\partial \theta\partial r}=-\frac{\partial^2 K}{\partial t\partial \theta},\tag{4.17}$$

$$\frac{\partial Q}{\partial \theta}=Q\cot\theta-\frac{3\Omega^+}{4\sin\theta},\quad Q\frac{\partial K}{\partial t}=\frac{1}{2}\frac{\partial G}{\partial \theta},\quad \frac{\partial^2 G}{\partial \theta^2}=2\frac{\partial K}{\partial t}\frac{\partial Q}{\partial \theta},\tag{4.18}$$

$$\frac{2Q\cos\theta}{\Omega^+}-\frac{3}{4}-\frac{Q^2}{(\Omega^+)^2}-\frac{1}{(\Omega^+)^2}=-\sin^2\theta,\tag{4.19}$$

$$2Q\cot\theta-\frac{3}{4}\frac{\Omega^+}{\sin\theta}-\frac{1+Q^2}{M}=-M.\tag{4.20}$$

Proof. This lemma can be proved by direct calculations. □

Remark 4.4. The relations given in Lemma 4.1 will play an important role in the future study on the geometry of the time-periodic space-time (4.6).

There are several important questions which deserve further study: (1) what is the topological structure of the time-periodic space-time (4.6)? (2) does the space-time (4.6) have a compact Cauchy surface? (3) an important point is to consider the structure of the maximal globally hyperbolic part of the space-time (4.6), the question is *whether this also exhibits time periodicity*. Problems (2) and (3) were suggested by Andersson [1].

5 Singularity behaviors and physical properties

This section is devoted to the analysis of singularities and the physical properties of the time-periodic solution (4.6).

For the metric (4.7), by a direct calculation, we have

$$g\triangleq\det(\eta_{\mu\nu})=-M^2K^4\frac{r^2}{(r-m)^2}\sin^2\theta.\tag{5.1}$$

Combing (4.8) and (5.1) gives

$$g = -(\Omega^+)^2 \sin^4\theta\,[r + m\ln|r-m| + \varepsilon\sin(t-r)]^4\,\frac{r^2}{(r-m)^2}. \tag{5.2}$$

Noting (4.4), we obtain from (5.2) that

$$r = 0, \quad r = m \quad \text{and} \quad r + m\ln|r-m| + \varepsilon\sin(t-r) = 0 \tag{5.3}$$

are singularities for the solution metric (4.6).

Remark 5.1. When $\theta = 0, \pi$, the determinant g of the metric $(\eta_{\mu\nu})$ also vanishes because of the use of polar coordinates. So $\theta = 0, \pi$ are not real singularities.

By a direct calculation, we have

$$R_{\alpha\beta\gamma\delta} = 0 \quad \text{and} \quad R^{\alpha\beta\gamma\delta} = 0, \quad \alpha,\beta,\gamma,\delta = 0,1,2,3. \tag{5.4}$$

This gives

$$R_{\mu\nu} = 0, \quad \mu,\nu = 0,1,2,3 \tag{5.5}$$

and

$$\|\mathbf{R}\| \triangleq R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 0. \tag{5.6}$$

(5.5) implies that the Lorentzian metric (4.6) is indeed a solution of the vacuum Einstein's field equation (4.1), and (5.6) implies that this solution does not have any essential singularity. Therefore, on the one hand, similarly to the Schwartzschild space-time, the cases $r = m$ and $r + m\ln|r-m| + \varepsilon\sin(t-r) = 0$ correspond to the *event horizon*; on the other hand, in contrast to the Schwartzschild space-time, the case $r = 0$ also corresponds to the "event horizon", which degenerates to a point. It is well-known that, in the Schwartzschild space-time, $r = 0$ corresponds to the *black hole*. In other words, the solution (4.6) describes an essentially regular space-time, it does not contain any essential singularity like black hole. It is an interesting topic to see how to cancel this kind of singularities by making some coordinate transformation. Therefore, we have

Property 5.1. The Lorentzian metric (4.6) describes a regular space-time, this space-time is Riemannian flat in the sense of (5.4), it does not contain any essential singularity. However it contains some non-essential singularities which correspond to event horizons.

Property 5.2. The non-essential singularities of the space-time (4.6) consist of three parts $r = 0$, $r = m$ and $r + m\ln|r-m| + \varepsilon\sin(t-r) = 0$. The case $r = 0$ is a degenerate event horizon; $r = m$ is a steady event horizon; and $r + m\ln|r-m| + \varepsilon\sin(t-r) = 0$ is the "time-periodic" event horizon.

According to the authors' knowledge, the degenerate event horizon and the time-periodic event horizon are two new phenomena in space-time geometry.

We next consider the time-periodic event horizon in detail. Without loss of generality, we may assume that ε and m are positive constants. We have two cases: $0 \leqslant r \leqslant m$ and $r > m$.

Case I. $0 \leqslant r \leqslant m$

Let

$$f(r;t) = r + m\ln|m-r| + \varepsilon\sin(t-r). \tag{5.7}$$

For any fixed $t \in \mathbb{R}$, it holds that

$$f(r;t) \longrightarrow -\infty \quad \text{as} \ \ r \to m \tag{5.8}$$

and

$$f(r;t) \longrightarrow \infty \quad \text{as} \ \ r \to \infty. \tag{5.9}$$

At $r = 0$, we consider

$$m\ln m + \varepsilon\sin t = 0, \tag{5.10}$$

i.e.,

$$\sin t = -\frac{m\ln m}{\varepsilon}. \tag{5.11}$$

It is obvious that (5.11) has a solution if and only if

$$\left|\frac{m\ln m}{\varepsilon}\right| \leqslant 1. \tag{5.12}$$

In what follows, we always assume the condition (5.12). Therefore, it follows from (5.11) that

$$f(0;t_k)=0, \tag{5.13}$$

where

$$t_k = 2k\pi + \arcsin\left\{-\frac{m\ln m}{\varepsilon}\right\}, \tag{5.14}$$

in which $k\in\mathbb{Z}$.

We now divide the discussion into two cases.

Case I-1. $0<m\leqslant 1$

In this case, we have

$$-\frac{m\ln m}{\varepsilon}\geqslant 0, \tag{5.15}$$

and then

$$\arcsin\left\{-\frac{m\ln m}{\varepsilon}\right\}\geqslant 0. \tag{5.16}$$

Therefore,

$$t_k = 2k\pi + \arcsin\left\{-\frac{m\ln m}{\varepsilon}\right\}, \quad k=0,1,2,\ldots. \tag{5.17}$$

Case I-2. $1<m$

In this case,

$$-\frac{m\ln m}{\varepsilon}<0, \tag{5.18}$$

and

$$\arcsin\left\{-\frac{m\ln m}{\varepsilon}\right\}< 0. \tag{5.19}$$

Thus, we shall take

$$t_k = 2(k+1)\pi + \arcsin\left\{-\frac{m\ln m}{\varepsilon}\right\}, \quad k=0,1,2,\ldots. \tag{5.20}$$

In both Case I-1 and Case I-2, by fixing $k\in\{0,1,2,\ldots\}$ and noting (5.8), we see that there exists a maximum $r_-\in[0,m)$ such that, for any given $r\in[0,r_-]$, the equation for t

$$f(r;t)=0 \tag{5.21}$$

has solutions. When $r=r_-$, we denote the solution by t_k^-. It holds that

$$f(r_-;t_k^-)=0. \tag{5.22}$$

Summarizing the above discussion, we observe that, for Case I, the time-periodic event horizons are given in Figure 2.

Case II. $r>m$

Similarly to the discussion of Case I, in this case the time-periodic event horizons are given in Figure 3.

In Figure 3, r_0 and r_+ are defined in the following way: noting (5.8) and (5.9), we see that there exists a minimum $r_0\in(m,\infty)$ and a maximum $r_+\in(m,\infty)$ such that, for any given $r\in[r_0,r_+]$, the equation for t

$$f(r;t)=0 \tag{5.23}$$

has solutions. In particular, when $r=r_0$ (resp. $r=r_+$), we denote the solution by t_k (resp. by t_k^+). That is to say, it holds that

$$f(r_0;t_k)=0 \quad\text{and}\quad f(r_+;t_k^+)=0. \tag{5.24}$$

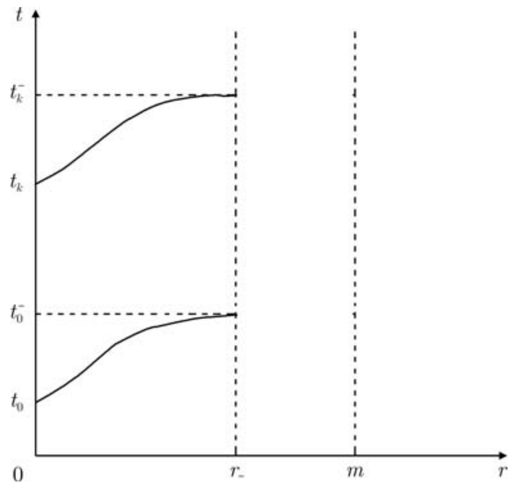

Figure 2 Time-periodic event horizons for Case I

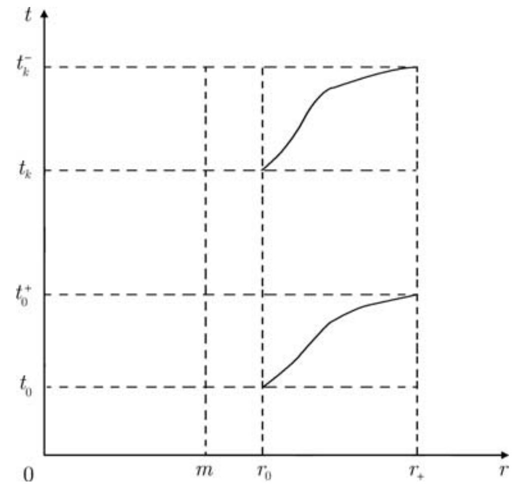

Figure 3 Time-periodic event horizons for Case II

Therefore, we have proved the following property.

Property 5.3. The non-essential singularities of the space-time (4.6) consists of three parts $r = 0$, $r = m$ and $r+m\ln|r-m|+\varepsilon\sin(t-r) = 0$. $r = 0$ is a degenerate event horizon, $r = m$ is a steady event horizon, and $r + m\ln|r-m| + \varepsilon\sin(t-r) = 0$ are the "time-periodic" event horizons. Time-periodic event horizons form and disappear in finite times, they propagate time-periodically.

On the other hand, by some elementary matrix transformations, the metric $(\eta_{\mu\nu})$ can be reduced to

$$(\hat{\eta}_{\mu\nu}) = \mathrm{diag}\left\{G, -\frac{M^2r^2}{G(r-m)^2}, -K^2, -K^2\sin^2\theta\right\}. \tag{5.25}$$

Note that (4.8) gives

$$(\hat{\eta}_{\mu\nu}) \sim \mathrm{diag}\left\{1+2\varepsilon\Omega^+\sin\theta\cos(t-r), -\frac{(\Omega^+)^2\sin^2\theta}{1+2\varepsilon\Omega^+\sin\theta\cos(t-r)}, -r^2, -r^2\sin^2\theta\right\}. \tag{5.26}$$

In (5.26), we have made use of the fact that, when r is large enough, it holds that $K \sim r$ because of the second equation in (4.8). (5.26) implies that the space-time (4.6) is not homogenous and asymptotically flat, more precisely not asymptotically Minkowski, because the first component in (5.26) depends strongly on the angle θ. Therefore, we have

Property 5.4. The space-time (4.6) is not homogenous and asymptotically flat.

Remark 5.2. Property 5.4 perhaps has some new applications in cosmology due to the recent WMAP data, since the recent WMAP data show that our Universe exists anisotropy (see [11]). This inhomogenous property of the new space-time (4.6) may provide a way to give an explanation of this phenomena.

Summarizing the above discussion gives the following theorem.

Theorem 5.1. *The vacuum Einstein's field equations have a time-periodic solution* (4.6)*, this solution describes a regular space-time, which has vanishing Riemann curvature tensor but is not homogenous and not asymptotically flat. This space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to three event horizons: a degenerate event horizon, a steady event horizon and time-periodic event horizons.*

6 Other examples

In this section, we illustrate that our method can be used to obtain other important solutions to the vacuum Einstein's field equations. Here we only briefly account for the known solutions, we will discuss in more details of some new solutions in the forthcoming papers [15, 16].

Ori's time-machine model. Ori [19] presented a class of curved-spacetime vacuum solutions which develop closed timelike curves at some particular moment, and then used these vacuum solutions to construct a time-machine model. His solution is given by

$$ds^2 = dx^2 + dy^2 - 2dzdt + [f(x,y,z) - t]dz^2, \tag{6.1}$$

where f is an arbitrary (properly periodic in z) satisfying $f_{xx} + f_{yy} = 0$. In the coordinates (z,t,x,y), the metric (6.1) belongs to both Type I and Type III metrics. The solution (6.1) can be obtained easily by using our method.

Gowdy's universes. In 1971 and 1975, Gowdy [7, 8] gave the Gowdy universes in the form

$$ds^2 = e^{-2U}\left[e^{2k}(d\rho^2 - dt^2) + \sin^2 t \sin^2 \rho d\varphi^2\right] + e^{2U}dz^2, \tag{6.2}$$

where U staisfies

$$(U_\rho \sin\rho)_\rho \sin t - (U_t \sin t)_t \sin\rho = 0. \tag{6.3}$$

In the coordinates (t,ρ,φ,z), the metric (6.2) belongs to the class of Type III. The solution (6.2) can be easily obtained by our method.

Khan-Penrose's solution. In 1971, Khan-Penrose [13] constructed a vacuum solution to the Einstein's field equations of the form

$$ds^2 = \frac{1-\eta^2}{(1-\eta^2)^{1/4}(1-\mu^2)^{1/4}}\left(\frac{d\mu^2}{1-\mu^2} - \frac{d\eta^2}{1-\eta^2}\right) + \sqrt{(1-\eta^2)(1-\mu^2)}\left(\frac{1-\eta}{1+\eta}dx^2 + \frac{1+\eta}{1-\eta}dy^2\right). \tag{6.4}$$

In the coordinates (η,μ,x,y), the metric (6.4) belongs to the class of Type III. Using our method, we can easily obtain the solution (6.4).

In fact, many classical exact solutions to the Einstein's field equations, for example, the Schwarzschild solution, the Kerr solution, etc. can be obtained by the method presented in this paper. Here we omit the details.

7 Summary and discussion

In this paper we present a general framework to find exact solutions to the Einstein's field equations (1.1). Using our method, we can construct some important exact solutions including the time-periodic solutions of the vacuum Einstein's field equations. These solutions enjoy some interesting properties, for example, although these solutions possess singularities, their Riemann curvature tensors vanish at any point in the these space-times, this implies that these space-times are Riemann flat. We also analyze the singularities of the time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times. In the series of works [15, 16], we will construct time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors keep finite or take the infinity at some points in these space-times, respectively. More precisely, the solutions constructed in [15, 16] contain *geometric singularities* or *physical singularities* or both singularities (see [15, 16] for the details).

We remark that, by using our method, we can obtain almost all known solutions to the Einstein's field equations. Our method can also be used to find exact solutions of the higher dimensional Einstein's field equations, which play an important role in string theory. The structures of these new space-times, the behaviors of their singularities and some new nonlinear phenomena appeared in the time-periodic solutions are very interesting and important. We expect some applications of these new phenomena and the time-periodic solutions in modern cosmology and general relativity.

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